Kontsevich's swiss cheese conjecture

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To Mary, Mom, and Dad

Abstract

We prove a conjecture of Kontsevich which states that if A is an E_{d-1} algebra then the Hochschild cohomology object of A is the universal E_d algebra acting on A. The notion of an E_d algebra acting on an E_{d-1} algebra was defined by Kontsevich using the swiss cheese operad of Voronov. We prove a homotopical property of the swiss cheese operad from which the conjecture follows.

1. *Introduction*

In [**Kon99**] Kontsevich conjectured that the Hochschild complex $\mathrm{Hoch}(A)$ of an E_{d-1} algebra A is the universal E_d algebra acting on A. The story surrounding this conjecture dates back to 1963, when Gerstenhaber published a paper [**Ger63**] stating that the Hochschild cohomology $\mathrm{HH}^*(A)$ of an associative algebra A has the following algebraic structure. There are maps of graded vector spaces

$$\cdot\,,\ \ [,]\colon \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(A) \to \mathrm{HH}^*(A)$$

where $\cdot$ is a commutative associative product of degree 0, and $[,]$ is a graded Lie bracket of degree -1. These maps satisfy a compatibility condition known as the Poisson identity,

$$[a \cdot b, c] = a \cdot [b, c] \pm b \cdot [a, c],$$

which states that $[-, c]$ is a (graded) derivation of the commutative product $\cdot$ for every $c \in \mathrm{HH}^*(A)$.

The discovery of Gerstenhaber is purely algebraic. The story becomes interesting for us when it is tied to topology. In [**Coh76**] (see also [**Sin06**]) Fred Cohen computed the homology operad $H_*(E_d)$ and found that it is a quotient of the free graded operad on two generators m of arity 2 and degree 0, and β of arity 2 and degree $d-1$. The relations on m and β state exactly that m is commutative and associative, that β is a Lie bracket, and that m and β satisfy the Poisson identity. This immediately shows that the Hochschild cohomology complex $\mathrm{HH}^*(A)$ is an algebra over the negatively graded homology of E_2. We must take negatively graded homology to get β to have degree -1 instead of $+1$.

Deligne then posed the conjecture that the action

$$H_{-*}(E_2) \circlearrowright \mathrm{HH}^*(A)$$

of the (negatively graded) homology of E_2 on the graded vector space $\mathrm{HH}^*(A)$ descends from a natural action at the level of chains. In other words, is there a natural algebra structure

$$\mathsf{Chains}(E_2) \circlearrowright \mathrm{CH}^*(A)?$$

I am being vague here about what I mean by $\mathsf{Chains}(E_2)$. It could be singular, cellular, or otherwise, so long as it computes the right homology. Already, this question is evidently in the realm of homotopy theory. In that case, we may as well relax the condition that A be an associative algebra. We might say that A is a homotopy associative algebra, known as an A_∞ algebra. However, the question is more interesting if we say that A is an E_1 algebra; thus making it clear that the question is fundamentally one about the relationship between the operads E_1 and E_2. Indeed, it allows us to further consider the question in light of the relationship between E_d and E_{d-1} algebras.

For any E_{d-1} algebra in a sufficiently rich category $\mathcal{C}$ we can make sense of its Hochschild cohomology. The Hochschild cohomology of A is denoted $\mathrm{Hoch}(A)$ and is an object of $\mathcal{C}$. This is a bit confusing, as it is the Hochschild *cochain complex* in the case where $\mathcal{C}$ is the category of differential graded vector spaces, but in a broader context it is best to just refer to it as the Hochschild cohomology of A.

The original Deligne conjecture where A is an E_1 algebra in the category of chain complexes has been solved several times [**Tam98**, **BF04**, **MS02**, **KS00**, **Vor00**] The generalized version where A is an E_d algebra in a general category like $\mathcal{C}$ has been proven by [**HKV06**, **Lur09**]. What we show here is that $\mathrm{Hoch}(A)$ is not just an E_d algebra, but comes equipped with a universal property. It is the universal E_d algebra acting on the E_{d-1} algebra A.

The notion of an E_d algebra acting on an E_{d-1} algebra was also introduced in [**Kon99**]. This notion uses the swiss cheese operad SC_d of Voronov [**Vor99**]. This is a two-colored operad which interpolates between E_d and E_{d-1}. A swiss cheese algebra is a pair (B, A) where B is an E_d algebra, A is an E_{d-1} algebra, and there is some extra structre compatible with these (definition 2.4). We refer to this extra structure as an *action* of B on A.

The case $d = 1$ is enlightening. For simplicity, let us work in the category of vector spaces. A (non-unital) E_0 algebra A in vector spaces is just a vector space with no extra data. The Hochschild cohomology in this case is $\mathrm{hom}(A, A)$, which is clearly an associative algebra. Moreover, to make A into a left module over an associative algebra B agrees with the swiss cheese notion of giving an action of the E_1 algebra B on the E_0 algebra A. Clearly, this is also equivalent to giving a map of algebras from B to the E_0 Hochschild cohomology of A

$$B \to \mathrm{Hoch}(A) = \mathrm{hom}(A, A).$$

The plan of the paper is as follows. In section 3 we fix our semantics for trees and define the cofibrant replacement functor known as the W construction. In section 2 we define the E_d and SC_d operads, and give a precise statement of the theorem we will prove. We also outline the method of proof. In section 4 we define Hochschild cohomology for E_{d-1} algebras and prove a simplified version of the swiss cheese conjecture using an operad denoted $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ which has nothing to do with E_d. In section 5 we use a homotopy theoretic result to recover the action of E_d up to homotopy and obtain an operad SC_d^1, which is equivalent to SC_d, but is better for the study of the swiss cheese conjecture. In section 6 we prove the main theorem of the paper. Finally, section 7 contains the proof of the homotopy theoretic result used in section 5.

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2. *The swiss cheese operad*

Fix a set K, a K-colored set is a pair $(I, i : I \to K)$ where I is a set and i is a map of sets, called the coloring. We will often denote such a colored set simply by I, leaving the coloring implicit. A map of colored sets is a map of sets which commutes with the colorings. The notation $\mathrm{Aut}(I)$ denotes the bijections on the set I which preserve its coloring.

Definition 2.1. Let $(S, \otimes)$ be a symmetric monoidal category and let K be a set. A K-colored operad $\mathcal{O}$ in $\mathcal{S}$ is a symmetric monoidal category enriched over $\mathcal{S}$ whose objects are K-colored finite sets, $I \to K$. We require each hom object $\mathcal{O}(I, J)$ to be equipped with the structure of a right $\mathrm{Aut}(I)$-module and a left $\mathrm{Aut}(J)$-module. On objects, the symmetric monoidal structure must be disjoint union of sets. Thus, we have induced maps of left $\mathrm{Aut}(I)$ and right $\mathrm{Aut}(J)$ modules

$$\coprod_{I \to J} \mathrm{Ind}^{\mathrm{Aut}(I)}_{\prod_J \mathrm{Aut}(I_j)} \left(\bigotimes_J \mathcal{O}(I_j, j) \right) \to \mathcal{O}(I, J)$$

which we require to be isomorphisms. In the notation above, given a map $I \to J$ and given $j \in J$, the set $I_j \subset I$ is the pre-image of j. Also, the K-colored set j is the composition $j \subset J \to K$. The composition in the category $\mathcal{O}$ must be compatible with the group actions, which means composition can be written as maps

$$\mathcal{O}(I, J) \otimes_{\mathrm{Aut}(I)} \mathcal{O}(K, I) \to \mathcal{O}(K, J)$$

which are morphisms of left $\mathrm{Aut}(J)$ and right $\mathrm{Aut}(K)$ modules.

Remark 2.2. We will have examples of operads where not all $\mathcal{O}(I, I)$ contain identity morphisms.

A simple example is a 1-colored operad $\mathcal{O}$, that is $K = *$. We have objects $\mathcal{O}(I, J) \in \mathcal{S}$ for each set pair of sets I and J. We denote $\mathcal{O}(\underline{n}, \underline{1})$ simply by $\mathcal{O}(n)$, where $\underline{n}$ is the set $\{1, \ldots, n\}$. Each $\mathcal{O}(n)$ is equipped with a right action of the symmetric group S_n. Composition in this category is determined by the maps $\mathcal{O}(\underline{n}, \underline{1}) \otimes \mathcal{O}(\underline{m}, \underline{n}) \to \mathcal{O}(\underline{m}, \underline{1})$, which we expand as

$$\mathcal{O}(n) \otimes \coprod_{\substack{m = m_1 + \cdots + m_n \\ m_i \geq 0}} \mathrm{Ind}^{S_m}_{S_{m_1} \times \cdots \times S_{m_n}} (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \to \mathcal{O}(m).$$

More commonly, the definition of a 1-colored operad says that for each $n \geq 0$ and for each $m_1, \ldots, m_n$, we have a morphism

$$\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \to \mathcal{O}(m),$$

where $m = m_1 + \cdots + m_n$, satisfying some equivariance associativity conditions.

Example 2.1. One of the primary examples we will use is the little discs operad E_d. Set $\bar{D}^d$ as the closed unit disc inside $\mathbb{R}^d$. Call a map $f : \bar{D}^d \to \bar{D}^d$ a *little d disc* if f is of the form $f(x) = rx + c$ for some $0 < r \leq 1$ and $c \in \mathbb{R}^d$. A point in $E_d(n)$ is an n-tuple $(f_1, \ldots, f_n)$ of little d disks whose images are disjoint. Each f_i determines $0 < r_i \leq 1$ and $c_i \in \mathbb{R}^d$. Thus we can consider $E_d(n)$ as a subspace of $\mathbb{R}^{n+dn}$. In fact, $E_d(n)$ is an open subset of $\mathbb{R}^{n+d}$ so, in particular, it is a smooth manifold. The operadic structure is given by composing little d discs as maps $\bar{D}^d \to \bar{D}^d$. The identity of E_d is the little d-disc $\mathsf{id} \colon \bar{D}^d \to \bar{D}^d$. This is the unital version E_d, so $E_d(0) = *$ and $E_d(1)$ consists of more than just the identity.

Definition 2.3. Suppose $\mathcal{O}$ is an operad in $\mathcal{S}$ and $\mathcal{C}$ is a symmetric monoidal category enriched over $\mathcal{S}$. An *algebra* over $\mathcal{O}$ in the category $\mathcal{C}$ is a symmetric monoidal functor $A : \mathcal{O} \to \mathcal{C}$.

Now we are ready to define the two-colored operad commonly known as swiss cheese. More precisely, we define a swiss cheese operad for each dimension $d \geq 0$, denoted SC_d. This operad interpolates between E_d and E_{d-1}.

We can think of the E_d operad as one of the colors of the swiss cheese operad. We denote this color by e. The other color is denoted h. This stands for *half disc*. The swiss cheese operad is built out of two types of spaces, one corresponding to each color. The first type SC_d^{e} has its output in the shape of a disc, just as does E_d. The second type SC_d^{h} has its output in the shape of a half-disc. It is given by d-dimensional discs and half-discs inside the unit d-dimensional half-disc. The half-discs give an E_{d-1} structure to the operad, and the interplay of the discs and half-discs defines a notion of compatibility between E_d and E_{d-1} structures.

