ABSENCE OF MAGNETISM IN CONTINUOUS-SPIN SYSTEMS WITH LONG-RANGE ANTIALIGNING FORCES

MAREK BISKUP[1,2] AND NICHOLAS CRAWFORD[3]

[1]*Department of Mathematics, UCLA, Los Angeles, California, U.S.A.*
[2]*School of Economics, University of South Bohemia, České Budějovice, Czech Republic*
[3]*Department of Industrial Engineering, Technion, Haifa, Israel*

ABSTRACT. We consider continuous-spin models on the d-dimensional hypercubic lattice with the spins σ_x *a priori* uniformly distributed over the unit sphere in $\mathbb{R}^n$ (with $n \geq 2$) and the interaction energy having two parts: a short-range part, represented by a potential Φ, and a long-range antiferromagnetic part $\lambda|x-y|^{-s}\sigma_x \cdot \sigma_y$ for some exponent $s > d$ and $\lambda \geq 0$. We assume that Φ is twice continuously differentiable, finite range and invariant under rigid rotations of all spins. For $d \geq 1$, $s \in (d, d+2]$ and any $\lambda > 0$, we then show that the expectation of each σ_x vanishes in all translation-invariant Gibbs states. In particular, the spontaneous magnetization is zero and block-spin averages vanish in all Gibbs states. This contrasts the situation of $\lambda = 0$ where the ferromagnetic nearest-neighbor systems in $d \geq 3$ exhibit strong magnetic order at sufficiently low temperatures. Our theorem extends an earlier result of A. van Enter ruling out magnetized states with uniformly positive two-point correlation functions.

1. INTRODUCTION

In the last couple of years, there has been renewed interest by mathematicians in the behavior of lattice models with spins interacting via long-range (e.g., dipole-dipole) interactions. This has partially been motivated by advances in quasi two-dimensional physics, but much of it derives from the theoretical challenge that these systems seem to pose to existing methods of proof. Indeed, long-range interactions are generally quite hard to handle and most of the techniques that control nearest-neighbor systems are of little use when short-range and long-range forces are mixed together.

For definiteness of further discussion, let us consider the system of O(n)-spins σ_x, with $x \in \mathbb{Z}^d$ and σ_x being *a priori* uniformly distributed over the unit sphere in $\mathbb{R}^n$. The interaction between the spins is described by the Hamiltonian

$$\mathscr{H}(\sigma) := -J \sum_{\langle x,y \rangle} \sigma_x \cdot \sigma_y + \sum_{x,y} \frac{\lambda}{|x-y|^s} \sigma_x \cdot \sigma_y. \tag{1.1}$$

Here the first sum goes over pairs of nearest neighbors in $\mathbb{Z}^d$, the long-range coupling strength obeys $\lambda \geq 0$ and the interaction is summable by the assumption $s > d$. The equation (1.1) defines the model with *scalar* long-range interaction; to get the *dipole* model one needs to change the second summand into $\sum_{i,j=1}^d K_{xy}(i,j)\sigma_x^i\sigma_y^j$ where $-K_{xy}(i,j)$ is the second partial derivative of the Coulomb potential at $x-y$ with respect to x_i and x_j. An intriguing feature of the dipole model is that the sign of the interaction, and its strength, depend sensitively on the orientation of the spins with respect to the vector connecting their spatial positions.

A key question concerning the model (1.1) is the existence of *stripe states*, i.e., Gibbs measures supported on configurations with alternating stripes of spins oriented in different directions. For certain 1D and 2D systems, existence of such states has been established mathematically in the papers by Giuliani, Lebowitz and Lieb [10–12], albeit only at zero temperature. Currently there seem to be no rigorous results concerning the stripe order at positive temperatures. Notwithstanding, Giuliani [9] recently completed an argument building on earlier work of Fröhlich, Simon and Spencer [6] and Fröhlich and Spencer [7] that establishes the existence of an *orientational* long-range order in the dipole-dipole system — albeit without the nearest-neighbor term.

The aim of this paper is to resolve a simpler question: the existence/absence of *magnetic* order. Our principal result is that, for the model in (1.1) with $n\ge 2$ and exponents $s\in(d,d+2]$, as soon as $\lambda>0$, the expectation of σ_x vanishes in all translation-invariant states at all positive temperatures. A consequence of this is that the spontaneous magnetization — defined by the derivative of the pressure with respect to the external field — vanishes as well, and so do the block-spin averages in all (translation-invariant or not) Gibbs states. This is somewhat surprising because when $\lambda=0$ and $d\ge 3$ (and $J>0$) the system shows a strong magnetic order at low temperatures (Fröhlich, Simon and Spencer [6]). Our theorem provides novel information even in dimensions $d=1$ and 2 because the Mermin-Wagner theorem does not apply to the whole range of exponents s we wish to consider.

The problem of magnetic order in model (1.1) has quite a long history. To our knowledge, it first appears in studies by van Enter [3,4] on the "instability" of phase diagrams (and validity of the Gibbs-phase rule) under "irrelevant" perturbations. Specifically, in [4] it was shown that certain natural magnetically-ordered states in short-range ferromagnetic spin systems are destabilized — in the sense of failing to minimize the Gibbs variational problem — by adding the above long-range antiferromagnetic interaction with exponents $d<s<d+2$. The subtle point is that the assumption made in [4] on the purported magnetized state μ is that of *clustering*; explicitly,

$$E_\mu(\sigma_0\cdot\sigma_x)-E_\mu(\sigma_0)\cdot E_\mu(\sigma_x)\underset{|x|\to\infty}{\longrightarrow}0. \tag{1.2}$$

Along with the assumption of translation invariance and non-vanishing value of $E_\mu(\sigma_x)$, this permits one to assume a uniform positive lower bound on $E_\mu(\sigma_x\cdot\sigma_y)$ for any x and y that are sufficiently far apart.

Unfortunately, the uniform positivity of $E_\mu(\sigma_x\cdot\sigma_y)$ is exactly what one should not expect in these kinds of systems. Indeed, if a typical configuration in such a state has a stripe structure — with "positively" oriented stripes slightly wider than the "negatively" oriented ones, in order to achieve non-zero magnetization — then the sign of $E_\mu(\sigma_x\cdot\sigma_y)$

will vary depending on where x and y land relative to the stripe boundaries. Ruling out such cases along the argument of [4] would require making further assumptions on how $E_\mu(\sigma_x \cdot \sigma_y)$ changes as x and y vary. And this would still not exclude the possibility of other structures — e.g., the *bubble* states or aperiodically modulated states.

