Almost reducibility for finitely differentiable $SL(2,\mathbb{R})$-valued quasi-periodic cocycles

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Abstract: Quasi-periodic cocycles with a diophantine frequency and with values in SL(2,R) are shown to be almost reducible as long as they are close enough to a constant, in the topology of k times differentiable functions, with k great enough. Almost reducibility is obtained by analytic approximation after a loss of differentiability which only depends on the frequency and on the constant part. As in the analytic case, if their fibered rotation number is diophantine or rational with respect to the frequency, such cocycles are in fact reducible. This extends Eliasson's theorem on Schrödinger cocycles to the differentiable case.

1 Introduction

This work is focused on $SL(2,\mathbb{R})$-valued quasi-periodic cocycles. We mean by quasi-periodic cocycle the fundamental solution of a linear system with quasi-periodic coefficients:

$$\forall (t,\theta)\in\mathbb{R}\times\mathbb{T}^d,\ \frac{d}{dt}X^t(\theta)=A(\theta+t\omega)X^t(\theta);\ X^0(\theta)=Id \tag{1}$$

where A is continuous on the d-dimensional torus $\mathbb{T}^d$, matrix-valued and $\omega\in\mathbb{R}^d$ is a rationally independent vector. In this case we say that X is the *cocycle associated to A*. In this paper we will have a particular interest in the case when A is $sl(2,\mathbb{R})$-valued since in this case it is possible to compute the fibered rotation number of the cocycle and have information on the rotational behaviour of the solutions of (1).

It is interesting to define an equivalence relation on cocycles as follow: if $A,B\in C^0(\mathbb{T}^d,gl(n,\mathbb{C}))$, one says that A and B are *conjugated in the sense of cocycles*, or just *conjugated*, if there exists a map Z which is continuous on the torus $2\mathbb{T}^d=\mathbb{R}^d/2\mathbb{Z}^d$ such that

$$\forall\theta\in 2\mathbb{T}^d,\ \frac{d}{dt}Z(\theta+t\omega)_{|t=0}=A(\theta)Z(\theta)-Z(\theta)B(\theta) \tag{2}$$

This kind of conjugation preserves some important dynamical invariants, as we will see later on. A natural question arises when dealing with a cocycle: can it be conjugated, in the sense of cocycles, to a system with constant coefficients? When it is so, one says that the solution cocycle is *reducible*. More precisely, a cocycle X solution of (1) is reducible if (2) holds for some constant B. In this case we also say that A is *reducible to B by Z*, which is equivalent to

$$\forall (t,\theta) \in \mathbb{R} \times 2\mathbb{T}^d, \; X^t(\theta) = Z(\theta + t\omega)^{-1} e^{tB} Z(\theta) \tag{3}$$

It is well known since the theory of Floquet that every periodic cocycle (i.e, in the notation above, when $d = 1$) is reducible (notice that we have allowed one period doubling in our definition of reducibility). However, the presence of at least two incommensurable frequencies in the coefficient of the system gives rise to non-reducible cocycles.
To mend this difficulty, some authors have considered the problem of almost reducibility of quasi-periodic cocycles. In some topology $\mathcal{C}$, a cocycle is said to be almost reducible if it can be conjugated, in the sense above, with Z of class $\mathcal{C}$, to another cocycle which is $\mathcal{C}$-arbitrarily close to a constant cocycle.
Many results about reducibility and almost reducibility of quasi-periodic cocycles have been obtained in the perturbative case, i.e the case when the vector ω satisfies a diophantine condition and (1) has the following form:

$$\frac{d}{dt} X^t(\theta) = (A + F(\theta + t\omega)) X^t(\theta); \; X^0(\theta) = 0 \tag{4}$$

where the coefficient $A + F(\theta + t\omega)$ is close enough to a constant, with a closeness condition related to the diophantine condition on ω:

$$|| \; F \; ||_{\mathcal{C}} \leq \epsilon(n, d, \omega, A) \tag{5}$$

Then, if $\mathcal{C}$ stands for some analytic topology, it is known that

- every cocycle is almost reducible ([4], [2])
- almost all cocycles are reducible, when considering a generic one-parameter family ([3], [7] completed with [1])
- reducible cocycles are dense ([2])
- in the $SO(3)$-valued case, also non reducible cocycles are dense ([5]).

In fact, [3] also investigates the link between reducibility and the rotational behaviour of the solutions, showing that Schrödinger cocycles are reducible if and only if their fibered rotation number either satisfies a diophantine condition or is rational with respect to ω; this result was extended to general $SL(2, \mathbb{R})$-valued cocycles in [6].

Here we shall adopt the perturbative framework, but in a finitely differentiable topology, a framework in which little is yet known. The aim of this work is to show that in the perturbative regime described by (4) and (5), for cocycles which are sufficiently smooth but finitely differentiable, say C^k for some $k \geq k_0(d, \omega, A)$, and have values in $SL(2, \mathbb{R})$, all cocycles are almost reducible in a finitely differentiable topology C^{k-D} with a loss of differentiability D which is independent of the initial regularity k; in fact, we state this theorem in such a way that it also holds for cocycles with values in other Lie groups. More precisely, we will prove, for G amongst $SL(2, \mathbb{C}), SL(2, \mathbb{R}), O(n), GL(n, \mathbb{C}), U(n)$, letting $\mathcal{G}$ be its Lie algebra:

Theorem 1.1 *Let $A \in \mathcal{G}$. There exists $k_0, D \in \mathbb{N}$ such that if $k \geq k_0$, there exists $\epsilon_0 > 0$ such that if $F \in C^k(\mathbb{T}^d, \mathcal{G})$ and $\| F \|_k \leq \epsilon_0$, then there exist $Z_\infty \in C^{k-D}(\mathbb{T}^d, G)$ and $\bar{A}_\infty \in C^{k-D}(\mathbb{T}^d, \mathcal{G})$ such that*

$$\forall \theta \in \mathbb{T}^d, \ \partial_\omega Z_\infty(\theta) = (A + F(\theta))Z_\infty(\theta) - Z_\infty(\theta)\bar{A}_\infty(\theta) \tag{6}$$

and $\bar{A}_\infty$ is the limit, in $C^{k-D}(\mathbb{T}^d, \mathcal{G})$, of reducible functions.

