3×3 MINORS OF CATALECTICANTS

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Abstract. Secant varieties to Veronese embeddings of projective space are classical varieties whose equations are not completely understood. Minors of catalecticant matrices furnish some of their equations, and in some situations even generate their ideals. Geramita conjectured that this is the case for the secant line variety of the Veronese variety, namely that its ideal is generated by the 3×3 minors of any of the "middle" catalecticants. Part of this conjecture is the statement that the ideals of 3×3 minors are equal for most catalecticants, and this was known to hold set-theoretically. We prove the equality of 3×3 minors and derive Geramita's conjecture as a consequence of previous work by Kanev.

1. Introduction

The secant varieties to rational normal curves are defined by ideals of minors of Hankel matrices, which are fairly well understood (see [GP82] or [Con98] and the references therein). However, surprisingly enough, we do not know in general the equations of the secant varieties to Veronese embeddings of higher projective spaces. Minors of catalecticant matrices (see Section 2.2) provide some equations for these secant varieties, but turn out not to be sufficient in many cases.

Determinantal loci of catalecticant matrices are of particular interest in their own right, but also via their connection to Hilbert functions of Gorenstein Artin algebras, the polynomial Waring problem, or configurations of points in projective space (see [IK99]). In [Ger99], Geramita gives a beautiful exposition of classical results about catalecticant varieties, and proposes several further questions (see also [IK99], Chapter 9). We recall the last one, which we shall answer affirmatively in Theorem 7.1. It is divided into two parts:

Q5a. Is it true that

$$I_3(Cat(2, d-2; n)) = I_3(Cat(t, d-t; n))$$

for all t with $2 \leq t \leq d-2$?

Q5b. Is it true that for $n \geq 3$ and $d \geq 4$

$$I_3(Cat(1, d-1; n)) \subsetneq I_3(Cat(2, d-2; n))?$$

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Here $Cat(t,d-t;n)$ denotes the t-th generic catalecticant (see Section 2.2), and $I_3(Cat(t,d-t;n))$ is the ideal generated by its 3×3 minors.

Geramita also conjectures that any of the catalecticant ideals $I_3(Cat(t,d-t;n))$, $2\leq t\leq d-2$, is the ideal of the secant line variety to the d-uple embedding of $\mathbb{P}^{n-1}$. This is also true, and is part of our main result:

Theorem 7.1. *Let K be a field of characteristic 0 and let $n,d\geq 2$ be integers. The following statements hold:*

(1) *For all t with $2\leq t\leq d-2$ one has*
$$I_3(Cat(2,d-2;n)) = I_3(Cat(t,d-t;n)).$$
(2) *If $d\geq 4$ then there is a strict inclusion*
$$I_3(Cat(1,d-1;n)) \subsetneq I_3(Cat(2,d-2;n)).$$
(3) *Any of the ideals $I_3(Cat(t,d-t;n))$, $2\leq t\leq d-2$, is the ideal of the first secant variety to the d-th Veronese embedding of $\mathbb{P}^{n-1}_K$.*

We can actually do a little better than that, namely describe the decomposition of the space of 3×3 minors of any catalecticant into irreducible representations of the general linear group, and hence calculate the number of generators of the ideal of the first secant variety to the d-th Veronese embedding of $\mathbb{P}^{n-1}$.

Theorem 7.2. *With the assumptions in Theorem 7.1 and writing $\mathbb{P}^{n-1}_K=\mathbb{P}V^*$ for some n-dimensional K-vector space V, the following statements hold:*

(1) *As $GL(V)$-representations,*
$$I_3(Cat(2,d-2;n))_3 \simeq S_{(3)}S_{(d)}V/\bigoplus_{\substack{i=0\\ i\neq 1}}^{d} S_{(3d-i,i)}V.$$
(2) *The number of generators of the ideal of the first secant variety to the d-th Veronese embedding of $\mathbb{P}^{n-1}_K$ is, whenever this ideal is nonzero (i.e. $n\geq 3$, or $n=2$ and $d\geq 4$), given by the formula*
$$\binom{\binom{n+d-1}{d}+2}{3}-\binom{n+2d-1}{2d}\cdot\binom{n+d-1}{d}+\binom{n+3d-2}{3d-1}\cdot n-\binom{n+3d-1}{3d}.$$

When $n=2$, it is well-known (see [GP82], [Eis86] for a proof) that
$$I_k(Cat(k-1,d-k+1;2)) = I_k(Cat(t,d-t;2))$$
for all t with $k-1\leq t\leq d-k+1$, and that any of these ideals is the ideal of the $(k-2)$-nd secant variety to the d-uple embedding of $\mathbb{P}^1$. This fact will turn out to be useful in the proof of Theorem 7.1. We note that part (3) of Theorem 7.1 is a consequence of (1) and

(2), combined with the result of Kanev ([Kan99]) which states that the ideal of the secant line variety to the Veronese variety is generated by the 3×3 minors of the first and second catalecticants.

Theorem 7.1 yields special cases of two general conjectures. One of them is implicit in Geramita's question Q4 from [Ger99]:

Conjecture 1.1. *For all $k, n \geq 2$, $d \geq 2k-2$ and t with $k-1 \leq t \leq d-k+1$, one has*

$$I_k(Cat(k-1, d-k+1; n)) = I_k(Cat(t, d-t; n)).$$

Moreover, the following inclusions hold:

$$I_k(Cat(1, d-1; n)) \subset I_k(Cat(2, d-2; n)) \subset \cdots \subset I_k(Cat(k-1, d-k+1; n)).$$

The other one is a conjecture by Sidman and Smith ([SS09]):

Conjecture 1.2. *Let k be a positive integer. If $X \subset \mathbb{P}^n$ is embedded by the complete linear series of a sufficiently ample line bundle, then the homogeneous ideal of the $(k-2)$-nd secant variety of X is generated by the $k \times k$-minors of a 1-generic matrix of linear forms.*

Conjecture 1.2 has been recently proved to be false for singular X ([BGL10]), but there are no known smooth counterexamples. The case $X = \mathbb{P}^r$ is a sufficiently interesting special case. Both conjectures 1.1 and 1.2 are known to be true for $k = 2$, by results of Pucci ([Puc98]) and Sidman and Smith ([SS09]). The argument in [Puc98] is rather long, so we will give a simplified proof in Section 6.

The main tool that we will be using in our proofs will be the construction of a series of *polarization* maps from certain representations W of the general linear group to representations W' of some symmetric group. These maps will be sections of certain *specialization* maps going in the opposite direction. Both the polarization and specialization maps will just be maps of vector spaces, depending on a partition λ and some choices: the choice of a Borel subgroup of GL_n and of a Young symmetrizer c_λ. The key point is that they will induce isomorphisms between the λ-highest weight space $\text{hwt}_\lambda(W) \subset W$ and the "λ-highest weight space" $\text{hwt}_\lambda(W') = c_\lambda \cdot W' \subset W'$, allowing us to go back and forth between GL-representations and representations of symmetric groups.

The structure of the paper is as follows. In Section 2 we introduce the basic notions from representation theory that will be needed later. We also give some background on catalecticant varieties and describe their relationship to Gorenstein Artin algebras, which motivates Conjecture 1.1. Section 3 deals with the secant varieties to the Veronese variety. We give a characterization of their equations as intersections of kernels of certain maps of representations (the *prolongations* in [Lan10]). In Section 4 we set up the "generic case": we introduce certain representations of symmetric groups which correspond by specialization to ideals of minors of catalecticant matrices; we also introduce the "generic prolongations". In Section 5 we describe the construction and properties of the polarization and specialization maps. We then illustrate our techniques in Section 6 by giving a simple proof of Pucci's result - Conjecture 1.1 in the case $k = 2$. Section 7 contains the proofs of the main results - theorems 7.1 and 7.2.

2. Preliminaries

2.1. **Representation theory.** In this section K will be a field of characteristic zero and G a group, either $GL(V)$, the group of invertible linear transformations of some vector space V, or S_N, the group of permutations of the set $\{1, \cdots, N\}$, for some positive integer N. For an introduction to the representation theory of the symmetric and general linear groups see [FH91] or [Mac79].

