Some estimates for Bochner-Riesz operators on the weighted Morrey spaces

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Abstract

In this paper, we will obtain some weighted strong type and weak type estimates of Bochner-Riesz operators $T_R^{(n-1)/2}$ on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $1 \le p < \infty$ and $0 < \kappa < 1$. We will also prove that the commutator formed by a $BMO(\mathbb{R}^n)$ function $b(x)$ and $T_R^\delta(\delta \ge (n-1)/2)$ is a bounded operator on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $1 < p < \infty$ and $0 < \kappa < 1$.

MSC(2000) 42B15; 42B35

Keywords: Bochner-Riesz operators; weighted Morrey spaces; commutator; A_p weights

1. Introduction

The Bochner-Riesz operators of order $\delta > 0$ in $\mathbb{R}^n(n \ge 2)$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$\big(T_R^\delta f\big)\hat{\ }(\xi) = \Big(1 - \frac{|\xi|^2}{R^2}\Big)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of f. The associated maximal Bochner-Riesz operator is defined by

$$T_*^\delta f(x) = \sup_{R>0} |T_R^\delta f(x)|.$$

These operators were first introduced by Bochner [2] in connection with summation of multiple Fourier series and played an important role in harmonic analysis. Let b be a locally integrable function on $\mathbb{R}^n$, for any given $R > 0$, the commutator of b and T_R^δ is defined as follows

$$[b, T_R^\delta]f(x) = b(x)T_R^\delta f(x) - T_R^\delta(bf)(x).$$

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The classical Morrey spaces $\mathcal{L}^{p,\lambda}$ were first introduced by Morrey in [9] to study the local behavior of solutions to second order elliptic partial differential equations. Recently, Komori and Shirai [7] considered the weighted version of Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators on these spaces.

The main purpose of this paper is to discuss the weighted boundedness of maximal Bochner-Riesz operator and commutator $[b, T_R^\delta]$ on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $1 < p < \infty$ and $0 < \kappa < 1$, where the symbol b belongs to BMO. We will also give the weighted weak type estimate of Bochner-Riesz operators on these spaces $L^{p,\kappa}(w)$ when $p = 1$ and $0 < \kappa < 1$. Our main results are stated as follows.

Theorem 1. *Let $\delta = (n-1)/2$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Then there exists a constant $C > 0$ independent of f such that*

$$\|T_*^\delta(f)\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Theorem 2. *Let $\delta = (n-1)/2$, $p = 1$, $0 < \kappa < 1$ and $w \in A_1$. Then for any given $R > 0$, all $\lambda > 0$ and any ball B, there exists a constant $C > 0$ independent of f such that*

$$w(\{x \in B : T_R^\delta f(x) > \lambda\}) \le \frac{C}{\lambda} \cdot \|f\|_{L^{1,\kappa}(w)} w(B)^\kappa.$$

Theorem 3. *Let $\delta \ge (n-1)/2$, $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Suppose that $b \in BMO$, then there exists a constant C independent of f such that*

$$\big\|[b, T_R^\delta]f\big\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

2. Definitions and Notations

First let us recall some standard definitions and notations of weight classes. A weight w is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere, $B = B(x_0, r)$ denotes the ball with the center x_0 and radius r. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w, we denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $w(B)$, where $w(B) = \int_B w(x)\,dx$.

We shall give the definitions of two weight classes as follows.

Definition 1 ([10])**.** *A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that*

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le C.$$

When $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|}\int_B w(x)\,dx \le C \mathop{ess\,inf}_{x\in B} w(x).$$

Definition 2 ([4])**.** *A weight function w belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality*

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r} \le C\left(\frac{1}{|B|}\int_B w(x)\,dx\right)$$

holds for every ball B in $\mathbb{R}^n$.

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. If $w \in A_p$ with $1 \le p < \infty$, then there exists $r > 1$ such that $w \in RH_r$.

We state the following results that we will use frequently in the sequel.

Lemma A ([4])**.** *Let $w \in A_p$, $p \ge 1$. Then, for any ball B, there exists an absolute constant C such that*

$$w(2B) \le Cw(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \le C\lambda^{np} w(B),$$

where C does not depend on B nor on λ.