Our approach overcomes these difficulties by working solely under the assumption of *ergodicity* with respect to spatial translations. Conceptually, we build on an earlier paper by Biskup, Chayes and Kivelson [2] showing that no magnetic order exists (at any temperature) in the *Ising*-spin version of the model once $\lambda > 0$ and $s \in (d, d+1]$. In fact, the method would establish the same result also for $\mathrm{O}(n)$ spins for all $s \le d+1$. Notwithstanding, as is also shown in [2], the proof cannot extend beyond this range because the Ising-spin version of (1.1) exhibits magnetic order at low temperatures as soon as $s > d+1$ and $\lambda \ll J$. The argument of [2] is based on a flip of all spins in a large box and a careful accounting of the change in energy caused thereby. A key technical challenge here is to find a way to achieve the same effect via a continuous — i.e., Mermin-Wagner like — deformation.

The rest of this note is organized as follows: In the next section (Section 2) we develop the necessary foundation for the statement of our main result. In Section 3, we give the main steps of the proof while deferring the technical claims to Sections 4 and 5.

2. Statement of the result

Consider the d-dimensional hypercubic lattice $\mathbb{Z}^d$ and let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. We will consider spin configurations $\sigma := (\sigma_x)_{x\in\mathbb{Z}^d}$ taking values in the product space $\Omega := (\mathbb{S}^{n-1})^{\mathbb{Z}^d}$. Let τ_x be the shift by x on Ω, which is defined by

$$(\tau_x\sigma)_z := \sigma_{x+z}, \qquad z \in \mathbb{Z}^d. \tag{2.1}$$

Let $\mathrm{SO}(n)$ denote the group of real orthogonal $n \times n$-matrices with unit determinant. For each $\mathsf{R} \in \mathrm{SO}(n)$, let $(\mathsf{R}\sigma)_x := \mathsf{R}\sigma_x$ denote the global (rigid) rotation of the spin configuration σ by matrix R. The definition of our model will require two objects: A function $\Phi\colon \Omega \to \mathbb{R}$ representing the short-range interaction and a kernel $(K_{xy})_{xy\in\mathbb{Z}^d}$ representing the coupling constants for the long-range interaction.

Assumptions 2.1 *(1) Suppose that there is $r \in \mathbb{N}$ such that $\Phi\colon \Omega \to \mathbb{R}$ depends only on $\{\sigma_x\colon x \in \Lambda_r\}$. Moreover, assume $\sigma \mapsto \Phi(\sigma)$ is C^2 (as a function on a smooth manifold) and*

$$\Phi \circ \mathsf{R} = \Phi, \qquad \mathsf{R} \in \mathrm{SO}(n). \tag{2.2}$$

(2) For any x, y we have $K_{xy} \ge 0$ and $K_{xy} = K_{0,y-x}$. Moreover, there is an $s > d$ such that

$$\lim_{|x|\to\infty} |x|^s K_{0x} \in (0, \infty). \tag{2.3}$$

(In particular, the limit exists.)

Let r be as in Assumption 2.1(1) above and, for each $N \in \mathbb{N}$, consider the block

$$\Lambda_N := [-N, N]^d \cap \mathbb{Z}^d. \tag{2.4}$$

The Hamiltonian $\mathscr{H}_N$ in Λ_N is then defined by

$$\mathscr{H}_N(\sigma) := \sum_{x\in\Lambda_{N+r}} \Phi\circ\tau_x(\sigma) + \lambda \sum_{\substack{x,y\colon x\neq y\\ \{x,y\}\cap\Lambda_N\neq\emptyset}} K_{xy}\,\sigma_x\cdot\sigma_y. \tag{2.5}$$

The conditions (1) and (2) ensure that the interaction is well-defined, shift-invariant — and so it will make sense to talk about translation invariant and ergodic Gibbs measures — and also invariant under simultaneous rotations of all spins, i.e., $\mathscr{H}_N(\mathsf{R}\sigma) = \mathscr{H}_N(\sigma)$. The model (1.1) is clearly a special case of (2.5).

We will need to invoke the formalism of infinite-volume Gibbs measures for which we refer the reader to the standard treatments by Georgii [8] or Simon [14]. We will only mention the features that are relevant for our problem. Let ν denote the uniform probability (Haar) measure on $\mathbb{S}^{n-1}$. The above Hamiltonian defines a finite-volume Gibbs specification γ_N with boundary condition $\bar\sigma\in\Omega$ via

$$\gamma_N(\mathrm{d}\sigma|\bar\sigma) := \frac{1}{Z_N(\bar\sigma)}\,\mathrm{e}^{-\beta\mathscr{H}_N(\sigma)}\prod_{x\in\Lambda_N}\nu(\mathrm{d}\sigma_x)\prod_{z\notin\Lambda_N}\delta_{\bar\sigma_z}(\mathrm{d}\sigma_z). \tag{2.6}$$

Here $\beta\ge 0$ denotes the inverse temperature and $Z_N(\bar\sigma)$ is the partition function. We say that a probability measure μ over Ω is a *Gibbs measure*, if for all events A and all $N\ge 1$,

$$E_\mu(\gamma_N(A|\cdot)) = \mu(A). \tag{2.7}$$

Here E_μ denotes expectation with respect to μ. We say that a measure μ is *translation invariant* if $\mu\circ\tau_x = \mu$ for all $x\in\mathbb{Z}^d$. The measure is *ergodic* if $\mu(A) = 0$ or 1 for all events A such that $\tau_x(A) = A$ for all $x\in\mathbb{Z}^d$.