For the remainder of this paper fix $K = \{\mathrm{e}, \mathrm{h}\}$. Let the colored set $\{1, \dots, n\} \sqcup \{1, \dots, m\} \to K$, which sends $\{1, \dots, n\}$ to e and $\{1, \dots, m\}$ to h, be denoted by $\underline{n} + \underline{m}$.

$$SC_d^{\mathrm{e}}(n, m) := SC_d(\underline{n} + \underline{m}, \underline{1} + \underline{0}) = \begin{cases} E_d(n) & m = 0 \\ \emptyset & m > 0. \end{cases}$$

In addition, we denote $SC_d(\underline{n} + \underline{m}, \underline{0} + \underline{1})$ by $SC_d^{\mathrm{h}}(n, m)$. This is the space of n discs and m half-discs inside the unit half-disc

$$\bar{D}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid |x| \leq 1 \text{ and } x_d \geq 0\}.$$

In other words it is the space of $(n + m)$-tuples

$$(f_1, \dots, f_n, g_1, \dots, g_m)$$

where each $f_i : \bar{D}^d \to \bar{D}_+^d$ is of the form $f_i(x) = r_i x + c_i$ for some $0 < r_i < 1$ and $c_i \in \mathbb{R}^d$. Each $g_i : \bar{D}_+^d \to \bar{D}_+^d$ is of the form $g_i(x) = r_i' x + c_i'$ for some $0 < r_i' \leq 1$ and $c_i \in \mathbb{R}^{d-1} \times 0$. The images of all the discs and all the half-discs must be disjoint. $SC_d^{\mathrm{h}}(n, m)$ is an open subset of $\mathbb{R}^{n+m+dn+(d-1)m}$.

A point in $SC_d^{\mathrm{h}}(n, m)$ is given by n labeled discs and m labeled half-discs in the unit half-disc where none of the discs or half-discs intersect and the half-discs all lie on the bottom. We allow the degenerate configuration when $(n, m) = (0, 1)$ which is the unit half-disc contained in itself. Note that we have $SC_d^{\mathrm{h}}(0, 0) = *$ and $SC_d^{\mathrm{h}}(1, 0)$ contains more than one point. Thus we are using the unital swiss cheese operad. This differs from Kontsevich in [**Kon99**] and Voronov in [**Vor99**].

Composition in SC_d is given by substituting discs and half-discs into each other as in figure 1. More precisely, we have maps

$$E_d(n) \times E_d(k_1) \times \cdots \times E_d(k_n) \to E_d(k_1 + \cdots + k_n)$$

and

$$\begin{aligned} SC_d^{\mathrm{h}}(n, m) \times E_d(k_1) \times \cdots \times E_d(k_n) \times \\ SC_d^{\mathrm{h}}(k_{n+1}, l_1) \times \cdots \times SC_d^{\mathrm{h}}(k_{n+m}, l_m) \to \\ SC_d^{\mathrm{h}}(k_1 + \cdots + k_{n+m}, l_1 + \cdots + l_m). \end{aligned}$$

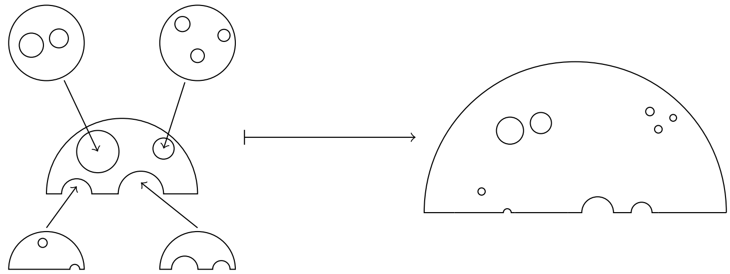
Notice that $SC_d^{\mathrm{h}}(0, m) = E_{d-1}(m)$ and that the restriction of SC_d to the spaces $SC_d^{\mathrm{h}}(0, -)$ is the operad E_{d-1}. This justifies the statement that SC_d interpolates between E_d and E_{d-1}.

If $\mathcal{C}$ is a category enriched over Top, we can consider an algebra over SC_d, which is a pair (B, A) of objects in $\mathcal{C}$ together with maps

$$E_d(n) \to \mathrm{map}(B^{\otimes n}, B)$$

and

$$SC_d^{\mathrm{h}}(n, m) \to \mathrm{map}(B^{\otimes n} \otimes A^{\otimes m}, A)$$



$$SC_d^{\mathrm{h}}(2,2) \times E_d(2) \times E_d(3) \times SC_d^{\mathrm{h}}(1,1) \times SC_d^{\mathrm{h}}(0,2) \to SC_d^{\mathrm{h}}(6,3)$$

FIGURE 1. *The operadic composition for SC_d when $d = 2$. Discs or half-discs can take inputs. The discs and half-discs should be labelled independently. We have omitted labels in the figure.*

The object B corresponds to the color e and the object A corresponds to h. Together these form a two-colored operad $\mathrm{End}(B, A)$ where

$$\mathrm{End}^{\mathrm{e}}(B,A)(n,m) = \mathrm{map}(B^{\otimes n} \otimes A^{\otimes m}, B)$$
$$\mathrm{End}^{\mathrm{h}}(B,A)(n,m) = \mathrm{map}(B^{\otimes n} \otimes A^{\otimes m}, A).$$

We will also make use of the fact that as SC_d algebra structure on (B, A) is given by a map of operads

$$SC_d \to \mathrm{End}(B, A).$$

One can see that a swiss cheese algebra (B, A) is an E_d algebra B, an E_{d-1} algebra A, and some compatibility between these structures. We refer to this compatiblity as an action of B on A. The following definition is due to Kontsevich [**Kon99**].

DEFINITION 2.4. Let B be an E_d algebra and A an E_{d-1} algebra. An action of B on A is the structure of a swiss cheese algebra on the pair (B, A) extending the given E_d and E_{d-1} structures.

With this we can informally state the conjecture proven in this paper.

THEOREM Kontsevich's Swiss Cheese Conjecture. *The Hochschild cohomology of an E_{d-1} algebra is the universal E_d algebra acting on it. In other words, giving a map of E_d algebras $B \to \mathrm{Hoch}(A)$ is equivalent to giving the structure of an SC_d algebra on the pair (B, A) extending the given E_d and E_{d-1} structures.*

First we will show how $\mathrm{Hoch}(A)$ is related to the swiss cheese operad. This requires dropping SC_d in favor of SC_d^{h1}, a simplified version of the swiss cheese operad which only remembers the spaces $SC_d^{\mathrm{h}}(0, m)$ and $SC_d^{\mathrm{h}}(1, m)$. The goal for the remainder of the paper will be to show that we can get from SC_d^{h1}, the one that understands Hochschild cohomology, back to SC_d, the operad defining actions of E_d algebras on E_{d-1} algebras.

This passage from SC_d^{h1} to SC_d happens in several steps. First, we freely extend SC_d^{h1} to $F(SC_d^{\mathrm{h1}})$ so that we can make sense of the spaces $F(SC_d^{\mathrm{h1}})(n, m)$ for $n > 1$. Next the most difficult part of the proof is a comparison result stating that $F(SC_d^{\mathrm{h1}})$ is weakly equivalent to SC_d^{h}, the operad obtained by forgetting about E_d.

Theorem. *Let SC_d^{h} be the two-colored operad obtained from SC_d by forgetting all outputs of color* e. *There is a canonical map $F(SC_d^{\mathrm{h1}}) \to SC_d^{\mathrm{h}}$ which is an equivalence of operads.*

The final step is to pass from SC_d^{h} to SC_d. The point is that the latter operad carries an action of E_d while the former does not. However, since they are equivalent we can, up to homotopy, pass the structure of the E_d action from one to the other. Once we have done this, we have a new operad SC_d^1 which is equivalent to SC_d. It is this operad which we will use to prove Kontsevich's conjecture. The e color of the operad is $\mathsf{E}_{\mathsf{d}-1}$, a cofibrant replacement of E_{d-1}.

Theorem (precise version of Kontsevich Swiss Cheese Theorem). *Let $\mathcal{C}$ be a symmetric monoidal model category tensored over* Top. *Assume that the category $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})$ inherits a model structure from the forgetful functor $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C}) \to \mathcal{C}$. Let A be an $\mathsf{E}_{\mathsf{d}-1}$ algebra in $\mathcal{C}$ which is fibrant and cofibrant as an object of $\mathcal{C}$. Let $\mathrm{Hoch}(A)$ denote the $\mathsf{E}_{\mathsf{d}-1}$ Hochschild cohomology of A. There is an equivalence of categories*

$$\mathsf{E}_\mathsf{d}\text{-alg}(\mathcal{C})_{/\mathrm{Hoch}(A)} \to \mathsf{SC}^1_\mathsf{d}\text{-alg}(\mathcal{C})_A$$

Where the category on the left has as its objects maps of E_d algebras $B \to \mathrm{Hoch}(A)$ and the category on the right has as objects SC^1_d algebras (B, A) where the induced $\mathsf{E}_{\mathsf{d}-1}$ structure on A agrees with the given $\mathsf{E}_{\mathsf{d}-1}$ structure on A.

2.1. *Outline of the proof*

Let Op_K denote the category of $K = \{\mathrm{e}, \mathrm{h}\}$-colored operads in Top. We use the notation $\underline{n} + \underline{m}$ to denote the K-colored set which is the disjoint union of

$$\{1, \ldots, n\} \to \{\mathrm{e}\} \text{ and } \{1, \ldots, m\} \to \{\mathrm{h}\}.$$

Any K-colored operad $\mathcal{O}$ is determined by spaces $\mathcal{O}(\underline{n} + \underline{m}, \underline{0} + \underline{1})$, which we denote by $\mathcal{O}^{\mathrm{h}}(n, m)$ and $\mathcal{O}(\underline{n} + \underline{m}, \underline{1} + \underline{0})$, which we denote by $\mathcal{O}^{\mathrm{e}}(n, m)$. We can think of the collection of spaces $\mathcal{O}^{\mathrm{h}}(n, m)$ as forming a K-colored operad whose e-colored output is always empty. We denote this operad simply by $\mathcal{O}^{\mathrm{h}}$. Denote the category of K-colored operads $\mathcal{O}$ satisfying $\mathcal{O}^{\mathrm{e}}(n, m) = \emptyset$ for all (n, m) by $\mathsf{Op}^\mathsf{h}_\mathsf{K}$. Furthermore, we can restrict to operads $\mathcal{O}$ where we also have $\mathcal{O}^{\mathrm{h}}(n, m) = \emptyset$ for $n \geq 2$. Denote this category by $\mathsf{Op}^{\mathsf{h1}}_\mathsf{K}$. There are forgetful functors (straight arrows) and a left adjoint (bent arrow). The images of SC_d under these functors is shown below.

$$\begin{array}{ccccc}
\mathsf{Op}_\mathsf{K} & \longrightarrow & \mathsf{Op}^\mathsf{h}_\mathsf{K} & \longrightarrow & \mathsf{Op}^{\mathsf{h1}}_\mathsf{K} \\
\mathsf{SC}_\mathsf{d} & \longmapsto & \mathsf{SC}^\mathsf{h}_\mathsf{d} & \longmapsto & \mathsf{SC}^{\mathsf{h1}}_\mathsf{d} \\
\mathsf{SC}^1_\mathsf{d} & \underset{\text{structure}}{\overset{\text{transfer of}}{\longleftarrow\!-\!-\!-}} & F(\mathsf{SC}^{\mathsf{h1}}_\mathsf{d}) & &
\end{array}$$

We begin with $\mathsf{SC}_\mathsf{d} \in \mathsf{Op}_\mathsf{K}$. This is a cofibrant replacement for SC_d. Specifically, it is $\mathrm{W}SC_d$, as defined in 3.1. We forget all the way down to $\mathsf{SC}^{\mathsf{h1}}_\mathsf{d}$. We will show that this operad controls Hochschild cohomology. We will prove an analogue of Kontsevich's swiss cheese theorem in this context. Next, we will take the free extension of $\mathsf{SC}^{\mathsf{h1}}_\mathsf{d}$ to an operad in $\mathsf{Op}^\mathsf{h}_\mathsf{K}$. The hard theorem of the paper shows that the canonical map $F(\mathsf{SC}^{\mathsf{h1}}_\mathsf{d}) \to \mathsf{SC}^\mathsf{h}_\mathsf{d}$ is a weak equivalence. The final step is to view SC_d as $\mathsf{SC}^\mathsf{h}_\mathsf{d}$ equipped with the extra structure of a right action of E_d. Then we use a transfer of structure argument to construct an operad SC^1_d which is $F(\mathsf{SC}^{\mathsf{h1}}_\mathsf{d})$ equipped with a homotopy action of E_d induced from the equivalence $F(\mathsf{SC}^{\mathsf{h1}}_\mathsf{d}) \to \mathsf{SC}^\mathsf{h}_\mathsf{d}$. We show that SC^1_d is equivalent to SC_d and prove the swiss cheese conjecture for this particular model of SC_d.