Lemma B ([5])**.** *Let $w \in RH_r$ with $r > 1$. Then there exists a constant C such that*

$$\frac{w(E)}{w(B)} \le C\left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E of a ball B.

A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|\,dx < \infty,$$

where $b_B = \frac{1}{|B|}\int_B b(y)\,dy$ and the supremum is taken over all balls B in $\mathbb{R}^n$.

Theorem C ([3,6])**.** *Assume that* $b \in BMO(\mathbb{R}^n)$*. Then for any* $1 \le p < \infty$*, we have*

$$\sup_B \left(\frac{1}{|B|} \int_B \big| b(x) - b_B \big|^p \, dx \right)^{1/p} \le C\|b\|_*.$$

Next we shall define the weighted Morrey space and give some results relevant to this paper. For further details, we refer the readers to [7].

Definition 3. *Let* $1 \le p < \infty$*,* $0 < \kappa < 1$ *and* w *be a weight function. Then the weighted Morrey space is defined by*

$$L^{p,\kappa}(w) = \{ f \in L^p_{loc}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty \},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) \, dx \right)^{1/p}$$

and the supremum is taken over all balls B *in* $\mathbb{R}^n$*.*

In [7], the authors proved the following result.

Theorem D. *If* $1 < p < \infty$*,* $0 < \kappa < 1$ *and* $w \in A_p$*, then the Hardy-Littlewood maximal operator* M *is bounded on* $L^{p,\kappa}(w)$*.*
If $p = 1$*,* $0 < \kappa < 1$ *and* $w \in A_1$*, then for all* $\lambda > 0$ *and any ball* B*, we have*

$$w(\{x \in B : Mf(x) > \lambda\}) \le \frac{C}{\lambda} \cdot \|f\|_{L^{1,\kappa}(w)} w(B)^\kappa.$$

We are going to conclude this section by giving two important results concerning the boundedness of Bochner-Riesz operators on the weighted L^p spaces. Given a Muckenhoupt's weight function w on $\mathbb{R}^n$, for $1 \le p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

Theorem E ([11])**.** *Let* $1 < p < \infty$*,* $w \in A_p$*. Then there exists a constant* $C > 0$ *such that*

$$\|T_*^{(n-1)/2}(f)\|_{L^p_w} \le C\|f\|_{L^p_w}.$$

Theorem F ([13])**.** *Let* $w \in A_1$*. Then there exists a constant* C *such that*

$$w\big(\{x \in \mathbb{R}^n : |T_1^{(n-1)/2} f(x)| > \lambda\}\big) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| w(y) \, dy.$$

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{A}{B} \le C$. Moreover, we will denote the conjugate exponent of $r > 1$ by $r' = r/(r-1)$.

3. Proofs of Theorems 1 and 2

The Bochner-Riesz operators can be expressed as convolution operators

$$T_R^\delta f(x) = (f * \phi_{1/R})(x),$$

where $\phi(x) = [(1-|\cdot|^2)_+^\delta]\hat{}\,(x)$. It is well known that the kernel ϕ can be represented as(see [8,12])

$$\phi(x) = \pi^{-\delta}\Gamma(\delta+1)|x|^{-(\frac{n}{2}+\delta)}J_{\frac{n}{2}+\delta}(2\pi|x|),$$

where $J_\mu(t)$ is the Bessel function

$$J_\mu(t) = \frac{(\frac{t}{2})^\mu}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{its}(1-s^2)^{\mu-\frac{1}{2}}\,ds.$$