Theorem 1.1 is about almost reducibility of differentiable cocycles. It easily implies density of reducible cocycles near a constant. The reason why it holds for those Lie groups is that it is based on another theorem which holds for many classical Lie groups (see [2]), but we can apply it here only when no period doubling is needed, that is to say, in the complex case or in the 2-dimensional case.

Focusing now on the 2-dimensional case, we will show that given a $sl(2, \mathbb{R})$-valued cocycle, if its fibered rotation number satisfies a diophantine condition or is rational with respect to ω, then it is in fact reducible, thus extending Eliasson's theorem of [3] to the differentiable case:

Theorem 1.2 *Let $A \in sl(2, \mathbb{R})$. There exists $k_0, D \in \mathbb{N}$ such that if $k \geq k_0$ and $F \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$, there exists $\epsilon_0 > 0$ such that if $\| F \|_k \leq \epsilon_0$ and the fibered rotation number $\rho(A+F)$ has the form $2\pi\langle m, \omega\rangle, m \in \mathbb{Z}^d$ or satisfies a diophantine condition with respect to ω:*

$$\exists \kappa > 0, \tau \geq \max(1, d-1), \ \forall m \in \mathbb{Z}^d \setminus \{0\}, \ | \rho(A+F) - 2\pi\langle m, \omega\rangle | \geq \frac{\kappa}{| m |^\tau}$$

then the cocycle associated to $A + F$ is reducible in $C^{k-D}(\mathbb{T}^d, SL(2, \mathbb{R}))$.

The demonstration of Theorem 1.1 relies essentially on a proposition shown in [2], which was used as an inductive lemma in a KAM scheme to show almost reducibility for some analytic and Gevrey cocycles. Here, we use it to get a good control on a sequence of analytic cocycles which, following an idea of Zehnder ([8]), are constructed in such a way that they approach a given differentiable cocycle. Since they are shown to be conjugated to something which becomes closer and closer to a constant, one finally gets almost reducibility for their limit in a topology with a finite loss of differentiability with respect to the initial topology.

The specificity of $SL(2, \mathbb{R})$, however, is that the eigenvalues of the constant part get closer to 0 every time that, in the KAM scheme, a resonance is removed. Thus, non reducibility implies that the fibered rotation number of the limit cocycle cannot be diophantine, and so, by invariance through conjugation in the sense of cocycles, neither can the fibered rotation number of the initial cocycle, which gives Theorem 1.2. We then easily get an application to Schrödinger cocycles inspired by [3].

1.1 Definitions and assumptions

Throughout this paper, we will make the following assumption.

Assumption: there exists $0 < \kappa < 1$ and $\tau \geq \max(1, d-1)$ such that

$$\forall m \in \mathbb{Z}^d \setminus \{0\}, \mid \langle m, \omega \rangle \mid \geq \frac{\kappa}{\mid m \mid^\tau} \tag{7}$$

The numbers κ and τ will be fixed from now on. This is a diophantine condition on ω. We shall define other types of diophantine conditions, which refer to the vector ω.

Definition: Let $z \in \mathbb{R}$; we say that z is diophantine with respect to $\omega \in \mathbb{R}^d$ and we write $z \in DC_\omega$ if there exists $\kappa' > 0, \tau' > \max(1, d-1)$ such that for all $m \in \mathbb{Z}^d \setminus \{0\}$,

$$\mid z - 2\pi \langle m, \omega \rangle \mid \geq \frac{\kappa'}{\mid m \mid^{\tau'}} \tag{8}$$

The following diophantine condition is also known as "second Melnikov condition" and refers to the spectrum of a matrix:

Definition: Let $A \in gl(n, \mathbb{C})$ and $\{\alpha_1, \ldots, \alpha_n\}$ its spectrum. Let $\kappa' > 0, N \in \mathbb{N}$; we say that A *has a* $DC_\omega^N(\kappa', \tau)$ *spectrum* if

$$\forall 1 \leq j, k \leq n, \forall m \in \mathbb{Z}^d \setminus \{0\}, \mid m \mid \leq N \Rightarrow \mid \mathrm{Im}(\alpha_j) - \mathrm{Im}(\alpha_k) - 2\pi \langle m, \omega \rangle \mid \geq \frac{\kappa'}{\mid m \mid^\tau} \tag{9}$$

If $A \in sl(2, \mathbb{R})$ with spectrum $\{\pm \alpha\}$, this reduces to

$$\forall m \in \mathbb{Z}^d \setminus \{0\}, \mid m \mid \leq N \Rightarrow \mid 2\,\mathrm{Im}(\alpha) - 2\pi \langle m, \omega \rangle \mid \geq \frac{\kappa'}{\mid m \mid^\tau} \tag{10}$$

Definition: We will denote by $\mathcal{M}_\omega$ the set of numbers which are rational with respect to ω, i.e

$$\mathcal{M}_\omega = \{2\pi \langle m, \omega \rangle, \ m \in \mathbb{Z}^d\} \tag{11}$$

It has a module structure, therefore it is sometimes called the *frequency module.*

Now we recall the definition of the fibered rotation number of an $SL(2, \mathbb{R})$-valued cocycle:

Definition: Let $A \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$. We will denote by $\rho(A)$ and refer to as the *fibered rotation number of the cocycle* X *associated to* A the number

$$\rho(A) = \lim_{t \to +\infty} \frac{1}{t} Arg(X^t(\theta)\phi)$$

where Arg is the variation of the argument.

Remark:

- This number does not depend on the choice of ϕ (see [3], appendix);

- If A and B are conjugated in the sense of cocycles, then $\rho(A) = \rho(B) + \langle m, \omega \rangle$ for some $m \in \frac{1}{2}\mathbb{Z}^d$;
- if A is reducible to B by some function Z, then $\rho(A)$ coïncides with the Floquet exponent of A i.e the modulus of the imaginary part of the eigenvalues of B (which is well-defined only modulo $\frac{1}{2}\mathcal{M}_\omega$).