3. *Trees and the W construction*

Here we will define our nomenclature for trees. The following definition is similar to one in [**KS00**].

Definition 3.1. A forest F is given by the following data

- A finite set of *extended vertices* $V_{\mathrm{ext}}(F)$ together with a subset $V_r(F)$ called the *root* vertices. Each root vertex corresponds to a *tree* in the forest.
- A partition $V_{\mathrm{ext}}(F) = V_r(F) \sqcup V(F) \sqcup V_t(F)$ of the extended vertices into roots, *vertices*, and *tails*.
- A function (or flow) $N\colon V_{\mathrm{ext}}(F) \to V_{\mathrm{ext}}(F)$.

This data is subject to the following conditions.

- The roots are sinks for the flow given by N. That is, $N(v) = v$ for all $v \in V_r(F)$ and $N^k(v) \in V_r(F)$ for all $v \in V_{\mathrm{ext}}(F)$ and $k \gg 1$.
- The tails are sources for the flow. That is, $V_t(F) \cap N(V_{\mathrm{ext}}(F)) = \emptyset$. Note that these are not necessarily the only sources.
- Each tree in the forest has a trunk. More precisely, for each $v \in V_r(F)$ there is a unique vertex $v' \in V(F) \smallsetminus V_r(F)$ such that $N(v') = v$.

An *edge* of F is a pair $(v, N(v))$ where $v \notin V_r(F)$. The set of edges will be denoted $E(F)$. The *internal edges* of F, denoted $E_i(F)$, are those edges $(v, N(v))$ where v and $N(v)$ both belong to $V(F)$.

Given a vertex $v \in V(F)$ its *outgoing* or *root* or *output* edge is $(v, N(v))$. It will also be denoted $\mathrm{out}(v)$. If $N(v) \in V_r(F)$ then the edge $(v, N(v))$ is called a *root edge*, or *outgoing edge* to the forest F. The set of all root edges to F is denoted $\mathrm{out}(F)$.

Given a vertex $v \in V(F)$, the set of $(v', N(v')) \in E(F)$ such that $N(v') = v$ is called the the set of *input* or *incoming* or *tail* or *leaf* edges to v. This set is denoted by $\mathrm{in}(v)$. The set of edges $(v', N(v'))$ with $v' \in V_t(F)$ is denoted $\mathrm{in}(F)$, the set of input edges to F.

The *adjacent* vertices to an edge $(v, N(v))$ are the elements of $\{v, N(v)\} \cap V(F)$.

Definition 3.2. A tree is a forest T where $V_r(T) = *$.

There is one tree with no vertices. The single input edge coincides with the single ouput edge. This tree is written $|$, it is called the *identity* tree.

Definition 3.3. A *colored forest* with colors given by a set K is a forest F together with a coloring of the edges, that is, a map $\mathrm{colcol} onE(F) \to K$. Each $\mathrm{in}(v), \mathrm{out}(v), \mathrm{in}(F), \mathrm{out}(F)$ becomes a K-colored set.

Definition 3.4. A morphism of forests $F \to F'$ is a map $V_{\mathrm{ext}}(F) \to V_{\mathrm{ext}}$ preserving the decompositions $V_{\mathrm{ext}} = V_r \sqcup V \sqcup V_t$, commuting with N, and preserving the coloring.

An isomorphism is a bijection on extended vertices. The group of isomorphisms $F \to F$ is denoted $\mathrm{Aut}(F)$. There is an evident homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(V_t(F))$, where automorphisms of $V_t(F)$ preserve the coloring.

Given a K-colored set $I \to K$ and $k \in K$, the groupoid of trees T with $\mathrm{in}(T) = I$ and $\mathrm{out}(T) \cong k$ as K-colored sets is denoted by $\mathsf{Trees}(I, k)$.

Given $T_1, (v, N(v)) = \epsilon \in \mathrm{in}(T_1)$ and T_2 with $\mathrm{col}(\mathrm{out}(T_2) = \mathrm{col}(\epsilon)$ we can define $T_1 \circ_\epsilon T_2$ by gluing the root vertex v_r of T_2 to $N(v)$ and the gluing $N^{-1}(v_r)$ to v. This is called *grafting* the tree T_2 to the tree T_1 at the edge ϵ.

3.1. *The W construction*

We outline the W construction of Boardman and Vogt [**BV73**]. In nice situations this gives a cofibrant replacement for an operad, as shown in [**BM06**]. We use $[0, \infty]$ as our edge labels as in [**Kon99**]. Suppose $\mathcal{O}$ is any K-colored operad in Top. Given a tree T, put

$$\mathcal{O}(T) = \prod_{v \in V(T)} \mathcal{O}(\mathrm{in}(v), \mathrm{out}(v))$$

with an appropriate action of $\mathrm{Aut}(T)$. Also define $W(T) = [0, \infty]^{E_i(T)}$, with the appropriate action of $\mathrm{Aut}(T)$. The underlying collection of the operad $\mathrm{W}\mathcal{O}$ is

$$\mathrm{W}\mathcal{O}(I, k) = \left(\coprod_{[T] \in \pi_0 \mathsf{Trees}(I,k)} (\mathcal{O}(T) \times W(T)) \times_{\mathrm{Aut}(T)} \mathrm{Aut}(\mathrm{in}(T)) \right) /\!\sim$$

where $\sim$ makes the identifications in 3.1. We write a representative as $(\alpha, \mathbf{t})$ where $\alpha = (\alpha_v \in \mathcal{O}(\mathrm{in}(v), \mathrm{out}(v))_{v \in V(T)}$ and $\mathbf{t} \colon E_i(T) \to [0, \infty]$ for some tree T.

Relations 3.1.
- If an internal edge $\epsilon = (v, N(v))$ satisfies $\mathbf{t}(\epsilon) = 0$, replace T by T'/ϵ, which is T with the edge ϵ collapsed. This forms a new vertex v' whose corresponding label in $\mathcal{O}$ is $\alpha_{N(v)} \circ_v \alpha_v$.
- If a vertex v satisfies $\alpha_v = 1_{\mathcal{O}^c}$, the identity for some color $c \in K$, then define T' by deleting v and joining the two edges on either side of v into one new edge ϵ. Set $\mathbf{t}(\epsilon) = \mathbf{t}(N^{-1}(v), v) + \mathbf{t}(v, N(v))$.

Composition in $\mathrm{W}\mathcal{O}$ is given by grafting trees and labeling the new internal edge by ∞. In [**BM06**] it is shown that, for one-colored operads $\mathcal{O}$, $\mathrm{W}\mathcal{O}$ is a cofibrant operad if $\mathcal{O}$ is cofibrant as a collection and the identity $1_{\mathcal{O}} : * \to \mathcal{O}(1)$ is a cofibration. The model structure being used here is the one where a map of operads $\mathcal{O} \to \mathcal{P}$ is a weak equivalence (respectively fibration) if it is a weak equivalence (respectively fibration) when regarded as a map of collections. The proofs transfer easily to the K-colored case. Each $\mathcal{O}(I, k)$ must be a cofibrant right $\mathrm{Aut}(I)$ module, and the identity for each color $1_{\mathcal{O}^k} : * \to \mathcal{O}(k, k)$ must be a cofibration.

4. *Hochschild Cohomology from swiss cheese*

For the remainder of the paper we replace E_{d-1}, E_d, and SC_d by cofibrant models given by the Boardman-Vogt W construction from section 3.1. We will denote these cofibrant replacements by $\mathsf{E_{d-1}}, \mathsf{E_d}$, and $\mathsf{SC_d}$. We also want to restrict our attention to swiss cheese algebras in categories where we can do homotopy theory.

Definition 4.1. A symmetric monoidal model category tensored over Top is a symmetric monoidal model category category $\mathcal{C}$, together with a monoidal Quillen functor $\mathsf{Top} \to \mathcal{C}$ (See [**Hov99**], definition 4.2.20).

In particular, $\mathcal{C}$ comes equipped with functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \otimes : \mathsf{Top} \times \mathcal{C} \to \mathcal{C}.$$
$$\underline{\hom} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C} \quad \mathrm{map} : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Top} \quad ()^{()} : \mathsf{Top}^{op} \times \mathcal{C} \to \mathcal{C}$$

The mapping spaces $\mathrm{map}(A, B)$ give $\mathcal{C}$ the structure of a category enriched over Top, so we can speak of $\mathsf{E_{d-1}}, \mathsf{E_d}$ and $\mathsf{SC_d}$ algebras in $\mathcal{C}$.

For any object A of $\mathcal{C}$, the functor $-\otimes A$ has right adjoints $\underline{\hom}(A, -) : \mathcal{C} \to \mathcal{C}$ and $\mathrm{map}(A, -) : \mathcal{C} \to \mathsf{Top}$. Also, for any X in Top, the functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ has right adjoint $(-)^X : \mathcal{C} \to \mathcal{C}$. This data satisfies Quillen's SM7 axiom ([**Hov99**] section 4.2). We assume that the unit of C is cofibrant.

Throughout the rest of this paper A will denote a fixed algebra over $\mathsf{E_{d-1}}$ in a fixed symmetric monoidal model category $\mathcal{C}$ tensored over Top. We will define the category of $\mathsf{E_{d-1}}$-A modules, which is a generalization of the category of $A \otimes A^{op}$ modules when A is an associative algebra.

4.1. *The category* $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})$

Consider $\mathsf{E_{d-1}}$ as an operad in $\mathsf{Op}^{\mathrm{h1}}_K$ denoted $\mathsf{E}^{\mathrm{h1}}_{\mathsf{d-1}}$ where $\mathsf{E}^{\mathrm{h1}}_{\mathsf{d-1}}(n, m) = \mathsf{E_{d-1}}(m)$ for $n = 0, 1$. Recall that all other morphism spaces are empty by definition of $\mathsf{Op}^{\mathrm{h1}}_K$ in section 2.1. An algebra over $\mathsf{E}^{\mathrm{h1}}_{\mathsf{d-1}}$ is a pair (M, A) together with maps

$$\mathsf{E_{d-1}}(m) \to \mathrm{map}(A^{\otimes m}, A) \quad \mathsf{E_{d-1}}(m) \to \mathrm{map}(A^{\otimes m-1} \otimes M, A)$$

making A into an $\mathsf{E_{d-1}}$ algebra, and defining a notion of an $\mathsf{E_{d-1}}$-A module structure on M.

