A UNIVERSAL ÉTALE LIFT OF A PROPER LOCAL EMBEDDING

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Abstract. To any finite local embedding of Deligne–Mumford stacks $g : Y \to X$ we associate an étale, universally closed morphism $F_{Y/X} \to X$ such that for the complement Y^2_X of the image of the diagonal $Y \to Y \times_X Y$, the stack $F_{Y^2_X/Y}$ admits a canonical closed embedding in $F_{Y/X}$, and $F_{Y/X} \times_X Y$ is a disjoint union of copies of $F_{Y^2_X/Y}$. The stack $F_{Y/X}$ has a natural functorial presentation, and the morphism $F_{Y/X} \to X$ commutes with base-change. The image of Y^2_X in Y is the locus of points where the morphism $Y \to g(Y)$ is not smooth. Thus for many practical purposes, the morphism g can be replaced in a canonical way by copies of the closed embedding $F_{Y^2_X/Y} \to F_{Y/X}$.

Introduction

Local embeddings of Deligne-Mumford stacks constitute a natural extension of the notion of closed embeddings of schemes. For example, the diagonal of a Deligne-Mumford a stack, and the natural morphism from its inertia stack to the stack itself, both belong to this class.

Many difficulties in extending classical algebraic geometry constructions from the category of schemes to stacks stem from the existence of such local embeddings. To solve this problem, one can rely on the local nature for the étale topology of these morphisms. Indeed, given a local embedding of algebraic stacks $g : Y \to X$, there exist étale atlases V_0 and U of Y and X respectively, and a closed embedding $V_0 \hookrightarrow U$ compatible with the morphism g. This local construction yields the notions of normal bundle of a local embedding as introduced by A. Vistoli ([V]), and deformation to the normal cone as introduced by A. Kresch ([K]), and consequently an intersection theory on smooth Deligne-Mumford stacks.

In [MM] we argued that a more refined étale presentation of the morphism $g : Y \to X$ is needed if such ubiquitous constructions like blow-ups are to be defined for local embeddings. For this purpose, given a proper local embedding $g : Y \to X$, we constructed an étale atlas U of X such that the fibre product $Y \times_X U$ is a union of étale atlases V_i of Y, each of which is embedded as a closed subscheme in U. The locus where the images W_i of V_i-s intersect pairwise is an étale atlas for the stack of non-smooth values of g. Moreover, the stratification determined by the number of intersecting components W_i indicates how far the morphism g is from being étale on the image over each point in $g(Y)$. The étale atlas U thus encodes essential information about the structure of g. In [MM] we set out to translate this information from étale atlases to stacks amenable to global constructions like e.g. blow-ups, or intersection rings. For a proper $g : Y \to X$,

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we found a pair of stacks Y' and X' with étale, universally closed morphisms $Y' \to Y$ and $X' \to X$, and a morphism $g' : Y' \to X'$ such that $Y' = Y \times_X X'$ is a disjoint union of stacks each embedded as a closed substack of X' via g'. However, our construction was not unique. Indeed, it depends on the choice of a suitable étale atlas U of X.

In this paper we introduce an étale, universally closed morphism $F_{Y/X} \to X$ which is intrinsically associated to the proper local embedding $g : Y \to X$, which has the desired properties listed above, and which commutes with base change. We give a functorial presentation of this stack and study its properties in more detail. As applications, this canonical definition provides grounds for extending other constructions from schemes to stacks. For example, we can now define compactifications of configuration spaces for stacks by extending W. Fulton and R. MacPherson's [FMcP] constructions in a coherent, natural way. Also, in our opinion the stack $F_{X/X\times X}$ provides a natural context for orbifold products like the ones defined by Edidin, Jarvis and Kimura in [EJK] for quotient Deligne-Mumford stacks. We will explore such applications in more detail in a sequel to this paper.

We start this article by discussing the case when $g : Y \to X$ is a morphism of Deligne–Mumford stacks which is finite and étale on its image. In [MM] we showed that such a morphism can be factored into an étale, universally closed morphism $F_{Y/X} \to X$ and an embedding $Y \hookrightarrow F_{Y/X}$, which identifies Y with the preimage of $g(Y)$ in $F_{Y/X}$, and such that $F_{Y/X} \setminus Y \cong X \setminus g(Y)$. In Proposition 1.2 we provide a detailed list of properties for $F_{Y/X}$, some of which will prove very useful in more general set-up. For example, property (10) will lead to a natural definition of a lift $F_{Y/X}$ in the case when g is a general proper local embedding, and Y is reducible.

For any morphism of Deligne–Mumford stacks $g : Y \to X$, the fibered product $Y \times_X Y$ represents the functor of isomorphisms in X of objects coming from Y: its objects over a scheme S are tuples (ξ_1, ξ_2, f), where ξ_1, ξ_2 are objects in $Y(S)$, and f is an isomorphism between $g(\xi_1)$ and $g(\xi_2)$. Let $\Delta : Y \to Y \times_X Y$ denote the diagonal morphism and let Y^2_X denote the complement of its image in $Y \times_X Y$. If g is finite and unramified, then so are the projections $Y^2_X \to Y$, and their image is the locus of points where g is not étale on its image. We can reiterate this construction with $(Y^2_X)^2_Y \to Y^2_X$. Here $(Y^2_X)^2_Y$ is isomorphic to the complement Y^3_X of all diagonals in $Y \times_X Y \times_X Y$, and as such it admits three different projections to Y^2_X. By successively reiterating this construction until we reach $Y^{n+1}_X = \emptyset$, we obtain a canonical network $\mathcal{N}^n(Y/X)$ of local embeddings, the last one of which is étale on its image. This network commutes with base change, and it encapsulates the local étale structure of the morphism $g : Y \to X$ in a way which is simultaneously comprehensive and non-redundant.

In a sequence of steps, the network $\mathcal{N}^n(Y/X)$ can be replaced by another network $\mathcal{N}^0(Y/X)$ where all morphisms are closed embeddings, and the objects admit étale, universally closed surjections to the objects of $\mathcal{N}^n(Y/X)$. The target of the new network is $F_{Y/X}$. Moreover, the other objects of $\mathcal{N}^0(Y/X)$ are also canonical lifts for the local embeddings contained in $\mathcal{N}^n(Y/X)$. Thus, for practical purposes the morphism g can be replaced by a set of copies of the closed embedding $F_{Y^2_X/Y} \to F_{Y/X}$. The functorial presentation and properties of $F_{Y/X}$ are listed in Theorem 1.21 and the Definition preceding it.

In [R], David Rydh constructed a different canonical lift $E_{Y/X}$ for any unramified morphism $g : Y \to X$: he showed that g has a universal factorization $Y \to E_{Y/X} \to X$, where the first morphism is a closed embedding i and the second is étale; moreover, $E_{Y/X}$ comes with an open immersion $j : X \to E_{Y/X}$ such that $i(Y)$ is the complement of $j(X)$ in $E_{Y/X}$. His construction works in a more general context than ours, and indeed it was meant to address the lack of an intrinsic presentation for our étale lift in [MM]. However $E_{Y/X}$ differs from $F_{Y/X}$ in its range of applicability. We would like to thank David Rydh for his useful observations.

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1. The universal lift of a local embedding

The stacks in this article are assumed to be algebraic in the sense of Deligne–Mumford, Noetherian, and all morphisms considered between them are of finite type.

1.1. The lift of a local embedding étale on its image.

Definition 1.1. Following [V], we will call local embedding any representable unramified morphism of finite type of stacks. A regular local embedding is a local embedding which is also locally a complete intersection.

Proposition 1.2. *Let $g : Y \to X$ be a proper morphism of stacks étale on its image. There exists an étale morphism $e_g : F_{Y/X} \to X$ together with an isomorphism $\phi : g(Y) \times_X F_{Y/X} \to Y$, such that*

(0) i) *the triangles in the following diagram are commutative*

$$\begin{tikzcd} Y \times_X F_{Y/X} \arrow[r] \arrow[d, "g \times id_{F_{Y/X}}"'] & Y \arrow[d, "g"] \\ g(Y) \times_X F_{Y/X} \arrow[r] \arrow[ur, "\phi"] & X, \end{tikzcd}$$

where the upper horizontal arrow is the projection on Y and the lower horizontal arrow is the restriction of e_g to $g(Y) \times_X F_{Y/X}$;

(0) ii) *Let $p_2 : g(Y) \times_X F_{Y/X} \to F_{Y/X}$ be the projection on the second factor and consider the closed embedding $i := p_2 \circ \phi^{-1}$. The restriction of e_g induces an isomorphism $F_{Y/X} \setminus i(Y) \to X \setminus g(Y)$.*

The following properties also hold:

(1) *For any stack Z, there is an equivalence of categories between $\mathrm{Hom}(Z, F_{Y/X})$ and the category of morphisms $Z \to X$ endowed with a section*

$$s : g(Y) \times_X Z \to Y \times_X Z$$

for the étale map $Y \times_X Z \to g(Y) \times_X Z$.