A (finite dimensional) *representation* of G is a left-module over the *group algebra* $K[G] = \{\sum_g a_g \cdot g : a_g \in K,\ a_g = 0 \ \text{ for almost all } \ g\}$, which is finite dimensional as a vector space over K. A representation is said to be *irreducible* if it has no nontrivial subrepresentations. Every representation decomposes as a direct sum of irreducible representations.

A *partition* λ of an integer N is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ with $N = \sum \lambda_i$. We write $\lambda = (\lambda_1, \lambda_2, \cdots)$. Alternatively, if μ is a partition having i_j parts equal to μ_j for all j, then we write $\mu = (\mu_1^{i_1} \cdot \mu_2^{i_2} \cdots)$. To a partition λ we associate a *Young diagram* which consists of left-justified rows of boxes, with λ_i boxes in the i-th row. We shall identify a partition λ with its Young diagram. A *tableau* is a filling of the Young diagram. The *canonical tableau* is the one that numbers the boxes consecutively from left to right, up to down. For $\lambda = (3, 3, 1) = (1^1 \cdot 3^2)$, the canonical tableau is

1	2	3
4	5	6
7		

2.1.1. *The general linear group: $G = GL(V)$.* Fixing a basis $x_1, \cdots, x_n$ of V we get an identification of G with the set of invertible $n \times n$ matrices. The G-representations for which elements of G act as matrices with polynomial entries are called *polynomial representations.* The polynomial irreducible G-representations are classified by partitions λ with at most n terms. They are the so called *Schur functors* $S_\lambda V$. Special cases are

$$S_{(d)}V = \mathrm{Sym}^d V, \quad \text{the } d\text{-th symmetric power of } V, \text{ and}$$

$$S_{(1^k)}V = \Lambda^k V, \quad \text{the } k\text{-th exterior power of } V.$$

Every polynomial representation W decomposes as a direct sum of $S_\lambda V$'s. We write

$$W = \bigoplus_\lambda W_\lambda, \quad \text{where } W_\lambda \simeq (S_\lambda V)^{m_\lambda} \ \text{ for some nonnegative integers } \ m_\lambda.$$

$m_\lambda = m_\lambda(W)$ is called the *multiplicity* of $S_\lambda V$ in W. We call W_λ the λ-*part* of the representation W. From now on, every representation will be assumed to be polynomial.

We consider the *maximal torus* $T \subset G$ consisting of the diagonal matrices in G and the *Borel subgroup* $B \subset G$ of upper-triangular matrices with ones on the diagonal. Given a G-representation W, a *weight vector* w with *weight* $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is a nonzero vector in W with the property that for any $t = \operatorname{diag}(t_1, \cdots, t_n) \in T$,

$$t \cdot w = t_1^{a_1} \cdots t_n^{a_n} w.$$

The vectors with this property form a vector space called the *a-weight space* of W, which we denote by $\operatorname{wt}_a(W)$.

A *highest weight vector* of a G-representation W is an element $w \in W$ invariant under the action of B. Such a w is automatically a weight vector. $W = S_\lambda V$ has a unique (up to scaling) highest weight vector w with corresponding weight $\lambda = (\lambda_1, \cdots, \lambda_n)$. In general, we define the λ-*highest weight space* of a representation W to be the set of highest weight vectors in W with weight λ, and denote it by $\operatorname{hwt}_\lambda(W)$. It is a vector space of dimension $m_\lambda(W)$.

2.1.2. *The symmetric group:* $G = S_N$. The irreducible representations of G are in one-to-one correspondence with the partitions of N. They are isomorphic to the left ideals $V_\lambda = K[G] \cdot c_\lambda$ for certain *Young symmetrizers* $c_\lambda \in K[G]$ constructed as follows. Given the canonical tableau T_λ of shape λ, we consider the subgroups of G

$$R_\lambda = \{g \in G : g \ \text{ preserves each row of } \ T_\lambda\},$$

$$C_\lambda = \{g \in G : g \ \text{ preserves each column of } \ T_\lambda\}$$

and define

$$a_\lambda = \sum_{g \in R_\lambda} g, \quad b_\lambda = \sum_{g \in C_\lambda} \operatorname{sgn}(g) \cdot g, \quad c_\lambda = a_\lambda \cdot b_\lambda,$$

where $\operatorname{sgn}(g)$ denotes the signature of the permutation g. Every G-representation W decomposes as a direct sum of V_λ's. We write

$$W = \bigoplus_\lambda W_\lambda, \quad \text{where } W_\lambda \simeq V_\lambda^{m_\lambda} \ \text{ for some nonnegative integers } \ m_\lambda.$$

$m_\lambda = m_\lambda(W)$ is called the *multiplicity* of V_λ in W. We call W_λ the λ-*part* of the representation W. We define, in analogy with the general linear group situation, the λ-*highest weight space* of W to be the vector space $\operatorname{hwt}_\lambda(W) = c_\lambda \cdot W \subset W$. It is a vector space of dimension m_λ.

2.2. **Catalecticant varieties.** Given a vector space V of dimension n over a field K of characteristic zero, and a basis $x_1, \cdots, x_n$ of V, we consider its dual space V^* with dual basis $e_1, \cdots, e_n$. For every positive integer d we get a basis of $S_{(d)}V^*$ consisting of *divided power monomials* $e^{(\alpha)}$ of degree d in the e_i's, as follows. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex of degree $|\alpha| = \alpha_1 + \cdots + \alpha_n = d$, then we write e^α for the monomial $e_1^{\alpha_1} \cdots e_n^{\alpha_n}$, and $e^{(\alpha)}$ for $e^\alpha/\alpha!$, where $\alpha! = \alpha_1! \cdots \alpha_n!$. For $a, b > 0$ with $a + b = d$ we get a *divided power multiplication* map $S_{(a)}V^* \otimes S_{(b)}V^* \to S_{(d)}V^*$, sending $e^{(\alpha)} \otimes e^{(\beta)}$ to $e^{(\alpha+\beta)}$. We can represent this via a multiplication table whose rows and columns are indexed by multiindices of degree a and b respectively, and whose entry in the (α, β) position is $e^{(\alpha+\beta)}$. The *generic catalecticant matrix* $Cat(a, b; n)$ is defined to be the matrix obtained from this multiplication table by replacing each $e^{(\alpha+\beta)}$ with the variable $z_{\alpha+\beta}$, where $(z_\gamma)_{|\gamma|=d} \subset S_{(d)}V$ is the dual basis to $(e^{(\gamma)})_{|\gamma|=d} \subset S_{(d)}V^*$.

One can also think of z_γ's as the coefficients of the generic form of degree d in the e_i's, $F = \sum z_\gamma e^{(\gamma)}$. Specializing the z_γ's we get an actual form $f \in S_{(d)}V^*$, and we denote the corresponding *catalecticant matrix* by $Cat_f(a, b; n)$. Any such form f is the *dual socle generator* of some Gorenstein Artin algebra A ([Eis95]) with socle degree d and Hilbert function

$$h_i(A) = \text{rank}(Cat_f(i, d - i; n)).$$

Macaulay's theorem on the growth of the Hilbert function of an Artin algebra ([BH93]) implies that if $h_i < r$ for some $i \geq r - 1$, then the function becomes nonincreasing from that point on. In particular, since A is Gorenstein, h is symmetric, so if $h_i < r$ for some $r - 1 \leq i \leq d - r + 1$, then we have

$$h_1 \leq h_2 \leq \cdots \leq h_{r-1} = h_r = \cdots = h_{d-r+1} \geq h_{d-r+2} \geq \cdots \geq h_d.$$

If we denote by $I_r(i) = I_r(Cat(i, d - i; n))$ the ideal of $r \times r$ minors of the i-th generic catalecticant, then the remarks above show that whenever $r - 1 \leq d - r + 1$ we have the following up-to-radical relations:

$$I_r(1) \subset \cdots \subset I_r(r - 1) = \cdots = I_r(d - r + 1) \supset \cdots \supset I_r(d - 1).$$

Conjecture 1.1 states that these relations hold exactly. We prove this in the case $r = 3$ in Theorem 7.1.

