FROM JANTZEN TO ANDERSEN FILTRATION VIA TILTING EQUIVALENCE

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ABSTRACT. The space of homomorphisms from a projective object to a Verma module in category $\mathcal{O}$ inherits an induced filtration from the Jantzen filtration on the Verma module. On the other hand there is the Andersen filtration on the space of homomorphisms from a Verma module to a tilting module as described in [9]. The tilting equivalence from [8] induces an isomorphism of these kinds of Hom-spaces. We will show that this equivalence even identifies both filtrations.

1. INTRODUCTION

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra with a Borel and a Cartan. In BGG-category $\mathcal{O}$ prominent objects are indecomposable projective and tilting modules.

Tilting modules were introduced in [2] as self dual Verma flag modules and the indecomposable tilting modules are classified by their highest weight. Let $\rho \in \mathfrak{h}^*$ be the halfsum of positive roots induced by $\mathfrak{b}$ and let $T = \mathbb{C}[\mathbb{C}\rho]$ denote the ring of regular functions on the line $\mathbb{C}\rho \subset \mathfrak{h}^*$. T is a quotient of the universal enveloping algebra $U(\mathfrak{h}) = S\mathfrak{h}^* =: S$. For a weight $\lambda \in \mathfrak{h}^*$ we get a $\mathfrak{b}$-module structure on $\mathbb{C}$ via the obvious map $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$. We denote this $\mathfrak{b}$-module by $\mathbb{C}_\lambda$. Now we can form the Verma module $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \in \mathfrak{g}$-mod and the deformed Verma module

$$\Delta_T(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes T) \in \mathfrak{g} - \mathrm{mod} - T$$

where tensor products without any specification are to be understood over $\mathbb{C}$. The T-structure on $\Delta_T(\lambda)$ is just multiplication from the right while the $\mathfrak{b}$-structure on $\mathbb{C}_\lambda \otimes T$ is the tensor product representation.

Taking the direct sum of the T-dual weight spaces and twisting the contragredient $\mathfrak{g}$-structure with a Chevalley automorphism, leads to the deformed dual Verma module $\nabla_T(\lambda) \in \mathfrak{g}$-mod-$T$. We will see that sending a generator of $\Delta_T(\lambda)$ to a generator of $\nabla_T(\lambda)$ induces an injective map

$$\mathrm{can} : \Delta_T(\lambda) \hookrightarrow \nabla_T(\lambda)$$

which forms a basis of the T-module $\mathrm{Hom}_{\mathfrak{g}-T}(\Delta_T(\lambda), \nabla_T(\lambda))$. Since T can be understood as a polynomial ring in one variable v we get the Jantzen filtration on $\Delta(\lambda)$ by taking the images of $\mathrm{can}^{-1}(\nabla_T(\lambda)v^i)$ for $i = 0, 1, 2, 3, ...$ under the surjection $\Delta_T(\lambda) \twoheadrightarrow \Delta(\lambda)$. For a projective object $P \in \mathcal{O}$ we get an induced filtration on $\mathrm{Hom}_{\mathfrak{g}}(P, \Delta(\lambda))$.

Let $\widehat{T}$ be the completion of T at the maximal ideal of 0. In this article we will also

introduce deformed tilting modules, which are certain $\mathfrak{g}$-$\widehat{T}$-bimodules corresponding to tilting modules in $\mathcal{O}$ after specializing with $\cdot \otimes_{\widehat{T}} \mathbb{C}$. Now let K be such a deformed tilting module, and consider the composition pairing

$$\operatorname{Hom}(\Delta_{\widehat{T}}(\lambda), K) \times \operatorname{Hom}(K, \nabla_{\widehat{T}}(\lambda)) \to \operatorname{Hom}(\Delta_{\widehat{T}}(\lambda), \nabla_{\widehat{T}}(\lambda)) \cong \widehat{T}$$

where all Hom-spaces are meant to be homomorphisms of $\mathfrak{g}$-$\widehat{T}$-bimodules. We will see that this is a nondegenerate pairing of free T-modules of finite rank and leads to an injection

$$\operatorname{Hom}(\Delta_{\widehat{T}}(\lambda), K) \hookrightarrow (\operatorname{Hom}(K, \nabla_{\widehat{T}}(\lambda)))^*$$

where $(\cdot)^*$ denotes the $\widehat{T}$-dual.

Taking the preimages of the $\widehat{T}$-submodules $(\operatorname{Hom}(K, \nabla_{\widehat{T}}(\lambda)))^* \cdot v^i$ under this embedding and applying $\cdot \otimes_{\widehat{T}} \mathbb{C}$ to these preimages defines the Andersen filtration on $\operatorname{Hom}_{\mathfrak{g}}(\Delta(\lambda), K \otimes_{\widehat{T}} \mathbb{C})$.

In [8] Soergel introduces the tilting functor t, which forms a contravariant equivalence of the category of modules with a Verma flag. The functor t takes projective modules to tilting modules and sends a Verma module $\Delta(\nu)$ to the Verma module $\Delta(-2\rho - \nu)$. So t induces an isomorphism of vector spaces

$$\operatorname{Hom}_{\mathfrak{g}}(P, \Delta(\lambda)) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{g}}(\Delta(-2\rho - \lambda), t(P))$$

which we denote by t as well.

In this paper we will prove that t even identifies the filtration induced by the Jantzen filtration on the left side with the Andersen filtration on the right side. With the arguments given in [9], this might give an alternative proof of the Jantzen conjectures about the semisimplicity of the subquotients of the Jantzen filtration to the one given in [1].

2. Preliminaries

This section contains some results about the deformed category $\mathcal{O}$ of a semisimple complex Lie algebra $\mathfrak{g}$ with Borel $\mathfrak{b}$ and Cartan $\mathfrak{h}$, which one can also find in [3] and [9]. By S we will denote the universal enveloping algebra of the Cartan $\mathfrak{h}$, which is equal to the ring of polynomial functions $\mathbb{C}[\mathfrak{h}^*]$. Let T be a commutative, associative, noetherian, unital, local S-algebra with structure morphism $\tau : S \to T$. We call T a *local deformation algebra*.

In this article we will mostly deal with the S-algebras $R = S_{(0)}$, the localisation of S at the maximal ideal of $0 \in \mathfrak{h}^*$, localisations $R_{\mathfrak{p}}$ of R at a prime ideal $\mathfrak{p}$ of height 1 or the residue fields of these rings $\mathbb{K}_{\mathfrak{p}} = R_{\mathfrak{p}}/R_{\mathfrak{p}}\mathfrak{p}$. To apply results of this section to both filtrations we will also be concerned with the power series ring $\mathbb{C}[[t]]$ in one variable and the quotient field Q of S. All of these rings are local deformation algebras.

2.1. Deformed category $\mathcal{O}$.

Let T be a local deformation algebra with structure morphism $\tau : S \to T$ and let $M \in \mathfrak{g}\text{-mod-}T$. For $\lambda \in \mathfrak{h}^*$ we set

$$M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \;\; \forall h \in \mathfrak{h}\}$$

where $(\lambda + \tau)(h)$ is meant to be an element of T.

We denote by $\mathcal{O}_T$ the full subcategory of all bimodules $M \in \mathfrak{g}$-mod-T such that $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, and with the properties that for every $m \in M$ the $\mathfrak{b}$-T-bimodule generated by m is finitely generated as a T-module and that M is finitely generated as a $\mathfrak{g}$-T-bimodule. For example for $T = \mathbb{C}$ we get for $\mathcal{O}_T$ just the usual BGG-category $\mathcal{O}$.

For $\lambda \in \mathfrak{h}^*$ we define the *deformed Verma module*

$$\Delta_T(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} T_\lambda$$

where T_λ denotes the $U(\mathfrak{b})$-T-bimodule T with $\mathfrak{b}$-structure given by the composition $U(\mathfrak{b}) \to S \xrightarrow{\lambda + \tau} T$.