(2) *The triple $(F_{Y/X}, e_g, \phi)$, with e_g étale and satisfying (0)i) and (0)ii) is uniquely defined up to unique 2-isomorphism.*

(3) *The morphism $e_g : F_{Y/X} \to X$ is universally closed.*

(4) *If $g : Y \to X$ is a closed embedding, then $F_{Y/X} \cong X$.*

(5) *If $g : Y \to X$ is étale and proper, and X is connected then $F_{Y/X} \cong Y$.*

(6) *For any morphism of stacks* $u : X' \to X$ *and* $Y' := Y \times_X X'$, *there exists a morphism* $F_u : F_{Y'/X'} \to F_{Y/X}$ *making the squares in the following diagram Cartesian:*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & F_{Y'/X'} \arrow[r] \arrow[d, "F_u"] & X' \arrow[d, "u"] \\ Y \arrow[r] & F_{Y/X} \arrow[r] & X. \end{tikzcd}$$

(7) *If* $h : Z \to Y$ *is proper and étale on its image, and* $g : Y \to X$ *is a closed embedding, then* $F_{Z/Y} \cong Y \times_X F_{Z/X}$. *In particular, there exists a natural étale morphism* $g_* : F_{Z/Y} \to F_{Z/X}$.

(8) *For any morphism* $h : Z \to Y$ *proper and étale on its image, the composition morphism* $g \circ i_h : F_{Z/Y} \to X$, *universally closed and étale on its image, comes with an étale map* $F_{F_{Z/Y}/X} \to X$ *satisfying properties (0) and (1). Moreover,*

$$F_{F_{Z/Y}/X} \cong F_{Z/F_{Y/X}}.$$

In particular, if the morphism $h : Z \to Y$ *is étale, then there exists a natural morphism* $h_* : F_{Z/X} \to F_{Y/X}$.

(9) *If* $h : Z \to Y$ *and* $g : Y \to X$ *are proper and étale on their images, and if* $g(h(Z)) \times_X Y \cong h(Z)$ *over* Y, *then there exists a morphism* $g_* : F_{Z/Y} \to F_{Z/X}$ *such that* $e_{g\circ h} \circ \bar{g} = g \circ e_h$. *In this case* $F_{F_{Z/Y}/F_{Z/X}} \cong F_{F_{Z/Y}/X} \cong F_{Z/F_{Y/X}}$.

(10) *Given any proper local embeddings* $g : Y \to X$ *and* $f : T \to X$, *and* $Z := Y \times_X T$, *there are natural isomorphisms*

$$F_{Y/X} \times_X F_{T/X} \cong F_{F_{Z/T}/F_{Y/X}} \cong F_{F_{Z/Y}/F_{T/X}} \cong F_{F_{Z/Y} \bigcup_Z F_{Z/T}/F_{Z/X}},$$

where $F_{Z/Y} \bigcup_Z F_{Z/T}$ *denotes the stack obtained by gluing the stacks* $F_{Z/Y}$ *and* $F_{Z/T}$ *along* Z.

Proof. An explicit étale groupoid presentation for a functor $F_{Y/X}$ which satisfies properties (0) and (1) was found in [MM], section 1.1. We briefly recall it here. One chooses an étale cover by a scheme $p : U \to X$ such that $Y \times_X U \cong V = V_1 \bigsqcup V_2$ where $V_1 = g(Y) \times_X U$. Let

$$S_{ij} := \text{ Im } (\phi_{ij} : V_i \times_Y V_j \to U \times_X U),$$

for the map ϕ_{ij} given as a composition

$$V_i \times_Y V_j \hookrightarrow V \times_Y V = V \times_Y (Y \times_X U) \cong V \times_X U \to U \times_X U.$$

A groupoid presentation of $F_{Y/X}$ is given by $[R' \rightrightarrows U]$

$$R' := (U \times_X U) \setminus (S_{12} \cup S_{21} \cup (S_{22} \setminus S_{11})) \cup \text{ Im } e.$$

To prove property (2), we note that for any triple (F', e', ϕ') satisfying the properties (0), there is a canonical section $g(Y) \times_X F' \to Y \times_X F'$ which, together with the map e', determine a unique morphism $u : F' \to F_{Y/X}$ such that $e \circ u = e'$, due to condition (1). Both e and e' are étale, and so u must be étale as well. On the other hand, e and e' induce isomorphisms $g(Y) \times_X F_{Y/X} \cong Y \cong g(Y) \times_X F'$ and $F_{Y/X} \setminus i(Y) \cong X \setminus g(Y) \cong F' \setminus i'(Y)$. Thus u is both étale and bijective, and so an isomorphism.

Property (3) follows from the valuation criterium in conjunction with property (1). Consider a complete discrete valuation ring R with field of fractions K, a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{u} & F_{Y/X} \\ \Big\downarrow \rho & & \Big\downarrow p \\ \mathrm{Spec}(R) & \xrightarrow{v} & X. \end{array}$$

The closed embedding $g(Y) \times_X \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ is either the empty embedding or an isomorphism. If empty, then u factors through $\mathrm{Spec}(K) \to F_{Y/X} \setminus Y \cong X \setminus g(Y)$ and so v also naturally yields $\mathrm{Spec}(R) \to X \setminus g(Y) \cong F_{Y/X} \setminus Y$. If an isomorphism, then the map v induces a natural morphism $\mathrm{Spec}(K) \cong g(Y) \times_X \mathrm{Spec}(K) \to Y$ whose composition with g is $u\rho$. As g is proper, there is a lift $\mathrm{Spec}(R) \to Y$, which yields a section $g(Y) \times_X \mathrm{Spec}(R) \to Y \times_X \mathrm{Spec}(R)$. This, together with the map $u \to \mathrm{Spec}(R) \to X$ give the data for a unique morphism $\mathrm{Spec}(R) \to F_{Y/X}$ as required.

Properties (4) and (5) are direct consequences of (2).

Property (6) was proven in [MM], Corollary 1.8. Alternatively, it follow immediately from (2). Indeed, consider a morphism of stacks $f : X' \to X$ and let $Y' := Y \times_X X'$, with the morphism $g' : Y' \to X'$ induced by g. Then the étale morphism $X' \times_X F_{Y/X} \to X'$ induced by e_g, together with the composition

$$\begin{aligned} g'(Y') \times_{X'} (X' \times_X F_{Y/X}) &\cong (g(Y) \times_X X') \times_{X'} (X' \times_X F_{Y/X}) \cong \\ &\cong g(Y) \times_X X' \times_X F_{Y/X} \cong X' \times_X Y \cong Y', \end{aligned}$$

satisfy properties (0) for the morphism $g' : Y' \to X'$ and so $X' \times_X F_{Y/X} \cong F_{Y'/X'}$. Thus

$$\begin{aligned} Y' &\cong g'(Y') \times_{X'} F_{Y'/X'} \cong (g(Y) \times_X X') \times_{X'} F_{Y'/X'} \cong \\ &\cong g(Y) \times_X F_{Y'/X'} \cong (g(Y) \times_X F_{Y/X}) \times_{F_{Y/X}} F_{Y'/X'} \cong Y \times_{F_{Y/X}} F_{Y'/X'}. \end{aligned}$$

To prove (7), we will construct a canonical morphism $F_{Z/Y} \to Y \times_X F_{Z/X}$, together with its inverse. To construct $F_{Z/Y} \to F_{Z/X}$, we first consider the composition $g \circ e_h : F_{Z/Y} \to Y \to X$. The canonical isomorphisms

$$\begin{aligned} g(h(Z)) \times_X F_{Z/Y} &\cong (h(Z) \times_X Y) \times_Y F_{Z/Y} \cong h(Z) \times_Y F_{Z/Y} \cong Z, \quad \text{and} \\ Z \times_X F_{Z/Y} &\cong (Z \times_X Y) \times_Y F_{Z/Y} \cong Z \times_Y F_{Z/Y}, \end{aligned}$$

together with the embedding $Z \to Z \times_Y F_{Z/Y}$ give a section $g(h(Z)) \times_X F_{Z/Y} \to Z \times_Y F_{Z/Y}$, and thus, according to (1), a map $F_{Z/Y} \to F_{Z/X}$. This, together with the étale map $F_{Z/Y} \to Y$ generate the desired morphism ${}_{Z/Y} \to Y \times_X F_{Z/X}$. Its inverse is also constructed via property (1) as follows: We consider the projection $Y \times_X F_{Z/X} \to Y$ together with the canonical section

$$h(Z) \times_Y (Y \times_X F_{Z/X}) \cong h(Z) \times_X F_{Z/X} \to Z \times_X F_{Z/X} \cong Z \times_Y (Y \times_X F_{Z/X}).$$