Notations: The usual operator norm will be denoted by $||\ .\ ||$. In the space $C^k(\mathbb{T}^d, gl(n,\mathbb{C}))$ of k times differentiable matrix-valued functions on the torus, we will use the norm

$$|| F ||_k = \sup_{k' \leq k; \theta \in \mathbb{T}^d} || d^{k'} F(\theta) ||$$

For any map $Z \in C^1(2\mathbb{T}^d, gl(n,\mathbb{C}))$ we will denote by $\partial_\omega Z$ the derivative of Z in the direction ω:

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega Z(\theta) = \frac{d}{dt} Z(\theta + t\omega)_{|t=0}$$

2 A lemma on analytic cocycles

We first recall a proposition which will be used as inductive step in the proof of Theorem 1.1. It was proved in [2] (Proposition 2.14).

Notations: In the following proposition, for $r > 0$ and any set E, we will denote by $C^\omega_r(\mathbb{T}^d, E)$ the space of functions which are analytic on a "strip" $\{z = (z_1, \dots, z_d) \in \mathbb{C}^d, | \operatorname{Im} z_1 | < r, \dots, | Imz_d | < r\}$, 1-periodic in $\operatorname{Re} z_1, \dots, \operatorname{Re} z_d$ and whose restriction on $\mathbb{R}^d$ has values in E. The writing $C^\omega_r(2\mathbb{T}^d, E)$ will stand for functions which are analytic on a strip and E-valued on $\mathbb{R}^d$, but only 2-periodic in $\operatorname{Re} z_1, \dots, \operatorname{Re} z_d$.

The norm in $C^\omega_r(2\mathbb{T}^d, gl(n,\mathbb{C}))$ will be written $|\ .\ |_r$.

We shall fix a Lie group G amongst $GL(n,\mathbb{C}), U(n), SL(2,\mathbb{C}), SL(2,\mathbb{R}), O(2)$ and denote by $\mathcal{G}$ its Lie algebra.

To simplify the statements, we shall use the following technical abreviations:

$$\left\{ \begin{array}{c} N(r,\epsilon) = \frac{1}{2\pi r} |\log \epsilon| \\ R(r,r') = \frac{1}{(r-r')^8} 80^4 (\frac{1}{2}n(n-1)+1)^2 \\ \kappa''(r,r',\epsilon) = \frac{\kappa}{n(8R(r,r')^{\frac{1}{2}n(n-1)+1} N(r,\epsilon))^\tau} \end{array} \right. \qquad (12)$$

Proposition 2.1 *Let*

- $A \in \mathcal{G}$,
- $r \leq \frac{1}{2}, r'' \in [\frac{95}{96}r, r[$,
- $\bar{A}, \bar{F} \in C^\omega_r(\mathbb{T}^d, \mathcal{G})$ *and* $\Psi \in C^\omega_r(2\mathbb{T}^d, G)$,

- $\epsilon = |\bar{F}|_r$,

There exists $C > 0$ depending only on n, d, κ, τ and there exists $D \in \mathbb{N}$ depending only on n, d, τ such that if

1. $\bar{A}$ *is reducible to* A *by* Ψ,
2. $$\epsilon \leq \frac{C}{(||A|| + 1)^D}(r - r'')^D \tag{13}$$
3. $|\Psi|_r \leq (\frac{1}{\epsilon})^{-\frac{1}{2}(r-r'')}$ *et* $|\Psi^{-1}|_r \leq (\frac{1}{\epsilon})^{-\frac{1}{2}(r-r'')}$,

then there exist

- $\epsilon' \in [\epsilon^{R(r,r'')^{n^2}}, \epsilon^{100}]$;
- $Z' \in C^\omega_{r''}(2\mathbb{T}^d, G)$,
- $\bar{A}', \bar{F}' \in C^\omega_{r''}(2\mathbb{T}^d, \mathcal{G})$,
- $\Psi' \in C^\omega_r(2\mathbb{T}^d, G)$,
- $A' \in \mathcal{G}$

satisfying the following properties:

1. $\bar{A}'$ *is reducible by* Ψ' *to* A',
2. $|\bar{F}'|_{r''} \leq \epsilon'$,
3. $|\Psi'|_{r''} \leq (\frac{1}{\epsilon'})^{\frac{1}{4}(r-r'')}$ *and* $|\Psi'^{-1}|_{r''} \leq (\frac{1}{\epsilon'})^{\frac{1}{4}(r-r'')}$,
4. $||A'|| \leq ||A|| + |\log \epsilon| \left(\frac{1}{r-r''}\right)^D$;
5. $$\partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}') \tag{14}$$
6. $$|Z' - Id|_{r''} \leq \frac{1}{C}\left(\frac{(1 + ||A||)|\log \epsilon|}{r - r''}\right)^D \epsilon^{1-4(r-r'')} \tag{15}$$

   and so does $(Z')^{-1} - Id$.

Moreover,

- *if* $\mathcal{G} = o(2)$ *or* $u(n)$*, the same holds with the weaker condition*

  $$\epsilon \leq C(r - r'')^D \tag{16}$$

  instead of (13);

- *if* $\mathcal{G} = sl(2,\mathbb{C})$ *or* $sl(2,\mathbb{R})$*, then either* $\Psi'^{-1}\Psi$ *is the identity or* $\| A' \| \leq \kappa''(r,r'',\epsilon) + \epsilon^{\frac{1}{2}}$*.*

The proof of Proposition 2.1 given in [2] also implies the following:

If A has a $DC_\omega^N(\kappa'',\tau)$ spectrum with $N = N(r,\epsilon)$ and $\kappa'' = \kappa''(r,r'',\epsilon)$, then $\Psi = \Psi'$.