3. Equations of the Secants to Veronese Varieties

Given a vector space U over a field K of characteristic zero, we write $\mathbb{P}(U)$ for the projective space of lines in U. For $0 \neq u \in U$, we denote by $[u]$ the corresponding line. If U comes with a basis B in which u has coordinates $(u_\beta)_{\beta \in B}$ then we also write $[u_\beta]_\beta$ for $[u]$. For some positive integer d, we consider the Veronese embedding

$$\nu_d : \mathbb{P}(V^*) \to \mathbb{P}(S_{(d)}V^*), \quad \text{given by} \quad [e] \mapsto [e^{(d)}].$$

Fixing bases $e_1, \cdots, e_n$ for V^* and $e^{(\alpha)} = e^\alpha/\alpha!$ for $S_{(d)}V^*$, ν_d is given as

$$[u_i]_i \mapsto [u^\alpha]_\alpha.$$

We define the *k-th secant variety* to the Veronese embedding of $\mathbb{P}V^*$ to be the closure of the set

$$\left\{ \left[\sum_{i=0}^{k} c_i v_i^{(d)} \right] : c_i \in K,\ v_i \in V^* \right\}$$

in $\mathbb{P}(S_{(d)}V^*)$, and denote it by $\sigma_{k+1}(\nu_d(\mathbb{P}V^*))$. Note that for $k = 0$ this is just the image of ν_d.

The homogeneous coordinate ring of $\mathbb{P}(S_{(d)}V^*)$ is $S = \mathrm{Sym}(S_{(d)}V)$, the symmetric algebra over $S_{(d)}V$. Using the basis $(z_\alpha) \subset S_{(d)}V$ dual to $(e^{(\alpha)}) \subset S_{(d)}V^*$ we can write S as the polynomial ring $K[z_\alpha]$. One very classical problem which is still open is to find the ideal $I \subset S$ of polynomials vanishing on $\sigma_k(\nu_d(\mathbb{P}V^*))$ (see [LO10] for the current state of the art). The following result is well-known (see [IK99] or [Lan10]).

Lemma 3.1. *For every $1 \leq i \leq d$ and $k \geq 1$, the ideal $I_{k+1}(Cat(i, d-i; n))$ is contained in the ideal of $\sigma_k(\nu_d(\mathbb{P}V^*))$.*

Given a positive integer r and a partition $\mu = (\mu_1, \cdots, \mu_t)$, we consider the set $\mathcal{P}_\mu$ of all partitions of $\{1, \cdots, r\}$ of shape μ, i.e.

$$\mathcal{P}_\mu = \left\{ A = \{A_1, \cdots, A_t\} : |A_i| = \mu_i \text{ and } \bigsqcup_{i=1}^{t} A_i = \{1, \cdots, r\} \right\}.$$

The set $\mathcal{OP}_\mu$ of ordered partitions of shape λ is defined analogously,

$$\mathcal{OP}_\mu = \left\{ A = (A_1, \cdots, A_t) : |A_i| = \mu_i \text{ and } \bigsqcup_{i=1}^{t} A_i = \{1, \cdots, r\} \right\}.$$

Definition 3.2. For a partition $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$ of r, we consider the map

$$\pi_\mu : S_{(r)}S_{(d)}V \longrightarrow \bigotimes_{j=1}^{s} S_{(i_j)}S_{(d \cdot \mu_j)}V,$$

given by

$$z_1 \cdots z_r \mapsto \sum_{A \in \mathcal{P}_\mu} \bigotimes_{j=1}^{s} \prod_{\substack{B \in A \\ |B| = \mu_j}} m(z_i : i \in B),$$

where

$$m : (S_{(d)}V)^{\otimes \mu_j} = S_{(d)}V \otimes S_{(d)}V \otimes \cdots \otimes S_{(d)}V \longrightarrow S_{(d \cdot \mu_j)}V$$

denotes the usual multiplication map.

Note that when $\mu=(1^r)$, π_μ is the identity map. We shall soon see that the polynomials vanishing on the secants to the Veronese variety may be characterized as the elements in the intersections of kernels of certain maps π_μ. In the next section we will consider analogous maps in the context of representations of symmetric groups.

Example 3.3. Take $d=4$, $\mu=(2,1)=(1^1\cdot 2^1)$, $z_1=z_{(3,1,0)}$, $z_2=z_{(2,1,1)}$, $z_3=z_{(2,0,2)}$ (recall that z_α denotes the dual of $e^{(\alpha)}$). Then

$$\begin{aligned}\pi_\mu(z_1z_2z_3)&=m(z_1,z_2)z_3+m(z_1,z_3)z_2+m(z_2,z_3)z_1\\&=z_{(5,2,1)}z_{(2,0,2)}+z_{(5,1,2)}z_{(2,1,1)}+z_{(4,1,3)}z_{(3,1,0)}.\end{aligned}$$

Proposition 3.4 (Prolongation, [Lan10])**.** *For a positive integer r, the polynomials of degree r vanishing on $\sigma_{k+1}(\nu_d(\mathbb{P}V^*))$ are precisely the elements of $S_{(r)}S_{(d)}V$ in the intersection of the kernels of the maps π_μ, where μ ranges over all partitions of r with at most $k+1$ parts.*

Remark 3.5. If $r\le k+1$ then the partition $\mu=(1^r)$ has at most $k+1$ parts. As noticed before, π_μ is then the identity map, proving that there are no polynomials of degree at most $k+1$ vanishing on the k-th secant of the Veronese (see [Lan10] for generalizations of this fact).

Proof of proposition 3.4. Consider a polynomial $f\in S_{(r)}S_{(d)}V$ vanishing on the variety $\sigma_{k+1}(\nu_d(\mathbb{P}V^*))$. We write

$$f=\sum_{\underline{\alpha}}c_{\underline{\alpha}}z^{\underline{\alpha}},$$

where each $\underline{\alpha}=\{\alpha_1,\cdots,\alpha_r\}$ is a multiset of multiindices $\alpha_1,\cdots,\alpha_r$, $|\alpha_i|=d$, and

$$z^{\underline{\alpha}}=z_{\alpha_1}z_{\alpha_2}\cdots z_{\alpha_r}.$$

Consider $k+1$ generic points $v_i\in\mathbb{P}V^*$, and a generic combination of $\nu_d(v_i)$

$$p=\sum_{i=0}^k a_i[v_i^\alpha]_\alpha\in\sigma_{k+1}(\nu_d(\mathbb{P}V^*)).$$

We have

$$\begin{aligned}0=f(p)&=\sum_{\underline{\alpha}=\{\alpha_1,\cdots,\alpha_r\}}c_{\underline{\alpha}}\prod_{j=1}^r\left(\sum_{i=0}^k a_iv_i^{\alpha_j}\right)\\&=\sum_{\underline{\alpha}}c_\alpha\cdot\left(\sum_{\mu_0+\cdots+\mu_k=r}a_0^{\mu_0}\cdots a_k^{\mu_k}\left(\sum_{\substack{A\in\mathcal{OP}_\mu\\A=(B_0,\cdots,B_k)}}v_0^{\sum_{j\in B_0}\alpha_j}\cdots v_k^{\sum_{j\in B_k}\alpha_j}\right)\right)\\&=\sum_{\mu_0+\cdots+\mu_k=r}a_0^{\mu_0}\cdots a_k^{\mu_k}\left(\sum_{\substack{\underline{\alpha},A\in\mathcal{OP}_\mu\\A=(B_0,\cdots,B_k)}}c_{\underline{\alpha}}\cdot v_0^{\sum_{j\in B_0}\alpha_j}\cdots v_k^{\sum_{j\in B_k}\alpha_j}\right).\end{aligned}$$

Now since $a_0, \cdots, a_k$ are generic, we must have for each $\mu = (\mu_0, \cdots, \mu_k)$ that

$$\sum_{\substack{\underline{\alpha}, A \in \mathcal{OP}_\mu \\ A=(B_0,\cdots,B_k)}} c_{\underline{\alpha}} \cdot v_0^{\sum_{j\in B_0} \alpha_j} \cdots v_k^{\sum_{j\in B_k} \alpha_j} = 0.$$

The v_i's are also generic, so in fact

$$\sum_{\substack{\underline{\alpha}, A \in \mathcal{OP}_\mu \\ A=(B_0,\cdots,B_k)}} c_{\underline{\alpha}} \cdot z_{\sum_{j\in B_0} \alpha_j} \otimes \cdots \otimes z_{\sum_{j\in B_k} \alpha_j} = 0.$$

Now since two ordered partitions in $\mathcal{OP}_\mu$ correspond to the same unordered partition in $\mathcal{P}_\mu$ if and only if they differ by permutations of parts with the same size, one can see that the left hand side of the above equality is precisely

$$\sum_{\underline{\alpha}} c_{\underline{\alpha}} \pi_\mu(z_{\alpha_1} \cdots z_{\alpha_r}) = \pi_\mu(f),$$

so $f \in \mathrm{Ker}(\pi_\mu)$.