Proof of (8): Consider now a morphism $h : Z \to Y$ proper and étale on its image. We will show that $e_g \circ e_{i \circ h} : F_{Z/F_{Y/X}} \to X$ is an 'etale lift for the composition $g \circ i_h : F_{Z/Y} \to X$. Note that $g \circ i_h$ is universally closed and étale on its image $\mathrm{Im}\ g \circ i_h = g(Y)$, though not necessarily separated. Let $i : Y \to F_{Y/X}$ be the natural embedding induced by g. Then $i \circ h : Z \to F_{Y/X}$ is the composition of a proper morphism étale on its image and a closed embedding. Due to (7) applied to this composition, there are Cartesian diagrams

$$\begin{tikzcd} F_{Z/Y} \arrow[r, "e_h"] \arrow[d] & Y \arrow[r] \arrow[d] & g(Y) \arrow[d] \\ F_{Z/F_{Y/X}} \arrow[r] & F_{Y/X} \arrow[r, "e_g"] & X, \end{tikzcd}$$

whose composition implies that $F_{Z/Y} \cong F_{Z/F_{Y/X}} \times_X g(Y)$ canonically and that $e_g \circ e_{i\circ h}$ induces $F_{Z/F_{Y/X}} \setminus F_{Z/Y} \cong X \setminus g(Y)$. We also check that $e_g \circ e_{i\circ h}$ satisfies property (1), namely that any map $T \to F_{Z/F_{Y/X}}$ is uniquely determined by a pair of maps $f : T \to X$ and a section $s : g(Y) \times_X T \to F_{Z/Y} \times_X T$. Indeed, such a pair, together with the composition $g(Y) \times_X T \to F_{Z/Y} \times_X T \to Y \times_X T$, determine in a first instance a map $T \to F_{Y/X}$, whose composition with e_g yields f. The restriction of s also yields a section $g(h(Z)) \times_X T \to Z \times_X T$ and so a sequence of morphisms over $F_{Y/X}$:

$$i(h(Z)) \times_{F_{Y/X}} T \to g(h(Z)) \times_X T \to Z,$$

and so a section $i(h(Z)) \times_{F_{Y/X}} T \to Z \times_{F_{Y/X}} T$. This determines a map $T \to F_{Z/F_{Y/X}}$ whose composition with $e_g \circ e_{i\circ h}$ yields f.

Proof of (9): The isomorphisms

$$g(h(Z)) \times_X F_{Z/Y} \cong (g(h(Z)) \times_X Y) \times_Y F_{Z/Y} \cong h(Z) \times_Y F_{Z/Y} \cong Z,$$

together with the composition $g\circ e_h$, give a morphism $F_{Z/Y} \to F_{Z/X}$. Clearly $F_{F_{Z/Y}/F_{Z/X}}$ satisfies properties (0) as an étale lift of $g \circ e_h$, hence the isomorphism $F_{F_{Z/Y}/F_{Z/X}} \cong F_{F_{Z/Y}/X}$.

Proof of (10): The first two isomorphisms are direct consequences of property (6). Indeed,

$$F_{F_{Y\times_X T/T}/F_{Y/X}} \cong F_{F_{Y/X}\times_X T/F_{Y/X}} \cong F_{Y/X} \times_X F_{T/X},$$

and similarly for $F_{F_{Y\times_X T/Y}/F_{T/X}}$. To prove the last isomorphism, we will first need to pinpoint the existence of a natural local embedding $F_{Z/Y} \bigcup_Z F_{Z/T} \to F_{Z/X}$. Indeed, via property (9), there exist compositions

$$F_{Z/T} \hookrightarrow F_{F_{Z/T}/X} \cong F_{Z/F_{T/X}} \to F_{Z/X} \text{ and } F_{Z/Y} \hookrightarrow F_{F_{Z/Y}/X} \cong F_{Z/F_{Y/X}} \to F_{Z/X}.$$

Indeed, the hypotheses necessary for property (9) hold because $Z = Y \times_X Y$. Moreover, the compositions above, together with the embeddings of Z into $F_{Z/T}$ and $F_{Z/Y}$, respectively, form a commutative diagram, which insure the existence of the morphism $F_{Z/Y} \bigcup_Z F_{Z/T} \to F_{Z/X}$ (conform [AGV], Appendix 1). Moreover, by construction this morphism is proper and a local embedding.

In a similar way we can check the existence of a closed embedding

$$j : F_{Z/Y} \bigcup_Z F_{Z/T} \to F_{Y/X} \times_X F_{T/X}.$$

Indeed, the isomorphisms $F_{Y/X} \times_X F_{T/X} \cong F_{F_{Z/T}/F_{Y/X}} \cong F_{F_{Z/Y}/F_{T/X}}$ implicitly state the existence of closed embeddings of $F_{Z/Y}$ and $F_{Z/T}$ into $F_{Y/X} \times_X F_{T/X}$, which commute with the embeddings of Z into $F_{Z/T}$ and $F_{Z/Y}$ respectively, and thus define the closed embedding j.

Furthermore, property (9) implies the existence of natural morphisms from $F_{Z/Y}$ and $F_{Z/T}$ to $F_{Z/X}$, which induce a natural morphism $F_{Z/Y}\bigcup_Z F_{Z/T}\to F_{Z/X}$.

The existence of a natural morphism $e: F_{Y/X}\times_X F_{T/X}\to F_{Z/X}$ follows from the universal property (1) of $F_{Z/X}$. Indeed, via the canonical étale morphism $F_{Y/X}\times_X F_{T/X}\to X$, there are natural morphisms

$$\mathrm{Im}\,(Z\to X)\times_X F_{Y/X}\times_X F_{T/X}\hookrightarrow\ \mathrm{Im}\,f\times_X F_{T/X}\times_X F_{Y/X}\to T\times_X F_{Y/X}\times_X F_{T/X},$$

and similarly

$$\mathrm{Im}\,(Z\to X)\times_X F_{T/X}\times_X F_{T/X}\hookrightarrow\ \mathrm{Im}\,g\times_X F_{Y/X}\times_X F_{T/X}\to Y\times_X F_{Y/X}\times_X F_{T/X},$$

forming a commutative diagram with the projections to $F_{Y/X}\times_X F_{T/X}$, and thus inducing a section

$$\mathrm{Im}\,(Z\to X)\times_X F_{Y/X}\times_X F_{T/X}\to Z\times_X F_{Y/X}\times_X F_{T/X}.$$

This proves the existence of the natural morphism $e: F_{Y/X}\times_X F_{T/X}\to F_{Z/X}$, which is étale because the natural maps from both its target and source to X are étale. We have thus obtained a diagram

$$\begin{tikzcd} & F_{Y/X}\times_X F_{T/X} \arrow[d, "e"] \\ F_{Z/Y}\bigcup_Z F_{Z/T} \arrow[ur, "j"] \arrow[r] & F_{Z/X}, \end{tikzcd}$$

which is commutative due to the natural choices of the morphisms and property (1).

By (0) and (2), it remains to show that the complement of $F_{Z/Y}\bigcup_Z F_{Z/T}$ in $F_{Y/X}\times_X F_{T/X}$ is naturally isomorphic to the complement of $\mathrm{Im}\,(F_{Z/Y}\bigcup_Z F_{Z/T}\to F_{Z/X})$ in $F_{Z/X}$, and that there is a natural isomorphism

$$\mathrm{Im}\,(F_{Z/Y}\bigcup_Z F_{Z/T}\to F_{Z/X})\times_{F_{Z/X}}(F_{Y/X}\times_X F_{T/X})\cong F_{Z/Y}\bigcup_Z F_{Z/T}.$$

These properties follow canonically from the definitions of the objects and morphisms involved.