By construction, functions Ψ and Ψ' also satisfy the following, in case $\mathcal{G} = sl(2,\mathbb{R})$:

If Ψ *satisfies*:

for all $A, A' \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\partial_\omega \Psi = A\Psi - \Psi A' \Rightarrow \rho(A) = \rho(A') + 2\pi\langle m,\omega\rangle$ for some $m \in \frac{1}{2}\mathbb{Z}^d$

then Ψ' *satisfies the same property:*

for all $A, A' \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\partial_\omega \Psi' = A\Psi' - \Psi' A' \Rightarrow \rho(A) = \rho(A') + 2\pi\langle m',\omega\rangle$ for some $m' \in \frac{1}{2}\mathbb{Z}^d$

with $m = m'$ *if and only if* $\Psi = \Psi'$*.*

3 Almost reducibility

First we need a numerical lemma:

Lemma 3.1 *Let* $C > 0, D, k \in \mathbb{N}, \epsilon_j = \frac{1}{j^k}$*. There exists a* k_1 *such that if* $k \geq k_1$*, then for all* $j \geq 2$*,*

$$C[j(j+1) \mid \log \epsilon_j \mid]^D \epsilon_j^{1-\frac{4}{j(j+1)}} \leq \frac{1}{j^2} \tag{17}$$

Proof: Equation (17) is equivalent to

$$C[j(j+1)k\log j]^D (\frac{1}{j})^{k(1-\frac{4}{j(j+1)})} \leq \frac{1}{j^2} \tag{18}$$

There exists k_1 such that for all $k \geq k_1, j \geq 2$,

$$C\frac{k^D}{j^{(\frac{k}{2}-3D)}} \leq \frac{1}{j^2} \tag{19}$$

so (17) holds. □

We will now state the main result for G among $GL(n,\mathbb{C}), U(n), SL(2,\mathbb{C}), SL(2,\mathbb{R}), O(2)$. We shall denote by $\mathcal{G}$ the Lie algebra associated to G.

Theorem 3.2 *Let* $A \in \mathcal{G}$*. There exists* $k_0, D' \in \mathbb{N}$ *only depending on* n, d, κ, τ, A *such that for all* $k \geq k_0$ *and* $F \in C^k(\mathbb{T}^d, \mathcal{G})$*, there exists* ϵ_0 *depending only on* n, d, κ, τ, A, k *such that if* $||F||_k \leq \epsilon_0$*, then there exist*

- $Z_\infty \in C^{k-D'}(\mathbb{T}^d, G)$,
- $\bar{A}_\infty \in C^{k-D'}(\mathbb{T}^d, \mathcal{G})$,
- *a sequence* $(\bar{A}_j)_{j\geq 1}$ *of functions in* $C^{k-D'}(\mathbb{T}^d, \mathcal{G})$,
- *a sequence* $(\Psi_j)_{j\geq 1}$ *of functions in* $C^{k-D'}(\mathbb{T}^d, G)$,
- *a sequence* $(A_j)_{j\geq 1}$ *of elements of* $\mathcal{G}$

such that

1. $\bar{A}_\infty$ *is the limit in* $C^{k-D'}(\mathbb{T}^d, \mathcal{G})$ *of the sequence* $\bar{A}_j$,
2. *the functions* $\bar{A}_j$ *are reducible to* A_j *by* Ψ_j,
3. $$\partial_\omega Z_\infty(\theta) = (A + F(\theta))Z_\infty(\theta) - Z_\infty(\theta)\bar{A}_\infty(\theta) \tag{20}$$

Moreover, in the case $\mathcal{G} = sl(2, \mathbb{R})$, *there exist*

- *a sequence* $(M_j)_{j\geq 1}$ *of elements of* $\frac{1}{2}\mathbb{Z}^d$
- *unbounded sequences* $(N_j)_{j\geq 1}$ *and* $(R_j)_{j\geq 1}$ *of integers*
- *and a sequence* $(\kappa_j)_{j\geq 1}$ *tending to 0*

such that

- *for all* $A_1, A_2 \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$ *and all* j,

$$\partial_\omega \Psi_j = A_1\Psi_j - \Psi_j A_2 \Rightarrow \rho(A_1) = \rho(A_2) + 2\pi \sum_{l=1}^{j} \langle M_l, \omega\rangle$$

- *either* A_j *has a* $DC_\omega^{N_j}(\kappa_j, \tau)$ *spectrum, which implies that* $M_j = 0$ *or* $M_{j-1} = 0$, *or* $R_{j-1}N_{j-1} < M_j \leq N_j$ *and in that case* $\sigma(A_j) \subset B(0, \kappa_{j-1})$;
- $\mid \rho(A_{j+1}) - (\rho(A_j) + 2\pi\langle M_j, \omega\rangle) \mid \leq \kappa_j$;
- *if there exists* $J \geq 1$ *such that* $M_j = 0$ *for all* $j \geq J$, *then* $A + F$ *is reducible.*

In this statement, properties 1, 2 and 3 are sufficient to get Theorem 1.1, but the other properties will be used in the application to $SL(2, \mathbb{R})$-valued cocycles.

Proof: • By [8], there exists a sequence $(F_j)_{j\geq 1}$, $F_j \in C^\omega_{\frac{1}{j}}(\mathbb{T}^d, \mathcal{G})$ and a universal constant C', such that

$$\begin{cases} ||F_j - F||_k \to 0 \text{ when } j \to +\infty \\ |F_j|_{\frac{1}{j}} \leq C'||F||_k \\ |F_{j+1} - F_j|_{\frac{1}{j+1}} \leq C'(\frac{1}{j})^k ||F||_k \end{cases} \tag{21}$$

Moreover, this sequence is obtained from F regardless of its regularity, i.e if $k \leq k'$ and $F \in C^{k'}$, then properties (21) hold with k' instead of k (since F_j is the convolution of F with a map which does not depend on k).

Let $C > 0, D$ be as in Proposition 2.1. One can assume $C \leq 1$. Recall that these numbers only depend on n, d, κ, τ, A. For all $r > r' > 0$, let

$$\epsilon_0'(r, r') = C(r - r')^D$$

For all $j \geq 1$, let

$$\epsilon_j = \epsilon_0'(\frac{1}{j}, \frac{1}{j+1})$$

Let k_1 be as in Lemma 3.1 and let $k_0 \geq k_1$ be a number depending only on n, d, κ, τ, A such that for all $j \geq 2$,