Conversely, the above calculations show that if $f \in \mathrm{Ker}(\pi_\mu)$ for all μ with at most $k+1$ parts, then f vanishes on $\sigma_{k+1}(\nu_d(\mathbb{P}V^*))$. □

Remark 3.6. In fact, in the previous proposition it suffices to take the intersection of the kernels of the maps π_μ, for μ a partition of r with exactly $k+1$ parts. This is because if λ is a partition obtained from μ by collecting together parts of μ, then π_λ factors through π_μ, so the kernel of π_λ is superfluous.

Corollary 3.7. *The polynomials of degree $k+2$ vanishing on the k-th secant variety to the Veronese variety are precisely the elements in the kernel of the map $\pi_{(1^k \cdot 2^1)}$.*

4. The "Generic" Case

Let $W_d^r = \mathrm{ind}_{S_r \sim S_d}^{S_N}(\mathbf{1})$ denote the induced representation of the trivial representation of the subgroup $S_r \sim S_d \subset S_N$, where $N = N(r,d) = r \cdot d$ and $S_r \sim S_d$ is the wreath product of S_r by S_d, regarded as a subgroup of S_N as follows. We think of S_N as the group of permutations of the set $\{1, \cdots, N\}$ and embed $S_d^r = S_d \times S_d \times \cdots \times S_d$ (r times) into S_N by letting the i-th copy of S_d act as the permutations of the set $\{d \cdot (i-1)+1, \cdots, d \cdot i\}$. Then $S_r \sim S_d$ is regarded as the normalizer of S_d^r inside S_N.

We shall think of W_d^r as the space of monomials

$$m = z_{\alpha_1} \cdots z_{\alpha_r}, \text{ where } \alpha_1, \cdots, \alpha_r \text{ is a partition of the set } \{1, \cdots, N\},$$
$$\text{with } |\alpha_i| = d \text{ for all } i = 1, \cdots, r.$$

An element σ of the symmetric group S_N acts on a monomial m as follows:

$$\sigma(m) = \sigma(z_{\alpha_1} \cdots z_{\alpha_r}) = z_{\sigma(\alpha_1)} \cdots z_{\sigma(\alpha_r)},$$

where for a subset $\alpha \subset \{1, \cdots, N\}$,

$$\sigma(\alpha) = \{\sigma(x) : x \in \alpha\}.$$

For $k \leq r$, a, b with $a+b = d$, and disjoint subsets $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k \subset \{1, \cdots, N\}$ with $|\alpha_i| = a$, $|\beta_i| = b$ for all $i = 1, \cdots k$, we let

$$[\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_k] = \det(z_{\alpha_i \cup \beta_j})_{1 \leq i,j \leq k}.$$

Fixing k, d and a, b with $a + b = d$, we define the *ideal of generic $k \times k$ minors of the a-th catalecticant* to be the collection, indexed by r, of subrepresentations $I_k^r(a, b) \subset W_d^r$ spanned by the expressions

$$[\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_k] \cdot z_{\gamma_1} \cdots z_{\gamma_{r-k}},$$

where $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k, \gamma_1, \cdots, \gamma_{r-k}$ form a partition of the set $\{1, \cdots, N\}$, with $|\alpha_i| = a$, $|\beta_i| = b$, $|\gamma_i| = d$. When r is understood from the context, we write $I_k(a, b)$ for the representation $I_k^r(a, b)$.

We would like to understand the decomposition into irreducible representations of all $I_k^r(a, b)$. This is of course a hopeless goal at this point, since not even the case $k = 1$, i.e. the decomposition of W_d^r, is understood in general. Nevertheless, we will be able to achieve our goal in the case of the representations $I_2^2(a, b)$ and $I_3^3(a, b)$. This will allow us to prove conjectures 1.1 and 1.2 in the special cases $k = 3$, $X = \mathbb{P}^n$, and to reprove Pucci's result (Theorem 6.1). We start with a general observation:

Proposition 4.1. *For any k, r, d, the subrepresentation $I_k^r(1, d-1) \subset W_d^r$ is the sum of the irreducible subrepresentations of W_d^r corresponding to partitions λ with at least k terms.*

Given a partition λ of N, we index the boxes of its Young diagram in the usual way: the i-th box is the one whose entry in the canonical tableau is equal to i. We shall identify a monomial $z_{\alpha_1} \cdots z_{\alpha_r}$ with a tableau of shape λ, having d entries equal to i in the positions indexed by the elements of the set α_i. For example, if $\lambda = (6, 2)$, $r = d = 3$, $m = z_{1,3,8} \cdot z_{2,4,7} \cdot z_{5,6,9}$, we write

$$m = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 1 & 2 & 3 & 3 \\ \hline 2 & 1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}.$$

Two tableaux differing by a permutation of the numbers $\{1, \cdots, r\}$ correspond to the same monomial, so we identify them.

$$m = z_{2,4,7} \cdot z_{5,6,9} \cdot z_{1,3,8} = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 1 & 3 & 1 & 2 & 2 \\ \hline 1 & 3 & 2 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}.$$

Given a tableau T, we write T_i for the entry in its i-th box. An element $\sigma \in S_N$ sends T to a tableau T' with $T'_i = T_{\sigma^{-1}(i)}$. If we write $\sigma = (1, 2, 4) \cdot (3, 5) \in S_9$ in cycle notation, we get

$$\sigma \cdot m = z_{2,5,8} \cdot z_{4,1,7} \cdot z_{3,6,9} = \begin{array}{|c|c|c|c|c|c|}\hline 2 & 1 & 3 & 2 & 1 & 3 \\ \hline 2 & 1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}.$$

Recall that c_λ is the Young symmetrizer corresponding to the canonical tableau of shape λ.

Lemma 4.2. *With the above conventions, we have*

(1) $c_\lambda \cdot m = 0$ *if m has repeated entries in some column.*
(2) $c_\lambda \cdot m_1 = \pm c_\lambda \cdot m_2$ *if m_1, m_2 differ by permutations within columns or by permutations of columns of the same size.*

Proof. If σ is a permutation preserving the columns of the canonical tableau, then $c_\lambda \cdot \sigma = \operatorname{sgn}(\sigma) \cdot c_\lambda$, so the first part of (2) follows. If m is a tableau with two equal entries in some column, say in the i-th and j-th boxes, then the transposition $\sigma = (i, j)$ preserves m. We get

$$c_\lambda \cdot m = c_\lambda \cdot (\sigma \cdot m) = (c_\lambda \cdot \sigma) \cdot m = -c_\lambda \cdot m,$$

so $c_\lambda \cdot m = 0$, proving (1). To finish the proof of (2), notice that if σ is a permutation of columns (of the same size) of a tableau m, then σ commutes with the symmetrizer b_λ and $a_\lambda \cdot \sigma = a_\lambda$, so $c_\lambda \cdot \sigma = c_\lambda$. □

Proof of Proposition 4.1. To prove the proposition, it's enough to show that

a) $c_\lambda \cdot m \in I_k^r(1, d-1)$ whenever λ has at least k terms and $m \in W_d^r$ is a monomial, and

b) $c_\lambda \cdot I_k^r(1, d-1) = 0$ when λ has less than k terms.

To prove a), consider a monomial $m = z_{\gamma_1} \cdots z_{\gamma_r} \in W_d^r$ and write $\lambda = (\lambda_1, \cdots, \lambda_t)$ with $k \leq t \leq r$, $\lambda_t \neq 0$. It's enough to prove that $c_\lambda \cdot m \in I_t^r(1, d-1)$, since $I_t^r(1, d-1) \subset I_k^r(1, d-1)$. If the tableau m has repeated entries in the first column, then $c_\lambda \cdot m = 0 \in I_k^r(1, d-1)$ by the preceding lemma. Otherwise, by rearranging the terms in the product $z_{\gamma_1} \cdots z_{\gamma_r}$, we may assume that the entries in the first column of m are $1, \cdots, t$.

