NORMALITY OF ADJOINTABLE MODULE MAPS

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ABSTRACT. Normality of bounded and unbounded adjointable operators are discussed. Suppose T is an adjointable operator between Hilbert C*-modules which has polar decomposition, then T is normal if and only if there exists a unitary operator $\mathcal{U}$ which commutes with T and T^* such that $T = \mathcal{U} T^*$. Kaplansky's theorem for normality of the product of bounded operators is also reformulated in the framework of Hilbert C*-modules.

1. Introduction and preliminary.

Normal operators may be regarded as a generalization of a selfadjoint operator T in which T^* need not be exactly T but commutes with T. They form an attractive and important class of operators which play a vital role in operator theory, especially, in spectral theory. In this note we will study bounded and unbounded normal module maps between Hilbert C*-modules which have polar decomposition. Indeed, for adjointable operator T between Hilbert C*-modules which has polar decomposition, we demonstrate that T is normal if and only if there exists a unitary operator $\mathcal{U}$ such that $T = \mathcal{U} T^*$. In this situation, $\mathcal{U} T \subseteq T \mathcal{U}$ and $\mathcal{U} T^* \subseteq T^* \mathcal{U}$ (compare [8, page 109] and [2, page 155]).

Suppose T, S are bounded adjointable operators between Hilbert C*-modules. Suppose T has polar decomposition and T and TS are normal operators. Then we show that ST is a normal operator if and only if S commutes with $|T|$. This fact has been proved by Kaplansky [9] in the case of Hilbert spaces.

Throughout the present paper we assume $\mathcal{A}$ to be an arbitrary C*-algebra. We deal with bounded and unbounded operators at the same time we simply denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $Dom(.)$, $Ker(.)$ and $Ran(.)$ for domain, kernel and range of operators, respectively.

Hilbert C*-modules are essentially objects like Hilbert spaces, except that inner product, instead of being complex-valued, takes its values in a C*-algebra. Although Hilbert C*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements

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do not hold. A (right) *pre-Hilbert C*-module* over a C*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module X equipped with an $\mathcal{A}$-valued inner product $\langle \cdot,\cdot \rangle : X \times X \to \mathcal{A}\,,\ (x,y) \mapsto \langle x,y \rangle$, which is $\mathcal{A}$-linear in the second variable y and has the properties:

$$\langle x,y \rangle = \langle y,x \rangle^*, \quad \langle x,x \rangle \geq 0 \ \text{ with equality only when } x=0.$$

A pre-Hilbert $\mathcal{A}$-module X is called a *Hilbert $\mathcal{A}$-module* if X is a Banach space with respect to the norm $\|x\| = \|\langle x,x \rangle\|^{1/2}$. A Hilbert $\mathcal{A}$-submodule W of a Hilbert $\mathcal{A}$-module X is an orthogonal summand if $W \oplus W^{\perp} = X$, where $W^{\perp}$ denotes the orthogonal complement of W in X. We denote by $\mathcal{L}(X)$ the C*-algebra of all adjointable operators on X, i.e., all $\mathcal{A}$-linear maps $T: X \to X$ such that there exists $T^*: X \to X$ with the property $\langle Tx,y \rangle = \langle x, T^*y \rangle$ for all $x,y \in X$. A bounded adjointable operator $\mathcal{V} \in \mathcal{L}(X)$ is called a *partial isometry* if $\mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{V}$. For the basic theory of Hilbert C*-modules we refer to the books [11, 15] and the papers [4, 13]

An unbounded regular operator on a Hilbert C*-module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed $\mathcal{A}$-linear map $t: Dom(t) \subseteq X \to X$ is called *regular* if it is adjointable and the operator $1+t^*t$ has a dense range. We denote the set of all regular operators on X by $\mathcal{R}(X)$. Recall that a densely defined operator $t: Dom(t) \subseteq X \to X$ is regular if and only if its graph is orthogonally complemented in $X \oplus X$ (see e.g. [5, Corollary 3.2]). If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint. Define $Q_t = (1+t^*t)^{-1/2}$ and $F_t = tQ_t$, then $Ran(Q_t) = Dom(t)$, $0 \leq Q_t = (1-F_t^*F_t)^{1/2} \leq 1$ in $\mathcal{L}(X)$ and $F_t \in \mathcal{L}(X)$ [11, (10.4)]. The bounded operator F_t is called the bounded transform of regular operator t. According to [11, Theorem 10.4], the map $t \to F_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(X) \to \{F \in \mathcal{L}(X) : \|F\| \leq 1 \ \text{ and } \ Ran(1-F^*F) \text{ is dense in } X\}.$$

Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t, some properties transfer to its bounded transform F_t, and vice versa. Suppose $t \in \mathcal{R}(X)$ is a regular operator, then t is called *normal* iff $Dom(t) = Dom(t^*)$ and $\langle tx,tx \rangle = \langle t^*x, t^*x \rangle$ for all $x \in Dom(t)$. t is called *selfadjoint* iff $t^* = t$ and t is called *positive* iff t is normal and $\langle tx,x \rangle \geq 0$ for all $x \in Dom(t)$. Remarkably, a regular operator t is normal (resp., selfadjoint, positive) iff its bounded transform F_t is normal (resp., selfadjoint, positive). Moreover, both t and F_t have the same range and the same kernel. If $t \in \mathcal{R}(X)$ then $Ker(t) = Ker(|t|)$ and $\overline{Ran(t^*)} = \overline{Ran(|t|)}$, cf. [5, 10]. If $t \in \mathcal{R}(X)$ is a normal operator then $Ker(t) = Ker(t^*)$ and $\overline{Ran(t)} = \overline{Ran(t^*)}$.

A bounded adjointable operator T has polar decomposition if and only if $\overline{Ran(T)}$ and $\overline{Ran(|T|)}$ are orthogonal direct summands [15, Theorem 15.3.7]. The result has been generalized in Theorem 3.1 of [6] for regular operators. Indeed, for $t \in \mathcal{R}(X)$ the following conditions are equivalent:

- t has a unique polar decomposition $t = \mathcal{V}|t|$, where $\mathcal{V} \in \mathcal{L}(X)$ is a partial isometry for which $Ker(\mathcal{V}) = Ker(t)$.
- $X = Ker(|t|) \oplus \overline{Ran(|t|)}$ and $X = Ker(t^*) \oplus \overline{Ran(t)}$.
- The adjoint operator t^* has polar decomposition $t^* = \mathcal{V}^*|t^*|$.
- The bounded transform F_t has polar decomposition $F_t = \mathcal{V}|F_t|$.

In this situation, $\mathcal{V}^*\mathcal{V}|t| = |t|$, $\mathcal{V}^*t = |t|$ and $\mathcal{V}\mathcal{V}^*t = t$, moreover, we have $Ker(\mathcal{V}^*) = Ker(t^*)$, $Ran(\mathcal{V}) = \overline{Ran(t)}$ and $Ran(\mathcal{V}^*) = \overline{Ran(t^*)}$.

The above facts and Proposition 2.2 of [5] show that each regular operator with closed range has polar decomposition.

Recall that an arbitrary C*-algebra of compact operators $\mathcal{A}$ is a c_0-direct sum of elementary C*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i, $i \in I$, cf. [1, Theorem 1.4.5]. If $\mathcal{A}$ is an arbitrary C*-algebra of compact operators then for every Hilbert $\mathcal{A}$-modules X, every densely defined closed operator $t : Dom(t) \subseteq X \to X$ is automatically regular and has polar decomposition, cf. [5, 6].

The stated results also hold for bounded adjointable operators, since $\mathcal{L}(X)$ is a subset of $\mathcal{R}(X)$. The space $\mathcal{R}(X)$ from a topological point of view are studied in [12, 14].

2. Normality

Proposition 2.1. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$ and $S \in \mathcal{L}(X)$ is an arbitrary operator which commutes with T and T^*. Then $\mathcal{V}$ and $|T|$ commute with S and S^*.*

Proof. It follows from the hypothesis that $(T^*T)S = S(T^*T)$ which implies $|T|S = S|T|$, or equivalently $|T|S^* = S^*|T|$. Using the commutativity of S with T and $|T|$, we get

$$(S\mathcal{V} - \mathcal{V}S)|T| = S\mathcal{V}|T| - \mathcal{V}|T|S = ST - TS = 0.$$

That is, $S\mathcal{V} - \mathcal{V}S$ acts as zero operator on $\overline{Ran(|T|)}$. If $x \in Ker(|T|) = Ker(\mathcal{V})$ then $|T|x = \mathcal{V}x = 0$, consequently $|T|Sx = S|T|x = 0$. Then $Sx \in Ker(|T|) = Ker(\mathcal{V})$, therefore, $S\mathcal{V} - \mathcal{V}S$ acts as zero operator on $Ker(|T|)$ too. We obtain

$$S\mathcal{V} - \mathcal{V}S = 0 \ \text{ on } \ X = Ker(|T|) \oplus \overline{Ran(|T|)}.$$

The statement $S^*\mathcal{V} - \mathcal{V}S^* = 0$ on $X = Ker(|T|) \oplus \overline{Ran(|T|)}$ can be deduced from the commutativity of S with T^* and $|T|$ in the same way. □