As in [9] we now introduce a functor

$$d = d_\tau : \mathfrak{g} \otimes T - \mathrm{mod} \longrightarrow \mathfrak{g} \otimes T - \mathrm{mod}$$

by letting $dM \subset \mathrm{Hom}_T(M, T)^\sigma$ be the sum of all deformed weight spaces in the space of homomorphisms of T-modules from M to T with its $\mathfrak{g}$-action twisted by an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ with $\sigma|_\mathfrak{h} = -\mathrm{id}$. We now set $\nabla_T(\lambda) = d\Delta_T(\lambda)$ for $\lambda \in \mathfrak{h}^*$. As in [9] one shows $d\nabla_T(\lambda) \cong \Delta_T(\lambda)$ and that tensoring with a finite dimensional representation E of $\mathfrak{g}$ commutes with d up to the choice of an isomorphism $dE \cong E$.

Proposition 2.1 ([9], Proposition 2.12.).   (1) *For all λ the restriction to the deformed weight space of λ together with the two canonical identifications $\Delta_T(\lambda)_\lambda \xrightarrow{\sim} T$ and $\nabla_T(\lambda)_\lambda \xrightarrow{\sim} T$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\lambda), \nabla_T(\lambda)) \xrightarrow{\sim} T$$

(2) *For $\lambda \neq \mu$ in $\mathfrak{h}^*$ we have $\mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\lambda), \nabla_T(\lambda)) = 0$.*

(3) *For all $\lambda, \mu \in \mathfrak{h}^*$ we have $\mathrm{Ext}^1_{\mathcal{O}_T}(\Delta_T(\lambda), \nabla_T(\lambda)) = 0$.*

Corollary 2.2 ([9], Corollary 2.13.). *Let $M, N \in \mathcal{O}_T$. If M has a Δ_T-flag and N a ∇_T-flag, then the space of homomorphisms $\mathrm{Hom}_{\mathcal{O}_T}(M, N)$ is a finitely generated free T-module and for any ring extension $T \to T'$ the obvious map defines an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_T}(M, N) \otimes_T T' \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{T'}}(M \otimes_T T', N \otimes_T T')$$

If $\mathfrak{m} \subset T$ is the unique maximal ideal in our local deformation algebra T we set $\mathbb{K} = T/\mathfrak{m}T$ for its residue field.

Theorem 2.3 ([3], Propositions 2.1 and 2.6.).   (1) *The base change $\cdot \otimes_T \mathbb{K}$ gives a bijection*

$$\left\{ \begin{array}{c} simple\ isomorphism \\ classes\ of\ \mathcal{O}_T \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} simple\ isomorphism \\ classes\ of\ \mathcal{O}_\mathbb{K} \end{array} \right\}$$

(2) *The base change $\cdot \otimes_T \mathbb{K}$ gives a bijection*

$$\left\{ \begin{array}{c} projective\ isomorphism \\ classes\ of\ \mathcal{O}_T \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} projective\ isomorphism \\ classes\ of\ \mathcal{O}_\mathbb{K} \end{array} \right\}$$

The category $\mathcal{O}_{\mathbb{K}}$ is the direct summand of the category $\mathcal{O}$ over the Lie algebra $\mathfrak{g} \otimes \mathbb{K}$ consisting of all objects, whose weights lie in the complex affine subspace $\tau + \mathfrak{h}^* = \tau + \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{K}}(\mathfrak{h} \otimes \mathbb{K}, \mathbb{K})$ for τ the restriction to $\mathfrak{h}$ of the map that makes $\mathbb{K}$ to a S-algebra.

Theorem 2.4 ([3], Propositions 2.4 and 2.7). *Let T be a local deformation algebra and $\mathbb{K}$ its residue field. Let $L_T(\lambda)$ be a simple object in $\mathcal{O}_T$.*

(1) *There is a projective cover $P_T(\lambda)$ of $L_T(\lambda)$ in $\mathcal{O}_T$. Every projective object in $\mathcal{O}_T$ is isomorphic to a direct sum of projective covers.*

(2) *$P_T(\lambda)$ has a Verma flag, i.e. a finite filtration with subquotients isomorphic to Verma modules, and for the multiplicities we have the BGG-reciprocity formula*

$$(P_T(\lambda) : \Delta_T(\mu)) = [\Delta_{\mathbb{K}}(\mu) : L_{\mathbb{K}}(\lambda)]$$

for all Verma modules $\Delta_T(\mu)$ in $\mathcal{O}_T$.

(3) *Let $T \to T'$ be a morphism of local deformation algebras and P projective in $\mathcal{O}_T$. Then $P \otimes_T T'$ is projective in $\mathcal{O}_{T'}$ and the natural transformation*

$$\mathrm{Hom}_{\mathcal{O}_T}(P, \cdot) \otimes_T T' \longrightarrow \mathrm{Hom}_{\mathcal{O}_{T'}}(P \otimes_T T', \cdot \otimes_T T')$$

is an isomorphism of functors from $\mathcal{O}_T$ to T'-mod.

Since Fiebig works over a complex symmetrizable Kac-Moody algebra, he has to introduce truncated subcategories and has to set some finiteness assumptions. In case of a finite dimensional semisimple Lie algebra we do not need these technical tools.

2.2. **Block decomposition.** Let T again denote a local deformation algebra and $\mathbb{K}$ its residue field.

Definition 2.5. *Let $\sim_T$ be the equivalence relation on $\mathfrak{h}^*$ generated by $\lambda \sim_T \mu$ if $[\Delta_{\mathbb{K}}(\lambda) : L_{\mathbb{K}}(\mu)] \neq 0$.*

Definition 2.6. *Let $\Lambda \in \mathfrak{h}^* / \sim_T$ be an equivalence class. Let $\mathcal{O}_{T,\Lambda}$ be the full subcategory of $\mathcal{O}_T$ consisting of all modules M such that every highest weight of a subquotient of M lies in Λ.*

Proposition 2.7 ([3], Proposition 2.8). *The functor*

$$\bigoplus_{\Lambda \in \mathfrak{h}^*/\sim_T} \mathcal{O}_{T,\Lambda} \quad \longrightarrow \quad \mathcal{O}_T$$
$$(M_\Lambda)_{\Lambda \in \mathfrak{h}^*/\sim_T} \quad \longmapsto \quad \bigoplus_{\Lambda \in \mathfrak{h}^*/\sim_T} M_\Lambda$$

is an equivalence of categories.

The isomorphism above is called *block decomposition*.
Later we will be especially interested in the case $T = R = S_{(0)}$ where $S_{(0)}$ denotes the localisation of S at the maximal ideal generated by $\mathfrak{h}$, i.e. the maximal ideal of $0 \in \mathfrak{h}^*$.
Since $\sim_R = \sim_{\mathbb{C}}$ the block decomposition of $\mathcal{O}_R$ corresponds to the block decomposition of the BGG-category $\mathcal{O}$ over $\mathfrak{g}$.

Let $\tau : S \to \mathbb{K}$ be the induced map that makes $\mathbb{K}$ into a S-algebra. Restricting to $\mathfrak{h}$ and extending with $\cdot \otimes_T \mathbb{K}$ yields a $\mathbb{K}$-linear map $\mathfrak{h} \otimes_T \mathbb{K} \to \mathbb{K}$, which we will also call τ. Let $\mathcal{R} \supset \mathcal{R}^+$ be the root system with positive roots according to our

data $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$. For $\lambda \in \mathfrak{h}_{\mathbb{K}}^* = \text{Hom}_{\mathbb{K}}(\mathfrak{h} \otimes_T \mathbb{K}, \mathbb{K})$ and $\check{\alpha} \in \mathfrak{h}$ the dual root to a root $\alpha \in \mathcal{R}$ we set $\langle \lambda, \check{\alpha} \rangle_{\mathbb{K}} = \lambda(\check{\alpha}) \in \mathbb{K}$. Let $\mathcal{W}$ be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

Definition 2.8. *For $\mathcal{R}$ the root system of $\mathfrak{g}$ and $\Lambda \in \mathfrak{h}^* / \sim_T$ we define*

$$\mathcal{R}_T(\Lambda) = \{\alpha \in \mathcal{R} | \langle \lambda + \tau, \check{\alpha} \rangle_{\mathbb{K}} \in \mathbb{Z} \subset \mathbb{K} \text{ for some } \lambda \in \Lambda\}$$

and call it the integral roots corresponding to Λ. Let $\mathcal{R}_T^+(\Lambda)$ denote the positive roots in $\mathcal{R}_T(\Lambda)$ and set

$$\mathcal{W}_T(\Lambda) = \{s_\alpha \in \mathcal{W} | \alpha \in \mathcal{R}_T^+(\Lambda)\}$$

We call it the integral Weyl group with respect to Λ.