Let $\alpha_i = \{1 + \lambda_1 + \cdots + \lambda_{i-1}\} \subset \gamma_i$, $i = 1, \cdots, t$, be singletons corresponding to the indices of the boxes in the first column of m, and let $\beta_i = \gamma_i \setminus \alpha_i$. We write the symmetrizer b_λ as $b \cdot b'$, with $b' = \sum_{\sigma \in C_1} \operatorname{sgn}(\sigma) \cdot \sigma$, where C_1 is the subgroup of C_λ consisting of permutations of the first column of the Young diagram of λ. We have

$$b' \cdot m = [\alpha_1, \cdots, \alpha_t | \beta_1, \cdots, \beta_t] \cdot z_{\gamma_{t+1}} \cdots z_{\gamma_r} \in I_t^r(1, d-1).$$

It follows that

$$c_\lambda \cdot m = (a_\lambda \cdot b) \cdot (b' \cdot m) \in I_t^r(1, d-1),$$

proving a).

To prove b) it suffices to show that $a_\lambda \cdot I_k^r(1, d-1) = 0$. Consider

$$u = [\alpha_1, \cdots, \alpha_k | \beta_1, \cdots, \beta_k] \cdot z_{\gamma_{k+1}} \cdots z_{\gamma_r} \in I_k^r(1, d-1)$$

with $|\alpha_i| = 1$, $|\beta_i| = d-1$. Since λ has less than k terms, there exist two sets α_i, α_j whose elements index boxes living in the same row of λ. Let σ denote the transposition of these two elements. We get $\sigma \cdot u = -u$ and $a_\lambda \cdot \sigma = a_\lambda$, thus

$$a_\lambda \cdot u = a_\lambda \cdot (\sigma \cdot u) = a_\lambda \cdot (-u)$$

yielding $a_\lambda \cdot u = 0$. □

Remark 4.3. Proposition 4.1 is the analogue in the setting of S_N-representations of Corollary 7.2.3 in [Wey03] or Theorem 5.2.3.6 in [Lan10].

We now want to construct analogous maps to the ones in Definition 3.2. For a partition $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$, we consider the S_N-representation $W_d^\mu = \boxtimes_{j=1}^s W_{d\mu_j}^{i_j}$, defined as follows. If we let $d_j = d \cdot i_j \cdot \mu_j$, each $W_{d\mu_j}^{i_j}$ is an S_{d_j}-representation, so tensoring together all these representations we get a representation W of $H = S_{d_1} \times \cdots \times S_{d_s}$. We regard H as a subgroup of S_N by choosing any partition $A_1 \sqcup \cdots \sqcup A_s$ of $\{1, \cdots, N\}$ into parts of sizes $d_1, \cdots, d_s$, and letting S_{d_i} act as the permutations of A_i. We define W_d^μ to be the S_N-representation induced from the representation W on H. The way we think of W_d^μ is as the space of monomials

$$\prod_{\alpha \in A} z_\alpha, \quad \text{for } A \in \mathcal{P}_\lambda,$$

where $\lambda = d\mu = ((d\mu_1)^{i_1} \cdots (d\mu_s)^{i_s})$ is the partition obtained from μ by multiplying its parts by d, and $\mathcal{P}_\lambda$ is as defined in Section 3.

Definition 4.4. For a partition $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$ of r, we consider the map

$$\pi_\mu : W_d^r \to W_d^\mu$$

defined by

$$z_{\alpha_1} \cdots z_{\alpha_r} \longrightarrow \sum_{A \in \mathcal{P}_\lambda} \bigotimes_{j=1}^{s} \prod_{\substack{B \in A \\ |B| = \mu_j}} z_{\cup_{i \in B} \alpha_i},$$

where $\lambda = d\mu$, as in the preceding paragraph.

Example 4.5. If $d = r = 3$ and $\mu = (2, 1)$ then

$$\pi_\mu(z_{\{1,3,8\}} \cdot z_{\{2,4,7\}} \cdot z_{\{5,6,9\}}) = z_{\{1,2,3,4,7,8\}} \cdot z_{\{5,6,9\}} + z_{\{1,3,5,6,8,9\}} \cdot z_{\{2,4,7\}} + z_{\{2,4,5,6,7,9\}} \cdot z_{\{1,3,8\}}.$$

Remark 4.6. We shall think of the intersection of the kernels of the maps π_μ, for μ a partition with (at most) $k+1$ parts, as "generic prolongations", in analogy with Proposition 3.4. Next section will make this analogy more precise.

5. Polarization and Specialization

Consider the vector space V with basis $x_1, \cdots, x_n$, and let W denote the $GL(V)$-representation $S_{(r)}S_{(d)}V$ (the composition of the Schur functors $S_{(r)}$ and $S_{(d)}$) for some positive integers r, d. Let $N = r \cdot d$ and consider the S_N-representation $W' = W_d^r$ described in the previous section. The character of W and the characteristic function of W' are equal as symmetric functions to the plethystic composition $h_r \circ h_d$ (here h_m denotes the m-th complete symmetric function, see [Mac79]). It follows that for each partition λ with at most n parts, the multiplicities of $S_\lambda V$ in W and V_λ in W' are the same. In this section we construct explicit maps inducing isomorphisms of vector spaces between $\mathrm{hwt}_\lambda(W)$ and $\mathrm{hwt}_\lambda(W')$ for all such λ.

We fix a partition λ of N with at most n parts and consider the weight space $\mathrm{wt}_\lambda(W) \subset W$.

Proposition 5.1. *There exist* polarization *and* specialization *maps*

$$P_\lambda : \mathrm{wt}_\lambda(W) \longrightarrow W', \quad Q_\lambda : W' \longrightarrow \mathrm{wt}_\lambda(W),$$

with the following properties:

1. *Q_λ is surjective.*
2. *P_λ is a section of Q_λ.*
3. *P_λ and Q_λ restrict to maps between* $\mathrm{hwt}_\lambda(W)$ *and* $\mathrm{hwt}_\lambda(W')$ *which are inverse to each other.*

Proof. We identify a permutation $\sigma \in S_N$ with the "tensor"

$$\sigma(1) \otimes \sigma(2) \otimes \cdots \otimes \sigma(N),$$

and consider the (regular) representation of S_N on the vector space R with basis consisting of the tensors σ for $\sigma \in S_N$. The action of S_N on R is given by

$$\sigma \cdot i_1 \otimes i_2 \otimes \cdots \otimes i_N = \sigma(i_1) \otimes \sigma(i_2) \otimes \cdots \otimes \sigma(i_N).$$

We also consider the vector space map $Q_\lambda : R \to V^{\otimes N}$ given by

$$i_1 \otimes i_2 \otimes \cdots \otimes i_N \mapsto g(i_1) \otimes g(i_2) \otimes \cdots \otimes g(i_N),$$

where $g : \{1, \cdots, N\} \to \{x_1, \cdots, x_n\}$ is the map sending i to x_j if the i-th box of λ is contained in the j-th row of λ (or equivalently if $\lambda_1 + \cdots + \lambda_{j-1} < i \le \lambda_1 + \cdots + \lambda_j$). The image of Q_λ is $\mathrm{wt}_\lambda(V^{\otimes N})$.

There is another (right) action of S_N on both the tensors in R and $V^{\otimes N}$, which we denote by $*$. It is given by

$$t_1 \otimes t_2 \otimes \cdots \otimes t_N * \sigma = t_{\sigma(1)} \otimes t_{\sigma(2)} \otimes \cdots \otimes t_{\sigma(N)},$$

and the map Q_λ defined above respects this action. The $*$ and $\cdot$ actions commute on R.

We view $S_r \sim S_d$ as a subgroup of S_N like in the previous section, and define the symmetrizer s by

$$s = \sum_{\sigma \in S_r \sim S_d} \sigma.$$

We can identify W' with $R * s$ and W with $V^{\otimes N} * s$, and note that Q_λ restricts to a map between these two spaces. In fact, the image of Q_λ lies in $W \cap \mathrm{wt}_\lambda(V^{\otimes N}) = \mathrm{wt}_\lambda(W)$. W and W' don't inherit the $*$ action of S_N, but W' is still an S_N-representation via the $\cdot$ action.