From the asymptotic properties of the Bessel function, we can deduce that $J_{\frac{n}{2}+\delta}(t) \le Ct^{\frac{n}{2}+\delta}$ when $0 < t \le 1$ and $J_{\frac{n}{2}+\delta}(t) \le Ct^{-\frac{1}{2}}$ when $t > 1$. Therefore, for $\delta \ge (n-1)/2$, we have

$$\big|\phi(x)\big| \le \frac{C}{(1+|x|)^{\frac{n+1}{2}+\delta}}. \tag{1}$$

Before proving our main theorems, we point out the following two facts.
(i) When $\delta > (n-1)/2$, it is well known that(see [8,12])

$$T_*^\delta(f)(x) \le C \cdot M(f)(x).$$

Then for this case, the conclusion of Theorems 1 and 2 follows immediately from Theorem D.
(ii) Let $f_t(x) = t^{-n}f(x/t)$. Then for any fixed $R > 0$, it is easy to verify that $T_R^{(n-1)/2}f(x) = (\phi * f_R)_{1/R}(x)$. Hence, by Theorem F, we can get

$$w\big(\{x \in \mathbb{R}^n : |T_R^{(n-1)/2}f(x)| > \lambda\}\big) \le \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(y)|w(y)\,dy. \tag{2}$$

On the other hand, E. M. Stein(see [12]) showed that when $n \ge 2$, then there exists a function $f \in L^1$ such that

$$\limsup_{R\to\infty}\big|T_R^{(n-1)/2}f(x)\big| = +\infty \quad \text{almost everywhere.}$$

Therefore the above inequality (2) can't hold for the maximal Bochner-Riesz operator $T_*^{(n-1)/2}$. Moreover, the corresponding result of Theorem 2 for $T_*^{(n-1)/2}$ is also unknown.

Proof of Theorem 1. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{_{2B}}$, $\chi_{_{2B}}$ denotes the characteristic function of $2B$. Since $T_*^\delta(\delta = (n-1)/2)$ is a sublinear operator, then we have

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B |T_*^\delta f(x)|^p w(x)\,dx\Big)^{1/p}\\
\le&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B |T_*^\delta f_1(x)|^p w(x)\,dx\Big)^{1/p}\\
&+\frac{1}{w(B)^{\kappa/p}}\Big(\int_B |T_*^\delta f_2(x)|^p w(x)\,dx\Big)^{1/p}\\
=&I_1+I_2.
\end{aligned}$$

Theorem E and Lemma A imply

$$\begin{aligned}
I_1 &\le C\cdot\frac{1}{w(B)^{\kappa/p}}\Big(\int_{2B} |f(x)|^p w(x)\,dx\Big)^{1/p}\\
&\le C\|f\|_{L^{p,\kappa}(w)}\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}}\\
&\le C\|f\|_{L^{p,\kappa}(w)}.
\end{aligned}\tag{3}$$

We now turn to estimate the term I_2. Note that when $\delta = (n-1)/2$, then by the estimate (1), we have $|\phi(x)| \le \frac{C}{|x|^n}$. We also observe that when $x \in B$, $y \in (2B)^c$, then $|y-x| \sim |y-x_0|$. Hence

$$\begin{aligned}
T_*^\delta f_2(x) &= \sup_{R>0}\big|f_2 * \phi_{1/R}(x)\big|\\
&\le C\cdot\sup_{R>0}\int_{(2B)^c}\frac{R^n}{(R|x-y|)^n}\big|f(y)\big|\,dy\\
&\le C\cdot\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|f(y)\big|\,dy.
\end{aligned}\tag{4}$$

It follows from Hölder's inequality and the condition A_p that

$$\begin{aligned}
\int_{2^{j+1}B}|f(y)|\,dy &\le \Big(\int_{2^{j+1}B}|f(y)|^p w(y)\,dy\Big)^{1/p}\Big(\int_{2^{j+1}B} w(y)^{-p'/p}\Big)^{1/p'}\\
&\le C\|f\|_{L^{p,\kappa}(w)}\cdot|2^{j+1}B|w(2^{j+1}B)^{(k-1)/p}.
\end{aligned}\tag{5}$$

Substituting the above inequality (5) into (4), we thus obtain

$$T^\delta_* f_2(x) \le C\|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^\infty w(2^{j+1}B)^{(k-1)/p}.$$

Consequently

$$I_2 \le C\|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^\infty \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}}.$$