□

Lemma 1.3. *Let $g: Y\to X$ be a proper local embedding of Noetherian stacks, with Y integral. Then there exists a stack $D_{Y/X}$ together with an étale epimorphism $e: Y\to D_{Y/X}$ and a proper local embedding $g_1: D_{Y/X}\to X$ of generic degree 1, such that $g=g_1\circ e$. Moreover, $D_{Y/X}$ is unique up to an isomorphism.*

Proof. A factorization of the morphism $g: Y\to X$ into an étale epimorphism $e: Y\to D_{Y/X}$ and a proper local embedding $g_1: D_{Y/X}\to X$ of generic degree one was constructed in [MM], Lemma 1.10. It remains to prove uniqueness up to an isomorphism. For this, we first recall the étale local structure of $D_{Y/X}$: There exists an étale cover of X by a scheme U such that $Y\times_X U=\bigsqcup_{i,a}V_i^a$, and for each i,a, the morphism $g_U: Y\times_X U\to U$ restricts to a closed embedding $V_i^a\hookrightarrow U$, with image W_i, such that $W_i\neq W_j$ if $i\neq j$. Let $W=\bigcup_i W_i$. There exists a canonical groupoid structure $[s_e,t_e: R_e:=\bigsqcup_i W_i\times_X U\rightrightarrows\bigsqcup_i W_i]$, and $D_{Y/X}$ is defined as its associated stack. The morphisms $e: Y\to D_{Y/X}$ and $g_1: D_{Y/X}\to X$, respectively, are determined by the canonical choice of maps $e_U:\bigsqcup_{i,a}V_i^a\to\bigcup_i W_i$ and $g_{1U}:\bigcup_i W_i\to U$, together with

$e_R : \bigsqcup_{i,a,j,b} V_i^a \times_Y V_j^b \to \bigsqcup_i W_i \times_X U$ and $g_{1R} : \bigsqcup_i W_i \times_X U \to U \times_X U$ at the level of relations.

Assume that $e' : Y \to Y'$ is another étale epimorphism, and that $f' : Y' \to X$ is a proper local embedding of generic degree one such that $f' \circ e' = g$. Let $\bigcup_i V_i' := Y' \times_X U$, with the induced morphism $f_U' : \bigcup_i V_i' \to U$, such that each V_i' is the preimage of W_i. As the induced morphism $e_U' : \bigsqcup_{i,a} V_i^a \to \bigcup_i V_i'$ is étale and surjective and sends each V_i^a to V_i', the components V_i' must be pairwise disjoint. We will construct an isomorphism of groupoids

$$\phi : \left[s_e, t_e : R_e := \bigsqcup_i W_i \times_X U \rightrightarrows \bigsqcup_i W_i \right] \to \left[s', t' : R' := \bigsqcup_i V_i' \times_{Y'} \bigsqcup_i V_i' \rightrightarrows \bigsqcup_i V_i' \right].$$

First consider any section $\sigma : \bigsqcup_i W_i \to \bigsqcup_{i,a} V_i^a$ of e_U and define $\phi_U := e_U' \circ \sigma$. As section of the étale morphism e_U, the map σ must be étale itself. In fact, it consists of a choice of an index a for each i, and an isomorphism $W_i \to V_i^a$. The map e_U' is étale and surjective, and it maps each V_I^a onto V_i'. Indeed, $\deg g_{1U} \circ e_U = \deg f_U' \circ e_U'$ while $\deg g_{1U}$, f_U' are both of generic degree one, so the image of each V_I^a under e_U' must be a dense open subset of V_i'. On the other hand, e_U' is also proper, so $e_U'(V_I^a) = V_i'$.

It follows that $\phi_U = e_U' \circ \sigma$ is étale and surjective as well. Moreover, $f_U' \circ \phi_U = g_{1U}$, so the degree of ϕ_U must be one. Thus ϕ_U is an isomorphism.

Let $R := U \times_X U$, and consider the first projection $s : R \to U$. As $R' \cong \bigsqcup_i V_i' \times_U R$, we can construct $\phi_R : R_e \to R'$ as the morphism uniquely defined by the conditions

$$f_R' \circ \phi_R = g_{1R} \text{ and } s' \circ \phi_R = \phi_U \circ s_e.$$

Similarly, a morphism $\psi_R : R' \to R_e$ can be defined by the conditions

$$g_{1R} \circ \psi_R = f_R' \text{ and } s_e \circ \psi_R = \phi_U^{-1} \circ s'.$$

We note that $\psi_R \circ \phi_R = \mathrm{id}_{R_e}$, as $g_{1R} \circ \psi_R \circ \phi_R = g_{1R}$ and $s_e \circ \psi_R \circ \phi_R = s_e$, and $R_e \cong \bigsqcup_i W_i \times_U R$. Similarly, $\phi_R \circ \psi_R = \mathrm{id}_{R'}$.

It remains to prove that the pair (ϕ_U, ϕ_R) is a morphism of groupoids. This is a slightly long, but direct check. Here we will prove the equality:

$$\phi_U \circ t_e = t' \circ \phi_R. \tag{1.1}$$

Let $i_e : R_e \to R_e$, $i' : R' \to R'$ and $i : R \to R$ denote the inverting maps of the groupoids $[R_e \rightrightarrows \bigsqcup_i W_i]$, $[R' \rightrightarrows \bigsqcup_i V_i']$ and $[R \rightrightarrows U]$ respectively, so that $i_e \circ s_e = t_e$, $i' \circ s' = t'$ and $i \circ s = t$. Composition with f_U' of the two terms in the equation (1.1) yields:

$$\begin{gathered} f_U' \circ \phi_U \circ t_e = g_{1U} \circ t_e, \text{ and} \\ f_U' \circ t' \circ \phi_R = f_U' \circ i' \circ s' \circ \phi_R = f_U' \circ i' \circ \phi_U \circ s_e = \\ = i \circ f_U' \circ \phi_U \circ s_e = i \circ g_{1U} \circ s_e = g_{1U} \circ i_e \circ s_e = g_{1U} \circ t_e. \end{gathered}$$

As f_U' is generically injective, the closed subset $\{x \in R_e;\ \ \phi_U \circ t_e(x) = t' \circ \phi_R(x)\}$ contains an open dense subset of R_e, so it must be the entire R_e.

All other compatibility relations follow directly by the same method as above.

□

Remark 1.4. We note that if $g : Y \to X$ was not separable, the uniqueness of a possible split $Y \to D_{Y/X} \to X$ would not be guaranteed. For example, if $p \neq q$ are natural numbers and Y is obtained by gluing pq copies of the space X along the complement of a point, then two possible choices for $D_{Y/X}$ would be obtained by gluing p, respectively q copies of the space X along the complement of that same point.

Proposition 1.5. *Let $g : Y \to X$ be a proper local embedding of Noetherian stacks, with Y integral. The stack $D_{Y/X}$ constructed above satisfies the following properties:*

(1) *For any morphism of stacks $u : X' \to X$ and $Y' := Y \times_X X'$, there exists a morphism $D_u : D_{Y'/X'} \to D_{Y/X}$ making the squares in the following diagram Cartesian:*

$$\begin{tikzcd} Y' \arrow[r, "e'"] \arrow[d] & D_{Y'/X'} \arrow[r, "f'"] \arrow[d, "D_u"] & X' \arrow[d, "u"] \\ Y \arrow[r, "e"] & D_{Y/X} \arrow[r, "f"] & X. \end{tikzcd}$$

(2) *If $h : Z \to Y$ is another proper local embedding of integral Noetherian stacks, then there exists a natural isomorphism*

$$D_{D_{Z/Y}/D_{Y/X}} \cong D_{Z/X},$$

where $D_{D_{Z/Y}/D_{Y/X}}$ is the stack associated to the composition $e \circ h_1$ of the local embedding of generic degree one $h_1 : D_{Z/Y} \to Y$ and the étale map $e : Y \to D_{Y/X}$.

(3) *There is an equivalence of categories between the category of commutative diagrams*

$$\begin{tikzcd} Y \arrow[r, "g"] & X \\ T \arrow[u, "v"] \arrow[r, "f"] & Z, \arrow[u, "u"'] \end{tikzcd}$$

with f étale, and that of pairs in $\mathrm{Hom}(Z, D_{Y/X}) \times \mathrm{Hom}(T, Z \times_{D_{Y/X}} Y)$ such that the induced morphism $T \to Z$ is étale.

The morphisms in the first category are given by Cartesian diagrams

$$\begin{tikzcd} T' \arrow[r] \arrow[d, "t"] & Z' \arrow[d, "z"] \\ T \arrow[r] & Z \end{tikzcd}$$

such that t and z commute with the given morphisms to Y and X, respectively.