Definition 4.2. Let A be an $\mathsf{E_{d-1}}$ algebra. An $\mathsf{E_{d-1}}$-A module is an object M of $\mathcal{C}$ together with an $\mathsf{E}^{\mathrm{h1}}_{\mathsf{d-1}}$ action on (M, A) extending the $\mathsf{E_{d-1}}$ structure on A.

Let $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})$ denote the category of $\mathsf{E_{d-1}}$-A modules. A morphism is a map of $\mathsf{E}^{\mathrm{h1}}_{\mathsf{d-1}}$ algebras which is the identity on A.

There is a forgetful functor $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C}) \to \mathcal{C}$ which has a left adjoint denoted by Free. We define fibrations and weak equivalences in $\mathsf{E_{d-1}}$-A-mod to be those maps which are fibrations and weak equivalences respectively when we forget down to $\mathcal{C}$. Under certain conditions, this will give $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}$ the structure of a model category. However, we will not concern ourselves with these details since we use a specific model (definition 4.3) which is well-defined whether or not $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})$, with the given fibrations and equivalences, forms a model category. In the case that there is a model category structure, our notion will agree with the usual notion of Hochschild cohomology.

Suppose M and N are $\mathsf{E_{d-1}}$-A modules. The hom-space of morphisms from M to N is defined as the equalizer

$$\mathrm{map}_{\mathsf{E_{d-1}}\text{-}A}(M, N) \to \mathrm{map}(M, N) \rightrightarrows \mathrm{map}(\mathrm{Free}(M), N).$$

The two parallel arrows are adjoint to the maps

$$\mathrm{map}(M, N) \otimes \mathrm{Free}(M) \to \mathrm{Free}(N) \to N$$
$$\mathrm{map}(M, N) \otimes \mathrm{Free}(M) \to \mathrm{map}(M, N) \otimes M \to N,$$

where we have denoted the underlying objects of $\mathcal{C}$ for the $\mathsf{E_{d-1}}$-A modules M and N by the same letters. The map $\mathrm{Free}(N) \to N$ is the counit of the forgetful-free adjunction between $\mathcal{C}$ and $\mathsf{E_{d-1}}$-A modules. We make the analogous definition for $\underline{\hom}_{\mathsf{E_{d-1}}\text{-}A}(M, N)$. It is the equalizer

$$\underline{\hom}_{\mathsf{E_{d-1}}\text{-}A}(M, N) \to \underline{\hom}(M, N) \rightrightarrows \underline{\hom}(\mathrm{Free}(M), N).$$

Recall that $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ is a two-colored operad whose e-colored output is always empty and where $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(n,m)=\emptyset$ for $n\geq 2$. An algebra over $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ is a pair (B,A) where A is an $\mathsf{E}_{\mathsf{d}-1}$ algebra and a there are maps

$$\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m)\to \mathrm{map}(B\otimes A^{\otimes m},A)$$

compatible with the $\mathsf{E}_{\mathsf{d}-1}$ structure on A. We want to define the "universal" object H_A such that (H_A,A) is an extension of the $\mathsf{E}_{\mathsf{d}-1}$ structure on A to an $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebra. Define objects $\bar{A}^{sc}$ and $A^{sc}=\bar{A}^{sc}/\sim$,

$$\bar{A}^{sc}=\coprod_{m\geq 0}\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m)\otimes_{S_m}A^{\otimes m}. \tag{4.1}$$

We can think of $\bar{A}^{sc}$ heuristically as $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,-)\otimes A^{\otimes -}$. Similarly, A^{sc} should be thought of as $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,-)\otimes_{\mathsf{E}_{\mathsf{d}-1}}A^{\otimes -}$. More precisely, there is a coequalizer

$$\coprod_{m,m'}\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m)\otimes\mathsf{E}_{\mathsf{d}-1}(\underline{m'},\underline{m})\otimes A^{\otimes m'}\rightrightarrows\bar{A}^{sc}\to A^{sc}, \tag{4.2}$$

where one of the arrows is given by the operadic composition on swiss cheese

$$\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m)\otimes\mathsf{E}_{\mathsf{d}-1}(\underline{m'},\underline{m})\to\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m'),$$

and the other by the $\mathsf{E}_{\mathsf{d}-1}$ structure on A.

$$\mathsf{E}_{\mathsf{d}-1}(\underline{m'},\underline{m})\otimes A^{\otimes m'}\to A^{\otimes m}$$

Observe that A itself is an $\mathsf{E}_{\mathsf{d}-1}$-$A$ module by repeating the algebra structure of A as the module structure of A. Figure 2 shows the relation $\sim$ such that $A^{sc}=\bar{A}^{sc}/\sim$.

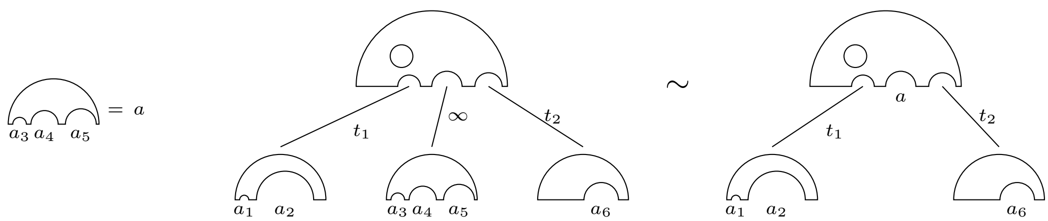


FIGURE 2. *The relations in A^{sc} come from the E_{d-1} algebra structure of A. If m is the map $A^{\otimes 3}\to A$ given by the swiss cheese element in $SC_d(0,3)$ in the figure, set $a=m(a_1,a_2,a_3)$. The edges t_1 and t_2 are less than ∞, so the relation does not apply to the vertices on the left and right.*

DEFINITION 4.3. Given an $\mathsf{E}_{\mathsf{d}-1}$ algebra $A\in\mathcal{C}$, let the *Hochschild cohomology object* of A be

$$\mathrm{Hoch}(A)=\underline{\mathrm{hom}}_{\mathsf{E}_{\mathsf{d}-1}\text{-}A}(A^{sc},A),$$

There is a map of $\mathsf{E}_{\mathsf{d}-1}$-$A$ modules $A^{sc}\to A$ given by the projection $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m)\to\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(0,m)=\mathsf{E}_{\mathsf{d}-1}(m)$ which forgets the single disc. The map $A^{sc}\to A$ applies this projection and then applies the $\mathsf{E}_{\mathsf{d}-1}$ algebra structure of A.

LEMMA 4.4. *Suppose the $\mathsf{E}_{\mathsf{d}-1}$ algebra A is cofibrant as an object of $\mathcal{C}$. The morphism $A^{sc}\to A$ exhibits A^{sc} as a cofibrant resolution of A in the category $\mathrm{Mod}^{A}_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})$.*

Proof. We postpone the proof that A^{sc} is cofibrant until section 7.1. The projections $\pi : \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m) \to \mathsf{E}_{\mathsf{d-1}}(m)$ define an acyclic fibration of operads $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}} \to \mathsf{E}_{\mathsf{d-1}}^{\mathrm{h1}}$ in the category $\mathsf{Op}_K^{\mathrm{h1}}$. The argument here will be made again in section 5. Since both $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ and $\mathsf{E}_{\mathsf{d-1}}^{\mathrm{h1}}$ are both cofibrant and fibrant and $\mathsf{Op}_K^{\mathrm{h1}}$ is tensored over Top we get a section $i : \mathsf{E}_{\mathsf{d-1}}(m) \to \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m)$ of the fibration π, which is compatible with the operad structures. This defines a morphism

$$A \simeq \{1_{\mathsf{E}_{\mathsf{d-1}}}\} \otimes A \to \mathsf{E}_{\mathsf{d-1}}(1) \otimes A \to \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,1) \otimes A \to A^{sc}$$

which splits the map $A^{sc} \to A$. Furthermore, we get a homotopy of operads $h : \mathsf{SC}_\mathsf{d}^{\mathrm{h1}} \otimes [0,1] \to \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ which gives a pair of commutative squares

$$\begin{array}{ccc}
\coprod_{m,m'} \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m) \otimes \mathsf{E}_{\mathsf{d-1}}(\underline{m}', \underline{m}) \otimes [0,1] \otimes A^{\otimes m'} & \rightrightarrows & \bar{A}^{sc} \otimes [0,1] \\
\downarrow & & \downarrow \\
\coprod_{m,m'} \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(1,m) \otimes \mathsf{E}_{\mathsf{d-1}}(\underline{m}', \underline{m}) \otimes A^{\otimes m'} & \rightrightarrows & \bar{A}^{sc},
\end{array}$$

which induces a homotopy $A^{sc} \times [0,1] \to A^{sc}$ interpolating between the identity and the composition $A^{sc} \to A \to A^{sc}$. □

Corollary 4.5. *The definition* (4.3) *for Hochschild cohomology of A agrees with the definitions given in [***HKV06***,* ***Fra08***] in the case that* $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d-1}}}(\mathcal{C})$ *inherits a model structure from $\mathcal{C}$ and A is cofibrant and fibrant as an object of $\mathcal{C}$. That is,*

$$\mathrm{Hoch}(A) = \underline{\mathrm{hom}}_{\mathsf{E}_{\mathsf{d-1}}\text{-}A}(A^c, A^f),$$

where A^c is a cofibrant replacement for A and A^f is a fibrant replacement for A as an $\mathsf{E}_{\mathsf{d-1}}$-$A$ module.

Proof. The model structure on $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d-1}}}(\mathcal{C})$ inherited from $\mathcal{C}$ Guarantees that A is fibrant since it is fibrant as an object of $\mathcal{C}$. Moreover, since A is cofibrant as an object of $\mathcal{C}$ lemma 4.4 shows that A^{sc} is a cofibrant replacement for A as an $\mathsf{E}_{\mathsf{d-1}}$-$A$ module. □

4.2. *An $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ Hochschild cohomology theorem*

We can prove a version of Kontsevich's swiss cheese conjecture for the operad $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$. That is, an $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ structure on the pair (B,A) is equivalent to a $\mathcal{C}$-morphism, $B \to \mathrm{Hoch}(A)$. In other words, $\mathrm{Hoch}(A)$ is the universal object of $\mathcal{C}$ acting on the $\mathsf{E}_{\mathsf{d-1}}$ algebra A through $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$.

Let $\mathcal{C}_{/\mathrm{Hoch}(A)}$ denote the over category of $\mathrm{Hoch}(A) \in \mathcal{C}$. Let $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}\text{-alg}(\mathcal{C})_A$ be the category whose objects are $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebras of the form (B,A) extending the $\mathsf{E}_{\mathsf{d-1}}$ structure on A. That is an object of $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}\text{-alg}(\mathcal{C})_A$ is an object $B \in \mathcal{C}$ together with a map of topological operads $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}} \to \mathrm{End}^{\mathrm{h1}}(B,A)$ making the following diagram commute

$$\begin{array}{ccc}
\mathsf{E}_{\mathsf{d-1}} & & \\
\downarrow & \searrow & \\
\mathsf{SC}_\mathsf{d}^{\mathrm{h1}} & \longrightarrow & \mathrm{End}^{\mathrm{h1}}(B,A)
\end{array}$$

A morphism is a map of spaces $B \to B'$ such that the map (B,A) to (B',A) which is the identity on A is a morphism of $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebras. Recall that $\mathrm{End}(B,A)$ is the K-colored endomorphism operad of the pair (B,A). The endomorphism operad $\mathrm{End}^{\mathrm{h1}}(B,A)$ remembers only the spaces $\mathrm{End}^{\mathrm{h}}(B,A)(n,m)$ for $n = 0,1$.