In order to define the notion of the spontaneous magnetization, pick a unit vector $\hat{\mathrm{e}}\in\mathbb{R}^n$ and consider the function $f\colon\mathbb{R}^n\to\mathbb{R}$ defined by

$$f(h) := \lim_{N\to\infty}\sup_{\bar\sigma}\frac{1}{N^d}\log\int\prod_{x\in\Lambda_N}\nu(\mathrm{d}\sigma_x)\exp\Bigl\{-\beta\mathscr{H}_N(\sigma)+\sum_{x\in\Lambda_N}h\cdot\sigma_x\Bigr\}, \tag{2.8}$$

where σ_x is (implicitly) fixed to $\bar\sigma_x$ for any $x\notin\Lambda_N$ inside $\mathscr{H}_N(\sigma)$. The limit exists by subadditivity arguments and is convex as a function of h. In addition, by the invariance of $\mathscr{H}_N$ and the measure ν with respect to rotations, f is independent of the choice of $\hat{\mathrm{e}}$. The convexity of f ensures the existence of the right derivative

$$m_\star := \frac{\mathrm{d}}{\mathrm{d}h^+}f(h)\Big|_{h=0}, \tag{2.9}$$

which by symmetry $\hat{\mathrm{e}}\leftrightarrow-\hat{\mathrm{e}}$ is positive. We will call $m_\star$ the *spontaneous magnetization*.

As is well known (see, e.g., Theorem 2.3(3) of [1]), for each unit vector $\hat{\mathrm{e}}$ there is a translation-invariant (and, in fact, ergodic) Gibbs measure μ such that $E_\mu(\sigma_x) = m_\star\hat{\mathrm{e}}$. Note that, in light of our remarks from Section 1, we are *not* assuming that μ is *extremal*, which would mean that $\mu(A) = 0$ or 1 for any event A that does not depend on the state of any finite number of σ_x's.

Our main result is now the following:

Theorem 2.2 *Suppose $d \geq 1$ and $s \in (d, d+2]$ and consider a model satisfying Assumptions 2.1. Then for any $\lambda > 0$ and any inverse temperature $\beta \geq 0$,*

$$E_\mu \sigma_x = 0, \qquad x \in \mathbb{Z}^d, \tag{2.10}$$

holds for every translation-invariant Gibbs measure μ. In particular, the spontaneous magnetization vanishes, $m_\star = 0$, and $h \mapsto f(h)$ is continuously differentiable at $h = 0$.

This statement is restricted to translation-invariant Gibbs measures. But a version of this conclusion is possible for all Gibbs measures:

Corollary 2.3 *Under the conditions of Theorem 2.2, if μ is any Gibbs measure, then*

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \sum_{x\in\Lambda_N} \sigma_x = 0, \qquad \mu\text{-a.s.}, \tag{2.11}$$

i.e., block-averages of the spins tend to zero in almost every sample from μ.

Note that these results do not preclude other types of long-range order (e.g., stripe states or an orientational order). A few additional remarks are in order:

Remarks 2.4 (1) As it not hard to check, the proof we constructed would work even if we assumed that $\sigma \mapsto \Phi(\sigma)$ — as a function on $(\mathbb{S}^{n-1})^{\Lambda_r}$ — has a Lipschitz-continuous derivative. However, we suspect that the theorem actually holds when Φ is just continuous (and perhaps even less). This is based on a similar observation that was made in the context of the Mermin-Wagner theorem by Ioffe, Shlosman and Velenik [13].

(2) We required that K_{xy} is asymptotic to $|x-y|^{-s}$, but it would suffice to assume that $x \mapsto K_{0x}$ is regularly varying with exponent s that obeys $s < d+2$. However, in the boundary case $s = d+2$ the slowly varying part becomes crucial for the result.

(3) Our proofs parallels (and in $d = 1$ actually uses) the proof of the Mermin-Wagner theorem. (This theorem states that, under precisely defined conditions, every Gibbs state for the model retains full invariance with respect to global rotations $\mathsf{R} \in \mathrm{SO}(n)$.) An interesting question is whether the introduction of antialigning long-range forces could restore global $\mathrm{SO}(n)$ symmetry even in the situations where the original reasoning for the Mermin-Wagner theorem no longer applies.

3. Main steps of the proof

Suppose $n \geq 2$ and fix a potential Φ and constants $\lambda > 0$ and $\beta > 0$. We will assume that $m_\star > 0$ and derive a contradiction. Let $\hat{\mathrm{e}}_i$ denote the i-th coordinate vector in $\mathbb{R}^n$ and let μ denote a translation-invariant Gibbs measure for which we have

$$E_\mu(\sigma_x) = m_\star \hat{\mathrm{e}}_1, \qquad x \in \mathbb{Z}^d. \tag{3.1}$$

As already mentioned, this measure exists by Theorem 2.3(3) of [1].

Now we pick two length scales L and a taking values in $\mathbb{D} := \{2^k : k \in \mathbb{N}\}$ with $L > a$, and consider a deformation of the spin configuration inside Λ_L that reverts the orientation of the first two components of the spin everywhere inside Λ_{L-a}. Explicitly, let

$\mathsf{R} \in \mathrm{SO}(n)$ be the rotation such that $\mathsf{R}\hat{e}_i = -\hat{e}_i$ for $i = 1, 2$ while $\mathsf{R}\hat{e}_i = \hat{e}_i$ for $i > 2$. We can view R as the endpoint of a continuous trajectory of maps

$$\mathsf{R}^\theta := \begin{pmatrix} \cos\theta & \sin\theta & 0 & \cdots & 0 \\ -\sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.2}$$

as θ varies from 0 to π or from 0 to $-\pi$. Next we define the "deformation angles"

$$\theta_x := \begin{cases} \pi a^{-1} \operatorname{dist}(x, \Lambda_L^{\mathrm{c}}), & \text{if } x \notin \Lambda_{L-a}, \\ \pi, & \text{if } x \in \Lambda_{L-a}, \end{cases} \tag{3.3}$$

where $\operatorname{dist}(x, y)$ is the ℓ_1-distance on $\mathbb{Z}^d$. These permit us to define the global *inhomogeneous* rotations $\mathsf{R}^\pm$ on the configuration space by

$$(\mathsf{R}^\pm \sigma)_x := \mathsf{R}^{\pm\theta_x} \sigma_x, \qquad x \in \mathbb{Z}^d. \tag{3.4}$$

Notice that $(\mathsf{R}^\pm\sigma)_x = (\mathsf{R}\sigma)_x$ for $x \in \Lambda_{L-a}$ while $(\mathsf{R}^\pm\sigma)_x = \sigma_x$ for $x \in \Lambda_L^{\mathrm{c}}$.

