# AREA MINIMIZERS AND BOUNDARY RIGIDITY OF ALMOST HYPERBOLIC METRICS

DMITRI BURAGO AND SERGEI IVANOV

Abstract. This paper is a continuation of our paper about boundary rigidity and filling minimality of metrics close to flat ones. We show that compact regions close to a hyperbolic one are boundary distance rigid and strict minimal fillings. We also provide a more invariant view on the approach used in the above mentioned paper.

## 1. Introduction and preliminaries

1.1. **Minimal surfaces vs. area minimizes: a preliminary discussion.** Before we proceed to the main results of this paper, we begin with a very general consideration. We want to formulate some open problems followed by a brief discussion. We begin with the following problem, which sounds extremely naturally, but we do not know any reference for it.

Let $M^N$ be a complete Riemannian manifold and $S$ a compact $n$-dimensional surface in $M$ with $\partial S \neq \emptyset$. Assume that $S$ is a convex set in the strongest possible sense, namely for every two points $p, q \in S$, there is a unique shortest path between $p$ and $q$ in $M$ and this shortest path lies in $S$. Is it true that $S$ is an area minimizer, namely that for any other surface $S_1$ such that $\partial S_1 = \partial S$, the $n$-dimensional area $\operatorname{vol}_n S_1$ of $S_1$ is greater than that of $S$?

At first glance, this seems to be a naive question that should be easy. However, apparently it is not and a solution would imply solutions of some notoriously difficult problems.

We want to emphasize the difference between minimal surfaces (in the variational sense), local area minimizers (that is, minimizing the area among all nearby surfaces) and global area minimizers. In the above question, $S$ is totally geodesic and hence is a minimal surface. Moreover one can show that (any proper subregion of) $S$ is a local minimizer (we do not give a proof here since we do not use this fact). The problem begins when we are looking for global minimality.

We know only two general methods of proving global minimality: constructing a projection or a calibrating form.

A projection is a map $P\colon M \to S$ such that it is the identity on $S$ and does not increase $n$-dimensional areas. This condition is equivalent to saying that the $n$-dimensional Jacobian of $P$ is no greater than 1 everywhere on $M$. (If in addition this Jacobian is strictly less than 1 outside $S$, the projection guarantees that $S$ is a unique area minimizer among the surfaces with the same boundary.) It is

2010 *Mathematics Subject Classification.* 53C24, 53C20, 53C65.

The first author was partially supported by NSF grants DMS-0604113 and DMS-0412166. The second author was partially supported by the Dynasty foundation and RFBR grants 08-01-00079-a and 09-01-12130-ofi-m.

very unlikely that such a projection exists for all minimizers even if there are no topological obstructions.

A more general method is constructing a calibration form, that is a closed $n$-form $\omega$ on $M$ such that the restriction of $\omega$ to $S$ is the $n$-dimensional volume form of $S$ and the norm of $\omega$ is less than 1 outside $S$. Then by Stokes' Formula we immediately solve the problem (for orientable surfaces).

However there is a difficulty with both methods. Imagine that there exists a surface $S_2$ such that its boundary $\partial S_2 = 10 \cdot \partial S$ (that is, $\partial S_2$ covers $\partial S$ 10 times) and $\mathrm{vol}_n S_2 < 10 \, \mathrm{vol}_n S$. In this case we say that $S$ is not stably minimizing. Such a surface can be an area minimizer but none of the above two methods can prove this. Such phenomena take place in a situation rather close to the one that will be discussed in this paper. Namely, regions of affine subspaces in normed spaces can be global minimizers but not stably minimizing surfaces (with respect to the Holmes-Thompson surface area [10]), see [3] and [4].

1.2. **Boundary rigidity and minimal fillings.** Now let us present our general set-up and explain why it is directly related to the above discussion. This paper is a continuation of [1] and we borrow large parts of introduction and formulations from there. We hope that we also can give a better and more invariant insight into what is done in [1], especially due to Proposition 3.6.

Let $M = (M^n, g)$ be a compact Riemannian manifold with boundary $\partial M$. Its *boundary distance function*, denoted by $bd_M$, is the restriction of the Riemannian distance $d_M$ to $\partial M \times \partial M$. The term "boundary rigidity" means that the metric is uniquely determined by its boundary distance function. More precisely,

**Definition 1.1.** $M$ is *boundary rigid* if every compact Riemannian manifold $M'$ with the same boundary and the same boundary distance function is isometric to $M$ via a boundary preserving isometry.

It is easy to construct metrics that are not boundary rigid. For example, consider a metric on a disc with a "big bump" around a point $p$, such that the distance from $p$ to the boundary is greater than the diameter of the boundary. Since no minimal geodesic between boundary points passes through $p$, a perturbation of the metric near $p$ does not change the boundary distance function.

Thus one has to impose restrictions on the metric in order to make the boundary rigidity problem sensible. One natural restriction is the following: a Riemannian manifold $M$ is called *simple* if the boundary $\partial M$ is strictly convex, every two points $x, y \in M$ are connected by a unique geodesic, and geodesics have no conjugate points (cf. [14]). A more general condition called SGM ("strong geodesic minimizing") was introduced in [6] in order to allow non-convex boundaries. Note that if $M$ is simple, then it is a topological disc.

The simplicity of $M$ can be seen from the boundary distance function. Indeed, the convexity of $\partial M$ is equivalent to a (local) inequality between boundary distances and intrinsic distances of $\partial M$. And if the boundary is convex, then the uniqueness of geodesics and the lack of conjugate points is equivalent to smoothness of the boundary distance function away from the diagonal. Thus if two Riemannian manifolds have the same boundary and the same boundary distance functions, then either both are simple or both are not.

**Conjecture 1.2** (Michel [14])**.** *All simple manifolds are boundary rigid.*

Pestov and Uhlmann [15] proved this conjecture in dimension 2. In higher dimensions, few examples of boundary rigid metrics are known. They are: regions in $\mathbb{R}^n$ [9], in the open hemisphere [14], in symmetric spaces of negative curvature (follows from the main result of [5]), and in product spaces of the form $N \times \mathbb{R}$ where $N$ is a simple $(n-1)$-dimensional Riemannian manifold [7]. We refer the reader to [8] and [15] for a survey of boundary rigidity, other inverse problems, and their applications.

One of the main results of [1] asserts that if $M$ is sufficiently close to a region in the Euclidean space, then $M$ is boundary rigid. In this paper we extend this to metrics close to the hyperbolic one. Namely we prove the following theorem:

**Theorem 1.3.** *If a metric in a region is sufficiently close to a hyperbolic one, then it is boundary rigid. More precisely, let $D \subset \mathbb{H}^n$ be a compact region with a smooth boundary. The there is a $C^r$-neighborhood (for a suitable $r$) of the standard hyperbolic metric on $D$ such that for every metric $g$ from this neighborhood, the space $M = (D, g)$ is boundary rigid.*

*Remark.* We do not track the number of derivatives required for our arguments to work. An interested reader can verify that one can take $r = 3$ for "a suitable $r$" in Theorem 1.3. This is worse than in [1] where the metric has to be only $C^2$-close to the Euclidean one.

The proof of Theorem 1.3 can be made to work for metrics close to the Euclidean one (in fact, the proof in this case is much easier), an outline of the argument can be found in [13]. While this approach proves a slightly weaker result than in [1] (namely $C^2$ is replaced by $C^3$), it provides a better insight into and allows some simplifications of the methods of [1].

We treat boundary rigidity as the equality case of the minimal filling problem discussed in [1], [3], and [11].

**Definition 1.4.** $M$ is a *minimal filling* if, for every compact Riemannian manifold $M'$ with $\partial M' = \partial M$, the inequality

$$d_{M'}(x, y) \geq d_M(x, y) \quad \text{for all } x, y \in \partial M$$

implies

$$\mathrm{vol}(M') \geq \mathrm{vol}(M).$$

We say that $M$ is a *strict minimal filling* if in addition the equality

$$\mathrm{vol}(M') = \mathrm{vol}(M)$$

implies that $M$ and $M'$ are isometric via an isometry that is identical on the boundary.

**Conjecture 1.5.** *Every simple manifold is a strict minimal filling.*

If $M$ is simple, then its volume is uniquely determined by its boundary distance function, namely there is an integral formula expressing $\mathrm{vol}(M)$ in terms of $bd_M$ and its first order derivatives (the Santaló formula [16], see also [9]). It is not clear though whether the formula is monotone in $bd_M$.

However the mere existence of this formula implies that every simple strict minimal filling is boundary rigid. Indeed, let $M$ be simple and a strict minimal filling, and suppose that $M'$ has the same boundary and the same boundary distance function as $M$. Since $M'$ has the same boundary distance function as a simple manifold, it is also simple, hence $\mathrm{vol}(M') = \mathrm{vol}(M)$ by the Santaló formula. Now the equality case in Definition 1.4 implies that $M'$ and $M$ are isometric.

Our approach to boundary rigidity is to prove a suitable partial case of Conjecture 1.5. In [1] we were able to carry out this plan for metrics close to a Euclidean one. Now we extend this method to metrics close to a hyperbolic one. Namely, we prove the following theorem:

**Theorem 1.6.** *If a metric $g$ on a region $D \subset \mathbb{H}^n$ is sufficiently close to the hyperbolic one, then $(D, g)$ is a strict minimal filling (and hence is boundary rigid).*

1.3. **Plan of the proof.** We deduce filling minimality of a metric from area minimality of its image in a suitable space. More precisely, given $M$ as in Theorem 1.6, we do the following:

(i) Construct a distance-preserving map $\Phi : M \to \mathcal{L} = L^\infty(S)$ where $S$ is a suitable measure space.

(ii) Define an $n$-dimensional surface area functional (referred to as the $n$-volume) for Lipschitz surfaces in $\mathcal{L}$ so that the following holds. First, every 1-Lipschitz map from a Riemannian manifold to $\mathcal{L}$ does not increase $n$-volumes. Second, the above map $\Phi$ preserves $n$-volumes.

(iii) Prove that $\Phi$ (regarded as a surface in $\mathcal{L}$) is an area minimizer, that is it has the least $n$-volume among all Lipschitz surfaces with the same boundary in $\mathcal{L}$. Moreover, $\Phi$ is a unique area-minimizer: every Lipschitz surface in $\mathcal{L}$ having the same boundary and the same $n$-volume as $\Phi$ is contained in the image of $\Phi$ (up to a set of zero area).

If there exist $\mathcal{L}$, $\Phi$ and a surface area functional satisfying (i)–(iii), then $M$ is a strict minimal filling. Indeed, consider a Riemannian $n$-manifold $M'$ such that $\partial M' = \partial M$ and $d_{M'} \geq d_M$ on $\partial M \times \partial M$. The latter implies that the map $\Phi|_{\partial M'} \to \mathcal{L}$ is 1-Lipschitz (with respect to the distance $d_{M'}$ on $\partial M'$). It is easy to see (cf. e.g. [12, Proposition 1.6]) that $\mathcal{L} = L^\infty(S)$ enjoys the following Lipschitz extension property: for any metric space $X$, any subset $Y \subset X$ and any 1-Lipschitz map $f : Y \to \mathcal{L}$ there exists a 1-Lipschitz map $\tilde{f} : X \to \mathcal{L}$ extending $f$. Substituting $X = M'$, $Y = \partial M'$ and $f = \Phi|_{\partial M'}$ yields a 1-Lipschitz map $\Phi' : M' \to \mathcal{L}$ such that $\Phi'|_{\partial M'} = \Phi|_{\partial M}$. Then by (ii) and (iii) we have

$$\mathrm{vol}(M') \geq \mathrm{vol}(\Phi') \geq \mathrm{vol}(\Phi) = \mathrm{vol}(M),$$

hence $M$ is a minimal filling. To prove strict filling minimality, observe that the equality in the above inequality implies that $\Phi'$ preserves $n$-volumes and the image of $\Phi'$ is contained in the image of $\Phi$. Thus there is a 1-Lipschitz volume-preserving map $\Phi^{-1} \circ \Phi$ from $M'$ to $M$. Such a map must be an isometry, thus $M$ is a strict minimal filling.

Actually our proof contains some extra technical details (in particular, we do not assume convexity of the boundary and work in some large ball in $\mathcal{L}$ rather than in the entire space). The proof with all these details is contained in Section 2.

For the purpose of proving boundary rigidity only, this strategy *should* work as follows. Imagine that for every simple manifold $M$ we construct (in a canonical way) an isometric embedding $\Phi_M : M \to \mathcal{L} = L^\infty(S)$ with the following properties:

1. The restriction of $\Phi_M$ to $\partial M$ depends only on the boundary distance function of $M$.

2. A notion of $n$-volume in $\mathcal{L}$ is defined so that these isometric embeddings $\Phi_M$ preserve $n$-volumes.

3. $\Phi_M$ is a strict area minimizer with respect to this $n$-volume in $\mathcal{L}$ (at least for the particular manifold $M$ for which we are trying to prove rigidity).

Given such a construction, the boundary rigidity of $M$ follows immediately from the fact that the boundary distance function uniquely determines the volume. Our proof is more complicated than this, since our definition of the $n$-volume in $\mathcal{L}$ depends on $M$ (still we have that $\Phi_M$ preserves $n$-volumes and any competitor map $\Phi_{M'}$ does not increase $n$-volumes).

There are many natural constructions satisfying the first of the above requirements. For instance, one could embed $M$ into $L^\infty(\partial M)$ by distance functions: for $x \in M$, let $\Phi(x) = d_M(x, \cdot)|_{\partial M}$. We use a slightly different embedding of the same type where $\mathcal{L}$ will be the $L^\infty$ of the ideal boundary of $\mathbb{H}^n$ rather than that of the boundary of $M$.