We now construct the polarization map P_λ. Assume that λ has t parts, $\lambda = (\lambda_1, \cdots, \lambda_t)$. We define $P_\lambda : \mathrm{wt}_\lambda(V^{\otimes N}) \to W'$ to be the unique map of S_N-representations (with respect to the $*$ action), which sends

$$x_1 \otimes x_1 \otimes \cdots \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_t, \quad \text{with each} \quad x_i \text{ appearing } \lambda_i \quad \text{times, to}$$
$$\frac{1}{\lambda_1! \cdots \lambda_t!} \sum_{\sigma \in S_{\lambda_1} \times \cdots \times S_{\lambda_t}} \sigma(1) \otimes \cdots \otimes \sigma(\lambda_1) \otimes \sigma(\lambda_1 + 1) \otimes \cdots \otimes \sigma(N),$$

where we regard the product of symmetric groups $S_{\lambda_1} \times \cdots \times S_{\lambda_t}$ as a subgroup of S_N by letting each S_{λ_i} act as the permutations of the subset

$$\{\lambda_1 + \cdots + \lambda_{i-1} + 1, \cdots, \lambda_1 + \cdots + \lambda_i\} \subset \{1, \cdots, N\}.$$

Note that

$$P_\lambda(x_1 \otimes x_1 \otimes \cdots \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_t) = \frac{1}{\lambda_1! \cdots \lambda_t!} a_\lambda \cdot 1 \otimes \cdots \otimes N,$$

and $Q_\lambda \circ P_\lambda$ is the identity. P_λ restricts to a map from $\mathrm{wt}_\lambda(W)$ to W' which is a section of Q_λ, so (1) and (2) hold.

It remains to prove that P_λ, Q_λ induce inverse isomorphisms between $\mathrm{hwt}_\lambda(W)$ and $\mathrm{hwt}_\lambda(W')$. The two vector spaces have the same dimension, so it suffices to prove that

a) $Q_\lambda(w') \in \mathrm{hwt}_\lambda(W)$ and

b) $P_\lambda(Q_\lambda(w')) = w'$ for all $w' \in \mathrm{hwt}_\lambda(W')$.

For any monomial $m \in W'$, and hence for any element $m \in W'$, we have

$$P_\lambda(Q_\lambda(m)) = \frac{1}{\lambda_1! \cdots \lambda_t!}(a_\lambda \cdot m),$$

thus

$$P_\lambda(Q_\lambda(a_\lambda \cdot m)) = \frac{1}{\lambda_1! \cdots \lambda_t!}(a_\lambda^2 \cdot m) = a_\lambda \cdot m.$$

Since $\mathrm{hwt}_\lambda(W') \subset a_\lambda \cdot W'$, b) follows.

To prove a) it's enough to show that $Q_\lambda(c_\lambda m)$ is fixed by the Borel, for any monomial m. Writing $m = t * s$ for some tensor $t = t_1 \otimes \cdots \otimes t_N \in R$, we get

$$Q_\lambda(c_\lambda m) = Q_\lambda(a_\lambda \cdot b_\lambda t * s) = \lambda_1! \cdots \lambda_t! Q_\lambda(b_\lambda t) * s.$$

Since the $*$ action commutes with the $GL(V)$ action on $V^{\otimes N}$, it suffices to show that $Q_\lambda(b_\lambda t)$ is invariant under the Borel. We can write $Q_\lambda(b_\lambda t)$ as a tensor product of exterior products

$$x_1 \wedge \cdots \wedge x_{\lambda'_i},$$

where λ'_i denotes the i-th part of λ', the conjugate partition to λ. Each such exterior product is invariant under the Borel, hence so is $Q_\lambda(b_\lambda t)$, concluding the proof. □

Example 5.2. Suppose $r = 2$, $d = 3$, $\dim(V) = 2$ and $\lambda = (4, 2)$. Consider the monomial $m = z_{\{1,2,3\}} \cdot z_{\{4,5,6\}} \in R$. Then

$$Q_\lambda(m) = z_{(3,0)} \cdot z_{(1,2)}, \text{ and}$$

$$\begin{aligned} P_\lambda(Q_\lambda(m)) = \frac{1}{4! \cdot 2!}(&3! \cdot 2! z_{\{1,2,3\}} \cdot z_{\{4,5,6\}} + 3! \cdot 2! z_{\{1,2,4\}} \cdot z_{\{3,5,6\}} \\ &+ 3! \cdot 2! z_{\{1,3,4\}} \cdot z_{\{2,5,6\}} + 3! \cdot 2! z_{\{2,3,4\}} \cdot z_{\{1,5,6\}}) \end{aligned}$$

$$= \frac{1}{4}(z_{\{1,2,3\}} \cdot z_{\{4,5,6\}} + z_{\{1,2,4\}} \cdot z_{\{3,5,6\}} + z_{\{1,3,4\}} \cdot z_{\{2,5,6\}} + z_{\{2,3,4\}} \cdot z_{\{1,5,6\}}),$$

where the 3! comes from the permutations of the entries of the index set of the first term of each monomial, and the 2! comes from the permutations of $\{5, 6\}$ in the index set of the second term of each monomial.

Notice that P_λ sends elements of $I_k^r(a, b)$ to elements of $I_k(Cat(a, b; n))$, and Q_λ sends homogeneous elements of degree r of $I_k(Cat(a, b; n))$ to $I_k^r(a, b)$. We get the following

Corollary 5.3. *If λ is a permutation with at most n parts, then the multiplicity of $S_\lambda V$ in the degree r part of $I_k(Cat(a, b; n))$ is the same as the multiplicity of V_λ in $I_k^r(a, b)$. Moreover, the polarization and specialization maps induce isomorphisms between the λ-highest weight spaces of $I_k(Cat(a, b; n))_r$ and $I_k^r(a, b)$.*

Remark 5.4. The highest weight spaces of the kernels of the maps π_μ in definitions 3.2 and 4.4 correspond to each other via the polarization and specialization maps (note that this gives a way to explain the inheritance principle of [Lan10]). It follows that in order to check that the the equations (of degree k) of the $(k-2)$-nd secant to the Veronese are precisely the $k \times k$-minors of the catalecticant matrices (in the cases we'll be interested in), it suffices to check their equality in the "generic" case.

6. 2 × 2 Minors

In this section we give two proofs of the following result of Pucci, which is the case $k = 2$ of Conjecture 1.1. The first proof works in arbitrary characteristic, while the second one is a characteristic zero proof meant to illustrate the methods we shall use in the case of 3×3 minors.

Theorem 6.1 ([Puc98]). *Let K be a field of arbitrary characteristic and let $n, d \geq 2$ be integers. Then for all t with $1 \leq t \leq d-1$ one has*

$$I_2(Cat(1, d-1; n)) = I_2(Cat(t, d-t; n)).$$

Proof in arbitrary characteristic. For multiindices m_1, m_2, n_1, n_2 we let

$$[m_1, m_2|n_1, n_2] = \begin{vmatrix} z_{m_1+n_1} & z_{m_1+n_2} \\ z_{m_2+n_1} & z_{m_2+n_2} \end{vmatrix}.$$

With this notation, we have the following identity for multiindices u_1, u_2, v_1, v_2, α_1, α_2, β_1, β_2:

$$\begin{aligned}[u_1 + u_2, v_1 + v_2|\alpha_1 + \alpha_2, \beta_1 + \beta_2] = [u_1 + \alpha_1, v_1 + \beta_1|u_2 + \alpha_2, v_2 + \beta_2] \\ + [u_1 + \beta_2, v_1 + \alpha_2|v_2 + \alpha_1, u_2 + \beta_1].\end{aligned} \tag{6.1}$$