$$\frac{C}{j^{k_0}} \leq \epsilon_j$$

Assume $k \geq k_0$ and let

$$\epsilon_j' = \frac{C}{j^k}$$

and

$$\alpha_j = \frac{4}{j(j+1)}$$

- *First step:* Assume that

$$C'||F||_k \leq \epsilon_2' = \frac{C}{2^k} \tag{22}$$

(notice that this condition on $||F||_k$ only depends on n, d, κ, τ, A, k). Then

$$|F_2|_{\frac{1}{2}} \leq \epsilon_2' \leq \epsilon_2$$

therefore, by Proposition 2.1, there exist

- $\epsilon_2'' \leq |F_2|_{\frac{1}{2}}^{100}$
- $Z_2 \in C^\omega_{\frac{1}{3}}(\mathbb{T}^d, G)$,
- $\bar{A}_2 \in C^\omega_{\frac{1}{3}}(\mathbb{T}^d, \mathcal{G})$
- $\bar{F}_2 \in C^\omega_{\frac{1}{3}}(\mathbb{T}^d, \mathcal{G})$
- $\Psi_2 \in C^\omega_1(\mathbb{T}^d, G)$
- $A_2 \in \mathcal{G}$

such that

1. $\bar{A}_2$ is reducible to A_2 by Ψ_2,

2. $| \bar{F}_2 |_{\frac{1}{3}} \leq \epsilon_2'' \leq \frac{1}{2}\epsilon_3'$,

3. $| \Psi_2 |_{\frac{1}{3}} \leq \left(\frac{1}{\epsilon_2''}\right)^{\alpha_2}$, as well as Ψ_2^{-1}

4. $$\partial_\omega Z_2 = (A + F_2)Z_2 - Z_2(\bar{A}_2 + \bar{F}_2) \tag{23}$$

5. $$|Z_2 - Id|_{\frac{1}{3}} \leq \frac{1}{C}\left(\frac{4}{\alpha_2}(1 + ||A||)|\log |F_2|_{\frac{1}{2}}|\right)^D |F_2|_{\frac{1}{2}}^{1-\alpha_2} \tag{24}$$

   as well as Z_2^{-1},

6. and if $\mathcal{G} = sl(2,\mathbb{R})$, Ψ_2 satisfies: for all $A, A' \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,

$$\partial_\omega \Psi_2 = A\Psi_2 - \Psi_2 A' \Rightarrow \exists M_1 \in \frac{1}{2}\mathbb{Z}^d, \rho(A) = \rho(A') + 2\pi\langle M_1, \omega\rangle \tag{25}$$

Property (24) implies

$$|Z_2 - Id|_{\frac{1}{3}} \leq \frac{1}{C}\left(\frac{4}{\alpha_2}(1 + ||A||)|\log \epsilon_2'|\right)^D (\epsilon_2')^{1-\alpha_2} \tag{26}$$

as well as for Z_2^{-1}. Lemma 3.1 then implies that $|Z_2 - Id|_{\frac{1}{3}} \leq \frac{1}{4}$.

- *Induction step:* Let $j \geq 2$. Suppose that there exists
  - $A_j \in \mathcal{G}$
  - $\Psi_j \in C_1^\omega(\mathbb{T}^d, G)$
  - $\bar{A}_j \in C^\omega_{\frac{1}{j+1}}(\mathbb{T}^d, \mathcal{G})$ et $\bar{F}_j \in C^\omega_{\frac{1}{j+1}}(\mathbb{T}^d, \mathcal{G})$
  - $\bar{Z}_j \in C^\omega_{\frac{1}{j+1}}(\mathbb{T}^d, G)$,

such that

1. $\bar{A}_j$ is reducible to A_j by Ψ_j

2. $| \Psi_j |_{\frac{1}{j+1}} \leq |\bar{F}_j|_{\frac{1}{j+1}}^{-\alpha_j}$

3. $|\bar{F}_j|_{\frac{1}{j+1}} \leq \frac{1}{2}\epsilon_{j+1}'$

4. $|\bar{Z}_j - Id|_{\frac{1}{j+1}} \leq \sum_{l=2}^{j} \frac{1}{l^2}$

5. if $\mathcal{G} = sl(2,\mathbb{R})$, for all $A, A' \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,

$$\partial_\omega \Psi_j = A\Psi_j - \Psi_j A' \Rightarrow \exists M_1, \dots, M_{j-1} \in \frac{1}{2}\mathbb{Z}^d, \rho(A') = \rho(A) + 2\pi\langle M_1 + \dots + M_{j-1}, \omega\rangle \tag{27}$$

6. and

$$\partial_\omega \bar{Z}_j = (A + F_j)\bar{Z}_j - \bar{Z}_j(\bar{A}_j + \bar{F}_j) \tag{28}$$

Then

$$\partial_\omega \bar{Z}_j = (A + F_{j+1})\bar{Z}_j - \bar{Z}_j(\bar{A}_j + \bar{Z}_j^{-1}(F_{j+1} - F_j)\bar{Z}_j + \bar{F}_j) \tag{29}$$

and moreover, by (21),

$$|\bar{Z}_j^{-1}(F_{j+1} - F_j)\bar{Z}_j + \bar{F}_j|_{\frac{1}{j+1}} \leq \frac{1}{2}\epsilon'_{j+1} + \frac{4}{j^k}C'||F||_k \tag{30}$$

which implies, by assumption (22), that

$$|\bar{Z}_j^{-1}(F_{j+1} - F_j)\bar{Z}_j + \bar{F}_j|_{\frac{1}{j+1}} \leq \epsilon'_{j+1} \tag{31}$$

so one can apply Proposition 2.1: denoting $\tilde{\epsilon}_j = |\bar{Z}_j^{-1}(F_{j+1} - F_j)\bar{Z}_j + \bar{F}_j|_{\frac{1}{j+1}}$, there exists

- $\epsilon''_{j+1} \leq \tilde{\epsilon}_j^{100}$
- $Z_{j+1} \in C^\omega_{\frac{1}{j+2}}(\mathbb{T}^d, G)$,
- $\bar{A}_{j+1} \in C^\omega_{\frac{1}{j+2}}(\mathbb{T}^d, \mathcal{G})$ and $\bar{F}_{j+1} \in C^\omega_{\frac{1}{j+2}}(\mathbb{T}^d, \mathcal{G})$
- $\Psi_{j+1} \in C^\omega_1(\mathbb{T}^d, G)$
- $A_{j+1} \in \mathcal{G}$