We will now use these rotations to express quantitatively the assumption of differentiability of the map Φ. Abusing the notation slightly, let R_x^t denote the inhomogeneous rotation of σ such that $(\mathsf{R}_x^t\sigma)_z = \sigma_z$ when $z \neq x$ and $(\mathsf{R}_x^t\sigma)_x = \mathsf{R}^t\sigma_x$. The map $t \mapsto \Phi(\mathsf{R}_x^t\sigma)$ is differentiable and the corresponding derivative is

$$D_x\Phi(\sigma) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathsf{R}_x^t\sigma)\Big|_{t=0}, \tag{3.5}$$

and, similarly, $D_xD_y\Phi(\sigma) := D_x(D_y\Phi)(\sigma)$. We also write

$$\|\Phi''\| := \sup_{\substack{v\in\ell^2(\mathbb{Z}^d) \\ \|v\|_2=1}} \sup_\sigma \Bigl| \sum_{z,z'} v_z v_{z'} D_z D_{z'} \Phi(\sigma) \Bigr|. \tag{3.6}$$

to denote a natural norm of the second derivative of Φ. Notice that, while the derivatives are defined using a specific one-parameter subgroup $\theta \mapsto \mathsf{R}^\theta$ of $\mathrm{SO}(n)$, the rotation invariance of Φ makes the specific choice of the subgroup immaterial.

Suppose now that $N > L + r$ and $L > a$. The entire argument is centered around the probability distribution of the *energy defect*,

$$\Delta_{L,a}(\sigma) := 2\mathscr{H}_N(\sigma) - \mathscr{H}_N(\mathsf{R}^+\sigma) - \mathscr{H}_N(\mathsf{R}^-\sigma), \tag{3.7}$$

which is independent of N as long as $N > L + r$. The reasons for consideration of both R^+ and R^- — inspired by some proofs of the Mermin-Wagner theorem (cf Fröhlich and Pfister [5]) and employed also by van Enter [4] — will become very apparent from the proof of a uniform bound on $\Delta_{L,a}$:

Lemma 3.1 *For all $L > a$ and all $\sigma \in \Omega$,*

$$\bigl|\Delta_{L,a}(\sigma)\bigr| \le U_{L,a}, \tag{3.8}$$

where

$$U_{L,a} := \|\Phi''\| \sum_{x\in\mathbb{Z}^d}\sum_{y\in\Lambda_r} (\theta_{x+y}-\theta_x)^2 + |\lambda| \sum_{x,y\colon x\neq y} K_{xy}\,(\theta_x-\theta_y)^2. \tag{3.9}$$

Proof. We will first deal with the long-range part of the interaction. Let φ_x denote the polar angle for the projection of σ_x to the subspace in $\mathbb{R}^n$ spanned by $\hat{\mathrm{e}}_1$ and $\hat{\mathrm{e}}_2$, and let s_x denote the projection of σ_x to the subspace of $\mathbb{R}^n$ spanned by $\hat{\mathrm{e}}_i$, $i=3,\dots,n$. Then

$$\sigma_x\cdot\sigma_y = s_x\cdot s_y + \sqrt{1-s_x^2}\,\sqrt{1-s_y^2}\,\cos(\varphi_x-\varphi_y). \tag{3.10}$$

Moreover, since the rotation of the spins occurs only in the $\hat{\mathrm{e}}_1,\hat{\mathrm{e}}_2$-plane, s_x is not changed when R^θ is applied to σ_x. Therefore,

$$\begin{aligned}\bigl|2\sigma_x\cdot\sigma_y &- (\mathsf{R}^+\sigma)_x\cdot(\mathsf{R}^+\sigma)_y - (\mathsf{R}^-\sigma)_x\cdot(\mathsf{R}^-\sigma)_y\bigr|\\ &\le \bigl|2\cos(\varphi_x-\varphi_y) - \cos(\varphi_x-\varphi_y+\theta_x-\theta_y)\\ &\qquad\qquad - \cos(\varphi_x-\varphi_y-\theta_x+\theta_y)\bigr|.\end{aligned} \tag{3.11}$$

It is now easy to check that the right hand side is smaller than $(\theta_x-\theta_y)^2$. Using this for all long-range terms in $\Delta_{L,a}$, we get the second term in (3.9).

In order to control the short-range contribution to $\Delta_{L,a}$, note that for each x, we can use invariance of Φ under $\mathrm{SO}(n)$ to write the corresponding term in the interaction as

$$2\Phi\circ\tau_x(\sigma) - \Phi\circ\tau_x\bigl(\mathsf{R}^{-\theta_x}\mathsf{R}^+\sigma\bigr) - \Phi\circ\tau_x\bigl(\mathsf{R}^{\theta_x}\mathsf{R}^-\sigma\bigr). \tag{3.12}$$

Abbreviate $\vartheta_z := \theta_z-\theta_x$ and, for $t\in[-1,1]$, let S^t denote the composition of the maps $\mathsf{R}_z^{t\vartheta_z}$ for all z. Then $\mathsf{R}^{-\theta_x}\mathsf{R}^+ = \mathsf{S}^1$ and $\mathsf{R}^{\theta_x}\mathsf{R}^- = \mathsf{S}^{-1}$ and, for $\Psi := \Phi\circ\tau_x$,

$$2\Psi(\sigma) - \Psi(\mathsf{S}^1\sigma) - \Psi(\mathsf{S}^{-1}\sigma) = -\int_0^1 \mathrm{d}t \int_{-t}^t \mathrm{d}u \sum_{z,z'} \vartheta_z\vartheta_{z'} D_z D_{z'}\Psi(\mathsf{S}^u\sigma). \tag{3.13}$$

The integrand is now bounded via

$$\Bigl|\sum_{z,z'}\vartheta_z\vartheta_{z'}D_zD_{z'}\Psi(\mathsf{S}^t\sigma)\Bigr| \le \|\Phi''\|\sum_{z\in\Lambda_r}\vartheta_{x+z}^2, \tag{3.14}$$

where we used that $D_zD_{z'}\Psi(\mathsf{S}^t\sigma)=0$ unless $z-x, z'-x\in\Lambda_r$. The integral over s and t then gives a factor of one; the claim then follows by summing the result over x. □

Our next observation will be concerned with the leading-order growth of $U_{L,a}$.