From [3] Corollary 3.3 it follows that

$$\Lambda = \mathcal{W}_T(\Lambda) \cdot \lambda \text{ for any } \lambda \in \Lambda$$

where we denote by $\cdot$ the ρ-shifted dot-action of the Weyl group.

Since most of our following constructions commute with base change, we are particularly interested in the case when $T = R_{\mathfrak{p}}$ is a localization of R at a prime ideal $\mathfrak{p}$ of height one. Applying the functor $\cdot \otimes_R R_{\mathfrak{p}}$ will split the deformed category $\mathcal{O}_T$ into generic and subgeneric blocks, which is content of the next

Lemma 2.9 ([4], Lemma 3). *Let $\Lambda \in \mathfrak{h}^* / \sim_R$ and let $\mathfrak{p} \in R$ be a prime ideal.*
  (1) *If $\check{\alpha} \notin \mathfrak{p}$ for all roots $\alpha \in \mathcal{R}_R(\Lambda)$, then Λ splits under $\sim_{R_{\mathfrak{p}}}$ into generic equivalence classes.*
  (2) *If $\mathfrak{p} = R\check{\alpha}$ for a root $\alpha \in \mathcal{R}_R(\Lambda)$, then Λ splits under $\sim_{R_{\mathfrak{p}}}$ into subgeneric equivalence classes of the form $\{\lambda, s_\alpha \cdot \lambda\}$.*

We recall that we denote by $P_T(\lambda)$ the up to isomorphism unique indecomposable projective object in $\mathcal{O}_T$ which is a projective cover of $L_T(\lambda)$. For an equivalence class $\Lambda \in \mathfrak{h}^* / \sim_T$, which is generic, i.e. $\Lambda = \{\lambda\}$, we get $P_T(\lambda) = \Delta_T(\lambda)$. If $\Lambda = \{\lambda, \mu\}$ and $\mu < \lambda$, we have $P_T(\lambda) = \Delta_T(\lambda)$ and there is a non-split short exact sequence in $\mathcal{O}_T$

$$0 \to \Delta_T(\lambda) \to P_T(\mu) \to \Delta_T(\mu) \to 0$$

In this case, every endomorphism $f : P_T(\mu) \to P_T(\mu)$ maps $\Delta_T(\lambda)$ to $\Delta_T(\lambda)$ since $\lambda > \mu$. So f induces a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta_T(\lambda) & \longrightarrow & P_T(\mu) & \longrightarrow & \Delta_T(\mu) & \longrightarrow & 0 \\
& & \downarrow f_\lambda & & \downarrow f & & \downarrow f_\mu & & \downarrow \\
0 & \longrightarrow & \Delta_T(\lambda) & \longrightarrow & P_T(\mu) & \longrightarrow & \Delta_T(\mu) & \longrightarrow & 0
\end{array}
$$

Since endomorphisms of Vermamodules correspond to elements of T, we get a map

$$
\begin{array}{rcl}
\chi : \text{End}_{\mathcal{O}_T}(P_T(\mu)) & \longrightarrow & T \oplus T \\
f & \longmapsto & (f_\lambda, f_\mu)
\end{array}
$$

For $\mathfrak{p} = R\check{\alpha}$ we define $R_\alpha = R_{\mathfrak{p}}$ for the localization of R at the prime ideal $\mathfrak{p}$.

Proposition 2.10 ([3], Corollary 3.5). *Let $\Lambda \in \mathfrak{h}^* / \sim_{R_\alpha}$. If $\Lambda = \{\lambda, \mu\}$ and $\lambda = s_\alpha \cdot \mu > \mu$, the map χ from above induces an isomorphism of R_α-modules*

$$\text{End}_{\mathcal{O}_{R_\alpha}}(P_{R_\alpha}(\mu)) \cong \{(t_\lambda, t_\mu) \in R_\alpha \oplus R_\alpha | t_\lambda \equiv t_\mu \mod \check{\alpha}\}$$

3. Tilting modules and tilting equivalence

For this chapter, we take for T a localisation of $R = S_{(0)}$ at a prime ideal $\mathfrak{p} \subset R$ and let $\mathbb{K}$ be its residue field. Let $\lambda \in \mathfrak{h}^*$ be such that $\Delta_{\mathbb{K}}(\lambda)$ is a simple object in $\mathcal{O}_{\mathbb{K}}$. Thus we have $\Delta_{\mathbb{K}}(\lambda) \cong \nabla_{\mathbb{K}}(\lambda)$ and the canonical map can $: \Delta_T(\lambda) \to \nabla_T(\lambda)$ becomes an isomorphism after applying $\cdot \otimes_T \mathbb{K}$. So by Nakayama's lemma we conclude that can was bijective already.

3.1. Deformed tilting modules.

Definition 3.1. *We denote by $\mathcal{K}_T$ the full subcategory of $\mathcal{O}_T$, which*
  (1) *includes the self-dual deformed Verma modules*
  (2) *is stable under tensoring with finite dimensional $\mathfrak{g}$-modules*
  (3) *is stable under forming direct sums and summands.*

For $T = S/S\mathfrak{h} = \mathbb{C}$ $\mathcal{K}_T$ is just the usual subcategory of tilting modules of category $\mathcal{O}$. In general, $\mathcal{K}_{\mathbb{K}}$ is the category of tilting modules of category $\mathcal{O}$ for the Lie algebra $\mathfrak{g} \otimes_T \mathbb{K}$, whose weights live in the affine complex subspace $\tau + \mathfrak{h}^* \subset \mathrm{Hom}_{\mathbb{K}}(\mathfrak{h} \otimes_T \mathbb{K}, \mathbb{K})$, where $\tau : \mathfrak{h} \otimes_T \mathbb{K} \to \mathbb{K}$ comes from the map, that makes $\mathbb{K}$ into an S-algebra.

Proposition 3.2. *The base change $\cdot \otimes_T \mathbb{K}$ gives a bijection*

$$\left\{ \begin{array}{c} \textit{isomorphism classes} \\ \textit{of } \mathcal{K}_T \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \textit{isomorphism classes} \\ \textit{of } \mathcal{K}_{\mathbb{K}} \end{array} \right\}$$

Proof. For $K, H \in \mathcal{K}_T$ with $K \otimes_T \mathbb{K} \cong H \otimes_T \mathbb{K}$ we conclude $K \cong H$ by Nakayama's lemma applied to the weight spaces, since the weight spaces of tilting modules are finitely generated and free over T. That shows injectivity.

For surjectivity we only have to show that every indecomposable tilting module in $\mathcal{O}_{\mathbb{K}}$ has an indecomposable preimage in $\mathcal{K}_T$. Since we are not working over a complete local ring we cannot apply the idempotent lifting lemma as in the proof of Proposition 3.4. in [9]. The proof works rather very similar to the proof of Theorem 6 in [6].

Let $K \in \mathcal{O}_{\mathbb{K}}$ be an indecomposable tilting module. For the sake of simplicity, we will assume the highest weight of K to be regular. The singular case is treated analogously.

If the highest weight λ of K is minimal in its equivalence class under $\sim_T$ then $K \cong \Delta_{\mathbb{K}}(\lambda) \cong \nabla_{\mathbb{K}}(\lambda)$ and we can take $\Delta_T(\lambda)$ as a preimage in $\mathcal{K}_T$.