The surface area functional in $\mathcal{L}$ is defined by a suitable Riemannian structure (that is, a family of $L^2$-compatible scalar products) on $\mathcal{L}$. The strict minimality of the surface $\Phi_M$ is proved by constructing a projection from $\mathcal{L}$ to this surface that strictly decreases $n$-volumes outside the surface. Note that a proof following this strategy never mentions the competitor manifold $M'$ and works exclusively with our simple manifold $M$.

A very important observation (Proposition 3.6) is that for a very natural variety of choices of $n$-volume forms on $\mathcal{L} = L^\infty(S)$ (induced by certain choices of Riemannian structures on $\mathcal{L}$), $\Phi_M$ is a minimal surface. This observation explains why there is a reasonable hope that this strategy could work. The difficult part is to construct an area-decreasing projection, and this is where we use the assumption that the metric of $M$ is close to a hyperbolic one.

*Remark.* To define $n$-volumes in $\mathcal{L}$ we use a somewhat artificial construction when a class of volumes is defined after we have an embedding of $M$. There are several nice notions of $n$-volumes for surfaces in $L^\infty$, which, unlike in our construction, do not depend on any base embedding or such. Unfortunately, we were not able to use any of those nice volumes directly.

*Acknowledgement.* We are grateful to Chris Croke, Bruce Klieiner, Bill Minicozzi, and Gunther Uhlmann for interesting discussions.

## 2. Proof of the theorems

The purpose of this section is to deduce the statements of Theorems from Proposition 2.1. This proposition asserts that there exist two maps with certain easy-to formulate properties. The rest of the paper is a rather technical construction of the maps along with verification of the properties.

Let $g_0$ denote the standard metric on $\mathbb{H}^n$. Let $D \subset \mathbb{H}^n$ be a region with a smooth boundary and $g$ a Riemannian metric on $D$ (assumed to be close to $g_{0|_D}$ in a suitable $C^r$ topology).

Fix a point $o \in \mathbb{H}^n$, this point will be referred to as the *origin.* By $B_o(R)$ we denote the ball of radius $R$ in $\mathbb{H}^n$ centered at $o$. Fix an $R > 0$ such that $D \subset B(R/5)$. The metric $g$ can be extended from $D$ to $\mathbb{H}^n$ so that the extension is smooth and coincide with $g_0$ outsides $B(R/2)$. Moreover the extension can be constructed in such a way that it converges to $g_0$ as $g$ converges to $g_0|_D$.

We denote the extension by the same letter $g$ and denote $M = (\mathbb{H}^n, g)$. Our goal is to prove that for any region $D \subset B(R/2)$ the space $(D, g) \subset M$ is a minimal filling and boundary rigid.

Let $S = S^{n-1}$ and $\mathcal{L} = L^\infty(S)$. For $r > 0$, let $\mathcal{B}(r)$ denote the ball of radius $r$ in $\mathcal{L}$ centered at the origin. Let $\mathcal{B} = \mathcal{B}(R)$ where $R$ is the radius fixed above.

The technical results established in the rest of the paper can be summarized as the following proposition.

**Proposition 2.1.** *If $g$ is sufficiently close to $g_0$, then there exists a distance-preserving map $\Phi : M \to \mathcal{L}$ such that $\Phi(o) = 0$ and a Lipschitz map $P_\sigma : \mathcal{B} \to M$ such that the following holds:*

*1. $P_\sigma \circ \Phi = id_M$.*

*2. For every Riemannian n-manifold $N$ and every 1-Lipschitz map $f : N \to \mathcal{B}$, the composition $P_\sigma \circ f : N \to M$ does not increase n-dimensional (Riemannian) volumes.*

*3. If, for $N$ and $f$ as above, the composition $P_\sigma \circ f : N \to M$ preserves n-volumes, then $f(N) \subset \Phi(M)$.*

Now we deduce the theorems from this proposition.

*Proof of Theorem 1.4.* Let $g$ be sufficiently close to $g_0$ so that the maps $\Phi$ and $P_\sigma$ from Proposition 2.1 exist. Let $g'$ be a metric on $D$ such that

$$d_{(D,g')}(x, y) \geq d_{(D,g)}(x, y)$$

for all $x, y \in \partial D$. Denote $M' = (D, g')$. Since $(D, g) \subset M$, we have $d_{(D,g)}(x, y) \geq d_M(x, y)$ for all $x, y \in D$, hence

$$d_{M'}(x, y) \geq d_M(x, y)$$

for all $x, y \in \partial D$. Since $\Phi$ is distance-preserving with respect to $d_M$, it follows that the restriction $\Phi|_{\partial D}$ is 1-Lipschitz with respect to the metric $d_{M'}$ restricted to $\partial D \times \partial D$. Therefore (cf. [12, Proposition 1.6] or [1, Proposition 4.9]) it admits a 1-Lipschitz extension $\Phi' : M' \to \mathcal{L}$. Furthermore this extension can be made so that $\Phi'(M') \subset \mathcal{B}$ by post-composing it with a cut-off map $\mathcal{L} \to \mathcal{L}$ given by

$$\mathrm{cutoff}(\varphi)(s) = \min\{R/2, \max\{-R/2, \varphi(s)\}\}, \qquad \varphi \in \mathcal{L},\ s \in S.$$

(The cut-off does not affect points in $\Phi'(\partial M')$ since $\Phi'|_{\partial M'} = \Phi|_{\partial D}$ and $\Phi(D)$ is contained in an $(R/2)$-ball centered at the origin.)

Consider a map $F = P_\sigma \circ \Phi' : M' \to M$. Since $\Phi'|_{\partial D} = \Phi_{\partial D}$, the first assertion of Proposition 2.1 implies that $F|_{\partial M'} = id_{\partial D}$, therefore $F(M') \supset D$. Then the second assertion of Proposition 2.1 implies that $\mathrm{vol}(M') \geq \mathrm{vol}(D, g)$. Thus $(D, g)$ is a minimal filing.

To prove that $(D, g)$ is a strict minimal filling, suppose that $\mathrm{vol}(M') = \mathrm{vol}(D, g)$. Then $F$ is volume-preserving, hence by the third assertion of Proposition 2.1 we have $\Phi'(M') \subset \Phi(M)$. Therefore $F$ can be written as $F = \Phi^{-1} \circ \Phi'$. Since $\Phi$ is distance-preserving and $\Phi'$ is 1-Lipschitz, it follows that $F : M' \to M$ is 1-Lipschitz. Since $F$ is 1-Lipschitz and volume-preserving, it follows easily that it is an isometry (cf. e.g. [1, Lemma 9.1]). Thus $(D, g)$ is a strict minimal filling. □

*Proof of Theorem 1.3.* Let $D$ and $g$ be as above, and let $g'$ be a Riemannian metric on $D$ inducing the same boundary distance function as $g$. Since $(D, g)$ is a strict minimal filling, it suffices to show that $\mathrm{vol}(D, g') = \mathrm{vol}(D, g)$. If $(D, g)$ had a convex boundary, this would follow from the Santaló formula for the volume of a simple Riemannian metric. Since we do not assume convexity, $(D, g)$ may fail to be simple. However it is a region in a simple manifold (namely in a large ball in $M$) and hence satisfies the SGM (Strong Geodesic Minimizing) condition introduced by C. Croke [6]. Then Lemma 5.1 from [6] implies the desired equality $\mathrm{vol}(D', g') = \mathrm{vol}(D, g)$. □

The rest of the paper is organized as follows. In Section 3 we consider a general class of constructions that work for any simple manifold $M$ (not necessarily almost hyperbolic). The main result of this section is Proposition 3.6 asserting that an isometric image of $M$ in $L^\infty$ is a minimal surface (in the variational sense) provided that the embedding of $M$ and the Riemannian structure on $\mathcal{L}$ used to define the surface area satisfy certain natural conditions.

In Section 4 we construct a distance-preserving map $\Phi : M \to \mathcal{L}$ (possessing all the nice properties that we need), a Riemannian structure $G$ in $\mathcal{B}$ used to define the surface area, and a (preliminary) projection $P : \mathcal{B} \to M$). This map does not increase $n$-volumes in the case of the hyperbolic metric (that is, when $g = g_0$). In the almost hyperbolic case this map can slightly expand $n$-volumes, the expansion coefficients are estimated in Sections 5 and 6. Finally, in Section 7 we compose $P$ with a family of shrinking maps $M \to M$ so that the resulting projection $P_\sigma$ strictly decreases $n$-volumes away from the surface $\Phi(M)$. This construction completes the proof of Proposition 2.1 and the theorems.

## 3. General computations

In this section we do not use the assumption that our metric is close to the hyperbolic one; everything here is valid for any Riemannian manifold $M = (M, g)$.

3.1. **Set-up and notation.** Recall that $S = S^{n-1}$ and $\mathcal{L} = L^\infty(S)$. We equip $S$ with a smooth probability measure; the integral of a function $f$ with respect to this measure will be written as $\int_S f(s)\, ds$.

Let $M = (M, g)$ be an $n$-dimensional Riemannian manifold. We denote by $UTM$ the unit tangent bundle of $M$ and by $UT_xM$ its fiber over $x \in M$. Every sphere $UT_xM$ is equipped with the standard Haar probability measure; the integral of a function $f$ with respect to this measure is denoted by $\int_{UT_xM} f(v)\, dv$. Note that the integration of these measures on fibers with respect to the Riemannian volume of $M$ yields the standard Liouville measure on $UTM$.

**Definition 3.1.** We say that a map $\Phi : M \to \mathcal{L}$ is a *special embedding* if there is a family $\{\Phi_s\}_{s\in S}$ of real-valued functions on $M$ (that will be referred to as *coordinate functions* of $\Phi$) such that

(1) For every $x \in M$, the image $\Phi(x)$ is a function $s \mapsto \Phi_s(x)$ on $S$ (more precisely, the element of $\mathcal{L}$ represented by this function).

(2) The function $(x, s) \mapsto \Phi_s(x)$ is smooth on $M \times S$.

(3) Every function $\Phi_s : M \to \mathbb{R}$ is distance-like, that is, $|\operatorname{grad} \Phi_s| \equiv 1$ on $M$.

(4) For every $x \in M$, the map $s \mapsto \operatorname{grad} \Phi_s(x)$ is a diffeomorphism between $S$ and $UT_xM$.

**Notation.** We denote the inverse of the above diffeomorphism $s \mapsto \operatorname{grad} \Phi_s(x)$ by $\alpha$. That is, $\alpha : UTM \to S$ is a map such that $\operatorname{grad} \Phi_{\alpha(v)}(x) = v$ for all $x \in M$

and $v \in UT_xM$. By $\alpha_x$ we denote the restriction of $\alpha$ to $UT_xM$, this map is a diffeomorphism from $UT_xM$ to $S$.

For every $x \in M$, let $\mu_x$ be the push-forward of the normalized Haar measure on $UT_xM$ under the diffeomorphism $\alpha_x : UT_xM \to S$. Then $\mu_x$ is smooth probability measure on $S$. We denote by $\lambda(x,s)$ the the density of $\mu_x$ at $s \in S$ with respect to the above fixed measure on $S$.

The second requirement of Definition 3.1 implies the image of $\Phi$ lies in the subspace $C^\infty(S) \subset \mathcal{L}$, $\Phi$ is a smooth map (even w.r.t. the $C^\infty$ topology in the target), and its derivative $d_x\Phi : T_xM \to \mathcal{L}$ is given by

$$d_x\Phi(v)(s) = d_x\Phi_s(v), \qquad v \in T_xM,\ s \in S. \tag{3.1}$$

Let $v_0 \in UT_xM$ and $T = d_x\Phi(v_0)$. We rewrite the above identity as

$$T(s) = d\Phi_s(v_0) = \langle \operatorname{grad} \Phi_s(x), v_0\rangle = \langle \alpha_x^{-1}(s), v_0\rangle.$$

Substituting $s = \alpha(v)$, $v \in UT_xM$ yields

$$T(\alpha(v)) = \langle v, v_0\rangle \qquad \text{if } v_0, v \in UT_xM \text{ and } T = d_x\Phi(v_0). \tag{3.2}$$

As usual, the image of the derivative $d_x\Phi : T_xM \to \mathcal{L}$ is denoted by $T_x\Phi$ and referred to as the tangent space of $\Phi$ at $x$, and the elements of $T_x\Phi$ are referred to as the tangent vectors of $\Phi$.

The density function $\lambda : M \times S \to \mathbb{R}$ is smooth and positive. The definitions imply that

$$\int_S f\, d\mu_x = \int_S f(s)\lambda(x,s)\, ds = \int_{UT_xM} f(\alpha(v))\, dv \tag{3.3}$$

for any measurable function $f : S \to \mathbb{R}$.

**Definition 3.2.** By a *scalar product* on $\mathcal{L}$ we mean an $L^2$-compatible scalar product, that is, a symmetric bilinear form $G$ on $\mathcal{L}$ satisfying

$$c\|u\|^2_{L^2(S)} \le G(u,u) \le C\|u\|^2_{L^2(S)}$$

for some positive constants $c$ and $C$. The norm of a scalar product is defined as the minimum possible value of $C$.

A *Riemannian metric* in a region $\mathcal{U} \subset \mathcal{L}$ is a smooth family $G = \{G_\varphi\}_{\varphi\in\mathcal{U}}$ of scalar products on $\mathcal{L}$. (The smoothness here is that of a map from $\mathcal{U} \subset L^\infty(S)$ to the space of $L^2$-compatible scalar products on $\mathcal{L}$ equipped with the above norm.)

**Definition 3.3.** Let $G$ be a Riemannian metric in a region $\mathcal{U} \subset \mathcal{L}$ containing $\Phi(M)$. We say that $G$ is *special* with respect to $\Phi$ if the following holds:

(1) For every $\varphi \in \mathcal{U}$, the scalar product $G_\varphi$ has the form

$$G_\varphi(X,Y) = n \int_S X(s)Y(s)\, d\nu_\varphi(s), \qquad X, Y \in \mathcal{L},$$

where $\nu_\varphi$ is a probability measure on $S$.

