The adjoint group of an Alexander quandle.

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To an abelian group M equipped with an automorphism T one can associate a quandle $A(M,T)$ called its Alexander quandle. It is given by the set M together with the quandle operation $*$ defined by $y * x = Ty + x - Tx$. To any quandle Q one can associate a group $\mathrm{Adj}(Q)$ called the *adjoint group* of Q. It is defined as the abstract group with one generator e_x for each $x \in Q$ and one relation $e_{y*x} = e_x^{-1} e_y e_x$ for each $x, y \in Q$.

It is the purpose of this note to show that the adjoint group of an Alexander quandle $Q(M,T)$ has an elegant description in terms of M and T, at least if the quandle is connected, which is the case if $1 - T$ is invertible. From this description one gets a similar description of the *fundamental group* of $Q(M,T)$ based at $0 \in M$. This note can be viewed as an exercise inspired by [2], to which we refer for motivation and definitions.

The adjoint group $A = \mathrm{Adj}(A(M,T))$ acts from the right on M by the formula $p \cdot e_x = p * x$. This defines a homomorphism ρ from A to the group G of quandle automorphisms of $A(M,T)$). Thus $p \cdot e_0^{-1} = T^{-1}p$ and $p \cdot e_0^{-1} e_x = p + x - Tx$. From one sees that

$$p \cdot e_0^{-1} e_x e_0^{-1} e_y = p + x - Tx + y - Ty = p \cdot e_0^{-1} e_{x+y}$$

Therefore

$$e_0^{-1} e_{x+y} = \gamma(x,y) e_0^{-1} e_x e_0^{-1} e_y \tag{1}$$

for some $\gamma(x,y) \in \mathrm{Adj}(Q)$ which acts trivially on M and thus is an element of $K = \ker(\rho)$. The group K is a central subgroup of A as explained in [2].

From the definition of $\gamma(x,y)$ we see that $\gamma(0,y) = 1$ and $\gamma(x,0) = 1$ for all x and y. Furthermore the formulas

$$\begin{aligned} e_0^{-1} e_{x+y+z} &= \gamma(x, y+z) e_0^{-1} e_x e_0^{-1} e_{y+z} \\ &= \gamma(x, y+z) e_0^{-1} e_x \gamma(y,z) e_0^{-1} e_y e_0^{-1} e_z \\ e_0^{-1} e_{x+y+z} &= \gamma(x+y, z) e_0^{-1} e_{x+y} e_0^{-1} e_z \\ &= \gamma(x+y, z) e_0^{-1} \gamma(x,y) e_x e_0^{-1} e_y e_0^{-1} e_z \end{aligned}$$

show that

$$\gamma(x, y+z)\gamma(y,z) = \gamma(x+y, z)\gamma(x,y) \text{ for all } x,y,z \in M \tag{2}$$

This shows that γ is a group 2-cocycle for the group M with values in K. We will not use this: our purpose is not to show that γ is a coboundary, but to show that it vanishes to a certain degree, by exploiting its relation with T. However if γ were a coboundary then in particular $\gamma(x, y)$ would be symmetric in x and y. This is one of the motivations to consider the map $\lambda\colon M \times M \to K$ defined by

$$\lambda(x, y) = \gamma(y, x)^{-1}\gamma(x, y) = [e_0^{-1}e_y, e_0^{-1}e_x] \tag{3}$$

The defining relation for A shows that $e_0e_xe_0^{-1} = e_{T^{-1}x}$ or equivalently $e_xe_0^{-1} = e_0^{-1}e_{T^{-1}x}$ for $x \in M$. So we can rewrite $e_{x+y} = \gamma(x, y)e_xe_0^{-1}e_y$ as $e_{x+y} = \gamma(x, y)e_0^{-1}e_{T^{-1}x}e_y$. In other words

$$e_ue_v = \gamma(Tu, v)^{-1}e_0e_{Tu+v} \text{ for all } u, v \in M \tag{4}$$

If we substitute this twice in the defining relation we find that

$$\begin{aligned}&\gamma(Tu, v)^{-1}e_0e_{Tu+v} = e_ue_v = e_ve_{Tu+v-Tv}\\ &= \gamma(Tv, Tu + v - Tv)^{-1}e_0e_{Tv+Tu+v-Tv}\end{aligned}$$