Proposition 3.2 *Assume $\lambda>0$. For each value of the ratio $\|\Phi''\|/\lambda>0$ there is a constant $c\in(0,1)$ such that if $c^{-1}\le a\le cL$, then*

$$c\lambda\mathcal{I}_{L,a} \le U_{L,a} \le c^{-1}\lambda\mathcal{I}_{L,a}, \tag{3.15}$$

where

$$\mathcal{I}_{L,a} := L^{d-1}\begin{cases} a^{d+1-s}, & \text{if } s<d+2,\\ a^{-1}\log a, & \text{if } s=d+2.\end{cases} \tag{3.16}$$

The proof of these bounds is relatively straightforward but, in order to stay focused on the main line of argument, we defer it to Section 4. The quantity $\mathcal{I}_{L,a}$ will play the role of the benchmark scale for all arguments that are to follow. Our next step is the connection between the above energy defect and positive magnetization:

Proposition 3.3 *Suppose $m_\star > 0$ and let μ be an ergodic Gibbs measure satisfying* (2.9)*. For each $\kappa > 0$ there is $c' \in (0,1)$ such that if $a, L \in \mathbb{D}$ obey $1/c' < a \le c'L$, then*

$$E_\mu\bigl(\Delta_{L,a}(\sigma)\bigr) \ge c'(m_\star^2 - \kappa)\,\mathcal{I}_{L,a}. \tag{3.17}$$

Again, to keep the main argument free of lengthy technical interruptions, we postpone the proof to Section 5. This estimate enters the main argument via:

Lemma 3.4 *Fix $\lambda > 0$ and let $c \in (0,1)$ be the constant from Proposition 3.2. Suppose that $c^{-1} \le a \le cL$. Then for each $\zeta \in [0, c\lambda)$,*

$$\mu(\Delta_{L,a} \ge \zeta\mathcal{I}_{L,a}) \ge \frac{E_\mu(\Delta_{L,a}) - \zeta\mathcal{I}_{L,a}}{(c\lambda - \zeta)\mathcal{I}_{L,a}}. \tag{3.18}$$

Proof. The absolute bound from Lemma 3.1 tells us

$$E_\mu(\Delta_{L,a}) \le \mu(\Delta_{L,a} \ge \zeta\mathcal{I}_{L,a})\bigl[U_{L,a} - \zeta\mathcal{I}_{L,a}\bigr] + \zeta\mathcal{I}_{L,a}. \tag{3.19}$$

Using (3.15) and $\zeta < c\lambda$, the claim now easily follows. □

The last essential ingredient we will need is the following fact:

Lemma 3.5 *For each $L > a$, any event A depending only on $\{\sigma_x \colon x \in \Lambda_{L-a}\}$, any Gibbs measure μ and any $t \in \mathbb{R}$ we have*

$$\mu\bigl(A \cap \{\Delta_{L,a} \ge t\}\bigr) \le \mathrm{e}^{-\frac12\beta t}\mu\bigl(\mathsf{R}(A)\bigr) \tag{3.20}$$

Proof. Let $N > L + r$ and abbreviate $A_t := A \cap \{\Delta_{L,a} \ge t\}$. Then for any $\sigma \in A_t$,

$$\mathrm{e}^{-\beta\mathscr{H}_N(\sigma)} \le \mathrm{e}^{-\frac12\beta t}\,\mathrm{e}^{-\frac12\beta\mathscr{H}_N(\mathsf{R}^+\sigma) - \frac12\beta\mathscr{H}_N(\mathsf{R}^-\sigma)} \tag{3.21}$$

It follows that

$$\gamma_N(A_t|\bar\sigma) \le \frac{\mathrm{e}^{-\frac12\beta t}}{Z_N(\bar\sigma)}\int_{A_t}\prod_{x\in\Lambda_N}\nu(\mathrm{d}\sigma_x)\,\mathrm{e}^{-\frac12\beta\mathscr{H}_N(\mathsf{R}^+\sigma) - \frac12\beta\mathscr{H}_N(\mathsf{R}^-\sigma)} \tag{3.22}$$

where we think of all σ_x with $x \notin \Lambda_N$ as fixed to $\bar\sigma_x$. Using the Cauchy-Schwarz inequality and $A_t \subset A$, the last integral is bounded by the product

$$\Bigl(\int_A\prod_{x\in\Lambda_N}\nu(\mathrm{d}\sigma_x)\,\mathrm{e}^{-\beta\mathscr{H}_N(\mathsf{R}^+\sigma)}\Bigr)^{1/2}\Bigl(\int_A\prod_{x\in\Lambda_N}\nu(\mathrm{d}\sigma_x)\,\mathrm{e}^{-\beta\mathscr{H}_N(\mathsf{R}^-\sigma)}\Bigr)^{1/2}. \tag{3.23}$$

But $\mathsf{R}^\pm$ alter only the spins inside Λ_L and since the product measure is $\mathsf{R}^\pm$-invariant, both integrals are equal to $Z_N(\bar\sigma)\gamma_L(\mathsf{R}(A)|\bar\sigma)$. As $\mathsf{R}^\pm(A) = \mathsf{R}(A)$,

$$\gamma_N(A_t|\bar\sigma) \le \mathrm{e}^{-\frac12\beta t}\gamma_L(\mathsf{R}(A)|\bar\sigma), \qquad \bar\sigma \in \Omega. \tag{3.24}$$

The claim is now proved by taking expectation with respect to μ. □

Now we are ready to begin the actual proof of our main result. Somewhat surprisingly, we will have to deal separately with two distinct cases:

CASE 1: $d \ge 2$ & $d < s \le d+2$.
CASE 2: $d = 1$ & $1 < s \le 3$.