(2) Every measure $\nu_\varphi$ has positive density separated away from zero; these densities depend smoothly on $\varphi$.

(3) If $\varphi = \Phi(x)$ for an $x \in M$, then $\nu_\varphi = \mu_x$.

**Lemma 3.4.** *If $G$ is a special Riemannian metric on $\mathcal{U} \subset \mathcal{L}$ with respect to $\Phi$, then $\Phi$ is an isometric immersion of $M$ to $(\mathcal{U}, G)$.*

*Proof.* Let $x \in M$, $\varphi = \Phi(x)$, $v_0 \in UT_xM$, $T = d\Phi(v_0)$. Then

$$G_\varphi(T,T) = n\int_S T^2\, d\mu_x = n\int_{UT_xM} T(\alpha(v))^2\, dv = n\int_{UT_xM} \langle v, v_0\rangle^2\, dv = 1,$$

hence the result. The first three equalities in this computation follow from Definition 3.3(3), (3.3) and (3.2), and the last one is a standard computation on the sphere. □

3.2. **Mean curvature.**

**Definition 3.5.** Let $G$ be a Riemannian metric in a region $\mathcal{U} \subset \mathcal{L}$, $\Phi : M \to \mathcal{U}$ a smooth surface, $x \in M$, $V \in \mathcal{L}$ a vector orthogonal to $T_x\Phi$ with respect to $G$. We define a quadratic form on $T_x\Phi$, called the *second fundamental form* with respect to $V$ and denoted by $\mathrm{I\!I}_V$, as follows.

Let $F$ be a smooth finite-dimensional submanifold of $\mathcal{U}$ containing a neighborhood of $\varphi = \Phi(x)$ in $\Phi(M)$ and such that $V$ is tangent to $F$ at $\varphi$. Let $T \in T_x\Phi$. Construct a smooth vector field $\tilde{T}$ tangent to $\Phi$ near $\varphi$ such that $\tilde{T}(\varphi) = T$. Then define

$$\mathrm{I\!I}_V^{\Phi,x}(T,T) = G(\nabla_T\tilde{T}, V)$$

where $\nabla$ is the Levi-Civita connection of the induced Riemannian metric on $F$.

The trace of the quadratic form $T \mapsto \mathrm{I\!I}_V(T,T)$ (with respect to the Euclidean structure on $T_x\Phi$ defined by $G = G_\varphi$) is called the *mean curvature* with respect to $V$ and denoted by $H_V^{\Phi,x}$.

It is easy to see (cf. e.g. (3.4) below) that the second fundamental form does not depend on the choice of the auxiliary submanifold $F$. The fact that is does not depend on $\tilde{T}$ follows from the standard (finite-dimensional) Riemannian geometry.

**Proposition 3.6.** *Let $\Phi : M \to \mathcal{L}$ be a special embedding and $G$ is a special Riemannian metric with respect to $\Phi$. Then for every $x \in M$ and every vector $V \in \mathcal{L}$ orthogonal to $T_x\Phi$ (with respect to $G$) one has*

$$H_V^{\Phi,x} = 0.$$

*Proof.* We use the notation from Definition 3.5. Extend $V$ and $T$ to commuting smooth vector fields $\tilde{V}$ and $\tilde{T}$ tangent to $F$ and defined in a neighborhood of $\Phi(x)$ in $F$. In addition, $\tilde{T}$ should be tangent to $\Phi(M)$. Then

$$\mathrm{I\!I}_V(T,T) = G(\nabla_T\tilde{T}, V) = G(\tilde{T},\tilde{V})'_T - \frac{1}{2}G(\tilde{T},\tilde{T})'_V$$

by Riemannian geometry on $F$. Here the notation like $(\dots)'_X$ denotes the derivative along a vector $X \in \mathcal{L}$; this derivative is well-defined whenever the argument is defined along any smooth curve in $\mathcal{L}$ with initial direction $X$. In our case this requirement is satisfied since $\tilde{T}$ and $\tilde{V}$ are defined along $F$. Differentiating the above $G$-products yields

$$G(\tilde{T},\tilde{V})'_T = G'_T(T,V) + G(\tilde{T}'_T, V) + (T, \tilde{V}'_T)$$

and

$$\frac{1}{2}G(\tilde{T},\tilde{T})'_V = \frac{1}{2}G'_V(T,T) + G(T,\tilde{T}'_V).$$

Since $\tilde{T}$ and $\tilde{V}$ commute, we have $\tilde{V}'_T = \tilde{T}'_V$, thus

$$\mathrm{I\!I}_V(T,T) = G'_T(T,V) + G(\tilde{T}'_T, V) - \frac{1}{2}G'_V(T,T). \tag{3.4}$$

We are going to compute the traces of the three terms of this sum separately. Let $v \in UT_xM$ be such that $T = d\Phi(v)$. We may assume that the vector field $\tilde{T}$ is chosen so that its trajectory through $\Phi(x)$ is the $\Phi$-image of a constant-speed geodesic $\gamma_v$ is $M$ such that $\gamma(0) = x$ and $\dot{\gamma}_v(0) = v$. Then the first term in (3.4) takes the form

$$\begin{aligned} G'_T(T,V) &= \frac{d}{dt}\Big|_{t=0} G_{\Phi(\gamma_v(t))}(T,V) = n \cdot \frac{d}{dt}\Big|_{t=0} \int_S T(s)V(s)\lambda(\gamma_v(t),s)\,ds \\ &= n \int_S V(s)T(s)L_s(v)\,ds \end{aligned}$$

where

$$L_s(v) = \frac{d}{dt}\Big|_{t=0} \lambda(\gamma_v(t),s). \tag{3.5}$$

(The second identity above follows from the special form of $G$ on $\Phi(M)$, cf. Definition 3.3.) Substituting $T = d\Phi(v)$ and using (3.1) yields

$$G'_T(T,V) = \int_S V(s) \cdot d\Phi_s(v) \cdot L_s(v)\,ds.$$

Hence

$$\operatorname{trace}_{T_x\Phi}\big[T \mapsto G'_T(T,V)\big] = n \int_S V(s) \cdot \operatorname{trace}_{T_xM}\big[v \mapsto d\Phi_s(v) \cdot L_s(v)\big]\,ds.$$

Recall that $|\operatorname{grad}\Phi_s| \equiv 1$ by the 3rd requirement of Definition 3.1. Fix an $s \in S$ and choose an orthonormal basis $(v_1, \dots, v_n)$ of $T_xM$ such that $v_1 = \operatorname{grad}\Phi_s(x)$. Then $d\Phi_s(v_1) = 1$ and $d\Phi_s(v_i) = 0$ for all $i > 1$. Hence

$$\operatorname{trace}_{T_xM}\big[v \mapsto d\Phi_s(v) \cdot L_s(v)\big] = \sum_{i=1}^n d\Phi_s(v_i) \cdot L_s(v_i) = L_s(v_1) = L_s(\operatorname{grad}\Phi_s(x)).$$

Thus

$$\operatorname{trace}_{T_x\Phi}\big[T \mapsto G'_T(T,V)\big] = n \int_S V(s) \cdot L_s(\operatorname{grad}\Phi_s(x))\,ds. \tag{3.6}$$

**Lemma 3.7.** *For every $x \in M$ and $s \in S$, one has*

$$L_s(\operatorname{grad}\Phi_s(x)) = -\lambda(x,s) \cdot \Delta\Phi_s(x)$$

*where $\Delta$ is the Riemannian Laplace operator on $M$.*

*Proof.* Consider a map $I : M \times S \to UTM$ given by $I(x,s) = \operatorname{grad}\Phi_s(X)$. Since $\Phi$ is a special map (cf. Definition 3.1(4)), $I$ is a diffeomorphism; its inverse $I^{-1} : UTM \to M \times S$ is given by $I^{-1}(v) = (x, \alpha(v))$ for $v \in UT_xM$.

Consider a vector field $W$ on $M \times S$ given by

$$W(x,s) = (\operatorname{grad}\Phi_s(x), 0).$$

The flow on $M \times S$ generated by this vector field is mapped by $I$ to the geodesic flow on $UTM$. This follows from the fact that every function $\Phi_s$ is distance-like (cf. Definition 3.1(3)) and hence the trajectories of its gradient flow are geodesics.

Substituting $v = \operatorname{grad}\Phi_s(x)$ into (3.5) yields

$$L_s(\operatorname{grad}\Phi_s(x)) = \lambda'_W(x,s)$$

where $\lambda'_W$ denotes the derivative of $\lambda$ along $W$.

Let $\mu$ denote the standard product measure on $M \times S$, that is, the product of the Riemannian volume form on $M$ and the standard measure (the one denoted by $ds$) on $S$. Then the measure $\lambda\mu$ (that is, the measure with density $\lambda$ with respect to $\mu$) is mapped by $I$ to the Liouville measure on $UTM$. Since the Liouville measure is preserved by the geodesic flow, the measure $\lambda\mu$ is preserved by the flow generated by $W$. Hence

$$\operatorname{div}_{\lambda\mu} W = 0$$

where div denotes the divergence (with respect to a given measure). On the other hand,

$$\operatorname{div}_{\lambda\mu} W = \operatorname{div}_\mu W + \lambda^{-1}\lambda'_W.$$

Thus

$$L_s(\operatorname{grad}\Phi_s(x)) = \lambda'_W(x,s) = -\lambda(x,s)\cdot \operatorname{div}_\mu W(x,s).$$

Since $W(x,s) = (\operatorname{grad}\Phi_s(x), 0)$ and $\mu$ is the product measure, we have

$$\operatorname{div}_\mu W(x,s) = \operatorname{div}_M \operatorname{grad}\Phi_s(x) = \Delta\Phi_s(x),$$

and Lemma 3.7 follows. □

With this lemma, (3.6) takes the form

$$\operatorname{trace}_{T_x\Phi}\big[T \mapsto G'_T(T,V)\big] = -n\int_S V(s)\cdot\lambda(x,s)\cdot\Delta\Phi_s(x)\,ds. \tag{3.7}$$

Now consider the second term of (3.4). Recall that $T = d_x\Phi(v)$ and $\tilde{T}$ contains the velocity field of the geodesic $\Phi\circ\gamma_v$. Hence

$$\tilde{T}'_T = \frac{d}{dt}\bigg|_{t=0} d\Phi(\dot\gamma_v(t)) = \frac{d^2}{dt^2}\bigg|_{t=0}\Phi(\gamma(t)).$$

The right-hand side is a function on $S$ whose value at $s \in S$ equals

$$\frac{d^2}{dt^2}\Big|_{t=0}\Phi_s(\gamma_v(t)) = D^2\Phi_s(v,v)$$

where $D^2$ denotes the Hessian w.r.t. the Riemannian metric of $M$. Hence

$$G(\tilde{T}'_T, V) = n\int_S D^2\Phi_s(v,v)\cdot V(s)\cdot\lambda(x,s)\,ds$$

(here we use the special form of $G$ at $\Phi(x)$, cf. Definition 3.3). Hence

$$\begin{aligned}\operatorname{trace}_{T_x\Phi}\big[T\mapsto G(\tilde{T}'_T,V)\big] &= n\int_S V(s)\cdot\lambda(x,s)\cdot\operatorname{trace}_{T_xM}[D^2\Phi_s(x)]\,ds\\ &= n\int_S V(s)\cdot\lambda(x,s)\cdot\Delta\Phi_s(x)\,ds.\end{aligned}\tag{3.8}$$

Adding together (3.7) and (3.8) yields

$$\operatorname{trace}_{T_x\Phi}\big[T\mapsto G'_T(T,V) + G(\tilde{T}'_T,V)\big] = 0. \tag{3.9}$$

It remains to get rid of the third term in (3.4). Let $(T_1,\dots,T_n)$ be an orthonormal basis of $T_x\Phi$. Then

$$\operatorname{trace}\big[T\mapsto G'_V(T,T)\big] = \sum_{i=1}^n G'_V(T_i,T_i) = n\sum_{i=1}^n\int_S T_i(s)^2(\rho'_V)(s)\,ds \tag{3.10}$$

where $\rho = \{\rho_\varphi\}_{\varphi\in\mathcal{U}}$ is the density of the measure $\nu_\varphi$ from Definition 3.3 regarded as a function on $U$ (with values in $L^\infty(S)$) and $\rho'_V$ is its derivative along $V$.

Let $T_i = d_x\Phi(v_i)$ where $v_i \in T_xM$ for $i = 1, \dots, n$. Since $\Phi$ is isometric by Lemma 3.4, $(v_1, \dots, v_n)$ is an orthonormal basis of $T_xM$. Then by (3.2) we have

$$\sum T_i(\alpha(v))^2 = \sum \langle v, v_i\rangle^2 = |v|^2 = 1$$

for all $v \in UT_xM$. Since $\alpha_x : UT_xM \to S$ is surjective, it follows that $\sum T_i(s)^2 = 1$ for all $s \in S$. Then (3.10) takes the form

$$\text{trace}\left[T \mapsto G'_V(T,T)\right] = n \int_S (\rho'_V)(s)\, ds = n \left(\int_S \rho\right)'_V = 0 \tag{3.11}$$

since the integral of $\rho$ is fixed to be 1 (cf. Definition 3.3(2)). Now (3.4), (3.9) and (3.11) imply that

$$H_V^{\Phi,x} = \text{trace}_{T_x\Phi}(\mathbf{II}_V^{\Phi,x}) = 0 - \frac{1}{2} \cdot 0 = 0.$$

This finishes the proof of Proposition 3.6. $\square$

3.3. **Jacobians and projections.**

**Definition 3.8.** Let $G$ be a scalar product on $\mathcal{L}$, $X$ an $n$-dimensional Euclidean space and $L : \mathcal{L} \to X$ a linear map bounded with respect to $G$ (or, equivalently, with respect to the standard $L^2$ scalar product). For an $n$-dimensional subspace $Y \subset \mathcal{L}$, we denote by $J_{G,Y}$ the Jacobian (that is, the $n$-dimensional volume expansion coefficient) of the restriction $L|_Y$ where $Y$ is regarded as a Euclidean space whose scalar product is the restriction of $G$.