PROPOSITION 4.1. *Let A be an $\mathsf{E_{d-1}}$ algebra in $\mathcal{C}$. There is an equivalence of categories*

$$\mathcal{C}_{/\mathrm{Hoch}(A)} \simeq \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}\text{-alg}(\mathcal{C})_A$$

Proof. The data of an $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebra structure on (B, A) extending the $\mathsf{E_{d-1}}$ algebra structure on A is a collection of maps of topological spaces for each $m \geq 0$,

$$\mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m) \to \mathrm{map}(B \otimes A^{\otimes m}, A) \tag{4.3}$$

the hom-tensor adjunction, on the one hand, says such data is equivalent to a collection of $\mathcal{C}$-morphisms in for each $m \geq 0$.

$$\mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m) \otimes B \otimes A^{\otimes m} \to A.$$

On the other hand, the hom-tensor adjunction also gives us that such data is equivalent to a morphism in $\mathcal{C}$,

$$B \to \underline{\mathrm{hom}}(\coprod_m \mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m) \otimes_{S_m} A^{\otimes m}, A) = \underline{\mathrm{hom}}(\bar{A}^{sc}, A). \tag{4.4}$$

We will show that the maps in (4.3) define an $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebra structure on (B, A) if and only if the morphism in (4.4) factors through $\mathrm{Hoch}(A)$. Since the $\mathsf{E_{d-1}}$ structure is fixed, to check that the maps (4.3) define an $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebra structure we only need to check that they are compatible with the compositions

$$\mathsf{E_{d-1}}(m) \times \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(\underline{1} + \underline{m}', \underline{0} + \underline{m}) \to \mathsf{SC}_\mathsf{d}^{\mathrm{h1}}(\underline{1} + \underline{m}'), \underline{0}, \underline{1}) = \mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m') \tag{4.5}$$

and

$$\mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m) \times \mathsf{E_{d-1}}(\underline{m}', \underline{m}) \to \mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m'). \tag{4.6}$$

We will show that the maps in (4.3) are compatible with the maps (4.6) if and only if (4.4) factors through $\underline{\mathrm{hom}}(A^{sc}, A)$:

$$B \to \underline{\mathrm{hom}}(A^{sc}, A) \to \underline{\mathrm{hom}}(\bar{A}^{sc}, A).$$

We will also show that the maps in (4.3) are compatible with the maps (4.5) if and only if (4.4) further factors through $\underline{\mathrm{hom}}_{\mathsf{E_{d-1}}\text{-}A}(A^{sc}, A)$:

$$B \to \mathrm{Hoch}(A) = \underline{\mathrm{hom}}_{\mathsf{E_{d-1}}\text{-}A}(A^{sc}, A) \to \underline{\mathrm{hom}}(A^{sc}, A)$$

Clearly then a sequence of morphisms $B \to B' \to \mathrm{Hoch}(A)$ corresponds to a sequence of maps

$$\mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m) \to \mathrm{map}(B' \otimes A^{\otimes m}, A) \to \mathrm{map}(B \otimes A^{\otimes m}, A),$$

and thus a morphism of $\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}$ algebras $(B, A) \to (B', A)$.

For now let us take $B = \mathrm{Hoch}(A) \to \mathrm{Hoch}(A)$ to be the identity map. Recall that $\mathrm{Hoch}(A)$ is defined as the equalizer

$$\underline{\mathrm{hom}}_{\mathsf{E_{d-1}}\text{-}A}(A^{sc}, A) \to \underline{\mathrm{hom}}(A^{sc}, A) \rightrightarrows \underline{\mathrm{hom}}(\mathrm{Free}\ A^{sc}, A), \tag{4.7}$$

where $\mathrm{Free}\ A^{sc}$ is the free $\mathsf{E_{d-1}}$-A module on the underlying $\mathcal{C}$-object of A^{sc}:

$$\mathrm{Free}\ A^{sc} = \coprod_{m \geq 0} \mathsf{E_{d-1}}(m+1) \otimes_{S_m} A^{\otimes m} \otimes A^{sc}.$$

Let $H = \mathrm{Hoch}(A)$. To show that the structure maps in (4.3) are compatible with (4.5) is equivalent to showing that the following diagram commutes

$$\begin{array}{ccc} \mathsf{E_{d-1}}(m) \otimes \mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m') \otimes H \otimes A^{\otimes m+m'-1} & \xrightarrow{\circ_i} & \mathsf{E_{d-1}}(m) \otimes A^{\otimes m} \\ \circ_i \big\downarrow & & \big\downarrow \\ \mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, m+m'-1) \otimes H \otimes A^{\otimes m+m'-1} & \longrightarrow & A. \end{array} \tag{4.8}$$

We can factor this diagram into two parts, the first is identity on H, so we remove it from the diagram:

$$\begin{array}{ccc}
\mathsf{E_{d-1}}(m) \otimes \mathsf{SC^h_d}(1, m') \otimes A^{\otimes m+m'-1} & \longrightarrow & \mathsf{E_{d-1}}(m) \otimes A^{\otimes i-1} \otimes A^{sc} \otimes A^{\otimes m-i} \\
\Big\downarrow & & \downarrow \\
 & & \text{Free } A^{sc} \\
 & & \downarrow \\
\mathsf{SC^h_d}(1, m+m'-1) \otimes A^{\otimes m+m'-1} & \longrightarrow & A^{sc}.
\end{array}$$

The second does involve H.

$$\begin{array}{ccc}
\mathrm{Hoch}(A) \otimes \text{Free } A^{sc} & \longrightarrow & \text{Free } A \\
\downarrow & & \downarrow \\
\mathrm{Hoch}(A) \otimes A^{sc} & \longrightarrow & A.
\end{array}$$

The first diagram commutes because A^{sc} is an $\mathsf{E_{d-1}}$-A module. The second commutes because $\mathrm{Hoch}(A)$ is an equalizer in (4.7).

Now to show that the structure maps (4.3) is compatible with (4.5) we show the following diagram commutes

$$\begin{array}{ccc}
\mathsf{SC^h_d}(1, m) \otimes \mathsf{E_{d-1}}(m') \otimes H \otimes A^{\otimes m+m'-1} & \xrightarrow{\circ_i} & \mathsf{SC^h_d}(1, m) \otimes H \otimes A^{\otimes m} \\
{\scriptstyle \circ_i}\downarrow & & \downarrow \\
\mathsf{SC^h_d}(1, m+m'-1) \otimes H \otimes A^{\otimes m+m'-1} & \longrightarrow & A.
\end{array}$$

This can be written as

$$\begin{array}{ccccc}
\mathsf{SC^h_d}(1, m) \otimes \mathsf{E_{d-1}}(m') \otimes H \otimes A^{m+m'-1} & \xrightarrow{\circ_i} & \mathsf{SC^h_d}(1, m) \otimes H \otimes A^{\otimes m} & & \\
{\scriptstyle \circ_i}\downarrow & & \downarrow & & \\
\mathsf{SC^h_d}(1, m+m'-1) \otimes H \otimes A^{m+m'-1} & \longrightarrow & A^{sc} \otimes H & \longrightarrow & A
\end{array}$$

This square commutes because A^{sc} is obtained from the coequalizer diagram (4.2).

Now suppose $\mathsf{SC^{h1}_d}$ acts on the pair (B, A) so that the induced $\mathsf{E_{d-1}}$ algebra structure on A is the one given. Let us first observe that the induced map $B \to \underline{\mathrm{hom}}(\bar{A}^{sc}, A)$ factors through $\underline{\mathrm{hom}}(A^{sc}, A)$. Indeed, the two arrows

$$\coprod_{m, m' \geq 0} \mathsf{SC^h_d}(1, m) \otimes \mathsf{E_{d-1}}(\underline{m}', \underline{m}) \otimes B \otimes A^{\otimes m'} \rightrightarrows A$$

are equal precisely because we assumed that the structure maps (4.3) are compatible with the maps (4.6). One of these arrows is given by the action of $\mathsf{E_{d-1}}$ on $\mathsf{SC^{h1}_d}$ and the other by the action of $\mathsf{E_{d-1}}$ on A. This implies that we get a map $B \to \underline{\mathrm{hom}}(A^{sc}, A)$. Finally, we observe that this map equalizes the arrows in (4.7) because (4.3) is compatible with (4.5), which is to say the diagram (4.8) commutes with H replaced by B. □

5. *From* $\mathsf{SC^{h1}_d}$ *to* $\mathsf{SC^1_d}$

In this section we use the model structure on $\mathsf{Op}^{\mathrm{h}}_K$ inherited from $\mathsf{Coll}^{\mathrm{h}}_K$. The latter category has its objects given by collections of spaces $P(n, m)$ for $n, m \geq 0$, each equipped with a right action of $S_n \times S_m$. Morphisms are collections of equivariant maps. In [**BM03**], we see that topological operads have a model category structure where a weak equivalence (resp. fibration) is a map of operads $\mathcal{O} \to \mathcal{P}$ where each $\mathcal{O}(n,m) \to \mathcal{P}(n,m)$ is a weak equivalence (resp. Serre fibration).

We know that $\mathsf{SC}_\mathsf{d}^\mathrm{h}$ is cofibrant since it is obtained as the W construction applied to a Σ-cofibrant, well-pointed operad SC_d^h [**BM06**]. Moreover, we see that every operad in Op_K^h is fibrant. We will prove the following theorem in section 7.2.

Theorem 5.1. *The natural map $F(\mathsf{SC}_\mathsf{d}^\mathrm{h1}) \to \mathsf{SC}_\mathsf{d}^\mathrm{h}$ is an acyclic cofibration of operads in Op_K^h.*

Thus we have a lift in the following diagram.

$$\begin{array}{ccc} F(\mathsf{SC}_\mathsf{d}^\mathrm{h1}) & \xrightarrow{\mathsf{id}} & F(\mathsf{SC}_\mathsf{d}^\mathrm{h1}) \\ {\scriptstyle \iota}\downarrow & \overset{p}{\nearrow} & \\ \mathsf{SC}_\mathsf{d}^\mathrm{h} & & \end{array}$$

Also, by the corner axiom (Quillen's SM7) for monoidal model categories tensored over topological spaces [**Spi01**] we have an acyclic fibration

$$\mathrm{map}(\mathsf{SC}_\mathsf{d}^\mathrm{h}, \mathsf{SC}_\mathsf{d}^\mathrm{h}) \xrightarrow{\iota^*} \mathrm{map}(F(\mathsf{SC}_\mathsf{d}^\mathrm{h1}), \mathsf{SC}_\mathsf{d}^\mathrm{h})$$

given by pre-composing with ι. Since both ιp and id live over ι, they must be homotopic. Let $h\colon \mathsf{SC}_\mathsf{d}^\mathrm{h} \otimes [0,\infty] \to \mathsf{SC}_\mathsf{d}^\mathrm{h}(n,m)$ be a homotopy with $h_0 = h(-,0) = \mathsf{id}$ and $h_\infty = \iota p$.