Now denote by $\bar{\lambda}$ the equivalence class of λ in $\mathfrak{h}^*/ \sim_T$ and let $\lambda = w \cdot \mu$ with $w \in \mathcal{W}_T(\bar{\lambda})$ and let μ be minimal in $\mathcal{W}_T(\bar{\lambda}) \cdot \lambda$. In addition, let $w = s_1 s_2 ... s_n$ be a minimal expression of w with simple reflections $s_i \in \mathcal{W}_T(\bar{\lambda})$. We denote by $\theta_i = \theta_i^{\mathbb{K}}$ the translation functor of $\mathcal{O}_{\mathbb{K}}$ through the s_i-wall. Then K is a direct summand of the tilting module

$$T = \theta_1 ... \theta_n \Delta_{\mathbb{K}}(\mu)$$

and we get a decomposition $T \cong K \oplus K'$. K' decomposes into indecomposable tilting modules with highest weights of the form $w' \cdot \mu$ and $l(w) > l(w')$ where l denotes the length of a Weyl group element. By using induction on the length of w, we get a preimage $\tilde{K} \in \mathcal{K}_T$ of K' together with a splitting inclusion

$$\tilde{K} \otimes_T \mathbb{K} \hookrightarrow T$$

By Nakayama's lemma this induces a splitting lift

$$\tilde{K} \hookrightarrow \theta_1^T ... \theta_n^T \Delta_T(\mu)$$

where θ_i^T denotes the T-deformed wall-crossing functor of $\mathcal{O}_T$ corresponding to $\theta_i^{\mathbb{K}}$. Finally, the cokernel of this inclusion is the indecomposable tilting module in $\mathcal{K}_T$ we were looking for. □

3.2. **Tilting functor.** Let $\mathcal{M}_T$ denote the subcategory of all modules in $\mathcal{O}_T$ admitting a Verma flag. We recall the map $\tau : S \to T$ which makes T into a S-algebra. We get a new S-algebra structure on T via $\tau \circ \gamma : S \to T$, where $\gamma : S \to S$ is the isomorphism given by $\gamma(h) = -h$ for all $h \in \mathfrak{h}$. We denote this new S-algebra by $\overline{T}$. Let $S_{2\rho}$ be the semiregular $U(\mathfrak{g})$-bimodule of [8]. If N is a $\mathfrak{g} \otimes T$-module which decomposes into weight spaces we get a $\overline{T}$-module

$$N^\star = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathrm{Hom}_T(N_\lambda, T)$$

Then we get a $\mathfrak{g} \otimes \overline{T}$-module via the $\mathfrak{g}$-action $(Xf)(v) = -f(Xv)$ for all $X \in \mathfrak{g}, f \in N^\star$ and $v \in N$.
Now we get a functor

$$t_T' : \mathcal{M}_T \longrightarrow \mathcal{M}_{\overline{T}}^{opp}$$

by setting $t_T'(M) = (S_{2\rho} \otimes_{U(\mathfrak{g})} M)^\star$.
For T a localisation of S at a prime ideal $\mathfrak{p}$ which is stable under γ, for a residue field of this or for the ring $\mathbb{C}[[t]]$ of formal power series coming from a line $\mathbb{C}\lambda \subset \mathfrak{h}^*$, γ is an automorphism $\gamma : T \xrightarrow{\sim} \overline{T}$ which induces an equivalence of categories

$$\gamma : \mathcal{M}_T \longrightarrow \mathcal{M}_{\overline{T}}$$

Theorem 3.3 ([4], section 2.6). *The functor $t_T = \gamma \circ (S_{2\rho} \otimes_{U(\mathfrak{g})} \cdot)^\star$ induces an equivalence of categories*

$$t_T : \mathcal{M}_T \longrightarrow \mathcal{M}_T^{opp}$$

which respects block decomposition, makes short exact sequences to short exact sequences and sends a Verma module $\Delta_T(\lambda)$ to the Verma module $\Delta(-2\rho - \lambda)$ for some weight $\lambda \in \mathfrak{h}^*$.

Proposition 3.4. *Let $\lambda \in \mathfrak{h}^*$. Then*

$$t_T(P_T(\lambda)) \cong K_T(-2\rho - \lambda)$$

where $P_T(\lambda)$ denotes the indecomposable projective cover of $L_T(\lambda)$ and $K_T(\mu)$ the up to isomorphism unique indecomposable deformed tilting module with highest weight $\mu \in \mathfrak{h}^*$.

Proof. We set $\mu = -2\rho - \lambda$. This proof is very similar to the proof of Proposition 3.1 of [7]. As we have seen already, the tilting module $K_T(\mu)$ can be described as the up to isomorphism unique indecomposable module in $\mathcal{O}_T$ with the properties:

  (1) $K_T(\mu)$ admits a Verma flag
  (2) $d(K_T(\mu))$ has a Verma flag
  (3) $K_T(\mu)_\mu$ is free of rank one over T
  (4) If γ is a weight of $K_T(\mu)$, we have $\gamma \leq \mu$.

Since t_T is fully faithful, we already conclude the indecomposability of $t_T(P_T(\lambda))$. Theorem 3.3 also tells us that $t_T(P_T(\lambda))$ has a Verma flag.
By BGG-reciprocity we get a short exact sequence

$$N \hookrightarrow P_T(\lambda) \twoheadrightarrow \Delta_T(\lambda)$$

where N has a Verma flag in which the occurring weights are strictly smaller than λ. By applying t_T we get a new short exact sequence

$$t_T(N) \twoheadleftarrow t_T(P_T(\lambda)) \hookleftarrow t_T(\Delta_T(\lambda))$$

By induction on the length of the Verma flag of N we conclude that the weights of $t_T(N)$ are strictly larger than μ. So the weight space $(t_T(P_T(\lambda)))_\mu$ is free of rank one, since $t_T(\Delta_T(\lambda)) \cong \Delta_T(\mu)$. Now, $P_T(\lambda)$ being projective and t_T being fully faithful, we get

$$\mathrm{Ext}^1_{\mathcal{O}_T}(\Delta_T(\delta), t_T(P_T(\lambda))) = 0 \ \ \forall\, \delta \in \mathfrak{h}^*$$

Now we set $D_T = t_T(P_T(\lambda))$ and consider the diagramm

$$
\begin{array}{ccccc}
\Delta_T(\mu) & \hookrightarrow & K_T(\mu) & \twoheadrightarrow & \mathrm{coker} \\
\| & & & & \\
\Delta_T(\mu) & \hookrightarrow & D_T & \twoheadrightarrow & \mathrm{coker}'
\end{array}
$$

Since coker has a Verma flag we conclude $\mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{coker}, D_T) = 0$ using induction on the length of a Verma flag of coker. So the restriction induces a surjection $\mathrm{Hom}_{\mathcal{O}_T}(K_T(\mu), D_T) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\mu), D_T)$ and we get a map $\alpha : K_T(\mu) \to D_T$ which induces the identity on $\Delta_T(\mu)$. For the same reason we also get a map $\beta : D_T \to K_T(\mu)$ with the same property.
Applying the base change functor $\cdot \otimes_T \mathbb{K}$ we get two maps $\varphi := (\beta \circ \alpha) \otimes \mathrm{id}_\mathbb{K} : K_\mathbb{K}(\mu) \to K_\mathbb{K}(\mu)$ and $\psi := (\alpha \circ \beta) \otimes \mathrm{id}_\mathbb{K} : D_T \otimes_T \mathbb{K} \to D_T \otimes_T \mathbb{K}$ which induce the identity on $\Delta_\mathbb{K}(\mu)$. We conclude that φ and ψ are not nilpotent and since $D_T \otimes_T \mathbb{K}$ and $K_\mathbb{K}(\mu)$ have finite length it follows from the Fitting lemma that φ and ψ are isomorphisms. Now by Nakayama's lemma applied to all weight spaces we conclude $K_T(\mu) \cong D_T$. $\qquad\square$

Lemma 3.5. *Let $T \to T'$ be a homomorphism of S-algebras where also T' is a localisation of S at a prime ideal, which is stable under γ, its residue field or the ring of formal power series $\mathbb{C}[[t]]$. Let $M, N \in \mathcal{M}_T$ and let M be projective in $\mathcal{O}_T$. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_T}(M, N) \otimes_T T' & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{T'}}(M \otimes_T T', N \otimes_T T') \\
\Big\downarrow{\scriptstyle t_T \otimes \mathrm{id}_{T'}} & & \Big\downarrow{\scriptstyle t_{T'}} \\
\mathrm{Hom}_{\mathcal{O}_T}(t_T(N), t_T(M)) \otimes_T T' & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{T'}}(t_{T'}(N \otimes_T T'), t_{T'}(M \otimes_T T'))
\end{array}
$$

commutes, where the horizontals are the base change isomorphisms and the verticals are induced by the tilting functors t_T resp. $t_{T'}$.