The ($n$-dimensional) Jacobian of $L$, denoted by $J_GL$, is the supremum of $J_{G,Y}$ where $Y$ ranges over all $n$-dimensional linear subspaces of $\mathcal{L}$.

Obviously $JL = 0$ if the rank of $L$ is less than $n$. If the rank equals $n$, then the supremum is attained when $Y$ is the orthogonal complement to $\ker L$. It follows that $J_GL$ depends continuously on $L$ and $G$, moreover this dependence is smooth on the set where $J_GL \neq 0$.

**Definition 3.9.** Let $G = \{G_\varphi\}_{\varphi\in\mathcal{U}}$ be a Riemannian metric in a region $\mathcal{U} \subset \mathcal{L}$, and let $P : \mathcal{U} \to M$ be an $L^2$-smooth map (see Definition 3.10 below). For an $\varphi \in \mathcal{U}$, we define the $n$-dimensional Jacobian of $P$ at $\varphi$, denoted by $J_GP(\varphi)$, as the Jacobian of the derivative $d_\varphi P : \mathcal{L} \to T_{P(\varphi)}M$ with respect to the scalar product $G_\varphi$ on $\mathcal{L}$ and the Riemannian scalar product on $T_{P(\varphi)}M$.

For an $n$-dimensional subspace $Y \subset T_\varphi\mathcal{L} = L$, we denote by $J_{G,Y}P$ the Jacobian of the restriction of $d_\varphi P$ to $Y$.

We will omit $G$ in these notations if the metric is clear from context.

**Definition 3.10.** We say that a map from a region $\mathcal{U} \subset \mathcal{L}$ to $M$ is $L^2$-smooth if it is differentiable, its derivative at every point $\varphi \in \mathcal{U}$ can be extended as a bounded linear map from $L^2$ to a fiber of $TM$ and this map from $L^2$ in its turn smoothly depends on $\varphi$.

Note that the Jacobian of an $L^2$-smooth map is a continuous function and moreover it is smooth in the complement of its zeros.

**Definition 3.11.** Let $\Phi : M \to \mathcal{U} \subset \mathcal{L}$ be a smooth isometric immersion with respect to a Riemannian metric $G$ on $\mathcal{U}$. We say that a map $P : \mathcal{U} \to M$ is a *projection* if it is $L^2$-smooth and

(1) $P \circ \Phi = id_M$ .

(2) For every $x \in M$, $d_{\Phi(x)}P(V) = 0$ for every vector $V \in \mathcal{L}$ orthogonal (with respect to $G$) to $T_x\Phi$.

**Proposition 3.12.** *Let $\Phi : M \to \mathcal{U} \subset \mathcal{L}$ be a smooth isometric immersion with respect to a Riemannian metric $G$ on $\mathcal{U}$ and $P : \mathcal{U} \to M$ a projection in the sense of Definition 3.11. Then, for every $x \in M$ and every $V \in \mathcal{L}$ orthogonal to $T_x\Phi$,*

$$d_{\Phi(x)}J_G P(V) = H_V^{\Phi,x}.$$

*In particular, if $\Phi$ is a special embedding and $G$ is a special metric with respect to $\Phi$, then*

$$d_{\Phi(x)}J_G P(V) = 0.$$

*Proof.* It suffices to verify this fact for a finite-dimensional Riemannian manifold $F$ in place of $(\mathcal{L}, G)$. Indeed, it suffices to compute the derivative of $JP$ along a smooth curve $\gamma$ in $\mathcal{U}$ starting at $\Phi(x)$ with initial velocity $V$. At every point $\gamma(t)$, there is a unique $n$-dimensional subspace $Y(t)$ of $\mathcal{L}$ where the maximum in the definition of the Jacobian is attained, and $Y(t)$ depends smoothly on $t$. Furthermore, $Y(0)$ is the tangent space of $T_x\Phi$. Therefore the derivative of the Jacobian can be computed within a smooth submanifold $F$ containing (locally) $\Phi(M)$ and $\gamma$ and such that every subspace $Y(t)$ is tangent to $F$ at $\gamma(t)$. It is easy to construct such a submanifold of dimension $n+1$.

In the finite-dimensional case, there exists a smooth variation $\{\Phi_t\}_{t\in(-\varepsilon,\varepsilon)}$ of $\Phi$ such that

(1) $P \circ \Phi_t = id_M$ for all $t$;
(2) $\frac{d}{dt}\big|_{t=0}\Phi_t(x) = V$;
(3) the image of $d_x\Phi_t$ is the subspace realizing the Jacobian of $P$.

The assumptions on $P$ imply that the variation field $\frac{d}{dt}\big|_{t=0}\Phi_t$ is orthogonal to our surface $\Phi$. Then $J_{\Phi_t(x)} = J\Phi_t(x)^{-1}$, hence

$$d_{\Phi(x)}JP(V) = -\frac{d}{dt}J\Phi_t(x).$$

The right-hand side is the first variation of the $n$-dimensional surface area which is known to be equal to $-H_V^{\Phi,x}$. ☐

3.4. **Surface area.** Let $N$ be a smooth $n$-dimensional manifold and $f : N \to \mathcal{L}$ a Lipschitz map. We say that $f$ is *weakly differentiable* at $p \in N$ if there exists a linear map $d_p^w f : T_pN \to \mathcal{L}$ (called the weak derivative of $f$ at $x$) such that, for every $u \in L^1(S)$ the map

$$x \mapsto \langle u, f(x)\rangle : N \to \mathbb{R}$$

is differentiable at $p$, and its derivative at $p$ is the linear map

$$v \mapsto \langle u, d_p^w f\rangle : T_pN \to \mathbb{R}.$$

Here the angle brackets denote the standard pairing between $L^1(S)$ and $\mathcal{L} = L^\infty(S)$.

It can be shown (cf. [1, §5] or [12, §3]) that every Lipschitz map $f : N \to \mathcal{L}$ is weakly differentiable a.e. on $N$. Moreover if $N$ is a Riemannian manifold and $f$ is 1-Lipschitz, then $d_p^w f : T_pN \to \mathcal{L}$ is a 1-Lipschitz linear map for a.e. $p \in N$.

Let $G$ be a Riemannian structure in a region of $\mathcal{L}$ containing $f(N)$. For every $x \in N$ where $f$ is differentiable, the weak derivative $d_x^w f$ determines a pull-back nonnegative definite quadratic form $f^*G(x) := (d_x^w f)^*(G_{f(x)})$. This quadratic form determines a nonnegative $n$-volume density on $T_xN$, denoted by $d\,\mathrm{vol}_G\, f(x)$. The $n$-volume of the surface $f$ with respect to $G$ is defined by

$$\mathrm{vol}_G(f) = \int_N d\,\mathrm{vol}_G\, f.$$

(There is a minor technical detail to show that the density under the integral is measurable. We leave this as an exercise to the reader. A reader who prefers to skip this exercise can replace the integral by the upper integral in the above definition).

**Lemma 3.13.** *Let $N$ be an $n$-dimensional Riemannian manifold and $f : N \to \mathcal{L}$ a 1-Lipschitz map. Suppose that $G$ is a special Riemannian metric (with respect to some $\Phi$, cf. Definition 3.3). Then $f$ does not increase $n$-volumes, that is,*

$$\mathrm{vol}_G(f) \le \mathrm{vol}(N).$$

*Proof.* This follows from the normalization condition of Definition 3.3(1), namely that the scalar product $G_\varphi$ at $\varphi \in \mathcal{L}$ has the form

$$G_\varphi(X,Y) = n \int_S X(s)Y(s)\, d\nu_\varphi(s), \qquad X, Y \in \mathcal{L},$$

where $\nu_\varphi$ is a probability measure on $S$.

Let $x \in N$ and $\varphi = f(x)$. Fix an orthonormal basis $e_1, \dots, e_n$ in $T_xN$ and define $X_i = d^w_x f(e_i)$. Since $d^w_x f$ is 1-Lipschitz, we have

$$\|d^w_x f(v)\|_{L^\infty} \le |v|$$

for any $v \in T_xN$. Substituting $v = \sum v_i e_i$ yields

$$\|v_1 X_1 + \dots + v_n X_n\|_{L^\infty} \le \sqrt{v_1^2 + \dots + v_n^2}$$

for any $v_1, \dots, v_n \in \mathbb{R}$. This implies that

$$X_1(s)^2 + \dots + X_n(s)^2 \le 1$$

for a.e. $s \in S$ (to see this, substitute $X_i(s)$ for $v_i$ in the previous inequality). Now

$$\mathrm{trace}(f^*G(x)) = \sum G_\varphi(X_i, X_i) = \sum n \int_S X_i(s)^2\, d\nu_\varphi(s) \le n \int_S d\nu_\varphi(s) = n.$$

Since $\mathrm{trace}(f^*G(x)) \le n$, we have $\det(f^*G(x)) \le 1$, thus the induced volume density on $T_xN$ is no greater that the Riemannian volume form. □

**Lemma 3.14.** *Let $N$ be an $n$-dimensional Riemannian manifold, and $f : N \to \mathcal{L}$ a 1-Lipschitz map. Suppose that $G$ is a special Riemannian metric in a region $\mathcal{U} \subset \mathcal{L}$ containing $f(N)$ and $P : \mathcal{U} \to M$ an $L^2$-smooth map (cf. Definition 3.10). Then*

$$\mathrm{vol}(P \circ f) \le \int_N J_G P(f(x))\, d\,\mathrm{vol}_N(x).$$

*Here $\mathrm{vol}(P \circ f)$ denotes the ordinary $n$-volume of a Lipschitz map (or, equivalently, the area of the image with multiplicities).*

*Proof.* The definitions of weak derivative and $L^2$-smoothness imply that $P$ and $f$ satisfy the chain rule:

$$d_x(P \circ f) = d_{f(x)} P \circ d^w_x f$$

for every $x \in N$ where the weak derivative $d^w_x f$ exists. It follows that the Jacobian of $d(P \circ f)$ is bounded above by the product of the Jacobians of $d_{f(x)}P$ and $d^w_x f$ (with respect to $G_{f(x)}$), and the latter Jacobian does not exceed 1 by the previous lemma. □

## 4. The construction

Now we return to the case of an almost hyperbolic metric. Recall that $g_0$ denotes the standard metric on $\mathbb{H}^n$, $o \in \mathbb{H}^n$ is a fixed origin. By $B_o(r)$ we denote the ball of radius $r$ in $\mathbb{H}^n$ centered at $o$. Let $S$ be the ideal boundary of $\mathbb{H}^n$; we identify $S$ with $S^{n-1}$ in a natural way (via the unit tangent space at the origin).

Recall that our metric $g$ is extended from a region $D \subset B_o(R/5)$ to the entire $\mathbb{H}^n$ so that $g \equiv g_0$ outside $B_o(R/2)$ and $g$ is close to $g_0$ on $\mathbb{H}^n$. We denote $M = (\mathbb{H}^n, g)$.

To simplify exposition, we do not track the dependence on $g$ and its derivatives in our constructions. We say that a dependence on $g$ is *smooth* if for every integer $k > 0$ there exists an $r > 0$ such that this dependence is $k$ times differentiable with respect to the $C^r$ norm on the space of metrics (more precisely, on a neighborhood of $g_0$ in the space of metrics).

Since $g$ and $g_0$ coincide outside a compact set, their boundaries at infinity are canonically identified. For every $s \in S$, let $\Phi_s : M \to \mathbb{R}$ be the Busemann function of a geodesic ray starting at $o$ towards $s \in \partial_\infty M = S$. Define a map $\Phi : M \to \mathcal{L}$ as in Definition 3.1, that is,

$$\Phi(x)(s) = \Phi_s(x), \qquad x \in M, \ s \in S.$$

**Lemma 4.1.** *1. $\Phi_x(s)$ depends smoothly on $x$, $s$ and $g$.*

*2. If $g$ is sufficiently close to $g_0$, then $\Phi$ is a special embedding in the sense of Definition 3.1.*

*Proof.* 1. The lemma becomes obvious once one realizes that in our situation the Busemann functions, which are usually defined using asymptotic constructions, are objects that can be defined in terms of a compact region where our metrics may be non-standard. Namely, rather than using the boundary at infinity, one could the boundary of the ball $B_o(R)$, then the Busemann functions turn into (normalized by their values at $o$) distance function to the horospheres tangent to this sphere, and these horospheres are the same for $g$ and $g_0$. After this observation the proof is straightforward.

2. The first three requirements of Definition 3.1 trivially follow from the construction and the first part of the lemma. The last requirement asserting that the map $s \mapsto \operatorname{grad} \Phi_s(x)$ is a diffeomorphism between $S$ and $UT_xM$ is trivial in the case $g = g_0$, and then the general case follows from the fact the derivatives of this map depend continuously on $g$. $\square$

We use notations introduced in Section 3 for special maps.

**Lemma 4.2.** *If $g = g_0$, then*

$$\lambda(x,s) = \frac{d\mu_x(s)}{d\mu_o(s)} = e^{-(n-1)\Phi_s(x)}$$

*for all $x \in M$, $s \in S$.*

*Proof.* This straightforward statement can be found in [5]. $\square$

Denote by $\mathcal{B} = \mathcal{B}(R)$ the ball of radius $R$ is $\mathcal{L}$ with respect to the $L^\infty$ norm.

**Definition 4.3.** We define a "projection" $P : \mathcal{B} \to M$ as follows. For every $\varphi \in \mathcal{L}$, introduce a vector field $W_\varphi$ on $M$ by

$$W_\varphi(x) = \int_S e^{n(\Phi_s(x)-\varphi(s))} \operatorname{grad} \Phi_s(x) \, d\mu_x(s)$$

and let $P(\varphi)$ be a point $x \in M$ such that

$$W_\varphi(x) = 0$$

provided that such a point exists and is unique.