We shall prove that $I_2(Cat(a, b; n)) \subset I_2(Cat(a+1, b-1; n))$ for $a + b = d$ and $1 \leq a \leq d-2$. This is enough to prove the equality of the 2×2 minors of all the catalecticants, since $I_2(Cat(1, d-1; n)) = I_2(Cat(d-1, 1; n))$. Since the ideal $I_2(Cat(a, b; n))$ is generated by minors $[m_1, m_2|n_1, n_2]$ with $|m_1| = |m_2| = a$ and $|n_1| = |n_2| = b$, it follows from 6.1 that it's enough to decompose m_1, m_2, n_1, n_2 as

$$m_1 = u_1 + u_2, \quad m_2 = v_1 + v_2, \quad n_1 = \alpha_1 + \alpha_2, \quad n_2 = \beta_1 + \beta_2,$$

in such a way that

$$\begin{aligned}|u_1| + |\alpha_1| = |v_1| + |\beta_1| = a + 1, \quad |u_2| + |\alpha_2| = |v_2| + |\beta_2| = b - 1, \\ |u_1| + |\beta_2| = |v_1| + |\alpha_2| = b - 1, \quad |v_2| + |\alpha_1| = |u_2| + |\beta_1| = a + 1,\end{aligned} \tag{6.2}$$

or

$$\begin{aligned}|u_1| + |\alpha_1| = |v_1| + |\beta_1| = a + 1, \quad |u_2| + |\alpha_2| = |v_2| + |\beta_2| = b - 1, \\ |u_1| + |\beta_2| = |v_1| + |\alpha_2| = a + 1, \quad |v_2| + |\alpha_1| = |u_2| + |\beta_1| = b - 1.\end{aligned} \tag{6.3}$$

If $a \leq 2b-2$, then we can find $0 \leq x, y \leq b-1$ with $x + y = a$. Choose any such x, y and decompose

$$m_1 = u_1 + u_2, \quad m_2 = v_1 + v_2, \quad \text{with } |u_2| = |v_1| = x \text{ and } |u_1| = |v_2| = y,$$

and

$$n_1 = \alpha_1 + \alpha_2, \quad n_2 = \beta_1 + \beta_2, \quad \text{with}$$
$$|\alpha_1| = x + 1,\ |\beta_1| = y + 1,\ |\alpha_2| = b - 1 - x \text{ and } |\beta_2| = b - 1 - y.$$

It's easy to see then that 6.2 is satisfied.

If $b \leq 2a+2$, then since $b \geq 2$ ($a \leq d-2$), we can find $1 \leq x, y \leq a+1$ with $x + y = b$. Choose any such x, y and decompose

$$n_1 = \alpha_1 + \alpha_2, \quad n_2 = \beta_1 + \beta_2, \quad \text{with } |\alpha_2| = |\beta_1| = x \text{ and } |\alpha_1| = |\beta_2| = y,$$

and

$$m_1 = u_1 + u_2, \quad m_2 = v_1 + v_2, \quad \text{with}$$

$$|u_1| = a + 1 - y,\ |v_1| = a + 1 - x,\ |u_2| = y - 1 \text{ and } |v_2| = x - 1.$$

It's easy to see then that 6.3 is satisfied.

If neither of $a \leq 2b - 2$ and $b \leq 2a + 2$ holds, then

$$a \geq 2b - 1 \geq 2(2a + 3) - 1 = 4a + 5,$$

so $0 \geq 3a + 5$, a contradiction. □

Proof in characteristic zero. By Corollary 5.3, it's enough to treat the "generic case". We want to show that for $a + b = d$, $N = 2d$, all S_N-subrepresentations $I_2(a, b) \subset W_d^2 = \operatorname{ind}_{S_d \sim S_2}^{S_N}(\mathbf{1})$ are the same. Clearly the trivial representation $V_{(N)}$ is not contained in any $I_2(a, b)$, so

$$I_2(a, b) \subseteq W_d^2 / V_{(N)} = \bigoplus_{i=1}^{\lfloor d/2 \rfloor} V_{(2\cdot(d-i), 2\cdot i)}, \quad \text{for all} \ \ a, b \text{ with } a + b = d.$$

(see [Mac79] for the formula of the decomposition of W_d^2 into irreducible representations; as the rest of the proof will show, we don't really need the precise description of this decomposition).

We will finish the proof by showing that all of the above inclusions are actual equalities. To see this, it's enough to prove that for any a, b with $a + b = d$, any partition λ with two parts, and any monomial $m = z_\alpha \cdot z_\beta$, $\alpha \sqcup \beta = \{1, \cdots, N\}$, we have $c_\lambda \cdot m \in I_2(a, b)$. Fix then such $a, b, \lambda = (\lambda_1, \lambda_2)$ and $m = z_\alpha \cdot z_\beta$.

Recall from Section 4 that we can identify m with a tableau of shape λ with 1's in the positions indexed by the elements of α, and 2's in the positions indexed by the elements of β. Recall also that if m has repeated entries in a column, then $c_\lambda \cdot m = 0$. Since permutations within columns of m can only change the sign of $c_\lambda \cdot m$, and permutations of the columns of m of the same size don't change the value of $c_\lambda \cdot m$, we can assume in fact that

$$m = z_{\{1, \cdots, d\}} \cdot z_{\{d+1, \cdots, N\}} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & \cdots & 2 & 2 & \cdots \\ \hline 2 & 2 & \cdots \\ \cline{1-3} \end{array}.$$

Consider the sets

$$\alpha_1 = \{2, \cdots, a + 1\},\ \alpha_2 = \{1, \cdots, d\} \setminus \alpha_1,\ \beta_1 \text{ and } \beta_2 = \{d + 1, \cdots, N\} \setminus \beta_1,$$

where β_1 is any subset with a elements of $\{d + 1, \cdots, N\}$ containing $\lambda_1 + 1$. It follows that

$$[\alpha_1, \beta_1 | \alpha_2, \beta_2] = z_{\alpha_1 \cup \alpha_2} \cdot z_{\beta_1 \cup \beta_2} - z_{\alpha_1 \cup \beta_2} \cdot z_{\alpha_2 \cup \beta_1} = m - m',$$

where m' is a tableau with two equal entries in the first column. We get

$$c_\lambda m = c_\lambda (m - m') = c_\lambda [\alpha_1, \beta_1 | \alpha_2, \beta_2] \in I_2(a, b),$$

completing the proof. □

Remark 6.2. The characteristic zero case is really much simpler than that: since all the partitions λ that show up have at most two parts, it suffices to prove the theorem when $n = 2$, but in this case all the catalecticant ideals are the same, as remarked in the introduction.

7. The Proofs

We are now ready to give an affirmative answer to questions Q5a and Q5b in the introduction.

Theorem 7.1. *Let K be a field of characteristic 0 and let $n, d \geq 2$ be integers. The following statements hold:*

(1) *For all t with $2 \leq t \leq d-2$ one has*
$$I_3(Cat(2, d-2; n)) = I_3(Cat(t, d-t; n)).$$
(2) *If $d \geq 4$ then there is a strict inclusion*
$$I_3(Cat(1, d-1; n)) \subsetneq I_3(Cat(2, d-2; n)).$$
(3) *Any of the ideals $I_3(Cat(t, d-t; n))$, $2 \leq t \leq d-2$, is the ideal of the first secant variety to the d-th Veronese embedding of $\mathbb{P}^{n-1}_K$.*

Proof. To prove (1), it suffices by Corollary 5.3 to show that $I_3^3(2, d-2) = I_3^3(t, d-t)$ for $2 \leq t \leq d-2$. We shall write I_3 instead of I_3^3 for the rest of the proof. The λ-highest weight spaces of all $I_3(t, d-t)$, $2 \leq t \leq d-2$, are all equal when λ has at most 2 parts. This follows by combining Corollary 5.3 with the fact that the theorem is known when $n = 2$. We shall prove that the λ-part of $I_3(t, d-t)$ is equal to the λ-part of W_3^3 for all t with $1 \leq t \leq d-1$ (we already know this when $t = 1$, by Proposition 4.1). This will imply (1) and the inclusion of (2). The reason why this inclusion is strict for $d \geq 4$ is because it is already strict for $n = 2$, and inheritance holds for catalecticant ideals by Corollary 5.3.

Consider a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with 3 parts, a monomial (tableau) $m \in W_d^3$ and integers $2 \leq a \leq b$ with $a + b = d$. We shall prove that $c_\lambda \cdot m \in I_3(a, b)$. We will see that if λ has only one entry in the second column, then $c_\lambda \cdot m = 0$, so let's assume this isn't the case for the moment. We may also harmlessly assume that m has no repeated entries in a column. Since permuting the numbers $1, 2, 3$ in the tableau m doesn't change m, and permutations within the columns of m preserve $c_\lambda \cdot m$ up to sign (Lemma 4.2), we may assume that m contains the subtableau

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

in its first two columns (there may or may not be a third box in the second column of λ).