such that

1. $\bar{A}_{j+1}$ is reducible by Ψ_{j+1} to A_{j+1},
2. $|\bar{F}_{j+1}|_{\frac{1}{j+2}} \leq \epsilon''_{j+1} \leq (\epsilon'_{j+1})^{100} \leq \frac{1}{2}\epsilon'_{j+2}$
3. $\mid \Psi_{j+1} \mid_{\frac{1}{j+2}} \leq (\epsilon''_{j+1})^{-\alpha_{j+1}} \leq |\bar{F}_{j+1}|_{\frac{1}{j+2}}^{-\alpha_{j+1}}$
4. $$\partial_\omega Z_{j+1} = (\bar{A}_j + \bar{Z}_j^{-1}(F_{j+1} - F_j)\bar{Z}_j + \bar{F}_j)Z_{j+1} - Z_{j+1}(\bar{A}_{j+1} + \bar{F}_{j+1}) \tag{32}$$
5. $$|Z_{j+1} - Id|_{\frac{1}{j+2}} \leq \frac{1}{C}\left(\frac{4}{\alpha_{j+1}}(1 + ||A||)|\log \tilde{\epsilon}_j|\right)^D (\tilde{\epsilon}_j)^{1-\alpha_{j+1}} \tag{33}$$

6. if $\mathcal{G} = sl(2,\mathbb{R})$, for all $A, A' \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,

$$\partial_\omega \Psi_{j+1} = A\Psi_{j+1} - \Psi_{j+1}A' \Rightarrow \exists M_1, \ldots, M_j \in \frac{1}{2}\mathbb{Z}^d, \rho(A') = \rho(A) + 2\pi\langle M_1 + \cdots + M_j, \omega\rangle \tag{34}$$

7. if A_j has a $DC_\omega^{N_j}(\kappa_j, \tau)$ spectrum, with $N_j = \frac{j+1}{2\pi} \mid \log \tilde{\epsilon}_j \mid$ and $\kappa_j = \frac{\kappa}{2[8R_j^2 N_j]^\tau}$ with $R_j = 4((j+1)(j+2))^8 80^4$, then $\Psi_{j+1} = \Psi_j$.
8. $\mid \rho(A_{j+1}) - \rho(A_j) + 2\pi\langle M_j, \omega\rangle) \mid \leq \kappa_j$.

Property (33) implies

$$|Z_{j+1} - Id|_{\frac{1}{j+2}} \leq \tilde{C}\left((j+2)^2 |\log \tilde{\epsilon}_j|\right)^D (\tilde{\epsilon}_j)^{1-\alpha_{j+1}} \tag{35}$$

so by lemma 3.1, $|Z_{j+1} - Id|_{\frac{1}{j+2}} \leq \frac{1}{(j+1)^2}$. Let $\bar{Z}_{j+1} = \bar{Z}_j Z_{j+1}$. Then

$$|\bar{Z}_{j+1} - Id|_{\frac{1}{j+2}} \leq \mid Z_{j+1} - Id \mid_{\frac{1}{j+2}} + \mid Z_{j+1} \mid_{\frac{1}{j+2}} \mid \bar{Z}_j - Id \mid_{\frac{1}{j+2}} \leq \sum_{l=2}^{j+1} \frac{1}{l^2} \tag{36}$$

Property (33) also implies

$$\begin{aligned} |Z_{j+1} - Id|_{\frac{1}{j+2}} &\leq \tilde{C}\left((j+2)^2 |\log \tilde{\epsilon}_j|\right)^D (\tilde{\epsilon}_j)^{1-\alpha_{j+1}} \\ &\leq \tilde{C}\left((j+2)^2 |\log(\epsilon'_{j+1})|\right)^D (\epsilon'_{j+1})^{1-\alpha_{j+1}} \\ &\leq \tilde{C}' k^{2D} (j+1)^{k(\alpha_{j+1}-1)+3D} \end{aligned} \tag{37}$$

- *Conclusion:* So for all $j \geq 2$, there exist
  - $\bar{Z}_j, Z_j \in C^\omega_{\frac{1}{j+1}}(\mathbb{T}^d, G)$,
  - $\bar{A}_j \in C^\omega_{\frac{1}{j+1}}(\mathbb{T}^d, \mathcal{G})$ and $\bar{F}_j \in C^\omega_{\frac{1}{j+1}}(\mathbb{T}^d, \mathcal{G})$
  - $\Psi_j \in C^\omega_1(\mathbb{T}^d, G)$,
  - $A_j \in \mathcal{G}$

such that

1. $\bar{Z}_j = Z_1 \ldots Z_j$
2. $\bar{A}_j$ is reducible to A_j by Ψ_j
3. $|\bar{F}_j|_{\frac{1}{j+1}} \leq \epsilon'_{j+1}$

4. $|\ \Psi_j\ |_{\frac{1}{j+1}} \leq |\bar{F}_j|_{\frac{1}{j+1}}^{-\alpha_j}$

5.
$$\partial_\omega \bar{Z}_j(\theta) = (A + F_j(\theta))\bar{Z}_j(\theta) - \bar{Z}_j(\theta)(\bar{A}_j(\theta) + \bar{F}_j(\theta)) \tag{38}$$

6.
$$|\bar{Z}_j - Id|_{\frac{1}{j+1}} \leq 1;\ |\bar{Z}_j^{-1} - Id|_{\frac{1}{j+1}} \leq 1 \tag{39}$$

7. and

$$|Z_j - Id|_{\frac{1}{j+1}} \leq \tilde{C}' k^{2D} j^{k(\alpha_j - 1) + 3D} \tag{40}$$

Moreover there exists a sequence $(M_j)_{j\geq 1}$ of elements of $\frac{1}{2}\mathbb{Z}^d$ such that for all $j \geq 1$,

- if A_j has a $DC_\omega^{N_j}(\kappa_j, \tau)$ spectrum, with $N_j = \frac{j+1}{2\pi} \mid \log \tilde{\epsilon}_j \mid$ and $\kappa_j = \frac{\kappa}{2[8R_j^2 N_j]^\tau}$ with $R_j = 4((j+1)(j+2))^8 80^4$, then $\Psi_{j+1} = \Psi_j$ and $M_j = 0$;
- for all $A, A' \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,

$$\partial_\omega(\Psi_{j+1}\Psi_j^{-1}) = A\Psi_{j+1}\Psi_j^{-1} - \Psi_{j+1}\Psi_j^{-1}A' \Rightarrow \rho(A) = \rho(A') + 2\pi\langle M_j, \omega\rangle \tag{41}$$

  and either $M_j = 0$ (if and only if $\Psi_j = \Psi_{j+1}$), or $M_{j-1} = 0$, or

$$R_{j-1}N_{j-1} < M_j \leq N_j$$

- $\mid \rho(A_{j+1}) - \rho(A_j) + 2\pi\langle M_j, \omega\rangle) \mid \leq \kappa_j$.