Proof. All composition factors of the tilting functor commute with base change in the sense of the lemma. $\qquad\square$

4. The Jantzen and Andersen filtrations

We fix a deformed tilting module $K \in \mathcal{K}_T$ and let $\lambda \in \mathfrak{h}^*$. The composition of homomorphisms induces a T-bilinear pairing

$$\mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\lambda), K) \times \mathrm{Hom}_{\mathcal{O}_T}(K, \nabla_T(\lambda)) \ \longrightarrow \ \mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\lambda), \nabla_T(\lambda)) \cong T$$

$$(\varphi, \psi) \hspace{4cm} \longmapsto \hspace{3cm} \psi \circ \varphi$$

For any T-module H we denote by H^* the T-module $\mathrm{Hom}_T(H, T)$. As in [9] section 4 one shows that for T a localization of S at a prime ideal $\mathfrak{p}$ or for $T = \mathbb{C}[[v]]$ our pairing is nondegenerate and induces an injective map

$$E = E_T^\lambda(K) : \mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\lambda), K) \longrightarrow (\mathrm{Hom}_{\mathcal{O}_T}(K, \nabla_T(\lambda)))^*$$

of finitely generated free T-modules.

If we take $T = \mathbb{C}[[v]]$ the ring of formal power series around the origin on a line $\mathbb{C}\delta \subset \mathfrak{h}^*$ not contained in any hyperplane corresponding to a reflection of the Weyl group, we get a filtration on $\mathrm{Hom}_{\mathcal{O}_T}(\Delta_T(\lambda), K)$ by taking the preimages of $(\mathrm{Hom}_{\mathcal{O}_T}(K, \nabla_T(\lambda)))^* \cdot v^i$ for $i = 0, 1, 2, ...$ under E.

Definition 4.1 ([9], Definition 4.2.). *Given $K_\mathbb{C} \in \mathcal{K}_\mathbb{C}$ a tilting module of $\mathcal{O}$ and $K \in \mathcal{K}_{\mathbb{C}[[v]]}$ a preimage of $K_\mathbb{C}$ under the functor $\cdot \otimes_{\mathbb{C}[[v]]} \mathbb{C}$, which is possible by 3.2 with $S \to \mathbb{C}[[v]]$ the restriction to a formal neighbourhood of the origin in the line $\mathbb{C}\rho$, then the image of the filtration defined above under specialization $\cdot \otimes_{\mathbb{C}[[v]]} \mathbb{C}$ is called the Andersen filtration on $\mathrm{Hom}_\mathfrak{g}(\Delta(\lambda), K_\mathbb{C})$.*

The Jantzen filtration on a Verma module $\Delta(\lambda)$ induces a filtration on the vector space $\mathrm{Hom}_\mathfrak{g}(P, \Delta(\lambda))$, where P is a projective object in $\mathcal{O}$. Now consider the embedding $\Delta_{\mathbb{C}[[v]]} \hookrightarrow \nabla_{\mathbb{C}[[v]]}(\lambda)$. Let $P_{\mathbb{C}[[v]]}$ denote the up to isomorphism unique projective object in $\mathcal{O}_{\mathbb{C}[[v]]}$ that maps to P under $\cdot \otimes_{\mathbb{C}[[v]]} \mathbb{C}$, which is possible by 2.3. Then we get the same filtration by taking the preimages of $v^i \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(P_{\mathbb{C}[[v]]}, \nabla_{\mathbb{C}[[v]]}(\lambda))$, $i = 0, 1, 2, ...$, under the induced inclusion

$$J = J_T^\lambda(P) : \mathrm{Hom}_{\mathcal{O}_T}(P_T, \Delta_T(\lambda)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_T}(P_T, \nabla_T(\lambda))$$

for $T = \mathbb{C}[[v]]$ and taking the images of these filtration layers under the map $\mathrm{Hom}_{\mathcal{O}_T}(P_T, \Delta_T(\lambda)) \twoheadrightarrow \mathrm{Hom}_\mathfrak{g}(P, \Delta(\lambda))$ induced by $\cdot \otimes_T \mathbb{C}$.

For what follows, we define $\mu' = -2\rho - \mu$ and $\lambda' = -2\rho - \lambda$. For the sake of readability we sometimes remove the T in the expression $(\cdot)^{*T}$.

Proposition 4.2. *Let $\lambda, \mu \in \mathfrak{h}^*$. Denote by $R = S_{(0)}$ the localization of S at 0. There exists an isomorphism $L = L_R(\lambda, \mu)$ which makes the diagram*

(4.1)
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \Delta_R(\mu)) & \xrightarrow{\ J\ } & \mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \nabla_R(\mu)) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle L} \\ \mathrm{Hom}_{\mathcal{O}_R}(\Delta_R(\mu'), K_R(\lambda')) & \xrightarrow{\ E\ } & (\mathrm{Hom}_{\mathcal{O}_R}(K_R(\lambda'), \nabla_R(\mu')))^* \end{array}$$

commutative. Here $J = J_R^\mu(P_R(\lambda))$ and $E = E_R^{\mu'}(K_R(\lambda'))$ come from the Jantzen and Andersen filtration defined above and $t = t_R$ denotes the isomorphism induced by the tilting functor.

Proof. If λ is not contained in the equivalence class of μ under $\sim_R = \sim_\mathbb{C}$ all Hom-spaces occurring in the diagram are 0 by block decomposition and the assertion of the proposition is true.

So, let us assume λ to be in the equivalence class of μ. It is easy to see that J and E commute with base change and we have also verified this property for t in lemma 3.5 already. Let $\mathfrak{p} \subset R$ be a prime ideal of height 1. We abbreviate $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{O}_{R_\mathfrak{p}}}, P = P_R(\lambda), K = K_R(\lambda'), \Delta = \Delta_{R_\mathfrak{p}}(\mu), \nabla = \nabla_{R_\mathfrak{p}}(\mu), \Delta' = \Delta_{R_\mathfrak{p}}(\mu')$ and $\nabla' = \Delta_{R_\mathfrak{p}}(\mu')$. After applying $\cdot \otimes_R R_\mathfrak{p}$ and using the base change isomorphisms of 2.4 and 2.2 to our diagram we get a Diagram of $R_\mathfrak{p}$-modules

$$\begin{array}{ccc}
\mathrm{Hom}(P \otimes_R R_{\mathfrak{p}}, \Delta) & \xrightarrow{\ J\ } & \mathrm{Hom}(P \otimes_R R_{\mathfrak{p}}, \nabla) \\
\Big\downarrow{\scriptstyle t} & & \\
\mathrm{Hom}(\Delta', K \otimes_R R_{\mathfrak{p}}) & \xrightarrow{\ E\ } & (\mathrm{Hom}(K \otimes_R R_{\mathfrak{p}}, \nabla'))^{*_{R_{\mathfrak{p}}}}
\end{array}$$

(4.2)

where we omit the index $R_{\mathfrak{p}}$ for t, J and E.