Let $\tilde{\mathsf{E}}_\mathsf{d}$ denote the W construction on $\tilde{E}_d$, which is E_d considered as an operad without identity. The underlying spaces are the same, we just forget that $\mathsf{id} \in E_d(1)$ is special. Thus, in $\tilde{\mathsf{E}}_\mathsf{d}$ we cannot delete a vertex labelled with the identity. Let $\mathsf{L}\tilde{\mathsf{E}}_\mathsf{d}$ denote the *levelling* of $\tilde{\mathsf{E}}_\mathsf{d}$. Given finite sets I, J, $\mathsf{L}\tilde{\mathsf{E}}_\mathsf{d}(I,J)$ is a quotient of

$$\coprod_{\substack{l \geq 0 \\ I_l = I, I_{l-1}, \ldots, I_1, I_0 = J}} E_d(I_1, I_0) \times [0,\infty] \times \cdots \times [0,\infty] \times E_d(I_l, I_{l-1})$$

by a relation which allows us to perform the composition $E_d(I_i, I_{i-1}) \times E_d(I_{i+1}, I_i) \to E_d(I_{i+1}, I_{i-1})$ if the coordinate corresponding to the intermediate factor of $[0,\infty]$ is 0. Composition in $\mathrm{W}\tilde{E}_d$ is given by concatenating sequences, setting the coordinate in $[0,\infty]$ between the two sequences to be ∞. Note that $\tilde{\mathsf{E}}_\mathsf{d}$ has no identity morphisms and there are no relations allowing us to delete the appearance of an identity in a sequence.

Define a map $\tilde{\mathsf{E}}_\mathsf{d}(I,J) \to \mathsf{L}\tilde{\mathsf{E}}_\mathsf{d}(I,J)$ by inducting on the height of the vertices in trees. A vertex in a tree has height k if it is separated from the root vertex by a shortest path of k edges. Given a morphism in $\tilde{\mathsf{E}}_\mathsf{d}(I,J)$ given as a labeled set of trees, look at the labels on the $|J|$ vertices of height 0 to get an element of $E_d(I_1, J)$ where I_1 is the set of incoming edges to the height 0 vertices.

Next, we get an element of $[0,\infty]$ by taking the smallest label, call it t_1 among the incoming edges to the height 0 vertices. We insert $\mathsf{id} \in E_d(1)$ into all edges longer than t_1, replacing an edge of length $t > t_1$ by an edge of length $t - t_1$ and an edge of length t_1. The result has all incoming edges to the height 0 vertices labeled t_1.

By induction, assume that all incoming edges to the height $0, \ldots, i-1$ vertices have the same length. The labels on the height i vertices define an element of $E_d(I_{i+1}, I_i)$, where I_i is the set of incoming edges to the height $i-1$ vertices. Let t_{i+1} be the length of the shortest incoming edge to the height i vertices and insert identitites into all longer incoming edges so that we get equal lengths (all t_{i+1}) on all incoming edges to height i vertices.

We can use $\mathsf{L}\tilde{\mathsf{E}}_\mathsf{d}$ as well as the maps h_t, p, ι and the action of $\tilde{E}_d$ on $\mathsf{SC}^\mathsf{h}_\mathsf{d}$ to define an action of $\tilde{\mathsf{E}}_\mathsf{d}$ on $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$. Let us construct a map

$$F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n,m) \times \tilde{\mathsf{E}}_\mathsf{d}(\underline{k},\underline{n}) \to F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(k,m) \tag{5.1}$$

for each $n, m, k \geq 0$. Given $\alpha \in \tilde{\mathsf{E}}_\mathsf{d}(\underline{k},\underline{n})$ we look at the levelling of α in $\mathsf{L}\tilde{\mathsf{E}}_\mathsf{d}(\underline{k},\underline{n})$. This is a sequence $n_l = k, n_{l-1} \dots, n_1, n_0 = n$ together with $\alpha_i \in \tilde{E}_d(\underline{n}_{i+1}, \underline{n}_i)$ and $t_i \in [0,\infty]$ then we get a chain of maps

$$F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n_0,m) \xrightarrow{\iota} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_0,m) \xrightarrow{\alpha_1} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_1,m) \xrightarrow{h_{t_1}} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_1,m) \xrightarrow{\alpha_2} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_2,m)$$
$$\xrightarrow{h_{t_2}} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_2,m) \xrightarrow{\alpha_3} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_3,m) \to \cdots \to \mathsf{SC}^\mathsf{h}_\mathsf{d}(k,m) \xrightarrow{p} F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n,m).$$

The maps $\mathsf{SC}^\mathsf{h}_\mathsf{d}(n_i,m) \xrightarrow{\alpha_{i+1}} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i+1},m)$ are defined by the action of E_d on $\mathsf{SC}^\mathsf{h}_\mathsf{d}$:

$$\mathsf{SC}^\mathsf{h}_\mathsf{d}(n_i,m) \times E_d(\underline{n}_{i+1},\underline{n}_i) \to \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i+1},m).$$

Let us check that the relations in $\tilde{\mathsf{E}}_\mathsf{d}$ are satisfied and that composition in $\tilde{\mathsf{E}}_\mathsf{d}$ corresponds to composition of maps of $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$. Since we have no relation regarding the identity of E_d, the only relation we consider is when a length is 0. Suppose $t_i = 0$ for some i. Then $h_0 = \mathsf{id}$ so our chain of arrows contains

$$\mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i-1},m) \xrightarrow{\alpha_i} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_i,m) \xrightarrow{\alpha_{i+1}} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i+1},m).$$

The composition of these two is equal to the map given by $\alpha_i\alpha_{i+1} \in \tilde{E}_d(\underline{n}_{i+1},\underline{n}_{i-1})$. This is because each $\mathsf{SC}^\mathsf{h}_\mathsf{d}(-,m)$ is a right E_d module.

Now suppose we have some $t_i = \infty$, so that $\alpha \in \tilde{\mathsf{E}}_\mathsf{d}(\underline{n}_l,\underline{n}_0)$ decomposes as $\beta^1\beta^2$ for some $\beta^1 \in \tilde{\mathsf{E}}_\mathsf{d}(\underline{n}_{i-1},\underline{n}_0)$ and $\beta^2 \in \tilde{\mathsf{E}}_\mathsf{d}(\underline{n}_l,\underline{n}_{i-1})$. The chain of compositions defining the action of α from $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n_0,m)$ to $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n_l,m)$ contains the following segment.

$$\cdots \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i-1},m) \xrightarrow{h_\infty} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i-1},m) \to \cdots$$

The action of $\beta^1\beta^2$ is computed by joining the chains for β^1 and for β^2. This joined chain agrees with the chain for α except for the segment above, which is replaced with the segment

$$\cdots \to \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i-1},m) \xrightarrow{p} F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n_{i-1},m) \xrightarrow{\iota} \mathsf{SC}^\mathsf{h}_\mathsf{d}(n_{i-1},m) \to \cdots$$

Since $h_1 = \iota p$, these chains of maps have the same composition.

Note that we have not allowed the identity relation in $\tilde{\mathsf{E}}_\mathsf{d}$ because we cannot guarantee that $h_t \circ h_s = h_{t+s}$.

5.1. $\mathsf{SC}^1_\mathsf{d} = F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d}) \ltimes \tilde{\mathsf{E}}_\mathsf{d}$

The operad $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$, the operad $\tilde{\mathsf{E}}_\mathsf{d}$ and the action of $\tilde{\mathsf{E}}_\mathsf{d}$ on $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$ define a two colored operad SC^1_d which can be thought of as a semi-direct product of $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$ and $\tilde{\mathsf{E}}_\mathsf{d}$. For any finite K-colored set $I \simeq (n,m)$ define

$$\mathsf{SC}^1_\mathsf{d}(I,\{\mathrm{e}\}) = F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})(n,m) \quad \mathsf{SC}^1_\mathsf{d}(I,\{\mathrm{h}\}) = \begin{cases} \tilde{\mathsf{E}}_\mathsf{d}(n) & m = 0 \\ \emptyset & m \neq 0 \end{cases}$$

Composition is defined using composition in $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$, composition in $\tilde{\mathsf{E}}_\mathsf{d}$ and using the action of $\tilde{\mathsf{E}}_\mathsf{d}$ on $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$ just defined in section 5. This is why we think of this operad as a sort of semi-direct product of operads. The homotopy h respects operadic composition in $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$, so the action of $\tilde{\mathsf{E}}_\mathsf{d}$ is compatible with the action of $F(\mathsf{SC}^{\mathrm{h1}}_\mathsf{d})$ on itself.

We will prove Kontsevich's conjecture using the operad SC^1_d. Thus, we need to show that SC^1_d is equivalent to the original swiss cheese operad SC_d. The argument is standard for the homotopy transfer of operadic structures. First, note that SC_d is equivalent to the semi-direct product of $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}$ and $\tilde{\mathsf{E}}_{\mathsf{d}}$ where the action of $\tilde{\mathsf{E}}_{\mathsf{d}}$ factors through the map $\tilde{\mathsf{E}}_{\mathsf{d}} \to \tilde{E}_d$ which sends all lengths of internal edges to zero. This is because the map $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}} \to SC_d^{\mathrm{h}}$ which collapses trees is a weak equivalence and respects the action of $\tilde{E}_d$.

Next, the action of $\tilde{\mathsf{E}}_{\mathsf{d}}$ on $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ can be extended to an action on all of $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}$. The sequence $\alpha_1, t_1, \cdots, t_l, \alpha_l$ acts via the composition

$$\begin{aligned} &\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_0, m) \xrightarrow{\alpha_1} \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_1, m) \xrightarrow{h_{t_1}} \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_1, m) \to \cdots \to \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_{l-1}, m) \\ &\qquad \xrightarrow{h_{t_l}} \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_{l-1}, m) \xrightarrow{\alpha_l} \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_l, m) \xrightarrow{\iota p} \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n_l, m). \end{aligned} \tag{5.2}$$

We can interpolate between this "fancy" action of $\tilde{\mathsf{E}}_{\mathsf{d}}$ on $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}$ and the "simple" action of $\tilde{\mathsf{E}}_{\mathsf{d}}$ on $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}$ discussed above. To do this, define for each $s \in [0, \infty]$ a homotopy $h^{[0,s]} \colon \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}} \otimes [0, \infty] \to \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}$ by setting $h_t^{[0,s]} = h_{\min(s,t)}$. Using this, we can define an action of $\tilde{\mathsf{E}}_{\mathsf{d}}$ on $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}} \otimes [0, \infty]_s$ by replacing h_{t_i} in (5.2) with $h_{t_i}^{[0,s]}$. Then, when $s = 0$ each $h_{t_i}^{[0,0]}$ is the identity, so the action is the "simple" one. When $s = 1$ we have $h_{t_i}^{[0,1]} = h_{t_i}$ so the action is the "fancy" one. Thus we have a diagram of equivalences

$$SC_d \leftarrow \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}} \ltimes_{\text{simple}} \tilde{\mathsf{E}}_{\mathsf{d}} \to (\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}} \otimes [0,1]) \ltimes \tilde{\mathsf{E}}_{\mathsf{d}} \leftarrow \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}} \ltimes_{\text{fancy}} \tilde{\mathsf{E}}_{\mathsf{d}} \to \mathsf{SC}_{\mathsf{d}}^{1}.$$

Thus we are justified in replacing SC_d by $\mathsf{SC}_{\mathsf{d}}^{1}$.