Since $w \in A_p$, then there exists $r > 1$ such that $w \in RH_r$. By using Lemma B, we can get

$$\frac{w(B)}{w(2^{j+1}B)} \le C\left(\frac{|B|}{|2^{j+1}B|}\right)^{(r-1)/r}.$$

Therefore

$$\begin{aligned} I_2 &\le C\|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^\infty \left(\frac{1}{2^{jn}}\right)^{(1-\kappa)(r-1)/pr} \\ &\le C\|f\|_{L^{p,\kappa}(w)}, \end{aligned} \tag{6}$$

where the last series is convergent since $(1-\kappa)(r-1)/pr > 0$. Combining the previous estimate (3) with (6) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we complete the proof of Theorem 1. □

Proof of Theorem 2. For any given $\lambda > 0$, we write

$$\begin{aligned} &w\big(\{x \in B : |T^\delta_R f(x)| > \lambda\}\big) \\ \le\, &w\big(\{x \in B : |T^\delta_R f_1(x)| > \lambda/2\}\big) + w\big(\{x \in B : |T^\delta_R f_2(x)| > \lambda/2\}\big) \\ =\, &I'_1 + I'_2. \end{aligned}$$

Lemma A and the inequality (2) yield

$$\begin{aligned} I'_1 &\le \frac{C}{\lambda}\int_{2B} |f(y)| w(y)\, dy \\ &\le \frac{C}{\lambda}\cdot \|f\|_{L^{1,\kappa}(w)} w(2B)^\kappa \\ &\le \frac{C}{\lambda}\cdot \|f\|_{L^{1,\kappa}(w)} w(B)^\kappa. \end{aligned} \tag{7}$$

It follows from the inequality (4) and the condition A_1 that

$$\big|T^\delta_R f_2(x)\big| \le C\cdot \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} \big|f(y)\big|\, dy$$

$$
\begin{aligned}
&\le C\cdot\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B}\big|f(y)\big|w(y)\,dy\\
&\le C\|f\|_{L^{1,\kappa}(w)}\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)^{1-\kappa}}\\
&= C\|f\|_{L^{1,\kappa}(w)}\cdot\frac{1}{w(B)^{1-\kappa}}\sum_{j=1}^{\infty}\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}}.
\end{aligned}
$$

Since $w\in A_1$, then $w\in RH_{r^*}$ with $r^*>1$. By Lemma B again, we obtain

$$
\frac{w(B)}{w(2^{j+1}B)}\le C\left(\frac{|B|}{|2^{j+1}B|}\right)^{(r^*-1)/r^*}.
$$

Hence

$$
\begin{aligned}
\big|T_R^{\delta}f_2(x)\big| &\le C\|f\|_{L^{1,\kappa}(w)}\cdot\frac{1}{w(B)^{1-\kappa}}\sum_{j=1}^{\infty}\left(\frac{1}{2^{jn}}\right)^{(1-\kappa)(r^*-1)/r^*}\\
&\le C\|f\|_{L^{1,\kappa}(w)}\cdot\frac{1}{w(B)^{1-\kappa}},
\end{aligned}
\tag{8}
$$

where in the last inequality we have used the fact that $(1-\kappa)(r^*-1)/r^*>0$.
If $\{x\in B:|T_R^{\delta}f_2(x)|>\lambda/2\}=\emptyset$, then the inequality

$$
I_2'\le\frac{C}{\lambda}\cdot\|f\|_{L^{1,\kappa}(w)}w(B)^{\kappa}
$$

holds trivially.
If $\{x\in B:|T_R^{\delta}f_2(x)|>\lambda/2\}\neq\emptyset$, then by the inequality (8), we have

$$
\lambda\le C\|f\|_{L^{1,\kappa}(w)}\cdot\frac{1}{w(B)^{1-\kappa}},
$$

which is equivalent to

$$
w(B)\le\frac{C}{\lambda}\cdot\|f\|_{L^{1,\kappa}(w)}w(B)^{\kappa}.
$$

Therefore

$$
I_2'\le w(B)\le\frac{C}{\lambda}\cdot\|f\|_{L^{1,\kappa}(w)}w(B)^{\kappa}.
\tag{9}
$$