We want to show that we get an isomorphism $L_{R_{\mathfrak{p}}}$ of $R_{\mathfrak{p}}$-modules as the missing right vertical of the upper diagram to make it commutative for every prime ideal $\mathfrak{p} \subset R$ of height 1. By block decomposition we get

$$P_R(\lambda) \otimes_R R_{\mathfrak{p}} \cong \bigoplus_{i=1}^{n} P_{R_{\mathfrak{p}}}(\lambda_i)$$

for certain indecomposable projective objects $P_{R_{\mathfrak{p}}}(\lambda_i) \in \mathcal{O}_{R_{\mathfrak{p}}}$ and $\lambda_i \in \overline{\lambda}$ where $\overline{\lambda}$ denotes the equivalence class of λ under $\sim_R = \sim_{\mathbb{C}}$. Since t is fully faithful and respects base change we also get a decomposition

$$K_R(\lambda') \otimes_R R_{\mathfrak{p}} \cong t_{R_{\mathfrak{p}}}(P_R(\lambda) \otimes_R R_{\mathfrak{p}}) \cong \bigoplus_{i=1}^{n} t_{R_{\mathfrak{p}}}(P_{R_{\mathfrak{p}}}(\lambda_i))$$

It is easy to see that J, E, t respect these decompositions. Hence, we get in formulas

$$J(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \Delta_{R_{\mathfrak{p}}}(\mu))) \quad \subset \quad \mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))$$

$$t(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \Delta_{R_{\mathfrak{p}}}(\mu))) \quad = \quad \mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(\Delta_{R_{\mathfrak{p}}}(\mu)), t(P_{R_{\mathfrak{p}}}(\lambda_i)))$$

$$E\left(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(\Delta_{R_{\mathfrak{p}}}(\mu)), t(P_{R_{\mathfrak{p}}}(\lambda_i)))\right) \quad \subset \quad \left(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(P_{R_{\mathfrak{p}}}(\lambda_i)), \nabla_{R_{\mathfrak{p}}}(\mu'))\right)^{*_{R_{\mathfrak{p}}}}$$

where we omit the index $R_{\mathfrak{p}}$ of our maps.

Let $Q = \mathrm{Quot}(S)$ be the quotient field of S. Since all deformed Verma modules over Q are simple, we get $\Delta_Q(\mu) \cong \nabla_Q(\mu)$ and $\Delta_Q(\mu') \cong \nabla_Q(\mu')$, respectively. Hence, after applying $\cdot \otimes_R Q$ to our diagram we get an isomorphism L_Q which makes the following diagram commutative

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_Q}(P_R(\lambda) \otimes_R Q, \Delta_Q(\mu)) & \xrightarrow{\ J_Q\ } & \mathrm{Hom}_{\mathcal{O}_Q}(P_R(\lambda) \otimes_R Q, \nabla_Q(\mu)) \\
\Big\downarrow{\scriptstyle t_Q} & & \Big\downarrow{\scriptstyle L_Q} \\
\mathrm{Hom}_{\mathcal{O}_Q}(t_Q(\Delta_Q(\mu)), t_Q(P_R(\lambda) \otimes_R Q)) & \xrightarrow{\ E_Q\ } & (\mathrm{Hom}_{\mathcal{O}_Q}(t_Q(P_R(\lambda) \otimes_R Q), \nabla_Q(\mu')))^{*_Q}
\end{array}$$

Let $\mathfrak{p} \subset R$ be a prime ideal of height one. So $\mathfrak{p} = R\gamma$ for an irreducible element $\gamma \in R$. If $\gamma \notin \mathbb{C}\check{\alpha}$ for all $\alpha \in \mathcal{R}^+$ or $\gamma \in \mathbb{C}\check{\alpha}$ for one $\alpha \in \mathcal{R}$ but with $\langle \mu + \rho, \check{\alpha} \rangle_{\mathbb{C}} \notin \mathbb{Z}$, then the block of μ in $\mathcal{O}_{R_{\mathfrak{p}}}$ is generic by 2.9 and we conclude $P_{R_{\mathfrak{p}}}(\mu) \cong \Delta_{R_{\mathfrak{p}}}(\mu) \cong \nabla_{R_{\mathfrak{p}}}(\mu)$. But in this case all maps $J_{R_{\mathfrak{p}}}$, $E_{R_{\mathfrak{p}}}$ and $t_{R_{\mathfrak{p}}}$ are isomorphisms and so we get the claimed $R_{\mathfrak{p}}$-isomorphism $L_{R_{\mathfrak{p}}}$ of our diagram (4.2). So we can assume $\mathfrak{p} = R\check{\alpha}$ for some $\alpha \in \mathcal{R}^+$ and $\langle \mu + \rho, \check{\alpha} \rangle_{\mathbb{C}} \in \mathbb{Z}$.

Now we want to show

$$L_Q(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_R(\lambda) \otimes_R R_{\mathfrak{p}}, \nabla_{R_{\mathfrak{p}}}(\mu))) \quad \cong \quad \left(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(P_R(\lambda)) \otimes_R R_{\mathfrak{p}}, \nabla_{R_{\mathfrak{p}}}(\mu'))\right)^{*R_{\mathfrak{p}}}$$

$$\subset \quad \left(\mathrm{Hom}_{\mathcal{O}_Q}(t(P_R(\lambda)) \otimes_R Q, \nabla_Q(\mu'))\right)^{*Q}$$

Since all our maps respect the decomposition of $P_R(\lambda) \otimes_R R_{\mathfrak{p}}$ we only have to prove this for the indecomposable summands of our decomposition. In formulas we want to show

(4.3) $\quad L_Q(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))) \quad \cong \quad \left(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(P_{R_{\mathfrak{p}}}(\lambda_i)), \nabla_{R_{\mathfrak{p}}}(\mu'))\right)^{*R_{\mathfrak{p}}}$

for all $i \in \{1, ..., n\}$. Both Hom-spaces in (4.3) are free and of the same rank over $R_{\mathfrak{p}}$. From the description of the projective covers in the generic and subgeneric case it follows

$$\mathrm{rk}_{R_{\mathfrak{p}}}(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))) \quad = \quad \dim_{\mathbb{K}}(\mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}(P_{\mathbb{K}}(\lambda_i), \nabla_{\mathbb{K}}(\mu)))$$

$$= \quad (P_{\mathbb{K}}(\lambda_i) : \Delta_{\mathbb{K}}(\mu)) \leq 1$$

where $\mathbb{K}$ denotes the residue field of $R_{\mathfrak{p}}$.

Now we will proceed in several steps to prove (4.3). From the choice of $\mathfrak{p}$ it follows that the equivalence class of μ under $\sim_{R_{\mathfrak{p}}}$ equals $\{\mu, s_\alpha \cdot \mu\}$ where again s_α is the reflection corresponding to the root $\alpha \in \mathcal{R}$ acting by the dot-action on $\mathfrak{h}^*$. If $\mu = s_\alpha \cdot \mu$ we are in the generic case and again our maps are isomorphisms, which proves the claim for this case. So let us further assume $\mu \neq s_\alpha \cdot \mu$.

Case 1. Let $\lambda_i \notin \{\mu, s_\alpha \cdot \mu\}$.
In this case, both Hom-spaces in (4.3) are zero by block decomposition and the claim is true.

Case 2. Let $\lambda_i = s_\alpha \cdot \mu > \mu$.
In this case we get $P_{R_{\mathfrak{p}}}(\lambda_i) \cong \Delta_{R_{\mathfrak{p}}}(\lambda_i) \neq \Delta_{R_{\mathfrak{p}}}(\mu) \cong \nabla_{R_{\mathfrak{p}}}(\mu)$ and therefore

$$\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \Delta_{R_{\mathfrak{p}}}(\mu)) \quad = \quad \mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))$$

$$= \quad \mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(\Delta_{R_{\mathfrak{p}}}(\mu)), t(P_{R_{\mathfrak{p}}}(\lambda_i)))$$

$$= \quad \left(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(P_{R_{\mathfrak{p}}}(\lambda_i)), \nabla_{R_{\mathfrak{p}}}(\mu'))\right)^* = 0$$

and we are done.