What makes these cases different is seen from the following remark:

Lemma 3.6 *Assume $L, a \in \mathbb{D}$ and suppose first CASE 1. Then there is a way to take $L, a \to \infty$ so that $L/a \to \infty$ and $\mathcal{I}_{L,a} \to \infty$. On the other hand, in CASE 2, for any fixed value of $a > 1$ the quantity $\mathcal{I}_{L,a}$ remains bounded in the limit $L \to \infty$.*

Proof. This is directly verified from the formula (3.16). □

Proof of Theorem 2.2, CASE 1. Suppose s and d are as stated above, and let $\lambda > 0$. Suppose $m_\star > 0$ and note that we must have $\beta > 0$. Let μ be a translation-invariant Gibbs measure obeying (2.10) and let c be as in Proposition 3.2. Pick $\kappa \in (0, m_\star^2)$ and let $c' \in (0,1)$ be as in Proposition 3.3. Set $c'' := \min\{c, c'\}$ and suppose $L, a \in \mathbb{D}$ are such that $1/c'' \le a \le c'' L$ for the rest of the argument.

Fix ζ such that $0 < \zeta < c'(m_\star^2 - \kappa)$ and $\zeta < c\lambda$. Proposition 3.3 and Lemma 3.4 yield

$$\mu(\Delta_{L,a} \ge \zeta \mathcal{I}_{L,a}) \ge \frac{c'(m_\star^2 - \kappa) - \zeta}{c\lambda - \zeta} > 0. \tag{3.25}$$

But this cannot hold uniformly in L and a because Lemma 3.5 and the fact that, by Lemma 3.6, $\mathcal{I}_{L,a} \to \infty$ when $L, a \to \infty$ along appropriate sequences — still subject to the aforementioned restrictions — allow us to make the left-hand side arbitrarily small. Hence $m_\star = 0$ as claimed.

The conclusion $m_\star = 0$ implies that the derivative in (2.9) vanishes and since $h \mapsto f(h)$ is even, f is thus continuously differentiable at $h = 0$. That this implies (2.10) is the consequence of standard thermodynamic arguments (see, e.g., [1, Theorem 2.5(2)]). □

Proof of Theorem 2.2, CASE 2. Here we will follow the proof of the Mermin-Wagner argument to show that every extremal Gibbs measure μ is actually R invariant. Let A be any event that depends only on $\{\sigma_x \colon x \in \Lambda_{L-a}\}$. Since $\mathcal{I}_{L,a}$ is bounded, so is $\Delta_{L,a}$, say, $\Delta_{L,a} \ge t$ for some $t \in \mathbb{R}$. Then Lemma 3.5 gives

$$\mu(A) \le \mathrm{e}^{\frac{1}{2}\beta t} \mu\bigl(\mathsf{R}(A)\bigr). \tag{3.26}$$

As this now extends to all events A, the fact that μ is extremal forces $\mu = \mu \circ \mathsf{R}$. In particular, $m_\star = 0$ and all other consequences follow as in CASE 1. □

Proof of Corollary 2.3. This is also quite standard, but we will for completeness sketch the main argument. Fix $\delta > 0$ and pick a unit vector $\hat{\mathrm{e}} \in \mathbb{R}^n$. By the exponential Chebyshev inequality, for any $h > 0$,

$$\mu\Bigl(\sum_{x \in \Lambda_N} \sigma_x \cdot \hat{\mathrm{e}} > \delta |\Lambda_N|\Bigr) \le \mathrm{e}^{-h\delta|\Lambda_N|} E_\mu\biggl(\exp\Bigl\{h \sum_{x \in \Lambda_N} \sigma_x \cdot \hat{\mathrm{e}}\Bigr\}\biggr). \tag{3.27}$$

Invoking (2.7), the definition (2.8) then yields

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mu\Bigl(\sum_{x\in\Lambda_N} \sigma_x \cdot \hat{\mathrm{e}} > \delta|\Lambda_N|\Bigr) \le f(h) - h\delta. \tag{3.28}$$

But $f(h) = o(h)$ as $h \downarrow 0$ by the fact that $m_\star = 0$ and so $f(h) - h\delta < 0$ once h is sufficiently small. The probability in (3.27) thus decays exponentially in $|\Lambda_N|$ and so the corresponding event occurs only for finitely many N, μ-a.s. As this holds for all $\delta > 0$ and all $\hat{\mathrm{e}}$, the claim follows. □

4. Estimates on interaction strength

In this section we will analyze the various contributions to the quantity $U_{L,a}$ that serves as the uniform upper bound on the energy defect. To keep our notations succinct, we will write this quantity as

$$U_{L,a} = \|\Phi''\| Q^+ + |\lambda| Q^-, \tag{4.1}$$

where

$$Q^+ := \sum_{x\in\mathbb{Z}^d} \sum_{y\in\Lambda_r} |\theta_{x+y} - \theta_x|^2 \tag{4.2}$$

and

$$Q^- := \sum_{x,y\colon x\ne y} K_{xy} |\theta_x - \theta_y|^2. \tag{4.3}$$

Then we have the following estimates:

Lemma 4.1 *There is $c_1 \in (1,\infty)$ such that for all L, a with $c_1 \le a \le c_1^{-1} L$,*

$$Q^+ \le c_1 L^{d-1} a^{-1}. \tag{4.4}$$

Lemma 4.2 *Let $s \in (d, d+2]$. There is $c_2 \in (0,1)$ such that for all L, a with $c_2^{-1} \le a \le c_2 L$,*

$$c_2 \mathcal{I}_{L,a} \le Q^- \le c_2^{-1} \mathcal{I}_{L,a}, \tag{4.5}$$

with $\mathcal{I}_{L,a}$ as in (3.16).