Combining the inequality (9) with (7), we finish the proof of Theorem 2. □

4. Proof of Theorem 3

Proof. As in the proof of Theorem 1, for any given $R>0$, we write

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B\big|[b,T_R^\delta]f(x)\big|^p w(x)\,dx\Big)^{1/p}\\
\le&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B\big|[b,T_R^\delta]f_1(x)\big|^p w(x)\,dx\Big)^{1/p}\\
&+\frac{1}{w(B)^{\kappa/p}}\Big(\int_B\big|[b,T_R^\delta]f_2(x)\big|^p w(x)\,dx\Big)^{1/p}\\
=&J_1+J_2.
\end{aligned}$$

By Theorem E and the previous pointwise inequality $T_*^\delta(f)(x)\le C\cdot M(f)(x)$, $\delta>(n-1)/2$, we have that for $\delta\ge(n-1)/2$, T_R^δ is bounded on L^p_w provided that $1<p<\infty$ and $w\in A_p$. Then by the well-known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez, Bagby, Kurtz and Pérez(see [1]), we see that $[b,T_R^\delta]$ is also bounded on L^p_w for all $1<p<\infty$ and $w\in A_p$. This together with Lemma A imply

$$\begin{aligned}
J_1&\le C\|b\|_*\cdot\frac{1}{w(B)^{\kappa/p}}\Big(\int_{2B}|f(x)|^p w(x)\,dx\Big)^{1/p}\\
&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \qquad (10)\\
&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}.
\end{aligned}$$

We turn to deal with the term J_2. When $\delta\ge(n-1)/2$, by the estimate (1), we thus have $|\phi(x)|\le\frac{C}{(1+|x|)^n}$. As before, we also get $|y-x|\sim|y-x_0|$ when $x\in B$ and $y\in(2B)^c$. Hence

$$\begin{aligned}
\big|[b,T_R^\delta]f_2(x)\big|&=\Big|\int_{(2B)^c}\big(b(x)-b(y)\big)\phi_{1/R}(x-y)f(y)\,dy\Big|\\
&\le C\big|b(x)-b_B\big|\cdot\int_{(2B)^c}\frac{|f(y)|}{|y-x_0|^n}\,dy\\
&\quad+C\int_{(2B)^c}\frac{|b(y)-b_B||f(y)|}{|y-x_0|^n}\,dy\\
&=\mathrm{I}+\mathrm{II}.
\end{aligned}$$

It follows immediately from the inequality (5) that

$$\begin{aligned}
\mathrm{I}&\le C\big|b(x)-b_B\big|\cdot\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|f(y)\big|\,dy\\
&\le C\|f\|_{L^{p,\kappa}(w)}\big|b(x)-b_B\big|\cdot\sum_{j=1}^\infty w(2^{j+1}B)^{(k-1)/p}.
\end{aligned}$$

Consequently

$$
\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B \mathrm{I}^p\, w(x)\,dx\Big)^{1/p}\\
\le& C\|f\|_{L^{p,\kappa}(w)}\frac{1}{w(B)^{\kappa/p}}\cdot\sum_{j=1}^\infty w(2^{j+1}B)^{(k-1)/p}\cdot\Big(\int_B\big|b(x)-b_B\big|^p w(x)\,dx\Big)^{1/p}\\
=& C\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^\infty\frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}}\cdot\Big(\frac{1}{w(B)}\int_B\big|b(x)-b_B\big|^p w(x)\,dx\Big)^{1/p}.
\end{aligned}
$$

Using the same arguments as in the proof of Theorem 1, we can see that the above summation is bounded by a constant. Hence

$$
\frac{1}{w(B)^{\kappa/p}}\Big(\int_B \mathrm{I}^p\, w(x)\,dx\Big)^{1/p}\le C\|f\|_{L^{p,\kappa}(w)}\Big(\frac{1}{w(B)}\int_B\big|b(x)-b_B\big|^p w(x)\,dx\Big)^{1/p}.
$$