• *Convergence:* Now we have to compute the topology in which the sequence $(\bar{Z}_j)$ defined above is Cauchy. Since

$$\begin{aligned} |\bar{Z}_j - \bar{Z}_{j+1}|_{\frac{1}{j+2}} &\leq |\bar{Z}_j|_{\frac{1}{j+1}} |Z_{j+1} - Id|_{\frac{1}{j+2}} \\ &\leq \tilde{C}' k^{2D} (j+1)^{k(\alpha_{j+1} - 1) + 3D} \end{aligned} \tag{42}$$

then for all $k' \in \mathbb{N}$,

$$||\bar{Z}_j - \bar{Z}_{j+1}||_{k'} \leq C_3 (j+1)^{k(\alpha_{j+1} - 1) + 3D + k' + 1} \tag{43}$$

for some C_3 independent of j, so the sequence $(\bar{Z}_j)$ is Cauchy in the $C^{k'}$ topology if there exists an j such that for all $j' \geq j$,

$$k' + 1 + k(\alpha_{j'+1} - 1) + 3D < 0 \tag{44}$$

Let $k' = k - 3D - 2$. If $j' > 4k$, then (44) holds, therefore $(\bar{Z}_j)$ is Cauchy in the $C^{k'}$ topology.

Let Z_∞ be the limit of $(\bar{Z}_j)$ in the $C^{k'}$ topology. Taking the limit in (38), one gets

$$\partial_\omega Z_\infty(\theta) = (A + F(\theta))Z_\infty(\theta) - Z_\infty(\theta)\bar{A}_\infty(\theta) \tag{45}$$

where $\bar{A}_\infty \in C^{k'}(\mathbb{T}^d, \mathcal{G})$ is the limit in $C^{k'}(\mathbb{T}^d, \mathcal{G})$ of functions $\bar{A}_j$ such that

$$\partial_\omega \Psi_j = \bar{A}_j \Psi_j - \Psi_j A_j \tag{46}$$

• *Reducibility:* If there exists $J \geq 1$ such that $\Psi_{j+1} = \Psi_j$ for all $j \geq J$, then, taking the limit in (46), one finds a matrix A_∞ satisfying

$$\partial_\omega \Psi_J = \bar{A}_\infty \Psi_J - \Psi_J A_\infty$$

so $\bar{A}_\infty$ is reducible, and therefore $A + F$ is reducible. □

4 Application to the fibered rotation number

Proposition 4.1 *Let $\bar{A} \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$. There exists k, D' only depending on $d, \kappa, \tau, \hat{\bar{A}}(0)$ and ϵ_0 only depending on $d, \kappa, \tau, \hat{\bar{A}}(0), k$ such that if $\bar{A} \in C^k(\mathbb{T}^d, sl(2,\mathbb{R}))$, if $\rho(\bar{A}) \in DC_\omega \cup \mathcal{M}_\omega$ and if $\| \bar{A} - \hat{\bar{A}}(0) \|_k \leq \epsilon_0$, then the cocycle associated to $\bar{A}$ is reducible in $C^{k-D'}(\mathbb{T}^d, SL(2,\mathbb{R}))$.*

Proof: We shall apply Theorem 3.2 with $A = \hat{\bar{A}}(0)$ and $F = \bar{A} - \hat{\bar{A}}(0)$. Let k_0, D' only depending on $d, \kappa, \tau, \hat{\bar{A}}(0)$ as in Theorem 3.2, $k \geq k_0$ and ϵ_0 only depending on $d, \kappa, \tau, \hat{\bar{A}}(0), k$ as in Theorem 3.2. If $\| \bar{A} - \hat{\bar{A}}(0) \|_k \leq \epsilon_0$, there exists $Z_\infty \in C^{k-D'}(\mathbb{T}^d, SL(2,\mathbb{R})), \bar{A}_\infty \in C^{k-D'}(\mathbb{T}^d, sl(2,\mathbb{R}))$ such that if

$$\partial_\omega Z_\infty = \bar{A} Z_\infty - Z_\infty \bar{A}_\infty$$

and $\bar{A}_\infty$ is the limit of a sequence of maps $(\bar{A}_j)$ which are reducible to A_j by Ψ_j.

Let $(M_j), (\kappa_j), (N_j)$ be sequences as in Theorem 3.2. We shall proceed by contradiction; suppose that the cocycle associated to $\bar{A}$ is not reducible: then there exists a sequence $(j_l)_{l\geq 1}$ such that for all l, $\Psi_{j_l+1} \neq \Psi_{j_l}$ (i.e $M_{j_l} \neq 0$). Now by definition of the sequence (M_j), for all j,

$$\rho(A_j) + 2\pi \sum_{j'=1}^{j-1} \langle M_{j'}, \omega \rangle = \rho(\bar{A}_j)$$

so we have

$$| \rho(\bar{A}_j) - \rho(\bar{A}_{j+1}) | = | \rho(A_{j+1}) - (\rho(A_j) - 2\pi\langle M_j, \omega \rangle) \leq \kappa_j$$

and therefore

$$| \rho(\bar{A}_j) - \rho(\bar{A}_\infty) | \leq \sum_{j' \geq j} \kappa_{j'}$$

Suppose $\rho(\bar{A}) \in DC_\omega$, then for all $m \in \mathbb{Z}^d \setminus \{0\}$,

$$| \rho(\bar{A}) - 2\pi \langle m, \omega \rangle | \geq \frac{\kappa'}{| m |^{\tau'}}$$

for some κ', τ'; now there exists $M \in \mathbb{Z}^d$ such that for all $m \in \mathbb{Z}^d \setminus \{0\}$,

$$| \rho(\bar{A}_\infty) - 2\pi \langle m, \omega \rangle | = | \rho(\bar{A}) - 2\pi \langle m - M, \omega \rangle |$$