Proof. Properties (1) and (2) are direct consequences of the definition of $D_{Y/X}$ and Lemma 1.3. The proof of property (3) is based on arguments also employed in the proof of same Lemma. Indeed, given an étale cover of X by a scheme U such that $Y \times_X U = \bigsqcup_{i,a} V_i^a$, and for each i, a, the morphism $g_U : Y \times_X U \to U$ restricts to a closed embedding $V_i^a \hookrightarrow U$, with image W_i, such that $W_i \neq W_j$ if $i \neq j$. Then $D_{Y/X} \times_X U \cong \bigsqcup_i W_i$. Also, $T \times_Y (\bigsqcup_{i,a} V_i^a)$, and as the morphism $f : T \to Z$ is étale, then $Z \times_X U \cong \bigsqcup V_i'$ for some V_i'-s such that for each i, the pullback of f restricts to maps $\bigsqcup_a V_i^a \to \bigsqcup V_i'$, and the pullback of u restricts to $V_i' \to W_i$. In particular, this induces a map $\bigsqcup V_i' \to \bigsqcup_i W_i$. A morphism of groupoids

$$[(\bigsqcup V_i') \times_Z (\bigsqcup V_i') \rightrightarrows \bigsqcup V_i'] \to [(\bigsqcup_i W_i) \times_{D_{Y/X}} (\bigsqcup_i W_i) \rightrightarrows \bigsqcup_i W_i]$$

can then be constructed by the exact same method as in the proof of the previous Lemma. □

1.2. In the next paragraphs we will work with simple categories whose objects are Noetherian stacks, and such that there exists at most one morphism between each pair of objects. We will discuss some additional properties below.

Definition 1.6. By extending the terminology of [L], we can define a poset of stacks as follows. We regard any poset $\mathcal{P}$ as a category, such that for any elements $I, J \in \mathcal{P}$, the set of morphisms $\mathrm{Morph}(I, J)$ consists of a unique element if $I \leq J$, and is empty otherwise. Then a poset of stacks is a contravariant functor from $\mathcal{P}$ to the category of sets. Any such poset of stacks $\mathcal{N} = \{\phi_J^I : Y_J \to Y_I\}_{I \subseteq J \in \mathcal{P}}$ where $\mathcal{P}$ is the power set of a finite set Λ, and the partial order is given by inclusion, will be called simply a network. In particular, a network will include a unique target $Y_\emptyset$, (and a source Y_Λ, possibly empty).

Definition 1.7. Let $\mathcal{N} = \{\phi_J^I : Y_J \to Y_I\}_{I \subseteq J \in \mathcal{P}}$ be a network of morphisms with target $X = Y_\emptyset$, and let $\mathcal{N}' = \{\phi_J'^I : Y_J' \to Y_I'\}_{I,J \in \mathcal{P}'}$ be another network with target $X' = Y_\emptyset$, where $\mathcal{P}' \subseteq \mathcal{P}$. A morphism of networks $F : \mathcal{N}' \to \mathcal{N}$ is a fully faithful functor from the category $\mathcal{N}'$ to the category $\mathcal{N}$, given by a set of morphisms $\{f_I : Y_I' \to Y_I\}_{I \in \mathcal{P}}$, such that $f_I \circ \phi_J'^I = \phi_J^I \circ f_J$. In particular, F includes a morphism between targets $f : X' \to X$.

We say that $\mathcal{N}' \cong \mathcal{N} \times_X X'$ if each of the commutative diagrams corresponding to the equalities $f_I \circ \phi_J'^I = \phi_J^I \circ f_J$ is Cartesian.

Given a network of closed embeddings, there is a natural way to glue any subset of objects $\{Y_I\}_{I \in \mathcal{Q}}$ into a stack $S_\mathcal{Q}$ as follows:

Lemma 1.8. *Let $\mathcal{N} = \{\phi_J^I : Y_J \to Y_I\}_{I \subseteq J \in \mathcal{P}}$ be a network of closed embeddings, with target X. Consider $\mathcal{Q} \subseteq \mathcal{P}$.*

a) There exists a stack $S_\mathcal{Q}$, and commutative diagrams

$$\begin{tikzcd} Y_{I\cup J} \arrow[r, "\phi^I_{I\cup J}"] \arrow[d, "\phi^J_{I\cup J}"'] & Y_J \arrow[d] \\ Y_I \arrow[r] & S_\mathcal{Q} \end{tikzcd}$$

for all $I, J \in \mathcal{Q}$, such that for any stack T, the natural functor

$$\mathrm{Hom}(S_\mathcal{Q}, T) \to \times_{\{\mathrm{Hom}(Y_{I\cup J}, T)\}_{I,J \in \mathcal{Q}}} \{\mathrm{Hom}(Y_J, T)\}_{J \in \mathcal{Q}}$$

is an equivalence of categories.

In particular, there exists a natural morphism $S_\mathcal{Q} \to Y_{(\bigcap_{I \in \mathcal{Q}} I)}$ compatible with the morphisms $\phi_I^{(\bigcap_{I \in \mathcal{Q}} I)}$, for all $I \in \mathcal{Q}$.

Such a stack is unique up to unique isomorphism.

b) For any morphism $X' \to X$, consider the network $\mathcal{N}' := \mathcal{N} \times_X X'$, with objects $Y'_I = Y_I \times_X X'$. Consider the stack $S'_{\mathcal{Q}}$, obtained by gluing the objects $\{Y'_I\}_{I\in\mathcal{Q}}$ in the network $\mathcal{N}'$. Then for all $I \in \mathcal{Q}$, the squares in the following diagram are Cartesian:

$$\begin{tikzcd} Y'_I \arrow[r] \arrow[d] & S'_{\mathcal{Q}} \arrow[r] \arrow[d] & Y'_{(\bigcap_{I\in\mathcal{Q}} I)} \arrow[d] \\ Y_I \arrow[r] & S_{\mathcal{Q}} \arrow[r] & Y_{(\bigcap_{I\in\mathcal{Q}} I)}. \end{tikzcd}$$

Proof. a) We will proceed by induction on the cardinality of $\mathcal{Q}$. If $|\mathcal{Q}| = 1$, then $S_{\mathcal{Q}} = Y_I$ for $I \in \mathcal{Q}$. Assume now that $S_{\mathcal{Q}}$ exists for any $\mathcal{Q}$ of a given cardinality. Fix such $\mathcal{Q}$ and let $J \notin \mathcal{Q}$. Note that if $J \supseteq I$ for some $I \in \mathcal{Q}$, then $S_{\mathcal{Q}\cup\{J\}} = S_{\mathcal{Q}}$. If this is not the case, let $\mathcal{Q}^J := \{I \bigcup J;\ \ I \in \mathcal{Q}\}$. Then by induction, $S_{\mathcal{Q}^J}$ exists and, moreover, there is a unique closed embedding $S_{\mathcal{Q}^J} \to S_{\mathcal{Q}}$ determined by the compositions $Y_{J\cup K} \to Y_K \to S_{\mathcal{Q}}$ for all $K \in \mathcal{Q}$. Gluing Y_J and $S_{\mathcal{Q}}$ along $S_{\mathcal{Q}^J}$ yields a stack satisfying the required properties (conform [AGV], Proposition A.1.1).

Part b) follows by standard category theoretical arguments. Indeed, since $Y'_I \cong Y_I \times_{Y_{(\bigcap_{I\in\mathcal{Q}} I)}} Y'_{(\bigcap_{I\in\mathcal{Q}} I)}$, it is enough to show that the right side of the diagram is Cartesian. Given two morphisms $T \to S_{\mathcal{Q}}$ and $T \to Y'_{(\bigcap_{I\in\mathcal{Q}} I)}$ commuting to the respective morphisms to $Y_{(\bigcap_{I\in\mathcal{Q}} I)}$, we can think of T as being obtained by gluing the objects $\{T \times_{S_{\mathcal{Q}}} Y_I\}_{I\in\mathcal{Q}}$ within the network whose objects are $\{T \times_{S_{\mathcal{Q}}} Y_J\}_{J\supseteq I \text{ for some } I\in\mathcal{Q}}$, and the target T. Each such object $T \times_{S_{\mathcal{Q}}} Y_J$ admits two natural morphisms, to Y_I and $Y'_{(\bigcap_{I\in\mathcal{Q}} I)}$, respectively, commuting to the respective morphisms to $Y_{(\bigcap_{I\in\mathcal{Q}} I)}$, and thus admits natural morphisms $T \times_{S_{\mathcal{Q}}} Y_J \to Y'_I \to S'_{\mathcal{Q}}$, for $J \supseteq I \in \mathcal{Q}$. By part a), there exists a unique natural morphism $T \to S'_Q$ compatible with $T \to S_{\mathcal{Q}}$ and $T \to Y'_{(\bigcap_{I\in\mathcal{Q}} I)}$.

□

Consider a proper local embedding of Noetherian stacks $g : Y \to X$, with Y integral. Starting from the flat stratification of g, in [MM], we constructed a network of local embeddings associated to g, and an étale lift $F_{Y/X} \to X$ which reflected the local étale structure of the morphism g. However, this lift was not canonical, as it depended of the choice of étale cover by a nice scheme U of X. We recall here the properties of U which were essential for the construction of $F_{Y/X}$.