Since $w\in A_p$, as before, there exists a number $r>1$ such that $w\in RH_r$. Then, by the reverse Hölder's inequality and Theorem C, we thus get

$$
\begin{aligned}
&\Big(\frac{1}{w(B)}\int_B\big|b(x)-b_B\big|^p w(x)\,dx\Big)^{1/p}\\
\le& C\cdot\frac{1}{w(B)^{1/p}}\Big(\int_B\big|b(x)-b_B\big|^{pr'}\,dx\Big)^{1/pr'}\Big(\int_B w(x)^r\,dx\Big)^{1/pr}\\
\le& C\cdot\Big(\frac{1}{|B|}\int_B\big|b(x)-b_B\big|^{pr'}\,dx\Big)^{1/pr'}\\
\le& C\|b\|_*.
\end{aligned}
\tag{11}
$$

So we have

$$
\frac{1}{w(B)^{\kappa/p}}\Big(\int_B \mathrm{I}^p\, w(x)\,dx\Big)^{1/p}\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}. \tag{12}
$$

On the other hand

$$
\begin{aligned}
\mathrm{II}\le& C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_B||f(y)|\,dy\\
\le& C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}||f(y)|\,dy\\
&+C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b_{2^{j+1}B}-b_B||f(y)|\,dy\\
=&\mathrm{III}+\mathrm{IV}.
\end{aligned}
$$

By using Hölder's inequality, we thus obtain

$$\begin{aligned}
&\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}||f(y)|\,dy \qquad (13)\\
\le&\Big(\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|^{p'}w^{-p'/p}(y)\,dy\Big)^{1/p'}\Big(\int_{2^{j+1}B}\big|f(y)\big|^{p}w(y)\,dy\Big)^{1/p}\\
\le&C\|f\|_{L^{p,\kappa}(w)}w(2^{j+1}B)^{\kappa/p}\Big(\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|^{p'}w^{-p'/p}(y)\,dy\Big)^{1/p'}.
\end{aligned}$$

Set $v(y)=w^{-p'/p}(y)=w^{1-p'}(y)$. Then we have $v\in A_{p'}$ because $w\in A_p$(see [3]). Following along the same lines as in the proof of (11), we can get

$$\Big(\frac{1}{v(2^{j+1}B)}\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|^{p'}v(y)\,dy\Big)^{1/p'}\le C\|b\|_*. \qquad (14)$$

Substituting the above inequality (14) into (13), we thus have

$$\begin{aligned}
\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}||f(y)|\,dy&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\cdot w(2^{j+1}B)^{\kappa/p}v(2^{j+1}B)^{1/p'}\\
&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\cdot|2^{j+1}B|w(2^{j+1}B)^{(\kappa-1)/p}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{w(B)^{\kappa/p}}\Big(\int_B \mathrm{III}^p\,w(x)\,dx\Big)^{1/p}&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}\frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}}\\
&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}. \qquad (15)
\end{aligned}$$

Finally, let's deal with the term IV. Since $b\in BMO(\mathbb{R}^n)$, then a simple calculation shows that

$$|b_{2^{j+1}B}-b_B|\le C\cdot j\|b\|_*.$$

Again, it follows from the inequality (5) that

$$\begin{aligned}
\mathrm{IV}&\le C\|b\|_*\sum_{j=1}^{\infty}j\cdot\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|f(y)\big|\,dy\\
&\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}j\cdot w(2^{j+1}B)^{(\kappa-1)/p}.
\end{aligned}$$

Therefore

$$\frac{1}{w(B)^{\kappa/p}}\Big(\int_B \mathrm{IV}^p\,w(x)\,dx\Big)^{1/p}\le C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}j\cdot\frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}}$$

$$\begin{aligned} &\leq C\|b\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}\frac{j}{2^{jn\theta}} \\ &\leq C\|b\|_*\|f\|_{L^{p,\kappa}(w)}, \end{aligned} \tag{16}$$

where $w \in RH_r$ and $\theta = (1-\kappa)(r-1)/pr$. Summarizing the estimates (15) and (16) derived above, we thus obtain

$$\frac{1}{w(B)^{\kappa/p}}\Big(\int_B \mathrm{II}^p\, w(x)\,dx\Big)^{1/p} \leq C\|b\|_*\|f\|_{L^{p,\kappa}(w)}. \tag{17}$$

Combining the inequalities (10), (12) with the above inequality (17) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we finally conclude the proof of Theorem 3. □

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