In the case $g = g_0$, the equation says that $x$ is a critical point of a function

$$F_\varphi(x) = \int_S e^{-n\varphi(s)} e^{\Phi_s(x)} ds.$$

It is easy to see that $F_\varphi$ is convex (and actually strictly quadratically convex) and grows to infinity as $x \to \infty$, and hence it has only one critical point where the minimum of $F_\varphi$ is attained.

**Lemma 4.4.** *If $g$ is sufficiently close to $g_0$, then:*

1. *$P$ is well-defined and smooth in a neighborhood of $\Phi(M)$ containing $\mathcal{B}$.*
2. *$P(\Phi(x)) = x$ for all $x \in M$.*
3. *$P(\mathcal{B}) \subset B_o(R_1)$ for some radius $R_1$ depending only on $n$ and $R$.*

*Remark.* We do not yet claim that $P$ is $L^2$-smooth (in the sense of Definition 3.10). This will be shown later.

*Proof of Lemma 4.4.* Substituting $s = \alpha(v)$, $v \in UT_xM$, we rewrite the equation $W_\varphi(x) = 0$ as

$$W_\varphi(x) = \int_{UT_xM} e^{n(\Phi_{\alpha(v)}(x) - \varphi(\alpha(v)))} v \, dv = 0.$$

This equation is trivially satisfied for $\varphi = \Phi(x)$, hence the second assertion of the lemma follows from the first one. This observation and the smoothness of our equation implies, by the implicit function theorem in $L^2$, that $P$ is well-defined and smooth in a neighborhood of $\Phi(M)$. Here are the details.

Consider a map $E : \mathcal{L} \to L^2(S)$ given by

$$E(\varphi)(s) = e^{-n\varphi(s)}.$$

Note that the set $E(\mathcal{B})$ is compact in $L^2$. Let $\varphi \in \mathcal{L}$ and $\psi = E(\varphi)$. Then the equation $W_\varphi(x) = 0$ takes the form

$$\int_S \psi(s) e^{n\Phi_s(x)} \operatorname{grad} \Phi_s(x) \, d\mu_x(s) = 0. \tag{4.1}$$

Since this equation is linear in $\psi$ and is satisfied for $\psi = E(\Phi(x))$, it is equivalent to

$$\int_S (\psi(s) - E \circ \Phi(x)(s)) e^{n\Phi_s(x)} \operatorname{grad} \Phi_s(x) \, d\mu_x(s) = 0$$

or, equivalently,

$$\int_S (\psi(s) - E \circ \Phi(x)(s)) e^{2n\Phi_s(x)} d_x(E \circ \Phi)(s) \, d\mu_x(s) = 0.$$

This can be interpreted as follows: the vector $\psi - E \circ \Phi \in L^2(S)$ belongs to the orthogonal complement to the tangent space $T_x(E \circ \Phi)$ of the surface $E \circ \Phi$ with respect to the scalar product defined by a measure $e^{2n\Phi_s(x)} d\mu_x(s)$ depending on $x$. Since $E \circ \Phi$ is a smooth surface in $L^2$ and the scalar products depend smoothly on $x$, this defines a smooth retraction $\tilde{P}$ from a neighborhood of $E(\Phi(M))$ to $E(\Phi(M))$. Our projection $P$ can be written as $P = (E \circ \Phi)^{-1} \circ \tilde{P} \circ \tilde{E}$ provided that $\tilde{P}$ is well-defined in a neighborhood of $E(\mathcal{B}) \cup E(\Phi(M))$.

It remains to verify that $x = \tilde{P}(\psi)$ satisfying (4.1) is unique and depends smoothly on $\psi$ in a neighborhood of $E(\mathcal{B}) \cup E(\Phi(M))$. Since $E(\mathcal{B})$ is compact and $\Phi$ is a smooth perturbation of the similar map for $g = g_0$, the implicit function theorem tells us that it suffices to verify this fact only in the case $g = g_0$. The uniqueness for $\psi \in E(\mathcal{B})$ is verified above, so it remains to see that the map is smooth in the $L^2$ sense.

Substituting the formula for the density of $\mu_x$ (Lemma 4.2) into our equation yields

$$\int_S \psi(s) e^{\Psi_s(x)} \operatorname{grad} \Phi_s(x)\, ds = 0,$$

or, equivalently,

$$\int_S \psi(s)\, d_x e^{\Phi_s}\, ds = 0. \tag{4.2}$$

We interpret this equation as follows: the vector $\psi \in L^2(S)$ is orthogonal to the tangent space (at the point corresponding to $x$) of the surface $Q$ in $L^2(S)$ parametrized by the map by $x \mapsto (s \mapsto e^{\Phi_s(x)})$ from $M = \mathbb{H}^n$ to $L^2(S)$.

Now we make an observation that exponents $e^{\Phi_s}$ of Busemann functions $\Phi_s$ on $\mathbb{H}^n$ form a set lying in an $(n+1)$-dimensional linear subspace. This follows from the fact that they satisfy a second order equation. It follows that $Q$ is contained in an $(n+1)$-dimensional linear subspace $Z \subset L^2(S)$. (In fact, $Q$ is a connected component of a hyperboloid of signature $(n,1)$ in $Z$.) Hence the tangent spaces of $Q$ are subspaces of $Z$. Therefore the equation (4.1) for $\psi$ is equivalent to the same equation for the orthogonal projection of $\psi$ to $Q$ in $L^2(S)$. Thus the map $\tilde{P}$ can be defined as follows: first, the argument $\psi$ is projected orthogonally to $Z$, then the resulting point is projected radially to $Q$, and the resulting point of $Q$ is mapped back to $\mathbb{H}^n$. The radial projection is well defined (within a suitable open cone) since no two elements of $Q$ are proportional to each other, and it is smooth since the resulting map is smooth in a neighborhood of $E \circ \Phi(M)$. $\square$

**Notation.** For every $x \in M$ and $s \in S$, define a linear operator $A_{x,s} : T_x M \to T_x M$ by

$$A_{x,s}(\xi) = e^{-n\Phi_s(x)} \lambda(x,s)^{-1} \nabla_\xi T_s \tag{4.3}$$

where $T_s$ is a vector field on $M$ given by

$$T_s(x) = e^{n\Phi_s(x)} \lambda(x,s) \operatorname{grad} \Phi_s(x) \tag{4.4}$$

and $\nabla_\xi$ denotes the Levi-Civita derivative along $\xi$.

Let $\varphi \in \mathcal{L}$ and $x = P(\varphi)$. Define a function $\rho_\varphi$ on $S$ by

$$\rho_\varphi(s) = e^{n(\Phi_s(x) - \varphi(s))} \tag{4.5}$$

and let $\overline{\rho}_\varphi$ be the same function normalized with respect to the measure $\mu_x$:

$$\overline{\rho}_\varphi = \frac{\rho}{\int_S \rho\, d\mu_x}. \tag{4.6}$$

Now define a linear operator $A_\varphi : T_x M \to T_x M$ by

$$A_\varphi = \int_S \overline{\rho}_\varphi(s) A_{x,s}\, d\mu_x(s). \tag{4.7}$$

**Lemma 4.5.** *Let $\varphi \in \mathcal{B}$, $x = P(\varphi)$ and $\delta \in T_\varphi \mathcal{L} \simeq \mathcal{L}$. Then*

$$d_\varphi P(\delta) = A_\varphi^{-1} \circ E_\varphi(\delta)$$

*where a linear map $E_\varphi : \mathcal{L} \to T_x M$ is given by*

$$E_\varphi(\delta) = n \int_S \delta(s) \overline{\rho}_\varphi(s) \operatorname{grad} \Phi_s(x) \, d\mu_x(s) \tag{4.8}$$

*Proof.* Let $\xi = d_\varphi(\delta)$. Differentiating the definition of $P$ yields

$$D_\varphi(W_\varphi(x))(\delta) + D_x(W_\varphi(x))(\xi) = 0$$

where $D_\varphi$ and $D_x$ are derivatives with respect to variables $\varphi$ and $x$ where the latter derivative is computed in normal coordinates centered at $x$. (It fact, the second term does not depend on the choice of local coordinates since the vector field vanishes at $x$.) Equivalently,

$$D_\varphi(W_\varphi(x))(\delta) + \nabla_\xi W_\varphi = 0. \tag{4.9}$$

Substituting the definition of $W_\varphi$ yields

$$\begin{aligned} D_\varphi(W_\varphi(x))(\delta) &= -n \int_S \delta(s) e^{n(\Phi_s(x) - \varphi(s))} \operatorname{grad} \Phi_s(x) \, d\mu_x(s) \\ &= -n \int_S \delta(s) \rho_\varphi(s) \operatorname{grad} \Phi_s(x) \, d\mu_x(s) \end{aligned}$$

where the second identity follows from the definition of $\rho$, cf. (4.5). Observe that

$$\begin{aligned} W_\varphi(x) &= \int_S e^{n(\Phi_s(x) - \varphi(s))} \operatorname{grad} \Phi_s(x) \lambda(x, s) \, ds \\ &= \int_S e^{-n\varphi(x)} T_s(x) \, ds, \end{aligned}$$

hence the second term of (4.9) takes the form

$$\begin{aligned} \nabla_\xi W_\varphi &= \int_S e^{-n\varphi(s)} \nabla_\xi T_s(x) \, ds = \int_S e^{-n\varphi(s)} \lambda(x,s)^{-1} \nabla_\xi T_s(x) \, d\mu_x \\ &= \int_S e^{n(\Phi_s(x) - \varphi(s))} A_{x,s}(\xi) \, d\mu_x = \int_S \rho_\varphi(s) A_{x,s}(\xi) \, d\mu_x. \end{aligned}$$

Now (4.9) takes the form

$$-n \int_S \delta(s) \rho_\varphi(s) \operatorname{grad} \Phi_s(x) \, d\mu_x(s) + \int_S \rho_\varphi(s) A_{x,s}(\xi) \, d\mu_x = 0,$$

or, equivalently,

$$n \int_S \delta(s) \overline{\rho}_\varphi(s) \operatorname{grad} \Phi_s(x) \, d\mu_x(s) = \int_S \overline{\rho}_\varphi(s) A_{x,s}(\xi) \, d\mu_x = A_\varphi(\xi).$$

Here we divided by the normalizing constant $\int_S \rho \, d\mu_x$ and substituted the definition of $A_\varphi$, cf. (4.7). Hence

$$\xi = A_\varphi^{-1} \left( n \int_S \delta(s) \overline{\rho}_\varphi(s) \operatorname{grad} \Phi_s(x) \, d\mu_x(s) \right)$$

and the assertion follows. $\square$

**Lemma 4.6.** *If $g = g_0$, then $A_{x,s} = id_{T_x M}$ for all $x \in M$, $s \in S$, and therefore $A_\varphi = id_{T_x M}$ for all $\varphi \in \mathcal{L}$ such that $P(\varphi) = x$.*

*Proof.* Substituting $\lambda(x,s) = e^{-(n-1)\Phi_s(x)}$ (cf. Lemma 4.2) into the definition of $A_{x,s}$ yields

$$A_{x,s}(\xi) = e^{-\Phi_s(x)}\nabla_\xi(e^{\Phi_s}\operatorname{grad}\Phi_s) = e^{-\Phi_s(x)}\nabla_\xi \operatorname{grad}(e^{\Phi_s}).$$

A straightforward computation shows that the Hessian of the function $e^{\Phi_s}$ is conformal with respect to $g$, more precisely, $D^2(e^{\Phi_s}) = e^{\Phi(s)} \cdot g$ (since the second derivative of $e^{\Phi_s}$ along the "radial" direction from $s$ is that of $e^t$ at $t = \Phi_s(x)$, and in orthogonal directions it is equal to the curvature of horospheres). Hence $\nabla_\xi \operatorname{grad}(e^{\Phi_s}) = e^{\Phi(s)} \cdot \xi$, and the first assertion follows.

Then the second assertion follows from the fact that $\int_S \overline{\rho}_\varphi \, d\mu_x = 1$. ☐

**Definition 4.7.** We introduce a Riemannian metric $G$ on $\mathcal{B}$ as follows: for every $\varphi \in \mathcal{B}$, he scalar product $G_\varphi$ on $T_\varphi\mathcal{L} = \mathcal{L}$ is defined by

$$G_\varphi(X,Y) = \int_S X(s)Y(s)\overline{\rho}_\varphi(s)\, d\mu_x(s), \qquad X, Y \in \mathcal{L}, \tag{4.10}$$

where $x = P(\varphi)$.

**Lemma 4.8.** *1. $G$ is a special metric (cf. Definition 3.3) with respect to $\Phi$.*

*2. $P$ is a projection (with respect to $\Phi$ and $G$) in the sense of Definition 3.11.*

*Proof.* 1. The first two requirements of Definition 3.3 follow immediately. To verify the third requirement, recall that $P \circ \Phi = id_M$ (cf. Lemma 4.4). Hence, for $\varphi = \Phi(x)$ where $x \in M$, Formula (4.5) takes the form

$$\rho_\varphi(s) = e^{n(\Phi_s(P(\varphi)) - \varphi(s))} = e^{n(\Phi(x)(s) - \varphi(s))} = e^0 = 1.$$

Therefore $\overline{\rho}_\varphi \equiv 1$ and the assertion follows.