Case 3. Let $\lambda_i = \mu < s_\alpha \cdot \mu$.
Then we have $\Delta_{R_{\mathfrak{p}}}(\mu) = \nabla_{R_{\mathfrak{p}}}(\mu)$ and also $t(P_{R_{\mathfrak{p}}}(\lambda_i)) \cong K_{R_{\mathfrak{p}}}(-2\rho - \lambda_i) \cong P_{R_{\mathfrak{p}}}(-2\rho - s_\alpha \cdot \mu)$. With these assumptions on λ_i and μ we get an injection

$$\varphi : t(\Delta_{\mathbb{K}}(\mu)) \hookrightarrow t(P_{\mathbb{K}}(\lambda_i))$$

with $E_{\mathbb{K}}(\varphi) \neq 0$, since over $\mathbb{K}$ in the subgeneric case there is no homomorphism from the Verma module with dominant highest weight to the nabla module of the same maximal weight factoring through the projective cover of the Verma module with antidominant highest weight. Comparing dimensions over $\mathbb{K}$, we yield an isomorphism

$$E_{\mathbb{K}} : \mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}(t(\Delta_{\mathbb{K}}(\mu)), t(P_{\mathbb{K}}(\lambda_i))) \xrightarrow{\sim} (\mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}(t(P_{\mathbb{K}}(\lambda_i)), \nabla_{\mathbb{K}}(\mu')))^*$$

By Nakayama's lemma, we conclude that $E_{R_{\mathfrak{p}}}$ is surjective and therefore an isomorphism.

We also have a map $P_{\mathbb{K}}(\lambda_i) \twoheadrightarrow \Delta_{\mathbb{K}}(\mu)$ and with similar arguments we get an isomorphism of vector spaces

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}(P_{\mathbb{K}}(\lambda_i), \Delta_{\mathbb{K}}(\mu)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}(P_{\mathbb{K}}(\lambda_i), \nabla_{\mathbb{K}}(\mu))$$

and Nakayama's lemma finishes this case.

Case 4. Let $\lambda_i = \mu > s_\alpha \cdot \mu$.
Then we have $P_{R_{\mathfrak{p}}}(\lambda_i) = \Delta_{R_{\mathfrak{p}}}(\lambda_i) = \Delta_{R_{\mathfrak{p}}}(\mu)$ and over the residue field $\mathbb{K}$ we yield $E_{\mathbb{K}}(\mathrm{id}) \neq 0$ und $J_{\mathbb{K}}(\mathrm{id}) \neq 0$. And again by Nakayama's lemma we get two isomorpisms

$$J: \quad \operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \Delta_{R_{\mathfrak{p}}}(\mu)) \qquad \xrightarrow{\sim} \quad \operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))$$

$$E: \quad \operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(\Delta_{R_{\mathfrak{p}}}(\mu)), t(P_{R_{\mathfrak{p}}}(\lambda_i))) \quad \xrightarrow{\sim} \quad \left(\operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(t(P_{R_{\mathfrak{p}}}(\lambda_i)), \nabla_{R_{\mathfrak{p}}}(\mu'))\right)^{*_{R_{\mathfrak{p}}}}$$

and the claim is true.

Case 5. Let $\lambda_i = s_\alpha \cdot \mu < \mu$ and set $\lambda_i' = -2\rho - \lambda_i$.
After base change with the residue field we yield

$$J_{\mathbb{K}}: \operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}(P_{\mathbb{K}}(\lambda_i), \Delta_{\mathbb{K}}(\mu)) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}(P_{\mathbb{K}}(\lambda_i), \nabla_{\mathbb{K}}(\mu))$$

On the left side we have the generator $P_{\mathbb{K}}(\lambda_i) \twoheadrightarrow \Delta_{\mathbb{K}}(\lambda_i) \hookrightarrow \Delta_{\mathbb{K}}(\mu)$ which becomes

$$P_{\mathbb{K}}(\lambda_i) \twoheadrightarrow \Delta_{\mathbb{K}}(\lambda_i) \hookrightarrow \Delta_{\mathbb{K}}(\mu) \twoheadrightarrow L_{\mathbb{K}}(\mu) \hookrightarrow \nabla_{\mathbb{K}}(\mu)$$

on the right side. But since this map is zero we get

$$J\left(\operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \Delta_{R_{\mathfrak{p}}}(\mu))\right) \subset \check{\alpha} \cdot \operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))$$

Now, we can lift the generator $j': P_{\mathbb{K}}(\lambda_i) \twoheadrightarrow \Delta_{\mathbb{K}}(\lambda_i) \hookrightarrow \Delta_{\mathbb{K}}(\mu)$ to a generator $j: P_{R_{\mathfrak{p}}}(\lambda_i) \to \Delta_{R_{\mathfrak{p}}}(\mu)$ of the $R_{\mathfrak{p}}$-module $\operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \Delta_{R_{\mathfrak{p}}}(\mu))$ via base change. Let also $l \in \operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))$ be a generator. Then we conclude $J_{R_{\mathfrak{p}}}(j) = cl$ for one $c \in R_{\mathfrak{p}}$ with $c \neq 0$. Multiplying with an appropriate invertible element of $R_{\mathfrak{p}}$ we can assume $c = (\check{\alpha})^m$ for $m \in \mathbb{N}$.
On $\Delta_{R_{\mathfrak{p}}}(\mu)$ we have the filtration which comes from the preimages of $(\check{\alpha})^i \nabla_{R_{\mathfrak{p}}}(\mu)$ under the inclusion $\Delta_{R_{\mathfrak{p}}}(\mu) \hookrightarrow \nabla_{R_{\mathfrak{p}}}(\mu)$. Denote by $\Delta_{\mathbb{K}}(\mu)_i$ the image of the i-th filtration layer under the surjection $\Delta_{R_{\mathfrak{p}}}(\mu) \twoheadrightarrow \Delta_{\mathbb{K}}(\mu)$, which is the i-th layer of the Jantzen filtration on $\Delta_{\mathbb{K}}(\mu)$. By the Jantzen sum formula we get

$$\sum_{i>0} \operatorname{ch}\Delta_{\mathbb{K}}(\mu)_i = \sum_{\substack{\alpha \in R^+, n \in \mathbb{N} \\ \langle \mu+\rho+\tau, \check{\alpha}\rangle_{\mathbb{K}} = n}} \operatorname{ch}\Delta_{\mathbb{K}}(\mu - n\alpha) = \operatorname{ch}\Delta_{\mathbb{K}}(s_\alpha \cdot \mu)$$

Since $\Delta_{\mathbb{K}}(s_\alpha \cdot \mu)$ is simple, we conclude $\Delta_{\mathbb{K}}(\mu)_1 \cong \Delta_{\mathbb{K}}(s_\alpha \cdot \mu)$ and $\Delta_{\mathbb{K}}(\mu)_2 = 0$.
If $J_{R_{\mathfrak{p}}}(j) \in (\check{\alpha})^2 \cdot \operatorname{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\lambda_i), \nabla_{R_{\mathfrak{p}}}(\mu))$ we would get $j'(P_{\mathbb{K}}(\lambda_i)) \subset \Delta_{\mathbb{K}}(\mu)_2 = 0$, i.e. $j' = 0$, which is a contradiction to the choice of j'. We conclude $J_{R_{\mathfrak{p}}}(j) = \check{\alpha}l$.