Let us first conclude the proof of the desired asymptotic for $U_{L,a}$:

Proof of Proposition 3.2. Notice that for all $s \le d+2$ the ratio $L^{d-1}a^{-1}/\mathcal{I}_{L,a}$ tends to zero in the limit when $L, a \to \infty$ with $a/L \to 0$ and so we can easily arrange that Q^+/Q^- is arbitrarily small by making a and L/a large enough. The claim follows. □

It remains to prove the two lemmas above:

Proof of Lemma 4.1. We have $|\theta_{x+y} - \theta_x| \le r a^{-1}$ for $x \in \Lambda_{L+r} \setminus \Lambda_{L-r-a}$ while the difference is zero in other cases. So the sum is at most of order $r^2(2r+a)L^{d-1}a^{-2}$. As r is fixed, this readily yields the claim. □

Proof of Lemma 4.2. To make expressions simple, let us agree to write $f \asymp g$ if the ratios f/g and g/f are bounded by universal constants depending only on d and s. We will assume that $K_{xy} \asymp |x-y|^{-s}$ because all finite-range deviations from this can be estimated by a term similar to Q^+. Abbreviate

$$R_N := \{z = (z_1, \dots, z_{d-1}) \in \mathbb{Z}^{d-1} \setminus \{0\} : |z_i| \le N\}. \tag{4.6}$$

We will consider quantities $Q_1, \dots, Q_4$ that collect the essential contributions from the sum constituting Q^-. To understand the notation, let $x \in \Lambda_L$ and $y \notin \Lambda_{L-a}$ — the other situations when $\theta_x - \theta_y \neq 0$ are reduce to this by interchanging x and y. Now fix a face of Λ_{L-a}, let u (roughly) correspond to the distance of x to this face and let t denote the corresponding distance of y. Finally let z stand for the projection of $y - x$ to the plane corresponding to this face.

There are four types of contributions we will need to distinguish. First those when $x \in \Lambda_{L-a}$ and $y \in \Lambda_L \setminus \Lambda_{L-a}$. This boils down to the quantity

$$Q_1 := \sum_{u=0}^{L} \sum_{t=1}^{a} \sum_{z \in \mathbb{Z}^{d-1}} \frac{({}^t/_a)^2}{[(u+t)^2 + |z|^2]^{s/2}}. \tag{4.7}$$

Next we will take the cases when $x \in \Lambda_{L-a}$ but $y \in \Lambda_L^c$. This will require estimating

$$Q_2 := \sum_{u=0}^{L} \sum_{t>a} \sum_{z \in \mathbb{Z}^{d-1}} \frac{1}{[(u+t)^2 + |z|^2]^{s/2}}. \tag{4.8}$$

The third instance to consider is when both x and y lie in $\Lambda_L \setminus \Lambda_{L-a}$. For this we need

$$Q_3 := \sum_{u=0}^{a} \sum_{t=u+1}^{a} \sum_{z \in R_N} \frac{\left(\frac{t-u}{a}\right)^2}{[(t-u)^2 + |z|^2]^{s/2}}, \tag{4.9}$$

where we will use $N := L$ in the proof of the lower bound in (4.5) and $N := \infty$ (and thus $R_\infty := \mathbb{Z}^{d-1}$) for the upper bound. And, finally, we also need to worry about the situations when $x \in \Lambda_L \setminus \Lambda_{L-a}$ and $y \in \Lambda_L^c$. This will require checking

$$Q_4 := \sum_{u=0}^{a} \sum_{t \ge 0} \sum_{z \in \mathbb{Z}^d} \frac{({}^u/_a)^2}{[(u+t)^2 + |z|^2]^{s/2}}. \tag{4.10}$$

We will now proceed to derive the asymptotic for all four terms.

We claim that, for any integer v with $1 \le v \le 2L$, we have

$$\sum_{z \in \mathbb{R}_L} \frac{1}{[v^2 + |z|^2]^{s/2}} \asymp \sum_{z \in \mathbb{Z}^{d-1}} \frac{1}{[v^2 + |z|^2]^{s/2}} \asymp v^{d-1-s}. \tag{4.11}$$

This permits us to write

$$Q_1 \asymp a^{-2} \sum_{u=0}^{L} \sum_{t=1}^{a} t^2 (u+t)^{d-1-s} \asymp a^{d+1-s}, \tag{4.12}$$

where we first summed over s assuming $s > d$ and then summed over t assuming that $d + 2 - s \ge 0$. Similarly we get

$$Q_2 \asymp \sum_{u=0}^{L} \sum_{t>a} (u+t)^{d-1-s} \asymp \begin{cases} a^{d+1-s}, & \text{if } s > d+1, \\ \log\left(\frac{L+a}{a}\right), & \text{if } s = d+1, \\ (L+a)^{d+1-s}, & \text{if } s < d+1. \end{cases} \tag{4.13}$$

Here we first summed over t and then distinguished the three possibilities depending on whether the remaining sum is divergent, logarithmically divergent and convergent.

For the remaining two terms we get the following: The fact that R_L and $\mathbb{Z}^{d-1}$ are interchangeable when absolute constants do not matter, we get

$$Q_3 \asymp a^{-2}\sum_{u=0}^{a}\sum_{t=u+1}^{a}(t-u)^{d+1-s} \asymp \begin{cases} a^{d+1-s}, & \text{if } s<d+2,\\ a^{-1}\log a, & \text{if } s=d+2,\end{cases} \tag{4.14}$$

where we only paid attention to the cases when $d<s\le d+2$. Finally we get

$$Q_4 \asymp a^{-2}\sum_{u=0}^{a}\sum_{t\ge 0}u^2(u+t)^{d-s-1} \asymp a^{d+1-s}, \tag{4.15}$$

where we again focussed on $s<d+3$.

Obviously, Q_3 is the dominant term for all $s\in(d,d+2]$. Since all terms contributing to Q^- are positive, we can now estimate

$$cQ_3 \le Q^- \le c^{-1}(Q_1+Q_2+Q_3+Q_4) \tag{4.16}$$

where $c=c(d)\in(0,1)$. Hereby the claim follows. □

5. Expected energy defect

Our final task is to establish Proposition 3.3. Fix $L,a\in\mathbb{D}$ with $L>a$ and recall the notation $\mathsf{R}^{\pm}$ for the inhomogeneous rotations from (3.4). For any x,y let

$$\Delta_{xy}(\sigma) := 2\sigma_x\cdot\sigma_y - (\mathsf{R}^+\sigma)_x\cdot(\mathsf{R}^+\sigma)_y - (\mathsf{R}^-\sigma)_x\cdot(\mathsf{R}^-\sigma)_y \tag{5.1}$$

denote the term corresponding to these vertices from the long-range part of the energy defect $\Delta_{L,a}$. Abbreviate

$$\widetilde{K}_{xy} := 4\sin^2\Bigl(\frac{\theta_x-\theta_y}{2}\Bigr)K_{xy} \tag{5.2}$$

and let P_{12} denote the orthogonal projection of $\mathbb{R}^n$ onto the linear span of $\hat{\mathrm{e}}_1,\hat{\mathrm{e}}_2$. We begin with a variation on Lemma 4.4 from [2]:

Lemma 5.1 *Suppose Assumption 2.1(2). For an integer $\ell\ge 1$, let V_1 and V_2 be two disjoint translates of Λ_ℓ. For each $\epsilon>0$ there is $\delta>0$ such that if $\operatorname{dist}(V_1,V_2)\ge \ell/\delta$ and $\ell/a<\delta$, then for all $\sigma\in\Omega$,*

$$\Bigl|\sum_{x\in V_1}\sum_{y\in V_2}K_{xy}\Delta_{xy}(\sigma) - m_1(\sigma)\cdot m_2(\sigma)\sum_{x\in V_1}\sum_{y\in V_2}\widetilde{K}_{xy}\Bigr| \le \epsilon\sum_{x\in V_1}\sum_{y\in V_2}\widetilde{K}_{xy}, \tag{5.3}$$

where $m_i(\sigma) := |\Lambda_\ell|^{-1}\sum_{x\in V_i}\mathsf{P}_{12}\sigma_x$ is the P_{12}-projection of the spin average in V_i.

Proof. As is easy to check from (3.2), we have

$$\Delta_{xy}(\sigma) = 4\sin^2\Bigl(\frac{\theta_x-\theta_y}{2}\Bigr)\,(\sigma_x\cdot\mathsf{P}_{12}\sigma_y) \tag{5.4}$$

and so $K_{xy}\Delta_{xy}(\sigma) = \widetilde{K}_{xy}(\sigma_x \cdot \mathsf{P}_{12}\sigma_y)$. Now pick $x_0 \in V_1$ and $y_0 \in V_2$. Assumption 2.1(2) ensures that, for each $\epsilon > 0$ there is $\delta > 0$ such that if

$$|x-y| \geq \delta^{-1}\max\{|x-x_0|, |y-y_0|\}, \tag{5.5}$$

then

$$\left|K_{xy} - K_{x_0y_0}\right| \leq \epsilon K_{x_0y_0}, \qquad x \in V_1,\, y \in V_2. \tag{5.6}$$

Since $|\theta_x - \theta_{x_0}| \leq a^{-1}|x-x_0| \leq {}^{\ell}/_{a} < \delta$, a similar bound holds also for $\widetilde{K}_{xy}$. The claim is now proved as in [2, Lemma 2.2]. □

Proof of Proposition 3.3. Consider a translation-invariant, ergodic Gibbs measure μ satisfying (2.9). For any $\epsilon > 0$, let

$$\mathcal{E}_\ell := \Bigl\{\sigma\colon \Bigl|\sum_{x\in\Lambda_\ell}\sigma_x - m_\star\hat{\mathrm{e}}_1|\Lambda_\ell|\Bigr| < \epsilon|\Lambda_\ell|\Bigr\}. \tag{5.7}$$

By the Spatial Ergodic Theorem, there exists $\ell_0 = \ell_0(\epsilon)$ such that for $\ell \geq \ell_0$ we have $\mu(\mathcal{E}_\ell) \geq 1-\epsilon$. Thus, if $\ell \geq \ell_0$ and V_1 and V_2 are disjoint translates of Λ_ℓ, then

$$\Bigl|E_\mu\bigl(m_1(\sigma)\cdot\mathsf{P}_{12}m_2(\sigma)\bigr) - m_\star^2\Bigr| < 5\epsilon. \tag{5.8}$$

Assuming that $\operatorname{dist}(V_1, V_2) \geq \ell/\delta$ and $\ell/a < \delta$, Lemma 5.1 shows

$$E_\mu\Bigl(\sum_{x\in V_1}\sum_{y\in V_2}K_{xy}\Delta_{xy}(\sigma)\Bigr) \geq (m_\star^2 - 6\epsilon)\sum_{x\in V_1}\sum_{y\in V_2}\widetilde{K}_{xy}. \tag{5.9}$$

Now consider a fixed partition of $\mathbb{Z}^d$ into blocks of side ℓ. Summing (5.9) over the blocks in the partition, and applying (5.4) one more time we get

$$\begin{aligned} E_\mu(\Delta_{L,a}) \geq\; &(m_\star^2 - 6\epsilon)\sum_{\substack{x,y\\ |x-y|\geq 2\ell/\delta}}\widetilde{K}_{xy} - \sum_{\substack{x,y\\ |x-y|\leq 2\ell/\delta}}\widetilde{K}_{xy} \\ \geq\; &(m_\star^2 - 6\epsilon)\sum_{x,y\colon x\neq y}\widetilde{K}_{xy} - 2\sum_{\substack{x,y\\ |x-y|\leq 2\ell/\delta}}\widetilde{K}_{xy}, \end{aligned} \tag{5.10}$$

where we used $0 < m_\star^2 - 6\epsilon < 1$. It remains to bound the terms on the right-hand side.

Using $\widetilde{K}_{xy} \geq (4/\pi^2)|\theta_x - \theta_y|^2 K_{xy}$ and Lemma 4.1, the first sum is at least a constant times $\mathcal{I}_{L,a}$. For the second sum we note that for all contributing x, y we have

$$\widetilde{K}_{xy} \leq K_{xy}|\theta_x - \theta_y|^2 \leq c_1\left(\frac{\ell}{\delta a}\right)^2, \tag{5.11}$$

where $c_1 := \sup K_{0,x}$. Moreover, $\widetilde{K}_{xy}$ is zero unless at least one of x and y lies in the annulus $\Lambda_L \setminus \Lambda_{L-a}$. This implies

$$\sum_{\substack{x,y\\ |x-y|\leq 2\ell/\delta}}\widetilde{K}_{xy} \leq c_2 L^{d-1}\frac{\ell^{d+2}}{\delta^{d+2}a} \tag{5.12}$$

for some c_2 proportional to c_1 above. If a is so large that one can find $\ell \ll \delta[\epsilon \log a]^{\frac{1}{d+2}}$ with $\ell \geq \ell_0$, then the right hand side is at most $\epsilon L^{d-1} a^{-1} \log a$. As this is much smaller than $\mathcal{I}_{L,a}$ for all $s \in (d, d+2]$, the claim follows. $\square$

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