6. *Kontsevich's swiss cheese conjecture*

We have shown that $\mathrm{Hoch}(A) = H$ is an algebra over $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}}$. Since $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ is the free extension of $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}}$ to an operad in Op_K^h, we then get an $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ structure on H

$$F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}}) \to \mathrm{End}^{\mathrm{h}}(H, A).$$

The level (n, m) component of this map is adjoint to

$$H^{\otimes n} \to \underline{\mathrm{hom}}\Big(\coprod_{m \geq 0} \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n, m) \otimes_{S_m} A^{\otimes m}, A\Big) =: \underline{\mathrm{hom}}(\bar{A}^{sc}(n), A). \tag{6.1}$$

Here $\bar{A}^{sc}(n)$ is defined analagous to $\bar{A}^{sc} = \bar{A}^{sc}$ in (4.1). In addition, if we take the $\tilde{\mathsf{E}}_{\mathsf{d}}$ action on $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ from (5.1), tensor with $A^{\otimes m}$ and apply $\underline{\mathrm{hom}}(-, A)$ we get

$$\tilde{\mathsf{E}}_{\mathsf{d}}(\underline{n}, \underline{1}) \otimes \underline{\mathrm{hom}}(\bar{A}^{sc}(n), A) \to \underline{\mathrm{hom}}(\bar{A}^{sc}(1), A). \tag{6.2}$$

Since the $\mathsf{E}_{\mathsf{d-1}}$ structure on A is preserved by the right action of $\tilde{\mathsf{E}}_{\mathsf{d}}$ the map above factors through H

$$\tilde{\mathsf{E}}_{\mathsf{d}}(\underline{n}, \underline{1}) \otimes \underline{\mathrm{hom}}(\bar{A}^{sc}(n), A) \to \mathrm{Hoch}(A).$$

Together, (6.1) and (6.2) define the $\tilde{\mathsf{E}}_{\mathsf{d}}$ structure on H:

$$\tilde{\mathsf{E}}_{\mathsf{d}}(\underline{n}, \underline{1}) \otimes H^{\otimes n} \to \tilde{\mathsf{E}}_{\mathsf{d}}(\underline{n}, \underline{1}) \otimes \underline{\mathrm{hom}}(\bar{A}^{sc}(n), A) \to H.$$

In fact, this makes (H, A) into an algebra over $\mathsf{SC}_{\mathsf{d}}^{1}$. The $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ and $\tilde{\mathsf{E}}_{\mathsf{d}}$ structures must be shown to be compatible. This is equivalent to showing that the following triangle commutes for all n, m, k.

$$\begin{array}{ccc} & & \mathrm{map}(H^{\otimes n} \otimes A^{\otimes m}, A) \\ & \nearrow & \downarrow \\ F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})(n, m) & \longrightarrow & \mathrm{map}(\tilde{\mathsf{E}}_{\mathsf{d}}(\underline{k}, \underline{n}) \otimes H^{\otimes k} \otimes A^{\otimes m}, A) \end{array} \tag{6.3}$$

The diagonal arrow is the $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ structure on (H, A). The vertical arrow is induced by the action on H just defined: $\tilde{\mathsf{E}}_{\mathsf{d}}(k, n) \otimes H^{\otimes k} \to H^{\otimes n}$. The horizontal arrow uses the $\tilde{\mathsf{E}}_{\mathsf{d}}$ action on $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ followed by the $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ structure on (H, A).

This triangle is given by a diagram of operads in $\mathsf{Op}_K^{\mathrm{h}}$. Since every composition of arrows in the diagram begins in $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ we can use the fact that $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ is freely generated by its $(0, m)$ and $(1, m)$ components. Thus we only need to show that the above diagram commutes when $n = 0$ or 1. In order to refer to this argument later, so we label it.

> In any diagram of spaces which is the (n, m) component of operads in $\mathsf{Op}_K^{\mathrm{h}}$ and which compares maps out of $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ the diagram commutes if and only if it commutes when $n = 0, 1$. (6.4)

To be precise, there is an operad in $\mathsf{Op}_K^{\mathrm{h}}$ which we will denote $\mathrm{End}^{\mathrm{h}}(\tilde{\mathsf{E}}_{\mathsf{d}} \otimes H, A)$. The components of this operad are

$$\mathrm{End}^{\mathrm{h}}(\tilde{\mathsf{E}}_{\mathsf{d}} \otimes H, A)(n, m) = \mathrm{map}\left(\left(\coprod_{k \geq 0} \tilde{\mathsf{E}}_{\mathsf{d}}(\underline{k}, \underline{n}) \otimes H^{\otimes k}\right) \otimes A^{\otimes m}, A\right). \tag{6.5}$$

Composition is given by substitution of maps in A and by tensoring together factors of the form $H^{\otimes k}$ and using the monoidal structure on the category $\tilde{\mathsf{E}}_{\mathsf{d}}$. This operad gives the lower right corner of the diagram. The upper right corner is given by $\mathrm{End}^{\mathrm{h}}(H, A)$.

The triangle commutes when $n = 0$ because the vertical map becomes the identity and the diagonal and horizontal maps agree. In the case $n = 1$ commutativity is given by the definition of the $\tilde{\mathsf{E}}_{\mathsf{d}}$ structure on H.

Let $\mathsf{SC}_{\mathsf{d}}^{1}\text{-alg}(\mathcal{C})_A$ be the category given by objects B of spaces together with the structure of an $\mathsf{SC}_{\mathsf{d}}^{1}$ algebra on (B, A) extending the given $\mathsf{E}_{\mathsf{d-1}}$ structure on A. That is, objects are objects B of $\mathcal{C}$ together with a morphism of operads $\mathsf{SC}_{\mathsf{d}}^{1} \to \mathrm{End}(B, A)$ preserving the $\mathsf{E}_{\mathsf{d-1}}$ structure on A,

$$\begin{array}{ccc} \mathsf{E}_{\mathsf{d-1}} & \longrightarrow & \mathrm{End}(B, A) \\ \downarrow & \nearrow & \\ \mathsf{SC}_{\mathsf{d}}^{1} & & \end{array}$$

where $\mathsf{E}_{\mathsf{d-1}}$ is considered as an $\{\mathrm{e}, \mathrm{h}\}$ colored operad by setting the e color to be the trivial operad. Also, let us define $\tilde{\mathsf{E}}_{\mathsf{d}}\text{-alg}(\mathcal{C})_{/\mathrm{Hoch}(A)}$ to be the category whose objects are maps of $\tilde{\mathsf{E}}_{\mathsf{d}}$ algebras $B \to \mathrm{Hoch}(A)$ and whose morphisms are commutative triangles.

Theorem 6.1. *There is an equivalence of categories*

$$\tilde{\mathsf{E}}_{\mathsf{d}}\text{-alg}(\mathcal{C})_{/\mathrm{Hoch}(A)} \simeq \mathsf{SC}_{\mathsf{d}}^{1}\text{-alg}(\mathcal{C})_A.$$

Proof.

Given a map of $\tilde{\mathsf{E}}_{\mathsf{d}}$ algebras $B \to \mathrm{Hoch}(A)$, the underlying map of objects of $\mathcal{C}$ gives an action of $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}}$ on the pair (B, A) by proposition (4.1). This freely extends to an $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ structure on (B, A). Thus we have an action of each color of $\mathsf{SC}_{\mathsf{d}}^{1}$ on (B, A). In order for this to give an $\mathsf{SC}_{\mathsf{d}}^{1}$ structure on (B, A) we must check that the $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})$ structure is compatible with $\tilde{\mathsf{E}}_{\mathsf{d}}$ structure.

$$\begin{array}{ccccccc}
F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(1,m)\otimes\tilde{\mathsf{E}_{\mathsf{d}}}(\underline{k},\underline{1})\otimes B^{\otimes k} & & & \longrightarrow & & & F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(k,m)\otimes B^{\otimes k}\\
\downarrow & \searrow & & & & \swarrow & \downarrow\\
 & & F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(1,m)\otimes\tilde{\mathsf{E}_{\mathsf{d}}}(\underline{k},\underline{1})\otimes H^{\otimes k} & \longrightarrow & F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(k,m)\otimes H^{\otimes k} & & \\
 & & \downarrow & & \downarrow & & \\
 & & F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(1,m)\otimes H & \longrightarrow & \underline{\mathrm{hom}}(A^{\otimes m},A) & & \\
 & \nearrow & & & & \searrow\!\!\!\!= & \\
F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(1,m)\otimes B & & & \longrightarrow & & & \underline{\mathrm{hom}}(A^{\otimes m},A)
\end{array}$$

FIGURE 3. *The outer square commutes if and only if 6.6 commutes when $n=1$.*

We need to show that the following triangle commutes for all n,m and k.

$$\begin{array}{ccc}
 & \nearrow & \mathrm{map}(B^{\otimes n}\otimes A^{\otimes m},A)\\
 & & \downarrow\\
F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(n,m) & \longrightarrow & \mathrm{map}(\tilde{\mathsf{E}_{\mathsf{d}}}(\underline{k},\underline{n})\otimes B^{\otimes k}\otimes A^{\otimes m},A)
\end{array} \tag{6.6}$$

The two compositions use either the $\tilde{\mathsf{E}_{\mathsf{d}}}$ action on B or the $\tilde{\mathsf{E}_{\mathsf{d}}}$ action on $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})$. By 6.4 the triangle (6.6) commutes because it does when $n=0,1$. The case $n=0$ simply states that the same $\mathsf{E}_{\mathsf{d-1}}$ structure on A is being used in both cases. The case $n=1$ is equivalent to the commutativity of the outer square in the diagram in figure 3. The upper trapezoid trivially commutes. The left trapezoid commutes because $B\to H$ is a map of $\tilde{\mathsf{E}_{\mathsf{d}}}$ algebras. The center square commutes by definition of the $\tilde{\mathsf{E}_{\mathsf{d}}}$ action on H. The bottom and right trapezoids commute by the definition of the $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})$ action on (B,A).

On the other hand, if (B,A) is an algebra over $\mathsf{SC}_{\mathsf{d}}^{1}$ then it is an algebra over $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})$, and thus over $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1}$. Again, by proposition 4.1 gives a map in $\mathcal{C}$, $B\to\mathrm{Hoch}(A)$. We have to check that this is a map of $\tilde{\mathsf{E}_{\mathsf{d}}}$ algebras.

We can decompose the diagram used to check this condition into two smaller squares.

$$\begin{array}{ccccc}
\tilde{\mathsf{E}_{\mathsf{d}}}(\underline{n},\underline{1})\otimes B^{\otimes n} & \longrightarrow & \tilde{\mathsf{E}_{\mathsf{d}}}(\underline{n},\underline{1})\otimes B^{\otimes n} & \longrightarrow & B\\
\downarrow & & \downarrow & & \downarrow\\
\tilde{\mathsf{E}_{\mathsf{d}}}(\underline{n},\underline{1})\otimes H^{\otimes n} & \longrightarrow & \tilde{\mathsf{E}_{\mathsf{d}}}(\underline{n},\underline{1})\otimes\underline{\mathrm{hom}}(\bar{A}^{sc}(n),A) & \longrightarrow & H
\end{array} \tag{6.7}$$

From (4.1) that the map of spaces $B\to\mathrm{Hoch}(A)$ is the unique map such that the induced morphism $\mathrm{End}^{\mathrm{h}}(H,A)\to\mathrm{End}^{\mathrm{h}}(B,A)$ makes the triangle on the left in (6.8) commute. Passing to $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})$ and examining each component (n,m) we get the commutative triangle on the right in (6.8).

$$\begin{array}{ccc}
 & \nearrow & \mathrm{End}^{\mathrm{h}}(H,A)\\
 & & \downarrow\\
\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1} & \longrightarrow & \mathrm{End}^{\mathrm{h}}(B,A)
\end{array}
\qquad
\begin{array}{ccc}
 & \nearrow & \mathrm{map}(H^{\otimes n}\otimes A^{\otimes m},A)\\
 & & \downarrow\\
F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})(n,m) & \longrightarrow & \mathrm{map}(B^{\otimes n}\otimes A^{\otimes m},A)
\end{array} \tag{6.8}$$

The lower edge of this triangle is adjoint to the arrow $B^{\otimes n}\to\underline{\mathrm{hom}}(\bar{A}^{sc}(n),A)$ in (6.7). The other edges give the composition $B^{\otimes n}\to H^{\otimes n}\to\underline{\mathrm{hom}}(\bar{A}^{sc}(n),A)$. This shows that the square on the left in (6.7) is commutative.