2. The fact that $P$ is $L^2$-smooth (cf. Definition 3.10) follows from its ordinary smoothness (cf. Lemma 4.4) and the explicit formula for its derivative (cf. Lemma 4.5). The first requirement of Definition 3.11 follows from Lemma 4.4. To verify the second one, consider $x \in M$, $\varphi = \Phi(x)$ and let $\delta \in \mathcal{L}$ be orthogonal to $T_x\Phi$ with respect to $G_\varphi$. By Lemma 4.5 it suffices to verify that $E_\varphi(\delta) = 0$. By (4.8) we have

$$E_\varphi(\delta) = n\int_S \delta(s)\overline{\rho}_\varphi(s)\operatorname{grad}\Phi_s(x)\, d\mu_x(s) = n\int_S \delta(s)\operatorname{grad}\Phi_s(x)\, d\mu_x(s)$$

since $\overline{\rho}_\varphi \equiv 1$ for $\varphi = \Phi(x)$. A substitution $s = \alpha(v)$, $v \in UT_xM$ (cf. (3.3)) yields

$$E_\varphi(\delta) = n\int_{UT_xM} \delta(\alpha(v))v\, dv$$

since $\operatorname{grad}\Phi_{\alpha(v)} = v$, cf. the definition of $\alpha$ in Section 3.

On the other hand, the assumption that $\delta$ is orthogonal to $T_x\Phi$ means that for every $v_0 \in T_xM$

$$\begin{aligned} 0 = G_\varphi(\delta, d_x\Phi(v_0)) &= \int_S \delta(s)\cdot d\Phi_s(v_0)\overline{\rho}_\varphi(s)\, d\mu_x(s) \\ &= \int_S \delta(s)\langle \operatorname{grad}\Phi_s(x), v_0\rangle\, d\mu_x(s) = \int_S \delta(\alpha(v))\langle v, v_0\rangle\, dv = \langle E_\varphi(\delta), v_0\rangle. \end{aligned}$$

(Here we again used the fact that $\rho_\varphi \equiv 1$ and the substitution $s = \alpha(v)$). Since $\langle E_\varphi(\delta), v_0\rangle = 0$ for all $v_0 \in T_xM$, we have $E_\varphi(\delta) = 0$ and the assertion follows. ☐

## 5. Estimating the Jacobian

Consider the Jacobian $J_Y P(\varphi) = J_{G,Y} P(\varphi)$ of $P$ with respect to $G$ where $\varphi \in \mathcal{B}$ and $Y$ is an $n$-dimensional subspace of $T_\varphi \mathcal{L} = \mathcal{L}$. Lemma 4.5 implies that

$$J_{G,Y} P(\varphi) = |\det A_\varphi|^{-1} J_{G_\varphi}(E_\varphi).$$

Therefore

$$J_G P(\varphi) = |\det A_\varphi|^{-1} J_{G_\varphi}(E_\varphi). \tag{5.1}$$

**Proposition 5.1.** *For every $\varphi \in \mathcal{B}$ the following holds.*
*1. $J_{G_\varphi}(E_\varphi) \le 1$.*
*2. $E_\varphi$ is $n$-Lipschitz with respect to $G_\varphi$.*

*Proof.* 1. We have to prove that $J := J_{G_\varphi,Y}(E_\varphi) \le 1$ for every $n$-dimensional subspace $Y \subset \mathcal{L}$. Choose an orthonormal basis $(\delta_1, \dots, \delta_n)$ in $Y$ and an orthonormal basis $(e_1, \dots, e_n)$ in $T_x M$ such that the matrix $(a_{ij})$ of $E_\varphi|_Y$ with respect to these bases is upper triangular (that is, $a_{ij} = 0$ for $i > j$) and $a_{ii} \ge 0$ for all $i$. Then

$$J = |\det(a_{ij})| = \prod_{i=1}^n a_{ii}.$$

Substituting $s = \alpha(v)$, $v \in T_x M$, yields

$$E_\varphi(\delta) = n \int_S \delta(s) \overline{\rho}_\varphi(s) \operatorname{grad} \Phi_s(x)\, d\mu_x(s) = n \int_{UT_x M} \delta(\alpha(v)) \rho(v) v\, dv \tag{5.2}$$

for every $\delta \in \mathcal{L}$, where

$$\rho(v) = \overline{\rho}_\varphi(\alpha(v)).$$

Note that

$$\int_{UT_x M} \rho(v)\, dv = \int_S \overline{\rho}_\varphi(s)\, d\mu_x(s) = 1. \tag{5.3}$$

Then

$$a_{ij} = \langle E_\varphi(\delta_i), e_j \rangle = n \int_{UT_x H^n} \delta_i(\alpha(v)) \langle v, e_j \rangle \rho(v)\, dv,$$

hence

$$\begin{aligned} J_Y(E_\varphi) = \prod_{i=1}^n a_{ii} &\le \left( \frac{1}{n} \sum_{i=1}^n a_{ii} \right)^n = \left( \sum_{i=1}^n \int_{UT_x M} \delta_i(\alpha(v)) \langle v, e_i \rangle \rho(v)\, dv \right)^n \\ &\le \left( \frac{1}{2} \sum_{i=1}^n \int_{UT_x M} \delta_i(\alpha(v))^2 \rho(v)\, dv + \frac{1}{2} \sum_{i=1}^n \int_{UT_x M} \langle v, e_i \rangle^2 \rho(v)\, dv \right)^n . \end{aligned} \tag{5.4}$$

where the first inequality is the arithmetic-geometric means inequality and the second one follows from Cauchy–Schwarz. Since $G_\varphi(\delta_i, \delta_i) = 1$, we have

$$\int_{UT_x M} \delta_i(\alpha(v))^2 \rho(v)\, dv = \int_S \delta_i(s)^2 \overline{\rho}_\varphi(s)\, ds = \frac{1}{n} G_\varphi(\delta_i, \delta_i) = \frac{1}{n}, \tag{5.5}$$

hence the first term of the sum in the right-hand side of (5.4) equals $\frac{1}{2}$. By (5.3),

$$\sum_{i=1}^n \int_{UT_x M} \langle v, e_i \rangle^2 \rho(v)\, dv = \int_{UT_x M} |v| \rho(v)\, dv = 1, \tag{5.6}$$

hence the first term of the sum in the right-hand side of (5.4) equals $\frac{1}{2}$. Thus the right-hand side of (5.4) equals 1, and the assertion follows.

2. The above argument shows that $\sum \langle E_\varphi(\delta_i), e_i\rangle \le n$ for every orthonormal $n$-frame $\{\delta_i\}$ in $(\mathcal{L}, G_\varphi)$ and every orthonormal basis $\{e_i\}$ in $T_xM$. It follows that $\langle E_\varphi(\delta), e\rangle \le n$ for any unit vectors $\delta \in (\mathcal{L}, G_\varphi)$ and $e \in T_xM$, hence the result. □

**Corollary 5.2.** *If $g = g_0$, then $P$ does not increase $n$-dimensional areas (with respect to $G$). Therefore every compact region in $\mathbb{H}^n$ is a minimal filling.*

*Proof.* Recall that $J_G P(\varphi) = |\det A_\varphi|^{-1} J_{G_\varphi}(E_\varphi)$, and in the case $g = g_0$ we have $\det A_\varphi = 1$ by Lemma 4.6. Hence $J_G P(\varphi) \le 1$ by Proposition 5.1, and the assertions follow. □

**Proposition 5.3.** *There is a positive constant $c_0 = c_0(R, n) > 0$ such that for every $g$ sufficiently close to $g_0$ the following holds: if $\varphi \in \mathcal{B}$, $Y \subset \mathcal{L}$ is an $n$-dimensional subspace, $\delta \in Y$ and $G_\varphi(\delta, \delta) = 1$, then*

$$J_{G_\varphi, Y}(E_\varphi) \le 1 - c_0 \|\delta - d\Phi(E_\varphi(\delta))\|^2_{L^2(S)}.$$

*Proof.* Choose an orthonormal basis $(\delta_1, \dots, \delta_n)$ in $Y$ such that $\delta_1 = \delta$. Then there exists an orthonormal basis $(e_1, \dots, e_n)$ in $T_xM$ such that the matrix $(a_{ij})$ of $E_\varphi|_Y$ with respect to these bases is upper triangular. We use notations and formulas from the proof of Proposition 5.1.

Let $m = \frac{1}{n}\sum_i a_{ii}$. Then

$$\begin{aligned} m &= \sum_{i=1}^n \int_{UT_xH^n} \delta_i(\alpha(v))\langle v, e_j\rangle \rho(v)\, dv \\ &= \frac{1}{2}\sum_{i=1}^n \int_{UT_xH^n} \delta_i(\alpha(v))^2 \rho(v)\, dv + \frac{1}{2}\sum_{i=1}^n \int_{UT_xH^n} \langle v, e_i\rangle^2 \rho(v)\, dv \\ &\qquad - \frac{1}{2}\sum_{i=1}^n \int_{UT_xH^n} (\delta_i(\alpha(v)) - \langle v, e_i\rangle)^2 \rho(v)\, dv \end{aligned}$$

By (5.5) and (5.6), the sum of the first two terms in the right-hand side equals 1 (cf. the proof of Proposition 5.1), hence

$$m = 1 - \frac{1}{2}\sum_{i=1}^n \int_{UT_xH^n} (\delta_i(\alpha(v)) - \langle v, e_i\rangle)^2 \rho(v)\, dv \tag{5.7}$$

Denote $\xi_i = d_x\Phi(e_i)$. Then, for every $v \in T_xM$,

$$\langle v, e_i\rangle = \langle \operatorname{grad} \Phi_{\alpha(v)}(x), e_i\rangle = d_x\Phi_{\alpha(v)}(e_i) = \xi_i(\alpha(v)),$$

hence

$$\int_{UT_xM} (\delta_i(\alpha(v)) - \langle v, e_i\rangle)^2 \rho(v)\, dv = \int_{UT_xM} (\delta_i(\alpha(v)) - \xi_i(\alpha(v)))^2 \rho(v)\, dv \ge 2c_1 \|\delta_i - \xi_i\|^2_{L^2}$$

for some $c_1 = c_1(n, R) > 0$. Here we used the fact that $\rho$ and the derivatives of $\alpha$ are uniformly bounded. Then (5.7) implies that

$$m \le 1 - c_1 \sum_{i=1}^n \|\delta_i - \xi_i\|^2_{L^2} \le 1 - c_1 \|\delta_1 - \xi_1\|^2_{L^2}. \tag{5.8}$$

Denote $J = J_{G_\varphi,Y}(E_\varphi)$, then $J = \prod_{i=1}^n a_{ii}$ as in the proof of Proposition 5.1. By the inequality between arithmetic and geometric means,

$$(5.9)\quad J^{1/n} = \left(\prod_{i<j}\sqrt{a_{ii}a_{jj}}\right)^{2/n(n-1)} \le \frac{2}{n(n-1)}\sum_{i<j}\sqrt{a_{ii}a_{jj}}$$

$$= \frac{2}{n(n-1)}\sum_{i<j}\frac{a_{ii}+a_{jj}-(\sqrt{a_{ii}}-\sqrt{a_{jj}})^2}{2} = \frac{1}{n}\sum_{i=1}^n a_{ii} - \frac{1}{n(n-1)}\sum_{i<j}(\sqrt{a_{ii}}-\sqrt{a_{jj}})^2$$

$$= m - \frac{1}{n(n-1)}\sum_{i<j}(\sqrt{a_{ii}}-\sqrt{a_{jj}})^2 \le m - \frac{1}{n(n-1)}\sum_{i=1}^n(\sqrt{a_{11}}-\sqrt{a_{ii}})^2.$$

Since $m = \frac{1}{n}\sum a_{ii} \le 1$ by (5.7), we have $a_{ii} \le n$ for all $i$. It follows that

$$|\sqrt{a_{11}}-\sqrt{a_{ii}}| \ge \frac{1}{2\sqrt{n}}|a_{11}-a_{ii}|.$$

Hence

$$\sum_{i=1}^n(\sqrt{a_{11}}-\sqrt{a_{ii}})^2 \ge \frac{1}{4n}\sum_{i=1}^n(a_{11}-a_{ii})^2 \ge \frac{1}{4}(a_{11}-m)^2$$

(the last inequality here follows from the fact that $\sum x_i^2 \ge n\big(\frac{1}{n}\sum x_i\big)^2$ where $x_i = a_{11}-a_{ii}$, $i = 1,\dots,n$). This and (5.9) imply

$$(5.10)\qquad J^{1/n} \le m - \frac{1}{4n(n-1)}(a_{11}-m)^2.$$

Since $m \le 1$, we have

$$m \le \frac{m+1-(m-1)^2}{2}$$

(this inequality is equivalent to $m^2 \le m$). Plugging this into (5.10) yields

$$J^{1/n} \le \frac{m+1}{2} - \frac{1}{2}(m-1)^2 - \frac{1}{4n(n-1)}(a_{11}-m)^2$$

$$\le \frac{m+1}{2} - \frac{1}{4n(n-1)}((m-1)^2+(a_{11}-m)^2) \le \frac{m+1}{2} - \frac{1}{8n(n-1)}(a_{11}-1)^2$$

(the last inequality here follows from the obvious one $x^2+y^2 \ge \frac{1}{2}(x+y)^2$ applied to $x = m-1$ and $y = a_{11}-m$). By (5.8) we have

$$\frac{m+1}{2} \le 1 - \frac{c_1}{2}\|\delta_1-\xi_1\|_{L^2}^2,$$

therefore

$$(5.11)\quad J^{1/n} \le 1 - \frac{c_1}{2}\|\delta_1-\xi_1\|_{L^2}^2 - \frac{1}{8n(n-1)}(a_{11}-1)^2 \le 1 - c_2(\|\delta_1-\xi_1\|_{L^2}^2 + (a_{11}-1)^2)$$

where $c_2 = \min\{\frac{c_1}{2}, \frac{1}{8n(n-1)}\}$. Recall that $E_\varphi(\delta) = E_\varphi(\delta_1) = a_{11}e_1$ by the choice of our bases, hence

$$d_x\Phi(E_\varphi(\delta)) = a_{11}\xi_1.$$

Therefore

$$(5.12)\quad \|\delta - d_x\Phi\circ E_\varphi(\delta)\|_{L^2}^2 = \|\delta - a_{11}\xi_1\|_{L^2}^2 \le 2(\|\delta-\xi_1\|_{L^2}^2 + (a_{11}-1)^2\|\xi_1\|_{L^2}^2)$$

(this is the inequality $\|x+y\|^2 \le 2(\|x\|^2+\|y\|^2)$ applied to vectors $x=\delta-\xi_1$ and $y=(1-a_{11})\xi_1$ in $L^2$). Recall that $\xi_1 = d_x\Phi(e_1)$ is a unit vector in the space $(\mathcal{L}, G_{\Phi(x)}) \subset L^2(\mu_x)$, hence $\|\xi_1\|^2_{L^2} \le M$ for some $M=M(n,R)>0$. (Here we use the fact that the densities of the measures $\mu_x$ are uniformly bounded.) We may assume that $M \ge 2$, then (5.12) implies that

$$\|\delta - d_x\Phi\circ E_\varphi(\delta)\|^2_{L^2} \le M(\|\delta_1-\xi_1\|^2_{L^2} + (a_{11}-1)^2)$$

Plugging this into (5.11) yields

$$J^{1/n} \le 1 - c_2 M^{-1}\|\delta - d_x\Phi\circ E_\varphi(\delta)\|^2_{L^2}$$

hence the result. □

## 6. Estimating the correction factor

It this section we estimate the correction factor $\det A_\varphi$ in (5.1).