Let $Y_n \hookrightarrow Y_{n-1} \hookrightarrow ... \hookrightarrow Y_1 \hookrightarrow Y_0 = Y$ be a filtration of Y consisting of the closures $\overline{g^{-1}(S_i)} \subseteq Y$, where $\{S_i\}_i$ is the flattening stratification for the morphism $Y \to g(Y)$.

Definition 1.9. Let $g : Y \to X$ be a proper local embedding of Noetherian stacks. An étale cover U of X is called suitable for the morphism g if the following properties hold:

(1) $g(Y) \times_X U = \bigcup_{l\in L} W_l$, where W_l are isomorphic, for all $l \in L$.
(2) For all $k = 0, ..., n$, and for some suitable choices of subsets $\mathcal{P}_k \subset \mathcal{P} := \mathcal{P}(L)$, we have $\bigcup_{I\in\mathcal{P}_k} W_I = g(Y_k) \times_X U$, where $W_I = \bigcap_{l\in I} W_l$, and $W_I \cong W_{I'}$ for all $I, I' \in \mathcal{P}_k$.

(3) For each I as above, there exist sets $\{V_I^a\}_{a\in A_I}$ mapping onto Y_k, with isomorphisms $V_I^a \to W_I$ standing over $g_k : Y_k \to g(Y_k)$, and satisfying
$$Y_k \times_X U = \bigcup_{I\in\mathcal{P}_k, a\in A_I} V_I^a.$$
Here $A_I = \bigsqcup_{k\in I} A_k$ and $V_I^a \subseteq V_k^a$ if $k \in I$ and $a \in A_k$.

Definition 1.10. Let X and Y be Noetherian stacks, with Y integral. Let $g : Y \to X$ be a proper local embedding of generic degree one. Let $U \to X$ be a suitable étale cover for g. We associate to g and U a network of local embeddings $\phi_J^I : Y_J \to Y_I$, one for each pair $I \subseteq J$, $I \in \mathcal{P}_i$ and $J \in \mathcal{P}_j$, as follows. For each $I \subseteq L$, for each distinct $i, j \in L$ and the uniquely associated indices $a \in A_i$, $b \in A_j$, we define
$$R_\emptyset := U \times_X U, \quad R_i := V_i^a \times_Y V_i^a \quad \text{ and } R_I := R_I = (\textstyle\prod_{i\in I})_{R_\emptyset} R_i,$$
and Y_I as the stack with groupoid presentation $[R_I \rightrightarrows V_I^a]$. We consider by convention $V_\emptyset^a = U$, such that $Y_\emptyset = X$. We note that
$$R_I \cong W_I \times_X W_I \setminus \bigcup_{j\neq i\in I} S_{ij}^{ab}, \text{ where } S_{ij}^{ab} := \text{ Im } (V_i^a \times_Y V_j^b \to W_i \times_X W_j).$$
Whenever $J \supseteq I$, the natural morphism between the groupoid presentations $[R_J \rightrightarrows V_J^a]$ and $[R_I \rightrightarrows V_I^a]$ induces the morphism of stacks $\phi_J^I : Y_J \to Y_I$. In particular, $\phi_I^I = \mathrm{id}_{Y_I}$. The space $Y_\emptyset = X$ will be called the target of the network.

Definition 1.11. Consider a proper local embedding of Noetherian stacks $g : Y \to X$, with Y integral. If g factors through an étale epimorphism $e : Y \to D_{Y/X}$ and a proper local embedding $g_1 : D_{Y/X} \to X$ of generic degree 1, then we define $Y_I := D_{Y/X,I} \times_{D_{Y/X}} Y$, for the network consisting of $\{D_{Y/X,I}, \varphi_J^I\}_{I\subseteq J\neq\emptyset}$ constructed as in the preceding definition, a target $Y_\emptyset = X$ and the morphisms $g_i : Y_i \to X$. The morphisms $\phi_J^I : Y_J \to Y_I$ are also obtained by pull-back from the network of $D_{Y/X}$.

Remark 1.12. Even though each space Y_I in the network of g and U is intrinsic to the morphism g ([MM], Corollary 2.8), the network itself depends on the choice of the suitable cover U, inasmuch as the number of copies of the same space Y_I can vary from network to network. For example, if we replace U by a disjoint union of m copies of U, where m is a positive integer, then the network of g, U is replaced by m of its copies, with the exception of the final target X which is unique. In the next proposition we will show that there is, however, a canonical choice of a minimal network for the morphism g, which will make the subsequent construction of an étale lift of g canonical, too.

Notation. Consider now a proper local embedding of Noetherian stacks $g : Y \to X$ of generic degree one, with Y integral. For every natural number n, we denote by $\prod_X^n Y$ the fibered product over X of n copies of Y. We denote by Δ_n the union of the images of all diagonal morphisms $\prod_X^m Y \to \prod_X^n Y$ for $m \le n$, and by Y^n the complement of Δ_n in $\prod_X^n Y$.

Lemma 1.13. *Y^n is a closed substack of $\prod_X^n Y$.*

Proof. We only need to check that the image of the diagonal morphism $Y \to Y \times_X Y$ is both open and closed in $Y \times_X Y$. Then, by induction on n we obtain that Δ_n is a union of connected components of $\prod_X^n Y$ for any $n > 1$. Indeed, since $g : Y \to X$, then so is $Y \to Y \times_X Y$. On the other hand, to prove that the image of this morphism is open, we choose any cover U of X suitable for the morphism g. Let $V = \bigsqcup_i V_i := Y \times_X U$, such that each V_i is imbedded as a closed subscheme of U. For any indices i, j as above, $V_i \times_X V_j$ is an étale cover of $Y \times_X Y$, and $(V_i \times_X V_j) \times_{Y \times_X Y} Y \cong V_i \times_Y V_j$. On the other hand,

$$\bigsqcup_j (V_i \times_Y V_j) = V_i \times_Y V \cong V_i \times_Y (Y \times_X U) \cong V_i \times_X U \cong \bigcup_j (V_i \times_X V_j),$$

and so $V_i \times_X V_j \cong (V_i \times_Y V_j) \bigsqcup (\bigsqcup_k (V_i \times_Y V_k) \bigcap (V_i \times_X V_j))$. □

Definition 1.14. Let n_g be the largest integer such that Y^{n_g} is non-empty. We denote by $\mathcal{N}(Y/X)$ the network made out of stacks $Y_J := Y^{|J|}$, for any $J \subseteq \{1, ..., n_g\}$, and of morphisms $\phi^I_J : Y_J \to Y_I$, defined by restrictions of the natural projections, for $I \subseteq J$. Here $Y_\emptyset = X$ and $\phi^\emptyset_i = g$ for any $i \in \{1, ..., n_g\}$. For a general proper local embedding g, let $\mathcal{N}(Y/X) := \mathcal{N}(D_{Y/X}/X) \times_{D_{Y/X}} Y$.

$\mathcal{N}(Y/X)$ will be called the canonical network of the the finite local embedding $g : Y \to X$.

Proposition 1.15. *a) There exists an étale cover U of X suitable for g such that $\mathcal{N}(Y/X)$ is the network of local embeddings associated to g, U.*

b) If $X' \to X$ is a morphism and $Y' \cong Y \times_X X'$, then $\mathcal{N}(Y'/X') \cong \mathcal{N}(Y/X) \times_X X'$.

Proof. Consider any étale covering U' of X suitable for g. At least one such cover exists, by Proposition 1.9 in [MM]. Let $\phi'^{I'}_{J'} : Y_{J'} \to Y_{I'}$ denote the morphisms in the associated network. By examining the respective groupoid presentations it can be proven ([MM], Corollary 2.8.) that the spaces $Y_{J'}$ are isomorphic to Y^n and, moreover, by the same proof, the morphisms are restrictions of projections as above. It remains to check that, after possibly "pruning" U' , the associated network has the required set of nodes and morphisms. Indeed, assume that $g(Y) \times_X U' = \bigcup_{l \in \{1,...,m\}} W_l$, with W_l as in Definition 1.9. If $m > n_g$, let $U := U' \setminus (\bigcup_{l=n_g+1}^m W_l)$. The induced map $U \to X$ is étale and also surjective, due to the maximality of n_g and to property (2) in Definition 1.9. As $W_I \cong W_{I'}$ whenever $|I| = |I'|$, then the network associated to U has exactly the right number of nodes and morphisms as $\mathcal{N}(Y/X)$. The second statement is due to the definition of $\mathcal{N}(Y/X)$ and Proposition 1.3. □

Lemma 1.16. *Let $g : Y \to X$ be a finite local embedding of Noetherian stacks, and let $\mathcal{N}(Y/X) = \{\phi^I_J : Y_J \to Y_I\}_{I \subseteq J \subseteq \{1,...,n_g\}}$ be its canonical network. Then for any integer k with $0 \le k < n_g$, the projection morphism and for any $K \subseteq L \subseteq \{1, ..., n_g\}$ with $|K| = k$ and $|L| = k + 1$, the morphism $\phi^K_L : Y_L \to Y_K$ is a finite local embedding with associated canonical network*

$$\mathcal{N}(Y_L/Y_K) = \{\phi^I_J : Y_J \to Y_I\}_{K \subseteq I \subseteq J \subseteq \{1,...,n_g\}}.$$

Here by convention $Y_0 = X$.