This implies that $\gamma(Tu, v) = \gamma(Tv, Tu + v - Tv)$ for $u, v \in M$, in other words

$$\gamma(x, y) = \gamma(Ty, x + y - Ty) \text{ for } x, y \in M \tag{5}$$

and in particular

$$\gamma(Ty, y - Ty) = 1 \text{ for } y \in M \tag{6}$$

We now switch to additive notation for K. From (5) and the cocycle relation we find

$$\begin{aligned}&\gamma(u, v) + \gamma(v - Tv, u)\\ &= \gamma(Tv, v - Tv + u) + \gamma(v - Tv, u)\\ &= \gamma(Tv + v - Tv, u) + \gamma(Tv, v - Tv)\end{aligned}$$

and in particular

$$\lambda(u, v) = \gamma(u, v) - \gamma(v, u) = -\gamma(v - Tv, u) \tag{7}$$

Thus if γ were symmetric then λ would vanish, and so would γ since $1 - T$ is assumed to be invertible.

Now we look at the consequences for λ of the cocycle condition for γ. If we substitute (7) in the cocycle condition for γ we find

$$\lambda((1-T)^{-1}(x+y), z) + \lambda((1-T)^{-1}x, y) = \lambda((1-T)^{-1}x, y+z) + \lambda((1-T)^{-1}y, z)$$

and putting $x = u - Tu$, $y = v - Tv$ this yields

$$\lambda(u + v, z) + \lambda(u, v - Tv) = \lambda(u, v - Tv + z) + \lambda(v, z) \tag{8}$$

On the other hand subtracting two instances of the cocycle condition for γ

$$\gamma(u, v+z) + \gamma(v, z) = \gamma(u+v, z) + \gamma(u, v)$$
$$\gamma(z, v+u) + \gamma(v, u) = \gamma(z+v, u) + \gamma(z, v)$$

we find

$$\lambda(u+v, z) + \lambda(u, v) = \lambda(u, v+z) + \lambda(v, z) \tag{9}$$

Substracting (9) from (8) we find

$$\lambda(u, v - Tv) - \lambda(u, v) = \lambda(u, v - Tv + z) - \lambda(u, v+z) \tag{10}$$

This means that the right hand side of (10) does not depend on z; in particular it has the same value for $z = -v$. Thus using the fact that $\lambda(u, 0) = 0$ we can rewrite (10) as

$$\lambda(u, -Tv) = \lambda(u, v - Tv + z) - \lambda(u, v+z) \tag{11}$$

Substituting $a = v + z$ and $b = -Tv$ this yields

$$\lambda(u, b) = \lambda(u, a+b) - \lambda(u, a) \tag{12}$$

We have just proved that λ is additive in its second coordinate. Since λ is skew-symmetric it is in fact bi-additive. Thus (7) and the invertibility of $1 - T$ imply that γ is bi-additive. Moreover using (6) we can simplify (5) to

$$\gamma(x, y) = \gamma(Ty, x) \text{ for all } x, y \tag{13}$$

This motivates the following definition and theorem.

Definition 1. Define $\tau\colon M \otimes M \to M \otimes M$ by the formula $\tau(x \otimes y) = Ty \otimes x$. Define $S(M, T)$ as $\operatorname{coker}(1-\tau)$. Thus γ can be viewed as a map from $S(T, M)$ to K. Finally define $F(M, T)$ as the set $\mathbf{Z} \times M \times S(M, T)$ with the multiplication given by

$$(k, x, \alpha)(m, y, \beta) = (k+m, T^m x + y, \alpha + \beta + [T^m x \otimes y])$$

Theorem 1. *The groups* $\operatorname{Adj}(A(M, T))$ *and* $F(M, T)$ *are isomorphic.*

Proof. We define $\phi\colon \operatorname{Adj}(A(M, T)) \to F(M, T)$ by setting $\phi(e_x) = (1, x, 0)$. To see that this is well defined we have to check the following:

$$\begin{aligned} &\phi(e_x)\phi(e_{y*x}) = (1, x, 0)(1, Ty + x - Tx, 0) \\ &= (2, Tx + (Ty + x - Tx), [Tx \otimes (Ty + x - Tx)]) \\ &= (2, Ty + x, [Ty \otimes x]) = (1, y, 0)(1, x, 0) = \phi(e_y)\phi(e_x) \end{aligned}$$

which is the case since $[Tx \otimes Ty] = [Ty \otimes x]$ and $[Tx \otimes Tx] = [Tx \otimes x]$.
We define $\psi\colon F(M, T) \to \operatorname{Adj}(A(M, T))$ by setting $\psi(k, x, \alpha) = e_0^{k-1} e_x \gamma(\alpha)^{-1}$.