**Proposition 6.1.** *There are positive constants $r$ and $C=C(n,R)$ such that for every $g$ sufficiently close to $g_0$ and every $\varphi\in\mathcal{B}$ one has*

$$\|A_\varphi - I\| \le C\cdot\|g-g_0\|_{C^r}\cdot\|\varphi-\Phi(P(\varphi))\|_{L^2(S)}$$

*where $I = id_{T_{P(\varphi)}M}$, and*

$$|\det A_\varphi - 1| \le C\cdot\|g-g_0\|^2_{C^r}\cdot\|\varphi-\Phi(P(\varphi))\|^2_{L^2(S)}.$$

*Proof.* Fix $\varphi\in\mathcal{B}$ and denote $p=P(\varphi)$, $I=id_{T_pM}$. Since $\varphi\in\mathcal{B}$, $|\varphi(s)|\le R$ for a.e. $s\in S$, and we may assume that $|\varphi(s)|\le R$ for all $s\in S$. Throughout the argument, we denote by $C(n,R)$ various positive constants depending only on $n$ and $R$. By Lemma 4.4, $p=P(\varphi)$ belongs to a ball of radius $R_1=C(n,R)$ centered at $o$. Hence $|\Phi_s(p)|\le C(n,R)$ and $C(n,R)^{-1}\le \frac{d\mu_p(s)}{ds}\le C(n,R)$ for all $s\in S$. We will use these estimates without explicit reference.

For every $t\in[0,1]$, define $\psi_t,\varphi_t\in L^\infty(S)$ by

$$\psi_t(s) = 1-t+t\cdot e^{n(\Phi_s(p)-\varphi(s))}$$

and

$$\varphi_t(s) = \Phi_s(p) - \frac{1}{n}\log\psi_t(s).$$

Then $\psi_t$ is linear in $t$ and

$$\psi_t(s) = e^{n(\Phi_s(p)-\varphi_t(s))}$$

for all $t\in[0,1]$. Note that $\varphi_t$ is a smooth function of $t$, $\varphi_0=\Phi(p)$ and $\varphi_1=\varphi$. Therefore $P(\varphi_t)$ is defined for all $t$ from a neighborhood of 0 in $[0,1]$, as well as for $t=1$. We are going to study $\det A_{\varphi_t}$ as a function of $t$.

**Lemma 6.2.** *$P(\varphi_t)=p$ for all $t$ such that $P(\varphi_t)$ is defined.*

*Proof.* By Definition 4.3 of $P$, the identity $P(\varphi_t)=p$ is equivalent to

$$\int_S \psi_t(s)\,\mathrm{grad}\,\Phi_s(p)\,d\mu_p(s) = 0.$$

The left-hand side is linear in $t$ and the equality holds for $t=0$ and $t=1$ (since $P(\varphi_0)=P(\Phi(p))=p$ and $P(\varphi_1)=P(\varphi)=p$). Therefore it holds for all $t$. □

**Lemma 6.3.** *$A_{\varphi_0}=I$.*

*Proof.* Since $P \circ \Phi = id_M$, we have $d_{\varphi_0} P \circ d_p \Phi = I$. By Lemma 4.5, we have $d_{\varphi_0} P = A_{\varphi_0}^{-1} \circ E_{\varphi_0}$, hence it suffices to verify that

$$E_{\varphi_0} \circ d_p \Phi = I. \tag{6.1}$$

Since $\varphi_0 \in \Phi(M)$, we have $\overline{\rho}_{\varphi_0} \equiv 1$, and the definition of $E_{\varphi_0}$, cf. (4.8), takes the form

$$E_\varphi(\delta) = n \int_S \delta(s) \operatorname{grad} \Phi_s(p)\, d\mu_p(s)$$

for all $\delta \in \mathcal{L}$. Substitute $\delta = d\Phi(v_0)$ where $v_0 \in T_p M$, that is,

$$\delta(s) = d\Phi_s(v_0) = \langle \operatorname{grad} \Phi_s(p), v_0 \rangle.$$

This yields

$$E_\varphi(d\Phi_s(v_0)) = n \int_S \langle \operatorname{grad} \Phi_s(p), v_0 \rangle \operatorname{grad} \Phi_s(p)\, d\mu_p(s) = n \int_{UT_p M} \langle v, v_0 \rangle v\, dv$$

where the second identity follows by substituting $s = \alpha(v)$, cf. (3.3). By the symmetry under rotations, the latter integral is a multiple of $v_0$. To find out the coefficient, observe that the scalar product of this integral with $v_0$ equals

$$n \int_{UT_p M} \langle v, v_0 \rangle^2\, dv = |v_0|^2.$$

Thus $E_\varphi(d\Phi_s(v_0)) = v_0$ and (6.1) follows. □

The definitions imply that $A_{\varphi_t}$ is a smooth function of $t$ (in fact, it is a rational function, see below).

**Lemma 6.4.** $\frac{d}{dt}\big|_{t=0} \det A_{\varphi_t} = 0$.

*Proof.* By (5.1) we have

$$\det A_{\varphi_t} = \frac{J_{G_{\varphi_t}}(E_{\varphi_t})}{J_G P(\varphi_t)}$$

At $t = 0$, all terms of this formula are equal to 1. Indeed, $\det A_{\varphi_0} = 1$ by Lemma 6.3, and the Jacobian of the projection $P$ equals 1 at the surface $\Phi(M)$. Since $P$ is a projection in the sense of Definition 3.11, Proposition 3.12 implies that the derivative of the denominator at $t = 0$ equals 0. By Proposition 5.1, the numerator attains its maximum at $t = 0$, hence its derivative at $t = 0$ also equals 0. Therefore the derivative of the fraction is zero. □

The definitions of $A_\varphi$ (cf. (4.5)–(4.7)) and $\psi_t$ yield that

$$A_{\varphi_t} = \frac{A(t)}{b(t)} \tag{6.2}$$

where $A(t)$ is an operator on $T_p M$ given by

$$A(t) = \int_S \psi_t(s) A_{p,s}\, d\mu_p(s)$$

and

$$b(t) = \int_S \psi_t(s)\, d\mu_p(s).$$

Since $\psi_t$ is linear in $t$, so are $A(t)$ and $b(t)$. The definition of $\psi_t$ implies that

$$C(n, R)^{-1} \le \psi_t(s) \le C(n, R)$$

for all $s \in S$, hence

$$C(n,R)^{-1} \le b(t) \le C(n,R) \tag{6.3}$$

for all $t \in [0,1]$. We rewrite $\psi_t$ as

$$\psi_t(s) = 1 + t\delta(s) \tag{6.4}$$

where

$$\delta(s) = e^{n(\Phi_s(p)-\varphi(s))} - 1.$$

Since $|\Phi_s(p) - \varphi(s)| \le C(n,R)$, we have

$$|\delta(s)| \le C(n,R) \cdot |\Phi_s(p) - \varphi(s)| = C(n,R) \cdot |\Phi(p)(s) - \varphi(s)|$$

for all $s \in S$. Hence

$$\|\delta\|_{L^2} \le C(n,R) \cdot \|\Phi(p) - \varphi\|_{L^2}. \tag{6.5}$$

Using (6.4), we rewrite $b(t)$ as

$$b(t) = \int_S (1 + t\delta(s))\, d\mu_p(s) = 1 + t\int_S \delta(s)\, d\mu_p(s). \tag{6.6}$$

In particular, $b(0) = 1$ and hence $A(0) = I$ by (6.2) Lemma 6.3. Similarly, we rewrite $A(t)$ as

$$A(t) = \int_S (1 + t\delta(s))A_{p,s}\, d\mu_p(s) = \int_S A_{p,s}\, d\mu_p(s) + t\int_S \delta(s) A_{p,s}\, d\mu_p(s).$$

Substituting $t = 0$ yields that the first term equals $A(0) = I$, thus

$$A(t) = I + t\int_S \delta(s) A_{p,s}\, d\mu_p(s) = b(t)I + t\Delta \tag{6.7}$$

where

$$\Delta = \int_S \delta(s)(A_{p,s} - I)\, d\mu_p(s).$$

(The second identity in (6.7) uses (6.6).) By Cauchy–Schwarz we have

$$\|\Delta\| \le C(n,R) \cdot \|\delta\|_{L^2} \cdot \|A_{p,s} - I\|_{L^2} \le C(n,R) \cdot \|\delta\|_{L^2} \cdot \|g - g_0\|_{C^r}$$

for some $r$. The second inequality follows from the fact that $A_{p,s}$ depends smoothly on $g$ and equals $I$ if $g = g_0$ (cf. Lemma 4.6). Substituting (6.5) yields

$$\|\Delta\| \le C(n,R) \cdot \|g - g_0\|_{C^r} \cdot \|\Phi(p) - \varphi\|_{L^2} \tag{6.8}$$

By (6.2) and (6.7) we have

$$A_{\varphi_t} = I + \frac{t}{b(t)}\Delta.$$

In particular, $A_\varphi = A_{\varphi_1} = I + \frac{1}{b(1)}\Delta$. Hence

$$\|A_\varphi - I\| = b(1)^{-1}\|\Delta\| \le C(n,R) \cdot \|g - g_0\|_{C^r} \cdot \|\Phi(p) - \varphi\|_{L^2}$$

by (6.3) and (6.8), and the first assertion of Proposition 6.1 follows. Furthermore, since $\|\Phi(p) - \varphi\|_{L^2} \le C(n,R)$, we may assume that $\|g - g_0\|_{C^r}$ is so small that

$$\|A_\varphi - I\| \le 1. \tag{6.9}$$

By Lemma 6.4,

$$0 = \frac{d}{dt}\Big|_{t=0} \det A_{\varphi_t} = \frac{d}{dt}\Big|_{t=0} \det\left(I + \frac{t}{b(t)}\Delta\right) = \operatorname{trace}\Delta$$

since $b(0) = 1$. Hence

$$\operatorname{trace}(A_\varphi - I) = b(1)^{-1} \operatorname{trace} \Delta = 0. \tag{6.10}$$

We need the following finite-dimensional lemma.

**Lemma 6.5.** *There is a constant $C = C(n) > 0$ such the following holds. For every $n \times n$ matrix $A$ such that $\operatorname{trace} A = 0$ and $\|A\| \le 1$, one has*

$$|\det(I + A) - 1| \le C\|A\|^2$$

*where $I$ is the identity matrix.*

*Proof.* Indeed, the differential of $\det(I+A)$ on the subspace $\{\operatorname{trace} A = 0\}$ is zero at the point $A = 0$, hence $\det(I+A)-1 \le C_1\|A\|^2$ for some constant $C_1$ and provided that $\|A\|^2 \le r$ for some positive $r > 0$ (this follows from Taylor the expansion for $\det(I + A)$). This proves the inequality in the $r$-neighborhood of $A = 0$. Since $\det(I + A)$ is bounded on the ball $\{\|A\| \le 1\}$ (since it is continuous and the ball is compact), the inequality is trivial for $A$ with $r \le \|A\| \le 1$, where one can use $\max\{\det(I + A) : \|A\| \le 1\}/r^2$ for the constant $C$. $\square$

Now the second assertion of Proposition 6.1 follows from (6.9), (6.10) and Lemma 6.5 applied to $A = A_\varphi - I$. $\square$

**Corollary 6.6.** *If $g$ is sufficiently close to $g_0$, then for every $\varphi \in \mathcal{B}$ the map $d_\varphi P$ is $2n$-Lipschitz with respect to the metric $G_\varphi$ on $T_\varphi\mathcal{L} = L$.*

*Proof.* Recall that $d_\varphi P = A_\varphi^{-1} \circ E_\varphi$, cf. Lemma 4.5. The first assertion of Proposition 6.1 implies that $\|A_\varphi^{-1}\| \le 2$ provided that $\|g - g_0\|_{C^r}$ is sufficiently small. Since $E_\varphi$ is $n$-Lipschitz by Proposition 5.1, it follows that $d_\varphi P$ is $2n$-Lipschitz. $\square$

## 7. A compression trick

Define a function $h : \mathcal{B} \to \mathbb{R}_+$ by

$$h(\varphi) = \|\varphi - \Phi(P(\varphi))\|_{L^2}$$

and let $\varepsilon(g) = \|g - g_0\|_{C^r}$ where $r$ is from Proposition 6.1. Assuming that $\varepsilon(g)$ is sufficiently small, we rewrite the assertions of Proposition 6.1 as follows:

$$\|A_\varphi^{-1} - I\| \le C(n, R)\varepsilon(g)h(\varphi)$$

and

$$|\det A_\varphi^{-1}| \le 1 + C(n, R)\varepsilon(g)h^2(\varphi).$$

Since $d_\varphi P = A_\varphi^{-1} \circ E_\varphi$ (Lemma 4.5) and $J_{G_\varphi}(E_\varphi) \le 1$ (Proposition 5.1), in follows that

$$J_\varphi P \le 1 + C(n, R)\varepsilon(g)h^2(\varphi) \le 2$$

provided that $\varepsilon(g)$ is sufficiently small.