Proof. The lemma is due to the existence of canonical isomorphisms $\prod^l_{\prod^k_X Y} \prod^{k+1}_X Y \cong \prod^{l+k}_X Y$, which commute with the projections $\prod^l_{\prod^k_X Y} \prod^{k+1}_X Y \to \prod^{l-1} l_{\prod^k_X Y} \prod^{k+1}_X Y$ and $\prod^{l+k}_X Y \to \prod^{l+k-1}_X Y$, respectively, and with the respective diagonal morphisms.

□

Definition 1.17. Consider a network $\mathcal{N}$ of proper local embeddings $\phi_J^I : Y_J \to Y_I$ for $I \subseteq J \in \mathcal{P}$ associated to a proper local embedding $g : Y \to X$, where by convention $Y_\emptyset = X$. We will briefly describe here an étale lift $\mathcal{N}^0$ of $\mathcal{N}$ which is a configuration stack, namely a network of closed embeddings $\mathcal{N}^0 = \{\phi_J^{I0} : Y_J^0 \hookrightarrow Y_I^0\}$, and a morphism $p^0 : \mathcal{N}^0 \to \mathcal{N}$ which is étale, in the sense that each of the constituent morphisms is étale. A detailed proof of the existence of this network based on étale coverings can be found in [MM], (Theorem 1.5). $\mathcal{N}^0$ is in fact the last of a sequence of networks $\{\mathcal{N}^i\}_{n_g \geq i \geq 0}$ constructed inductively, where $\mathcal{N}^{n_g} := \mathcal{N}$, and for each index i,

(1) the morphisms $\phi_J^{Ii} : Y_J^i \hookrightarrow Y_I^i$ are closed embeddings for all I, J such that $J \supseteq I \in \mathcal{P}_k$ with $k \leq i$;
(2) there is an étale morphism $\mathcal{N}^{i-1} \to \mathcal{N}^i$.

The sequence is constructed as follows. Assume that $\mathcal{N}^i$ with the property (2) above has been defined. For each $I \in \mathcal{P}$, we denote by S_I^i the stack obtained by gluing all stacks Y_J^i satisfying $J \supset I$ like in Lemma 1.8. Each S_I^i comes with a map $S_I^i \to Y_I^i$ which is in fact proper and étale on its image, and the set of all these maps defines a map of networks $\mathcal{S}^i \to \mathcal{N}^i$. With the notations from Proposition 1.2, we define $\mathcal{N}^{i-1} := F_{\mathcal{S}^i/\mathcal{N}^i}$ in the obvious sense: each $Y_I^{i-1} = F_{S_I^i/Y_I^i}$, the étale lift of $S_I^i \to Y_I^i$. Thus there exists a natural étale map $\mathcal{N}^{i-1} \to \mathcal{N}^i$. We note that for $I \in \mathcal{P}_k$ with $k \geq i$, the morphism $S_I^i \to Y_I^i$ is already a closed embedding, so $Y_I^{i-1} = Y_I^i$, while for $I \in \mathcal{P}_{i-1}$ and $J \supseteq I$, we have $Y_J^{i-1} = Y_J^i \hookrightarrow S_I^i \hookrightarrow Y_I^{i-1}$, a closed embedding.

Definition 1.18. Consider a proper local embedding $g : Y \to X$. Let $\{Y^a\}_a$ denote the set of irreducible components of Y, and for each a let $\mathcal{N}^i(Y^a/X) = \{\phi_J^{a,I,i} : Y_J^{a,i} \hookrightarrow Y_I^{a,i}\}_{J \supseteq I; I,J \in \mathcal{P}(\Lambda^a)}$ denote the networks associated to the restriction of g on Y^a, as in the previous definition. Here $0 \leq i \leq |\Lambda^a|$, and $|\Lambda^a|$ is the largest number such that $(Y^a)^n$ is nonempty for all $n \leq |\Lambda^a|$. We define the canonical network of the local embedding g by

$$\mathcal{N}(Y/X) = \mathcal{N}^{\sum_a |\Lambda^a|}(Y/X) := \times_X \{\mathcal{N}(Y^a/X)\}_a = \times_X \{\mathcal{N}^{|\Lambda^a|}(Y^a/X)\}_a.$$

The objects of this network are fibered products over X of factors $Y^a_{I^a}$ for all a, where $I^a \subseteq \Lambda^a$. All the networks $\mathcal{N}^i(Y/X)$ for $0 \leq i \leq |\Lambda^a|$ are constructed inductively by the process outlined in the previous definition.

Proposition 1.19. *Consider a proper local embedding $g : Y \to X$, and let $\{Y^a\}_a$ denote the set of irreducible components of Y. With the notations from the previous definitions,*

$$\mathcal{N}^0(Y/X) \cong \times_X \{\mathcal{N}^0(Y^a/X)\}_a,$$

the fiber product over X of all the networks $\mathcal{N}^0(Y^a/X)$.

Proof. The proof relies on induction after the number of irreducible components, as well as decreasing induction after the step i in the construction of the networks $\mathcal{N}^i(Y/X)$, and largely on property (10) in Proposition 1.2. The induction after the number of irreducible components reduces to proving the proposition for $Y = Z \bigcup T$. Denote

$$\mathcal{N}(Y/X) = \mathcal{N}^{m+n}(Y/X) := \mathcal{N}(Z/X) \times_X \mathcal{N}(T/X) = \mathcal{N}^m(Z/X) \times_X \mathcal{N}^n(T/X),$$

and $\mathcal{N}^i(Z/X) = \{\phi_J^{I,i} : Z_J^i \hookrightarrow Z_I^i\}_{J \supseteq I; I,J \in \mathcal{P}(\Lambda)}$, with $|\Lambda| = m$, while $\mathcal{N}^j(Z/X) = \{\phi_B^{A,j} : T_B^j \hookrightarrow T_A^j\}_{B \supseteq A; A,B \in \mathcal{P}(\Gamma)}$ with $|\Gamma| = n$. Our induction hypothesis will be that for a fixed k integer, $0 \leq k \leq m+n$, the objects of the network $\mathcal{N}^k(Y/X)$ are of the form

(1) $Z_I^{|I|} \times_X T_A^{|A|}$, if $|I| + |A| \geq k$, and
(2) $F_{S^{k+1}_{I\cup A}/(Z_I \times_X T_A)}$ otherwise.

Here the notations are consistent with Definition 1.17, and the index $(k+1)$ refers to the naturally corresponding objects in the $k+1$-th network of $g : Y \to X$. Thus (2) is a direct consequence of the definition of the objects in $\mathcal{N}^k(Y/X)$, together with property (8) in Proposition 1.2 applied successively to the compositions $S^{l+1}_{I\cup A} \hookrightarrow S^l_{I\cup A} \to F_{S^{l+1}_{I\cup A}/Z_I \times_X T_A}$ for $l \geq k+1 > |I| + |A|$. Condition (1) is clearly satisfied when $k = m+n$. If satisfied for a fixed k, then for any $I \in \mathcal{P}(\Lambda)$ and $A \in \mathcal{P}(\Gamma)$ there is a Cartesian diagram

$$\begin{tikzcd} S_I^{|I|+1} \times_X S_A^{|A|+1} \arrow[r] \arrow[d] & S_I^{|I|+1} \times_X T_A \arrow[d] \\ Y_I \times_X S_A^{|A|+1} \arrow[r] & Y_I \times_X T_A. \end{tikzcd}$$

whenever $|I| + |A| = k-1$. Gluing in the network $\mathcal{N}^k(Y/X)$ yields

$$S^k_{I\bigcup A} = (S_I^{|I|+1} \times_X T_A^{|A|}) \bigcup_{S_I^{|I|+1}\times_X S_A^{|A|+1}} (Y_I^{|I|} \times_X S_A^{|A|+1}) =$$
$$= (F_{S_I^{|I|+1}\times_X S_A^{|A|+1}/S_I^{|I|+1}\times_X T_A}) \bigcup_{S_I^{|I|+1}\times_X S_A^{|A|+1}} (F_{S_I^{|I|+1}\times_X S_A^{|A|+1}/Y_I\times_X S_A^{|A|+1}}),$$

(in accord with property (8), Proposition 1.2). Thus by (2) above and properties (10), (8) in Proposition 1.2, when $|I| + |A| = k-1$,

$$(Y_I \times_X T_A)^{k-1} \cong F_{S^k_{I\bigcup A}/Y_I\times_X T_A} \cong$$
$$\cong (F_{S_I^{|I|+1}\times_X T_A/Y_I\times_X T_A}) \times_{Y_I\times_X T_A} (F_{Y_I\times_X S_A^{|A|+1}/Y_I\times_X T_A}) \cong$$
$$\cong F_{S_I^{|I|+1}/Y_I} \times_X F_{S_A^{|A|+1}/T_A} \cong Y_I^{|I|} \times_X T_A^{|A|}.$$

(Here $F_{S_I^{|I|+1}/Y_I} \cong Y_I^{|I|}$ due to property (8) in Proposition 1.2, applied successively to the compositions $S_I^{l+1} \hookrightarrow S_I^l \to Y_I^l$ for $l > |I|$.) This ends the proof of the induction step.