Corollary 2.2. *Suppose $T \in \mathcal{L}(X)$ is a normal operator which admits the polar decomposition $T = \mathcal{V}|T|$ then $\mathcal{V}$ and $|T|$ commute with the operators $T, T^*, \mathcal{V}$ and $\mathcal{V}^*$. In particular, $\mathcal{V}$ is a unitary operator on $\overline{Ran(T)} = \overline{Ran(T^*)}$.*

The results follow from Proposition 2.1, Proposition 3.7 of [11] and the fact that $\mathcal{V}\mathcal{V}^*T = \mathcal{V}^*\mathcal{V}T = T$.

Corollary 2.3. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$. Then T is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ commuting with $|T|$ such that $T = \mathcal{U}T^*$. In this situation, $\mathcal{U}$ also commutes with T and T^*.*

Proof. Suppose T is a normal operator then $Ker(T) = Ker(T^*)$ and $\overline{Ran(T)} = \overline{Ran(T^*)}$. For every $x \in X = Ker(T) \oplus \overline{Ran(T^*)}$ we define

$$\mathcal{U}x = \begin{cases} 0 & \text{if } x \in Ker(T) \\ \mathcal{V}x & \text{if } x \in \overline{Ran(T^*)}, \end{cases}$$

$$\mathcal{W}x = \begin{cases} 0 & \text{if } x \in Ker(T^*) \\ \mathcal{V}^*x & \text{if } x \in \overline{Ran(T)}. \end{cases}$$

Then $\langle \mathcal{U}x, y\rangle = \langle x, \mathcal{W}y\rangle$ for all $x, y \in X$, that is, $\mathcal{W} = \mathcal{U}^*$. Corollary 2.2 implies that $\mathcal{U}\mathcal{U}^* = \mathcal{U}^*\mathcal{U} = 1$ on X and $T = \mathcal{U}|T|$. Commutativity of $\mathcal{U}$ with T, T^* and $|T|$ follows from the commutativity of $\mathcal{V}$ with T, T^* and $|T|$.

Conversely, suppose $T = \mathcal{U}|T|$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ which commutes with $|T|$. Then $T^* = |T|\mathcal{U}^*$ and so $TT^* = \mathcal{U}\,|T|\,|T|\mathcal{U}^* = |T|\mathcal{U}\,|T|\mathcal{U}^* = T^*T$. □

Corollary 2.4. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$. Then T is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $T = \mathcal{U}T^*$. In this situation, $\mathcal{U}$ commutes with T and T^*.*

Proof. Suppose T is a normal operator then $|T| = |T^*| = \mathcal{V}T^*$ and so $T = \mathcal{V}|T| = \mathcal{V}|T^*| = \mathcal{V}^2T^*$. For $x \in X$ we define

$$\mathcal{U}x = \begin{cases} 0 & \text{if } x \in Ker(T) \\ \mathcal{V}^2x & \text{if } x \in \overline{Ran(T^*)}. \end{cases}$$

As in the proof of Corollary 2.3, $\mathcal{U}$ is unitary and $T = \mathcal{U}T^*$. Commutativity of $\mathcal{U}$ with T and T^* follows from the commutativity of $\mathcal{V}$ with T and T^*.

Conversely, suppose $T = \mathcal{U}T^*$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$. Then $T^* = (\mathcal{U}T^*)^* = T\,\mathcal{U}^*$ and so $T^*T = T\,\mathcal{U}^*\,\mathcal{U}\,T^* = T\,T^*$. □

If the normal operator $T \in \mathcal{L}(X)$ has closed range, one can find shorter proof for the above result.

Theorem 2.5. *Suppose $t \in \mathcal{R}(X)$ admits the polar decomposition $t = \mathcal{V}|t|$. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U}t^*$. In this situation, $t\,\mathcal{U} = \mathcal{U}\,t$ and $t^*\,\mathcal{U} = \mathcal{U}\,t^*$ on $Dom(t) = Dom(t^*)$.*

Proof. Recall that t admits the polar decomposition $t = \mathcal{V}|t|$ if and only if its bounded transform F_t admits the polar decomposition $F_t = \mathcal{V}|F_t|$, furthermore, t is a normal operator if and only if its bounded transform F_t is a normal operator.

Suppose t is a normal operator then there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $tQ_t = F_t = \mathcal{U}F_t^* = \mathcal{U}F_{t^*} = \mathcal{U}\,t^*Q_{t^*} = \mathcal{U}\,t^*Q_t$. Since $Q_t : X \to Ran(Q_t) = Dom(t)$ is invertible, we obtain $t = \mathcal{U}t^*$.