So, for all $m \in \mathbb{Z}^d \setminus \{M\}$ and all l big enough,

$$| \rho(A_{j_l}) + 2\pi \sum_{j'=1}^{j_l - 1} \langle M_{j'}, \omega \rangle + \sum_{j' \geq j_l} \kappa_{j'} - 2\pi \langle m, \omega \rangle | \geq \frac{\kappa'}{| m - M |^{\tau'}} \tag{47}$$

Now by definition of the sequence (j_l), for all l,

$$| \rho(A_{j_l}) - 2\pi \langle M_{j_l}, \omega \rangle | < \kappa_{j_l}$$

therefore

$$| \rho(A_{j_l}) + 2\pi \sum_{j'=1}^{j_l - 1} \langle M_{j'}, \omega \rangle + \sum_{j' \geq j_l} \kappa_{j'} - 2\pi \langle M_{j_l} - \sum_{j'=1}^{j_l - 1} M_{j'}, \omega \rangle | < 2 \sum_{j' \geq j_l} \kappa_{j'} \tag{48}$$

If one lets

$$\kappa'_l = 2 \sum_{j' \geq j_l} \kappa_{j'} [| M_{j_l} - \sum_{j'=1}^{j_l - 1} M_{j'} | + | M |]^\tau \tag{49}$$

then

$$\kappa'_l > 2 \sum_{j' \geq j_l} \kappa_{j'} (R_{j_l - 1} N_{j_l - 1} - (j_l - 1) N_{j_l - 1} + | M |)^\tau > 0 \tag{50}$$

and

$$| \rho(A_{j_l}) + 2\pi \sum_{j'=1}^{j_l - 1} \langle M_{j'}, \omega \rangle + \sum_{j' \geq j_l} \kappa_{j_l} - 2\pi \langle M_{j_l} - \sum_{j'=1}^{j_l - 1} M_{j'}, \omega \rangle | < \frac{\kappa'_l}{| M_{j_l} - \sum_{j'=1}^{j_l - 1} M_{j'} - M |^\tau} \tag{51}$$

The sequence κ'_l also satisfies

$$\kappa'_l \leq 2 \sum_{j' \geq j_l} \kappa_{j'} [\sum_{j'=1}^{j_l} N_{j'} + | M |]^\tau \leq c(j_l N_{j_l} + | M |) \sum_{j' \geq j_l} \kappa_{j'} \leq c'(j_l \frac{1}{R_{j_l}^{2\tau}} + | M | \sum_{j' \geq j_l} \kappa_{j'})$$

where c, c' do not depend on l. For l big enough, (51) contradicts (47) since $(M_j)_{j\geq 1}$ is unbounded and κ'_l tends to 0.

In the case when $\rho(\bar{A})$ is rational with respect to ω, since $\bar{A}$ and $\bar{A}_\infty$ are conjugated, $\rho(\bar{A}_\infty)$ is also rational with respect to ω. Therefore, (47) still holds, with $\kappa' = \kappa$ and $\tau' = \tau$, for M such that $\rho(\bar{A}_\infty) = \langle M, \omega \rangle$, so one is led to the same contradiction. □

Let us now consider the cocycle associated to the Schrödinger equation

$$\frac{d}{dt}X(t) = (A_\lambda + F(t\omega))X(t) \tag{52}$$

where $A_\lambda = \begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}$ and $F(t\omega) = \begin{pmatrix} 0 & V(t\omega) \\ 0 & 0 \end{pmatrix}$ with $V \in C^k(\mathbb{T}^d)$ with k to be determined later on.

Theorem 4.2 *There exists k_0 only depending on d, κ, τ such that if $k \geq k_0$ and if $V \in C^k(\mathbb{T}^d)$, there exists ϵ_0 only depending on d, κ, τ, k such that if $\| V \|_k \leq \epsilon_0$, then the cocycle which is solution of* (52) *is*

- *almost reducible for all λ,*
- *reducible for all λ such that $\rho(A_\lambda + F) \in DC_\omega \cup \mathcal{M}_\omega$.*

Proof: • First case: $\lambda \in [-2, 2]$. The norm of A_λ is then bounded independently of λ so it is enough to apply Theorem 3.2 with $A = A_\lambda$ and F as above to deduce almost reducibility; to infer reducibility if $\rho(A_\lambda + F) \in DC_\omega \cup \mathcal{M}_\omega$, apply Proposition 4.1 with $\bar{A} = A_\lambda + F$.

• Second case: $|\lambda| > 2$. Letting $Y(t) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{\lambda}}{2} \\ \frac{1}{2} & \frac{\sqrt{\lambda}}{2} \end{pmatrix} X(t)$, one has

$$\begin{aligned} Y'(t) &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{\lambda}}{2} \\ \frac{1}{2} & \frac{\sqrt{\lambda}}{2} \end{pmatrix} \begin{pmatrix} 0 & V(t\omega) - \lambda \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} \end{pmatrix} Y(t) \\ &= (\tilde{A}(\lambda) + \tilde{F}(\lambda, t\omega))Y(t) \end{aligned} \tag{53}$$

with

$$\tilde{A}(\lambda) = \begin{pmatrix} 0 & -\sqrt{\lambda} \\ \sqrt{\lambda} & 0 \end{pmatrix}$$

and

$$\tilde{F}(\lambda, t\omega) = \begin{pmatrix} -\frac{V(t\omega)}{2\sqrt{\lambda}} & \frac{V(t\omega)}{2\sqrt{\lambda}} \\ -\frac{V(t\omega)}{2\sqrt{\lambda}} & \frac{V(t\omega)}{2\sqrt{\lambda}} \end{pmatrix}$$

Thus, one can apply Theorem 3.2 with $A = \tilde{A}(\lambda)$ and $F(t\omega) = \tilde{F}(\lambda, t\omega)$ to get almost reducibility if V is bounded in the C^k topology by some constant depending only on d, κ, τ, k. One can also apply Proposition 4.1 with $\bar{A} = \tilde{A}(\lambda) + \tilde{F}(\lambda, t\omega)$ to get reducibility in the case when $\rho(A_\lambda + F) \in DC_\omega \cup \mathcal{M}_\omega$, since $\rho(A_\lambda + F) = \rho(\tilde{A}(\lambda) + \tilde{F}(\lambda, .))$. □

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