Observe that the function $h^2$ is smooth on $\mathcal{B}$ and its derivative is given by

$$d_\varphi h^2(\delta) = 2\langle\varphi - \Phi(P(\varphi)), \delta - d\Phi(dP(\delta))\rangle_{L^2}$$

for all $\varphi \in \mathcal{B}$, $\delta \in T_\varphi\mathcal{L} = \mathcal{L}$. Hence

$$|d_\varphi h^2(\delta)| \le 2h(\varphi)\|\delta - d\Phi(dP(\delta))\|_{L^2}. \tag{7.1}$$

For a constant $c > 0$ define $F_c : L^\infty(S) \to H^n \times \mathbb{R}_+$ by $F_c(\varphi) = (P(\varphi), ch(\varphi))$. Note that $F_c$ is smooth on $\mathcal{B} \setminus \Phi(M)$.

**Lemma 7.1.** *For every $R > 0$, there exists a $c_1 = c_1(n, R) > 0$ such that for every positive $c < c_1$ and every $\varphi \in \mathcal{B} \setminus \Phi(M)$, the $n$-dimensional Jacobian of $F_c$ at $\varphi$ with respect to $G$ is bounded above by*

$$1 + C(n, R)\varepsilon(g)h^2(\varphi).$$

*Proof.* Let $Y$ be an $n$-dimensional subspace of $T_\varphi\mathcal{L}$ equipped with our scalar product $G_\varphi$. Denote the $n$-dimensional Jacobian of $dF_c|_Y$ by $J$. Choose an orthonormal basis $(\delta_1, \dots, \delta_n)$ in $Y$ such that $d_\varphi h(\delta_i) = 0$ for $i \geq 2$. Then choose an orthonormal basis $e_1, \dots, e_n$ in $T_{P(\varphi)}M$ as in the proof of Proposition 5.3, namely so that the matrix $(a_{ij})$ of $d_\varphi P|_Y$ w.r.t. these bases is upper triangular and its diagonal elements $a_{ii}$ are nonnegative. Then

$$J = \sqrt{a_{11}^2 + t^2} \cdot \prod_{i=2}^{n} a_{ii}$$

where

$$t = d_\varphi(ch)(\delta_1) = \frac{c}{2h(\varphi)} d_\varphi h^2(\delta_1).$$

By 7.1 we have

$$|t| \leq c \cdot \|\delta_1 - d\Phi \circ d_\varphi P(\delta_1)\|_{L^2} \tag{7.2}$$

By Corollary 6.6 we have

$$a_{ii} \leq |d_\varphi P(\delta_i)| \leq 2n \tag{7.3}$$

for all $i$, hence

$$\|\delta_1 - d\Phi \circ d_\varphi P(\delta_1)\|_{L^2} = \|\delta_1 - a_{11} d\Phi(e_1)\|_{L^2} \leq C(n, R).$$

Therefore we may assume that $c_1$ is so small that (7.2) implies that $|t| < (2n)^{-n}$. Consider two cases.

*Case 1:* $a_{11} < (2n)^{-n}$. Then $\sqrt{a_{11}^2 + t^2} \leq \sqrt{2}(2n)^{-n}$, hence

$$J = \sqrt{a_{11}^2 + t^2} \cdot \prod_{i=2}^{n} a_{ii} \leq \sqrt{2}(2n)^{-n} \prod_{i=2}^{n} a_{ii} \leq \sqrt{2}(2n)^{-1} < 1.$$

Here the second inequality follows from (7.3).

*Case 2:* $a_{11} \geq (2n)^{-n}$. Then

$$J = \sqrt{a_{11}^2 + t^2} \cdot \prod_{i=2}^{n} a_{ii} \leq \left(a_{11} + \frac{t^2}{2a_{11}}\right) \prod_{i=2}^{n} a_{ii} = J_Y P \left(1 + \frac{t^2}{2a_{11}^2}\right) \leq J_Y P + (2n)^{2n} t^2.$$

Here we used that $J_Y P = \prod_i a_{ii}$ and $J_Y P \leq 2$. Since $d_\varphi P = A_\varphi^{-1} \circ E_\varphi$ (Lemma 4.5), we have

$$J_Y P = \det A_\varphi^{-1} J_Y(E_\varphi) \leq J_Y(E_\varphi) + C(n, R)\varepsilon(g)h^2(\varphi).$$

For the first term we use the estimate

$$J_Y(E_\varphi) \leq 1 - c_0\|\delta_1 - d\Phi \circ E_\varphi(\delta_1)\|_{L^2}^2$$

from Proposition 5.3, thus

$$J \leq 1 - c_0\|\delta_1 - d\Phi \circ E_\varphi(\delta_1)\|_{L^2}^2 + (2n)^{2n} t^2 + C(n, R)\varepsilon(g)h^2(\varphi) \tag{7.4}$$

By (7.2) and the triangle inequality in $L^2$,

$$|t| \leq c(\|\delta_1 - d\Phi \circ E_\varphi(\delta_1)\|_{L^2} + \|d\Phi \circ (d_\varphi P - E_\varphi)(\delta_1)\|_{L^2})$$

We estimate the second term using Proposition 6.1:

$$|(d_\varphi P - E_\varphi)(\delta_1)| = |(A^{-1} - I) \circ E_\varphi(\delta_1)| \le C(n,R)\varepsilon(g)h(\varphi)$$

since $|E_\varphi(\delta_1)| \le n$, cf. Proposition 5.1. Therefore

$$|t| \le c\|\delta_1 - d\Phi \circ E_\varphi(\delta_1)\|_{L^2} + C(n,R)\varepsilon(g)h(\varphi),$$

hence

$$t^2 \le 2c^2\|\delta_1 - d\Phi \circ E_\varphi(\delta_1)\|_{L^2}^2 + 2(C(n,R)\varepsilon(g)h(\varphi))^2$$

Substituting this into (7.4) yields

$$J \le 1 - (c_0 - 2(2n)^{2n}c^2)\|\delta_1 - d\Phi \circ E_\varphi(\delta_1)\|_{L^2}^2 + C(n,R)\varepsilon(g)h^2(\varphi)$$

We may assume that $c_1$ is chosen so small that $c_0 - 2(2n)^{2n}c_1^2 \ge 0$, then

$$J \le 1 + C(n,R)\varepsilon(g)h^2(\varphi)$$

and the lemma follows. □

For $t \in [0,1]$ define a "homothety" $A_t : M \to M$ by

$$A_t(x) = \exp_o(t \cdot \exp_o^{-1}(x).$$

Clearly $A_t$ is a smooth map and it is $t$-Lipschitz due to nonpositive curvature of $M$. For a small $\sigma > 0$, define a map $Q_\sigma : M \times \mathbb{R}_+$ by

$$Q_\sigma(x,h) = A_{(1+\sigma h^2)^{-1}}(x).$$

**Lemma 7.2.** *If $x \in M$ is such that $\operatorname{dist}_M(o,x) < (4\sigma)^{-1/2}$, then the $n$-dimensional Jacobian of $Q_\sigma$ at $(x,h)$ is no greater than $(1+\sigma h^2)^{-1}$.*

*Proof.* Due to the Rauch Comparison Theorem, the $n$-dimensional Jacobian of $Q_\sigma$ does not exceed that of the similar map for $\mathbb{R}^n$, namely

$$(x,h) \mapsto (1+\sigma h^2)^{-1}x, \qquad x \in \mathbb{R}^n, h \in \mathbb{R}_+.$$

The latter equals

$$(1+\sigma h^2)^{-(n+1)}\sqrt{(1+\sigma h^2)^2 + (2\sigma h|x|)^2}.$$

If $|x|$ does not exceed $(4\sigma)^{-1/2}$, one easily sees that the expression under the square root is no greater than $(1+\sigma h^2)^3$, hence the result. □

Now define $P_\sigma : \mathcal{B} \to M$ by

$$P_\sigma(\varphi) = Q_\sigma(P(\varphi), \sigma h(\varphi)) = A_{(1+\sigma^3 h^2(\varphi))^{-1}}(P(\varphi))$$

where $P$ and $h$ are the same as above. Note that the second formula implies that $P_\sigma$ is smooth.

**Proposition 7.3.** *For every $R > 0$ there exist $\sigma > 0$, $c > 0$ and $\varepsilon > 0$ such that the $n$-dimensional Jacobian $J(\varphi) := J_G P_\sigma(\varphi)$ of $P_\sigma$ with respect to $G$ at any point $\varphi \in \mathcal{B}$ satisfies*

$$J(\varphi) \le 1 - c \cdot h^2(\varphi).$$

*provided that $\varepsilon(g) = \|g - g_0\|_{C^r} < \varepsilon$.*

*Proof.* Choose $\sigma$ so that $\sigma < c_1(n,R)$ from Lemma 7.1 and $(4\sigma)^{-1/2} > \operatorname{diam}(P(\mathcal{B}))$. For a point $\varphi \in \Phi(M)$, we have $d_\varphi Q_\sigma = d_\varphi P$ since $d_\varphi h^2 = 0$, and therefore $J(\varphi) = J_G P(\varphi) = 1$.

Now consider a point $\varphi \notin \Phi(M)$. The map $P_\sigma$ is a composition of the map $F_\sigma : \mathcal{B} \to M \times \mathbb{R}_+$ whose Jacobian is estimated in Lemma 7.1 and the map $Q_\sigma$ whose Jacobian is estimated in Lemma 7.2. These estimates yield

$$J \leq \frac{1 + C(n,R)\varepsilon(g)h^2(\varphi)}{1 + \sigma(\sigma h(\varphi))^2} \leq 1 - 2c \cdot h^2(\varphi) + C(n,R)\varepsilon(g)h^2(\varphi)$$

for a suitable $c = c(\sigma, n, R)$. Choosing $\varepsilon < \frac{c}{C(n,R)}$ yields the desired inequality. $\square$

Now we are in position to complete the proof of Proposition 2.1. Let $\varepsilon$ and $\sigma$ be as in Proposition 7.3. Assuming that $\|g - g_0\|_{C^r} < \varepsilon$, consider the map $P_\sigma : \mathcal{B} \to M$ constructed above. By Lemma 3.14, for any Riemannian $n$-manifold $N$ and any 1-Lipschitz map $f : N \to \mathcal{B}$ we have

$$\operatorname{vol}(P_\sigma \circ f) \leq \int_N J_G P_\sigma(f(x))\, d\operatorname{vol}_N(x).$$

Then the inequality for $J$ from Proposition 7.3 implies that $\operatorname{vol}(P_\sigma \circ f) \leq \operatorname{vol}(N)$. Moreover in the case of equality we have $h(\varphi) = 0$ for all $\varphi \in f(N)$, hence $f(N) \subset \Phi(M)$. Thus the map $P_\sigma$ possesses the properties claimed in Proposition 2.1.

## References

[1] D. Burago and S. Ivanov, *Boundary rigidity and filling volume minimality of metrics close to a flat one.* Ann. of Math. (2) **171** (2010), no. 2, 1183–1211.

[2] D. Burago and S. Ivanov, *On asymptotic volume of tori.* Geom. Funct. Anal., (5) **5** (1995), 800 – 808.

[3] D. Burago and S. Ivanov, *On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume.* Ann. of Math. (2) **156** (2002), 891–914.

[4] D. Burago and S. Ivanov, *Gaussian images of surfaces and ellipticity of surface area functionals.* Geom. Funct. Anal., **14** (2004), 469-490.

[5] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geom. Funct. Anal., **5** (1995), 731–799.

[6] C. Croke, *Rigidity and the distance between boundary points*, J. Diff. Geom. **33** (1991), 445–464.

[7] C. Croke and B. Kleiner, *A rigidity theorem for simply connected manifolds without conjugate points*, Ergodic Theory Dynam. Systems **18** (1998), no. 4, 807–812.

[8] C. Croke, *Rigidity theorems in Riemannian geometry*, in "Geometric Methods in Inverse Problems and PDE Control", C. Croke, I. Lasiecka, G. Uhlmann, and M. Vogelius eds., Springer, 2004.

[9] M. Gromov, *Filling Riemannian manifolds*, J. Diff. Geom. **18** (1983), 1–147.

[10] R. D. Holmes and A. C. Thompson, *N-dimensional area and content in Minkowski spaces*, Pacific J. Math. **85** (1979), 77-110.

[11] S. Ivanov. *On two-dimensional minimal fillings*, St.-Petersburg Math. J, **13** (2002), 17–25.

[12] S. Ivanov, *Volumes and areas of Lipschitz metrics*, St. Petersburg Math. J. **20** (2009), no. 3, 381–405.

[13] S. Ivanov. *Volume comparison via boundary distances*, Proc. ICM 2010, vol.2, 769–784.

[14] R. Michel, *Sur la rigidité imposeée par la longuer des géodésiques*, Invnt. Math. **65** (1981), 71–83.

[15] L. Pestov and G. Uhlmann, *Two-dimensional compact simple Riemannian manifolds are boundary distance rigid*, Ann. of Math. (2) **161** (2005), 1093–1110.

[16] L. A. Santaló, *Integral geometry and geometric probability*, Encyclopedia Math. Appl., Addison-Wesley, Reading, MA, 1976.

Dmitri Burago: Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA
*E-mail address*: burago@math.psu.edu

Sergei Ivanov: St.Petersburg Department of Steklov Mathematical Institute, Fontanka 27, St.Petersburg 191023, Russia
*E-mail address*: svivanov@pdmi.ras.ru