It follows that $m = z_{\gamma_1} z_{\gamma_2} z_{\gamma_3}$, with $\gamma_1 = \{1, 2, \cdots\}$, $\gamma_2 = \{\lambda_1 + 1, \lambda_1 + 2, \cdots\}$, $\gamma_3 = \{\lambda_1 + \lambda_2 + 1, \cdots\}$, $|\gamma_i| = d$. Consider subsets $\alpha_i \subset \gamma_i$, $|\alpha_i| = a$ satisfying the conditions

$$1, 2 \in \alpha_1, \quad \lambda_1 + 1 \in \alpha_2, \lambda_1 + 2 \notin \alpha_2, \quad \lambda_1 + \lambda_2 + 1, \lambda_1 + \lambda_2 + 2 \notin \alpha_3,$$

and let $\beta_i = \gamma_i \setminus \alpha_i$, for $i = 1, 2, 3$. We have

$$[\alpha_1, \alpha_2, \alpha_3 | \beta_1, \beta_2, \beta_3] = m + \sum_{j=1}^{5} \pm m_j,$$

where each m_j is a monomial with repeated entries in one of the first two columns. It follows that

$$c_\lambda \cdot m = c_\lambda \cdot [\alpha_1, \alpha_2, \alpha_3 | \beta_1, \beta_2, \beta_3] \in I_3(a, b),$$

which is what we wanted to prove.

To see that $c_\lambda \cdot m = 0$ for all monomials m when $\lambda = (3d - 2, 1, 1)$, it suffices to notice that if σ is the transposition of the $(3d-1)$-st and $3d$-th boxes of λ (the 2nd and 3rd boxes in the first column of λ), then $\sigma \cdot m$ and m are the same up to permutations of the columns of m of size 1 (and permutations of $1, 2, 3$, the entries of the tableau m). It follows that

$$c_\lambda \cdot m = c_\lambda \cdot (\sigma \cdot m) = (c_\lambda \cdot \sigma) \cdot m = -c_\lambda \cdot m,$$

so $c_\lambda \cdot m = 0$, as desired. Alternatively, see [Mac79] for a description of the decomposition of W_d^3 into a sum of irreducible representations.

As mentioned in the introduction, part (3) follows from (1), (2) and the result of Kanev ([Kan99], see also [Lan10], Corollary 6.4.2.4). We include a proof for completeness: by Proposition 4.1 and Corollary 5.3, the modules in $S_{(r)} S_{(d)} V$ corresponding to partitions λ with at least 3 parts are contained in $I_3(1, d-1; n)$, hence also in $I_3(t, d-t; n)$ for all t, and in the ideal of $\sigma_2(\nu_d(\mathbb{P}V^*))$; it remains to check that the modules corresponding to partitions with at most 2 parts in the ideal of $\sigma_2(\nu_d(\mathbb{P}V^*))$ are the same as those in $I_3(t, d-t; n)$ for $2 \leq t \leq d-2$, but this follows by inheritance from the case of the rational normal curve ($n = 2$). □

We now give a more precise description of the degree 3 part of the ideal of the secant line variety to the Veronese variety.

Theorem 7.2. *With the assumptions in Theorem 7.1 and writing $\mathbb{P}_K^{n-1} = \mathbb{P}V^*$ for some n-dimensional K-vector space V, the following statements hold:*

(1) *As $GL(V)$-representations,*

$$I_3(Cat(2, d-2; n))_3 \simeq S_{(3)} S_{(d)} V / \bigoplus_{\substack{i=0 \\ i \neq 1}}^{d} S_{(3d-i,i)} V.$$

(2) *The number of generators of the ideal of the first secant variety to the d-th Veronese embedding of $\mathbb{P}^{n-1}_K$ is, whenever this ideal is nonzero (i.e. $n \geq 3$, or $n = 2$ and $d \geq 4$), given by the formula*

$$\binom{\binom{n+d-1}{d}+2}{3} - \binom{n+2d-1}{2d}\cdot\binom{n+d-1}{d} + \binom{n+3d-2}{3d-1}\cdot n - \binom{n+3d-1}{3d}.$$

Proof. By Corollary 3.7 and part (3) of Theorem 7.1, it follows that $I_3(Cat(2, d-2; n))_3$ is the kernel of the map

$$\pi_{(2,1)} : S_{(3)}S_{(d)}V \longrightarrow S_{(2d)}V \otimes S_{(d)}V \simeq \bigoplus_{i=0}^{d} S_{(3d-i,i)}V,$$

given on monomials by

$$z_\alpha z_\beta z_\gamma \mapsto z_{\alpha+\beta}z_\gamma + z_{\alpha+\gamma}z_\beta + z_{\beta+\gamma}z_\alpha.$$

It follows that in order to prove (1) it suffices to show that the image of $\pi_{(2,1)}$ contains all the irreducible representations in the decomposition of $S_{(2d)}V \otimes S_{(d)}V$, except for $S_{(3d-1,1)}V$. We will prove the corresponding statement in the generic case, namely for the map

$$\pi_{(2,1)} : W^3_d \longrightarrow W^{(2,1)}_d = V_{(2d)} \boxtimes V_{(d)}.$$

(see Definition 4.4). $V_{(3d-1,1)}$ does not occur in the decomposition of W^3_d (see [Mac79] or the proof of the similar statement for $V_{(3d-2,1,1)}$ on the previous page). The trivial representation $V_{(3d)}$ is in the image of $\pi_{(2,1)}$, since the image of $c_{(3d)} \cdot m$ is nonzero for some (any) monomial m.

Consider now a partition $\lambda = (3d-i, i)$ with $2 \leq i \leq d$, and let m be the monomial

$$m = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3 \\ \hline 2 & 3 & \cdots & 3 \\ \cline{1-4}\end{array},$$

i.e. m has entries $1, 2$ in the first column, $1, 3$ in each of the 2nd to the i-th column, and $(d-i)$ 1's, $(d-1)$ 2's and $(d-i+1)$ 3's in the remaining columns. $\pi_{(2,1)}(m)$ is a sum of three monomials with corresponding tableaux obtained from the tableau of m by letting two of the three entries $1, 2, 3$ equal. Making $1, 2$ or $1, 3$ equal yields a tableau with repeated entries in the 1st or 2nd column, i.e. one that is killed by the symmetrizer c_λ. It follows that

$$\pi_{(2,1)}(c_\lambda \cdot m) = c_\lambda \cdot m',$$

where

$$m' = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & \cdots & 1 & 2 & \cdots & 2 & 2 & \cdots & 2 \\ \hline 2 & 2 & \cdots & 2 \\ \cline{1-4}\end{array} = z_{\{1,\cdots,d\}} \cdot z_{\{d+1,\cdots,3d\}}.$$

Now it is not hard to check that

$$c_\lambda \cdot m' = d! \cdot z_{\{1,\cdots,d\}} \cdot z_{\{d+1,\cdots,3d\}} + \text{other terms},$$

i.e. it is nonzero. This shows that V_λ occurs as a subrepresentation in the image of $\pi_{(2,1)}$, finishing the proof of part (1).

To see why (2) is true, note that since V has dimension n, $S_{(d)}V$ has dimension $\binom{n+d-1}{d}$, hence

$$\dim(S_{(3)}S_{(d)}V) = \binom{\binom{n+d-1}{d}+2}{3}.$$

By Pieri's rule

$$S_{(2d)}V \otimes S_{(d)}V \simeq \bigoplus_{i=0}^{d} S_{(3d-i,i)}V$$

and

$$S_{(3d-1,1)}V \simeq S_{(3d-1)}V \otimes V / S_{(3d)}V.$$

It follows that

$$\begin{aligned}\dim(I_3(Cat(2, d-2; n))_3) = \dim(S_{(3)}S_{(d)}V) - \dim(S_{(2d)}V \otimes S_{(d)}V) \\ + \dim(S_{(3d-1)}V \otimes V) - \dim(S_{(3d)}V),\end{aligned}$$

and a simple calculation based on the fact that $\dim(S_{(r)}V) = \binom{n+r-1}{r}$ yields the desired formula. □

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