□

Definition 1.20. Let $g : Y \to X$ be a proper local embedding. For each positive integer k, let Y_X^k denote the complement of all the diagonals in the k-th fibered product of Y over X. Let n be the largest integer such that Y_X^n is non-empty. We define the functor $F_{Y/X} : \mathbf{Sch}_{/X} \to \mathbf{Sets}$ as follows: For any scheme T and any morphism $T \to X$ given by an object $\alpha \in X(T)$, we consider the set of all tuples $((T_i, \beta_i, f_i)_i)_{i\in\{1,\dots,n\}}$, where

(1) T_i are closed subschemes of T such that for $I \subseteq \{1, ..., n\}$, the intersections $T_I = \bigcap_{i\in I} T_i$ (where by convention $T_\emptyset = T$) satisfy
$$T_I \times_X g(Y_X^k) = \bigcup_{J\supseteq I;|J|=k+|I|} T_J,$$
(2) $\beta_i \in Y(T_i)$ are objects whose pullbacks to any of the subsets T_I are pairwise distinct (non-isomorphic), and
(3) f_i is an isomorphism between $g(\beta_i)$ and $\alpha_{|T_i}$.

Theorem 1.21. *Let $g : Y \to X$ be a proper local embedding. The functor $F_{Y/X}$ is a stack. Moreover, there exists a unique morphism $F_{Y/X} \to X$, étale and universally closed, with the following properties:*

1. *$F_{Y/X} \times_X g(Y) \cong S_{Y/X}$, where $S_{Y/X} = S_{\{1,2,...,n\}}$ is the stack constructed by gluing the stacks $\{F_{Y_{ij}/Y_i}\}_{i \neq j; i,j \in \{1,...,n\}}$ within the network $\mathcal{N}^0(Y/X)$. Furthermore, $F_{Y/X} \setminus S_{Y/X} \cong X \setminus Y$, and the étale morphism $F_{Y/X} \to X$ is uniquely (up to a unique isomorphism) defined by these properties.*
2. *For each object Y_I in $\mathcal{N}(Y/X)$,*
$$Y_I \times_X F_{Y/X} \cong \bigsqcup_{|I_0|=|I|} F_{Y_{I_1}/Y_{I_0}},$$
where $I_1 \supset I_0$ is a fixed choice such that $|I_1| = |I_0| + 1$.
3. *If $g : Y \to X$ is a closed embedding, then $F_{Y/X} \cong X$.*
4. *If $g : Y \to X$ is étale and proper, and X is connected then $F_{Y/X} \cong Y$.*
5. *For any morphism of stacks $u : X' \to X$ and $Y' := Y \times_X X'$, there exists a morphism $F_u : F_{Y'/X'} \to F_{Y/X}$ making the squares in the following diagram Cartesian:*
$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & F_{Y'/X'} \arrow[r] \arrow[d, "F_u"] & X' \arrow[d, "u"] \\ Y \arrow[r] & F_{Y/X} \arrow[r] & X. \end{tikzcd}$$
6. *If $h : Z \to Y$ is proper and étale on its image, and $g : Y \to X$ is a closed embedding, then $F_{Z/Y} \cong Y \times_X F_{Z/X}$. In particular, there exists a natural étale morphism $g_* : F_{Z/Y} \to F_{Z/X}$.*
7. *If $g : Y \bigcup T \to X$ is a local embedding, then*
$$F_{Y/X} \times_X F_{T/X} \cong F_{Y \bigcup T/X}.$$

Proof. With the notations from Definitions 1.14 and 1.17, we will show that $F_{Y/X} = X^0$, the target of the network $\mathcal{N}^0(Y/X)$. More generally, if $I, J \subset \{1, ..., n\}$ are such that $J \supset I$ and $|J| = |I|+1$, then we will show by decreasing induction on I that $F_{Y_J/Y_I} = Y_I^{|I|}$, the target of the network $\mathcal{N}^0(Y_J/Y_I)$. We recall that $\mathcal{N}^0(Y_J/Y_I)$ is a subnetwork of $\mathcal{N}^0(Y/X)$, due to Lemma 1.16 and Definition 1.17.

As a first step, we notice that when $|I| = n-1$, the functor F_{Y_J/Y_I}, coincides with the one in Proposition 1.2, and it is thus a stack satisfying all the required properties. Consider now a general I and assume that $F_{Y_K/Y_J} = Y_J^{|J|}$ for all $K \supset J \supset I$ with $|K| = |J| + 1 = |I| + 2$. Consider now a scheme T_I with a morphism $T_I \to Y_I$ and a set of data $(T_{I \bigcup \{i\}}, \beta_{I \bigcup \{i\}}, f_{I \bigcup \{i\}})_{i \notin I}$ like in Definition 1.20. Thus

$$T_I \times_{Y_I} \phi_J^I(Y_J) \cong \bigcup_{i \notin I} T_{I \bigcup \{i\}}. \tag{1.2}$$

Moreover, for each J as above, the set of data consisting in

β_J together with the tuple$(T_{J \bigcup \{i\}}, \phi_{J \bigcup \{i\}}^{I \bigcup \{i\}*} \beta_{I \bigcup \{i\}}, \phi_{J \bigcup \{i\}}^{I \bigcup \{i\}*} f_{I \bigcup \{i\}})_{i \notin J}$

determine an element in $F_{Y_K/Y_J}(T)$ and thus by the induction hypothesis, a morphism $T_J \to Y_J^{|J|}$, making the following diagrams commutative:

$$\begin{array}{ccc} T_{J\bigcup\{i\}} & \longrightarrow & Y^{|J|}_{J\bigcup\{i\}} \\ \downarrow & & \downarrow \\ T_J & \longrightarrow & Y_J^{|J|}, \end{array}$$

for all $i \notin J$. Composition with the closed embeddings $Y_J^{|J|} \hookrightarrow S_I^{|J|}$, for $S_I^{|J|}$ like in Definition 1.17, yields maps $T_J \to S_I^{|J|}$ for all J as above, which glue to $\bigcup_{i\notin I} T_{I\bigcup\{i\}} \to S_I^{|J|}$. This, together with equation (1.2) and Proposition 1.2, insure the existence of a natural morphism $T_I \to Y_I^{|I|}$ compatible with the data $(T_{I\bigcup\{i\}}, \beta_{I\bigcup\{i\}}, f_{I\bigcup\{i\}})_{i\notin I}$. This ends the induction step.

From here, properties (1) and (2), and (6) follow from the construction of the network $\mathcal{N}(Y/X)$, together with Proposition 1.2. Properties (2) and (3) are direct consequences of (1). Property (4) also follows the construction of the network $\mathcal{N}(Y/X)$, together with properties listed in Proposition 1.15 b), Lemma 1.8, b) and Proposition 1.5, part (1). Property (7) is a consequence of Proposition 1.19.

□

Example 1.22. If $g : Y \to X$ is proper and étale on its image, then $F_{Y/X}$ coincides with the stack defined in Proposition 1.2.

Example 1.23. For a separated Deligne-Mumford stack X, the diagonal morphism $\Delta : X \to X \times X$ is a finite local embedding. Then $X \times_{X\times X} X$ is the inertia stack $I^1(X)$ of X, representing objects of X with their isomorphisms. Similarly, the higher inertia stack $I^n(X)$ is defined as the n-th order product of X over $X \times X$. With notations from 1.14, the objects of the canonical stack of $\Delta : X \to X \times X$ are the components $I_0^n(X)$ of the inertia stacks obtained after removing all the previous components which are images of $I^k(X)$ for $k < n$, as well as X itself, through diagonal morphisms.

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