Now we will observe the lower horizontal of the diagram. We have $t_{R_{\mathfrak{p}}}(P_{R_{\mathfrak{p}}}(\lambda_i)) = K_{R_{\mathfrak{p}}}(\lambda_i')$ and by our assumptions we get $s_\alpha \cdot (\lambda_i') = \mu' < \lambda_i'$. But since μ' is minimal in its equivalence class under $\sim_{R_{\mathfrak{p}}}$ we get an isomorphism $P_{\mathbb{K}}(\mu') \cong K_{\mathbb{K}}(\lambda_i')$ and then an inclusion $\gamma': \Delta_{\mathbb{K}}(\mu') \hookrightarrow K_{\mathbb{K}}(\lambda_i')$. Lifting this map with base change 2.2

we get a basis $\gamma \in \mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(\Delta_{R_{\mathfrak{p}}}(\mu'), K_{R_{\mathfrak{p}}}(\lambda_i'))$. But every map $\Delta_{\mathbb{K}}(\mu') \to \nabla_{\mathbb{K}}(\mu')$ which factors through $K_{\mathbb{K}}(\lambda_i')$ is zero already. So we yield that

$$E_{\mathbb{K}} : \mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}(\Delta_{\mathbb{K}}(\mu'), K_{\mathbb{K}}(\lambda_i')) \to (\mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}(K_{\mathbb{K}}(\lambda_i'), \nabla_{\mathbb{K}}(\mu')))^*$$

is the zero map. If we now choose a basis δ of $\left(\mathrm{Hom}_{\mathcal{O}_{R_{\mathfrak{p}}}}(K_{R_{\mathfrak{p}}}(\lambda_i'), \nabla_{R_{\mathfrak{p}}}(\mu'))\right)^*$ and eventually multiply it with an appropriate invertible element of $R_{\mathfrak{p}}$, we conclude $E_{R_{\mathfrak{p}}}(\gamma) = (\check{\alpha})^n \delta$ for $n > 0$.
By 2.10 we get

$$\begin{aligned}
\mathrm{End}_{\mathcal{O}_{R_{\mathfrak{p}}}}(K_{R_{\mathfrak{p}}}(\lambda_i')) &\cong \mathrm{End}_{\mathcal{O}_{R_{\mathfrak{p}}}}(P_{R_{\mathfrak{p}}}(\mu')) \\
&\cong \{(x,y) \in R_{\mathfrak{p}} \oplus R_{\mathfrak{p}} \mid x \equiv y \bmod \check{\alpha}\}
\end{aligned}$$

where for $\varphi \in \mathrm{End}_{\mathcal{O}_{R_{\mathfrak{p}}}}(K_{R_{\mathfrak{p}}}(\lambda_i'))$ x and y are given by the induced maps on the short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta_{R_{\mathfrak{p}}}(\lambda_i') & \longrightarrow & K_{R_{\mathfrak{p}}}(\lambda_i') & \longrightarrow & \Delta_{R_{\mathfrak{p}}}(\mu') & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle x\cdot}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle y\cdot} & & \downarrow \\
0 & \longrightarrow & \Delta_{R_{\mathfrak{p}}}(\lambda_i') & \longrightarrow & K_{R_{\mathfrak{p}}}(\lambda_i') & \longrightarrow & \Delta_{R_{\mathfrak{p}}}(\mu') & \longrightarrow & 0
\end{array}$$

If for φ we choose the homomorphism that corresponds to the tupel $(x,y) = (0, \check{\alpha})$ the map on the cokernels of the diagram will factor through the middle and we yield a map

$$\Delta_{R_{\mathfrak{p}}}(\mu') \to K_{R_{\mathfrak{p}}}(\lambda_i') \to \Delta_{R_{\mathfrak{p}}}(\mu') = \nabla_{R_{\mathfrak{p}}}(\mu')$$

and the image of this map is $\check{\alpha}\nabla_{R_{\mathfrak{p}}}(\mu')$. So we get $n = 1$. This proves the last case.

For now, we showed that for every prime ideal $\mathfrak{p} \subset R$ of height 1 we find an isomorphism $L_{R_{\mathfrak{p}}}$ that makes the diagram (4.2) commutative. Since E, J and t respect base change we get an isomorphism which we will also call $L = L_{R_{\mathfrak{p}}}$ that makes

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \Delta_R(\mu)) \otimes_R R_{\mathfrak{p}} & \xrightarrow{\quad J \quad} & \mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \nabla_R(\mu)) \otimes_R R_{\mathfrak{p}} \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle L} \\
\mathrm{Hom}_{\mathcal{O}_R}(t(\Delta_R(\mu)), t(P_R(\lambda))) \otimes_R R_{\mathfrak{p}} & \xrightarrow{\quad E \quad} & (\mathrm{Hom}_{\mathcal{O}_R}(t(P_R(\lambda)), \nabla_R(\mu')))^* \otimes_R R_{\mathfrak{p}}
\end{array}$$

commutative.
But then we also get an isomorphism L_R which is the restriction of the Q-linear map L_Q to $\mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \nabla_R(\mu))$ and makes the diagram

$$\begin{array}{ccc}
\bigcap_{\mathfrak{p} \in \mathfrak{P}} \mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \Delta_R(\mu)) \otimes_R R_{\mathfrak{p}} & \xrightarrow{\quad J \quad} & \bigcap_{\mathfrak{p} \in \mathfrak{P}} \mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), \nabla_R(\mu)) \otimes_R R_{\mathfrak{p}} \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle L} \\
\bigcap_{\mathfrak{p} \in \mathfrak{P}} \mathrm{Hom}_{\mathcal{O}_R}(t(\Delta_R(\mu)), t(P_R(\lambda))) \otimes_R R_{\mathfrak{p}} & \xrightarrow{\quad E \quad} & \bigcap_{\mathfrak{p} \in \mathfrak{P}} (\mathrm{Hom}_{\mathcal{O}_R}(t(P_R(\lambda)), \nabla_R(\mu')))^* \otimes_R R_{\mathfrak{p}}
\end{array}$$

commutative, where $\mathfrak{P}$ denotes the set of all prime ideals of R of height one. Since all Hom-spaces of this diagram are finitely generated free R-modules, it equals (4.1).

$\square$

Corollary 4.3. *Let $P \in \mathcal{O}$ be a projective object, $\mu \in \mathfrak{h}^*$ and let $t = t_{\mathbb{C}}$ be the tilting functor. Then $K = t(P)$ is a tilting object in $\mathcal{O}$ and the isomorphism $t :$ $\mathrm{Hom}_{\mathfrak{g}}(P, \Delta(\mu)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(\Delta(-2\rho - \mu), K)$ induced by the tilting functor identifies the filtration induced by the Jantzen filtration with the Andersen filtration.*

Proof. Consider the restriction map $S = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[[v]]$ to an infinitesimal line through the origin not contained in a reflection hyperplane corresponding to a root in $\mathcal{R}$. Since this map factors through R, we get a homomorphism $R \to \mathbb{C}[[v]]$ of S-algebras. Since the maps from the proposition above commute with base change $\cdot \otimes_R \mathbb{C}[[v]]$, we get a commuting diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(P_{\mathbb{C}[[v]]}, \Delta_{\mathbb{C}[[v]]}(\mu)) & \xrightarrow{\ J\ } & \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(P_{\mathbb{C}[[v]]}, \nabla_{\mathbb{C}[[v]]}(\mu)) \\
\Big\downarrow t & & \Big\downarrow L \\
\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(\Delta_{\mathbb{C}[[v]]}(\mu'), K_{\mathbb{C}[[v]]}) & \xrightarrow{\ E\ } & \left(\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(K_{\mathbb{C}[[v]]}, \nabla_{\mathbb{C}[[v]]}(\mu'))\right)^*
\end{array}
$$

where $P_{\mathbb{C}[[v]]}$ (resp. $K_{\mathbb{C}[[v]]}$) denote the unique (up to isomorphism) deformed projective (resp. tilting) module in $\mathcal{O}_{\mathbb{C}[[v]]}$ that map to P (resp. K) under $\cdot \otimes_{\mathbb{C}[[v]]} \mathbb{C}$, $\mu' = -2\rho - \mu$ and $(\cdot)^*$ means the $\mathbb{C}[[v]]$-dual. Since $L(v^i \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(P_{\mathbb{C}[[v]]}, \nabla_{\mathbb{C}[[v]]}(\mu))) = v^i(\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}[[v]]}}(K_{\mathbb{C}[[v]]}, \nabla_{\mathbb{C}[[v]]}(\mu')))^*$, the preimages of these submodules under J resp. E are identified by t. Applying $\cdot \otimes_{\mathbb{C}[[v]]} \mathbb{C}$ to the left vertical of the diagram yields the claim. $\qquad\square$

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