To see that ψ is a homomorphism we have to check the following:

$$\begin{aligned}
&\psi(k,x,\alpha)\psi(m,y,\beta) = e_0^{k-1}e_x\gamma(\alpha)^{-1}e_0^{m-1}e_y\gamma(\beta)^{-1} \\
&= e_0^{k-1}e_xe_0^{m}e_0^{-1}e_y\gamma(\alpha)^{-1}\gamma(\beta)^{-1} = e_0^{k-1}e_0^{m}e_{T^mx}e_0^{-1}e_y\gamma(\alpha+\beta)^{-1} \\
&= e_0^{k+m-1}e_{T^mx+y}\gamma(T^mx\otimes y)^{-1}\gamma(\alpha+\beta)^{-1} \\
&= \psi(k+m, T^mx+y, \alpha+\beta+[T^mx\otimes y])
\end{aligned}$$

which is the case $e_ze_0^{-1}e_y = e_{z+y}\gamma[z\otimes y]^{-1}$ for $z = T^mx$ by (1).
From $\psi(\phi(e_x)) = \psi(1,x,0) = e_x$ we see that $\psi\phi = 1$. The other composition requires more work; first we compute

$$\begin{aligned}
&\phi(\gamma[u\otimes v]^{-1}) = \phi(e_{u+v}^{-1}e_ue_0^{-1}e_v) = (1,u+v,0)^{-1}(1,u,0)(1,0,0)^{-1}(1,v,0) \\
&= (-1,-T^{-1}(u+v),[(u+v)\otimes(u+v)])(1,u,0)(-1,0,0)(1,v,0) \\
&= (-1,-T^{-1}(u+v),[(u+v)\otimes(u+v)])(1,u+v,[u\otimes v]) = (0,0,[u\otimes v])
\end{aligned}$$

which shows that $\phi(\gamma(\alpha)^{-1}) = (0,0,\alpha)$ for all α. From this we get

$$\phi(\psi(k,x,\alpha)) = \phi(e_0^{k-1})\phi(e_x)\phi(\gamma(\alpha)^{-1}) = (k-1,0,0)(1,x,0)(0,0,\alpha) = (k,x,\alpha)$$

so we find that $\phi\psi = 1$. □

For any quandle Q there is a unique homomorphism $\epsilon\colon \mathrm{Adj}(Q) \to \mathbf{Z}$ such that $\epsilon(e_x) = 1$ for all $x \in Q$; the kernel is denoted by $\mathrm{Adj}(Q)^o$. It is clear that $\epsilon(\alpha) = 0$ for all α, so $\epsilon(\psi(k,x,\alpha)) = k$. Therefore under ψ the subgroup $\mathrm{Adj}(A(M,T))^o$ of $\mathrm{Adj}(A(M,T))$ corresponds to the subgroup $F(M,T)^o$ of $F(M,T)$ consisting of the triples $(0,x,\alpha)$. Note that on $F(M,T)^o$ the multiplication simplifies to

$$(0,x,\alpha)(0,y,\beta) = (0,x+y,\alpha+\beta+[x\otimes y])$$

For any quandle the fundamental group based at $q \in Q$ is defined as $\pi_1(Q,q) = \{g \in \mathrm{Adj}(Q)^o \mid q\cdot g = q\}$. For these definitions we refer to [2]. In order to describe this in terms of (M,T) for the case $Q = A(M,T)$ we need to translate the action of $\mathrm{Adj}(A(M,T))$ on M into an action of $F(M,T)$ on M.

One can easily check that $0\cdot\psi(k,x,\alpha) = x - Tx$ for all k, x and α. This implies that $0\cdot(0,x,\alpha) = 0$ if and only if $x = 0$, which means that $\pi_1(A(M,T),0)$ is isomorphic to $S(M,T)$.

Example 1. Let $\mathbf{F}$ be a field, let $M = \mathbf{F}[t]/(t^2+at+b)$ and let T be multilpication by the class of t. Then T is an automorphism if $b \neq 0$ and $A(M,T)$ is connected if $1+a+b \neq 0$. In this case $S(M,T)$ isomorphic to $K/(b^2+ab-a-1)$. Thus $A(M,T)$ is simply connected if $b^2+ab-a-1 \neq 0$. The entry for $\mathbf{F} = \mathbf{Z}/(3)$ and $f(t) = t^2-t+1$ in the table on page 49 of [1] is not compatile with this, but it is a misprint.

References

[1] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. 355 No. 10 (2003), 3947-3989.
Also math.GT/9903135.

[2] M. Eisermann, Quandle coverings and their Galois correspondence, http://www-fourier.ujf-grenoble.fr/~eiserm or math.GT/0612459.