Conversely, suppose $t = \mathcal{U}t^*$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$. Then, in view of Remark 2.1 of [6], we have $t^* = (\mathcal{U}t^*)^* = t^{**}\,\mathcal{U}^* = t\,\mathcal{U}^*$ on $Dom(t^*)$ and so $t^*t = t\,\mathcal{U}^*\,\mathcal{U}\,t^* = t\,t^*$.

According to Corollary 2.4 and the first paragraph of the proof, the unitary operator $\mathcal{U}$ commutes with F_t and F_t^*. Thus for every polynomial p we have $\mathcal{U}\,p(F_t^*F_t) = p(F_t^*F_t)\,\mathcal{U}$ and so for every continuous function $p \in \mathbf{C}[0,1]$ we have $\mathcal{U}\,p(F_t^*F_t) = p(F_t^*F_t)\,\mathcal{U}$. In particular, $\mathcal{U}\,(1 - F_t^*F_t)^{1/2} = (1 - F_t^*F_t)^{1/2}\,\mathcal{U}$ which implies $\mathcal{U}\,Q_t = Q_t\,\mathcal{U}$. This fact together with the equality $F_t\,\mathcal{U} = \mathcal{U}F_t$ imply that $t\,\mathcal{U}\,Q_t = tQ_t\,\mathcal{U} = \mathcal{U}\,tQ_t$. Again by invertibility of the map $Q_t : X \to Ran(Q_t) = Dom(t)$ we obtain $t\,\mathcal{U} = \mathcal{U}\,t$ on $Dom(t)$. To demonstrate the second equality we have $\mathcal{U}^*\,t = \mathcal{U}^*\,\mathcal{U}\,t^* = t^*$ which yields $t^*\,\mathcal{U} = (\mathcal{U}^*\,t)^* = t^{**} = t = \mathcal{U}\,t^*$ on $Dom(t^*)$. □

The preceding theorem can also be reformulated in terms of densely defined closed operators on Hilbert C*-modules over C*-algebras of compact operators, or in terms of densely defined closed operators on Hilbert spaces.

Corollary 2.6. *Suppose $t \in \mathcal{R}(X)$ has closed range. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U}t^*$. In this situation, $t\,\mathcal{U} = \mathcal{U}\,t$ and $t^*\,\mathcal{U} = \mathcal{U}\,t^*$ on $Dom(t) = Dom(t^*)$.*

The proof immediately follows from Theorem 2.5, Proposition 2.2 of [5] and Theorem 3.1 of [6].

Consider two normal operators T and S on a Hilbert space it is known that, in general, TS is not normal. Historical notes and several versions of the problem are investigated in [7]. Kaplansky has shown that it may be possible that TS is normal while ST is not. Indeed, he has shown that if T and TS are normal, then ST is normal if and only if S commutes with $|T|$, cf. [9]. We reformulate his result for bounded adjointable operator on Hilbert C*-modules. For this aim we need the Fuglede-Putnam theorem for bounded adjointable operators on Hilbert C*-modules. Using Theorem 4.1.4.1 of [3] for the unital C*-algebra $\mathcal{L}(X)$, we obtain:

Theorem 2.7. (Fuglede-Putnam) *Assume that T, S and A are bounded adjointable operator in $\mathcal{L}(X)$. Suppose T and S are normal and $TA = AS$, then $T^*A = AS^*$.*

Theorem 2.8. *Let $T, S \in \mathcal{L}(X)$ be such that T and TS are normal and T has polar decomposition. ST is normal if and only if S commutes with $|T|$.*

Proof. Suppose ST and T are normal operators and $A = TS$ and $B = ST$, then $AT = TB$. In view of the Theorem 2.7, $A^*T = TB^*$, that is, $S^*T^*T = T\,T^*S^*$, and taking into account the normality of T, we find S^* commutes with T^*T. Therefore, $S^*|T| = |T|S^*$ and so S commutes with $|T|$ by the Fuglede-Putnam theorem.

Conversely, suppose S commutes with $|T|$. Then the normal operator T has a representation $T = \mathcal{U}\,|T|$ in which $\mathcal{U} \in \mathcal{L}(X)$ is unitary and commutes with $|T|$. Therefore,

$$\mathcal{U}^*\,TS\,\mathcal{U} = \mathcal{U}^*\,\mathcal{U}\,|T|\,S\,\mathcal{U} = S|T|\,\mathcal{U} = S\,\mathcal{U}\,|T| = ST.$$

The operator ST is normal as an operator which is unitary equivalent with the normal operator TS. □

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