Consider the square on the right in (6.7). We have assumed that the triangle in (6.6) commutes. The square commutes because the triangle commutes when $n=1$. Put simply, the square compares the action of $\tilde{\mathsf{E}_{\mathsf{d}}}$ on B with the action of $\tilde{\mathsf{E}_{\mathsf{d}}}$ on $F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}1})$. We have assumed that (B,A) is an $\mathsf{SC}_{\mathsf{d}}^{1}$ algebra, so we have assumed that these actions are compatible.

We have shown that the left and right compositions in the following diagram factor through the vertical functors to give the dotted arrows:

$$\begin{array}{ccccc} \tilde{\mathsf{E}_{\mathsf{d}}}\text{-alg}(\mathcal{C})_{/\mathrm{Hoch}(A)} & \longleftarrow\cdots\cdots\cdots\longrightarrow & & & \mathsf{SC}^1_{\mathsf{d}}\text{-alg}(\mathcal{C})_A \\ \downarrow & & & & \downarrow \\ \mathcal{C}_{/\mathrm{Hoch}(A)} & \overset{\sim}{\longleftrightarrow} & \mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}}\text{-alg}(\mathcal{C})_A & \overset{\sim}{\longleftrightarrow} & F(\mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}})\text{-alg}(\mathcal{C})_A \end{array} \tag{6.9}$$

Proposition 4.1 shows that the lower left horizontal map is an equivalence. The lower right horizontal arrow is an equivalence by the free-forgetful adjunction between algebras over $\mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}}$ and over $F(\mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}})$. □

7. *Cofibration and weak equivalence proofs*

In the following two sections we prove theorem 5.1, as well as the cofibrance of the E_{d-1}-A module A^{sc} from lemma 4.4.

7.1. *Cofibration proofs*

In this section we prove some of the main theorems of the paper concerning cofibrant properties of Swiss cheese. We prove that A^{sc} from section 4 is a cofibrant $\mathsf{E}_{\mathsf{d}-1}$-$A$-module. We also prove that $F(\mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}}) \to \mathsf{SC}^{\mathrm{h}}_{\mathsf{d}}$ is a cofibration of operads in $\mathsf{Op}^{\mathrm{h}}_K$.

We will use the ideas in Berger and Moerdijk's proof that W$\mathcal{O}$ is cofibrant for any well-pointed, Σ-cofibrant operad $\mathcal{O}$ [**BM06**]. The two proofs in this section are not much more than Berger and Moerdijk's proof adapted to fit this situation. The proofs use the same technique. The first proof, that A^{sc} is a cofibrant $\mathsf{E}_{\mathsf{d}-1}$-$A$ module, is given with a good bit of detail. The second proof is lighter on the details and a bit heavier on the intuition.

As a setup for the proof that A^{sc} is cofibrant, define a sequence of categories

$$\cdots \to (\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C}))_k \to \cdots \to \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_1 \to \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_0 = \mathcal{C}$$

as well as functors $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C}) \to \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_k$ compatible with the sequence above. When $k = 0$ this will be the usual forgetful functor $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C}) \to \mathcal{C}$. Moreover, each $\mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_k$ has a model structure where the fibrations (respectively equivalences) are the maps which are fibrations (respectively equivalences) when we forget down to $\mathcal{C}$. If $M \in \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_k$ then we say M is an $(\mathsf{E}_{\mathsf{d}-1}$-$A)_k$ module.

We will define a sequence $A^{sc}_k, A^{sc+}_k \in \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_k$ for $k \geq 0$ and maps of objects in $\mathcal{C}$, $A^{sc}_k \to A^{sc+}_k \to A^{sc}_{k+1}$ such that, as an object of $\mathcal{C}$, $A^{sc} = \mathrm{colim}_k\, A^{sc}_k$. This data satisfies the following conditions

$$\begin{array}{l} - \ A^{sc}_0 \text{ is cofibrant in } \mathcal{C}. \\ - \ A^{sc}_k \text{ is an } (\mathsf{E}_{\mathsf{d}-1}\text{-}A)_k \text{ module.} \\ - \ A^{sc+}_k \text{ is the free extension of } A^{sc}_k \text{ to an } (\mathsf{E}_{\mathsf{d}-1}\text{-}A)_{k+1} \text{ module.} \\ - \ A^{sc+}_k \to A^{sc}_k \text{ is a cofibration in } \mathcal{C}. \\ - \ \text{Every decomposable in } A^{sc}_k \text{ comes from } A^{sc+}_k. \\ - \ \text{A morphism } A^{sc} \to M \text{ is a map of } \mathsf{E}_{\mathsf{d}-1}\text{-}A \text{ modules if each } A^{sc}_k \to M \text{ is} \\ \quad \text{a map of } (\mathsf{E}_{\mathsf{d}-1}\text{-}A)_k \text{ modules.} \end{array} \tag{7.1}$$

Let $\mathsf{Trees}^k_{\mathrm{h}}(n, m)$ denote the groupoid of K-colored trees with $\leq k$ internal edges, n input edges of color e, m input edges of color h, and where every vertex has output edge of color h. Also let $\mathsf{Trees}_{\mathrm{h}}(n, m)$ denote the space of all such trees with no bound on the number of internal edges.

We can recharacterize the condition that M is an $\mathsf{E_{d-1}}$-A module in terms of the trees in $\mathsf{Trees_h}(n,m)$ where $0 \le n \le 1$ and $m \ge 0$. Given any such tree T define

$$\mathsf{E_{d-1}}(T) = \prod_{v \in V(T)} \mathsf{E_{d-1}}(\mathrm{in}(v)) \qquad M(T) = \bigotimes_{\epsilon \in \mathrm{in}(T)} M_\epsilon,$$

where the coloring of in(v) is forgotten in the expression $\mathsf{E_{d-1}}(\mathrm{in}(v))$ and

$$M_\epsilon = \begin{cases} M & \epsilon \text{ of color e} \\ A & \epsilon \text{ of color h} \end{cases}$$

The $\mathsf{E_{d-1}}$-A module structure on M is equivalent to a collection of maps for each $T \in \mathsf{Trees_h}(\le 1, m)$,

$$\mathsf{E_{d-1}}(T) \otimes M(T) \to M$$

compatible with grafting trees. We have $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})_0 = \mathcal{C}$. Inductively define the category $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})_k$ to be given by sequences $M = (M_i)_{0 \le i \le k}$ and structure maps

$$\mathsf{E_{d-1}}(T) \otimes M_i(T) \to M_{i+j+1}(T)$$

for each $T \in \mathsf{Trees}^j_\mathsf{h}(\le 1, m)$ and $i + j + 1 \le k$. We require these maps to be compatible with grafting of trees so long as all trees involved have $\le k$ internal edges. Explicitly this means that if $T = T_1 \circ_\epsilon T_2$, where $\epsilon \in \mathrm{in}(T_1)$ and T_i has $\le k_i$ internal edges, then the composition

$$\begin{aligned} \mathsf{E_{d-1}}(T) \otimes M_i(T) &\xrightarrow{\sim} \mathsf{E_{d-1}}(T_1) \otimes (\mathsf{E_{d-1}}(T_2) \otimes M_i(T_2))(T_1) \\ &\to \mathsf{E_{d-1}}(T_1) \otimes M_{i+k_1+1}(T_1) \to M_{i+k_1+k_2+2} \end{aligned}$$

must agree with structure map defined by T so long as $i + k_1 + k_2 + 2 \le k$.

A morphism $f\colon M' \to M$ in $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})_k$ is a sequence of maps $f_i\colon M'_i \to M_i$ making the appropriate diagrams of structure maps commute. The forgetful functor

$$\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})_k \to \mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})_{k-1}$$

forgets M_k, and the functor

$$\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C}) \to \mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})_k$$

sends M to the constant sequence $M_i = M$.

We say that every decomposable in M comes from M' (via the map $f\colon M' \to M$) if f factors the "decomposables map" for M:

$$\begin{array}{ccc} & & M'_{i+k_1+k_2+2} \\ & \nearrow & \downarrow \\ \coprod_{T_1,\epsilon,T_2} \mathsf{E_{d-1}}(T_1) \otimes (\mathsf{E_{d-1}}(T_2) \otimes M_i(T_2))(T_1) & \longrightarrow & M_{i+k_1+k_2+2} \end{array}$$

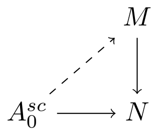
The disjoint union runs over all trees T_1, T_2 with at most k_1, k_2 internal edges respectively and input edges $\epsilon \in \mathrm{in}(T_1)$. The grafted tree $T_1 \circ_\epsilon T_2$ lives in $\mathsf{Trees_h}(\le 1, -)$.

Proof of Lemma 4.4, part 1: A^{sc} is a cofibrant $\mathsf{E_{d-1}}$-A module.

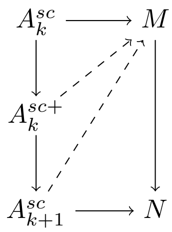
Suppose for the moment that we are given A^{sc}_k, A^{sc+}_k satisfying (7.1). Given an acyclic fibration $M \to N$ of $\mathsf{E_{d-1}}$-A modules, and a diagram

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ A^{sc} & \longrightarrow & N \end{array} \tag{7.2}$$

we restrict to A_0^{sc} and get a lift of $(\mathsf{E_{d-1}}\text{-}A)_0$ modules (objects of $\mathcal{C}$).



Now by induction we have a commutative diagram



which, excluding A_k^{sc}, is a commutative diagram of $(\mathsf{E_{d-1}}\text{-}A)_{k+1}$ modules. Since A_k^{sc+} is the free extension of A_k^{sc} to a $(\mathsf{E_{d-1}}\text{-}A)_{k+1}$ module, there is a unique lift of $(\mathsf{E_{d-1}}\text{-}A)_{k+1}$ modules $A_k^{sc+} \to M$ which makes the upper triangle a commutative diagram of $(\mathsf{E_{d-1}}\text{-}A)_k$ modules. We also get a lift out of A_{k+1}^{sc} since the map $A_k^{sc+} \to A_{k+1}^{sc}$ is a cofibration in $\mathcal{C}$. Any lift as objects of $\mathcal{C}$ will be a lift as $(\mathsf{E_{d-1}}\text{-}A)_{k+1}$ modules since every decomposable in A_{k+1}^{sc} comes from A_k^{sc+}. Since $A^{sc} = \operatorname{colim}_k A_k^{sc}$ we get a lift in the original diagram (7.2). Also, the map $A^{sc} \to M$ is a map of $\mathsf{E_{d-1}}\text{-}A$ modules by the last condition in (7.1).

Now we show how to get A_k^{sc} and A_k^{sc+}. For a tree $T \in \mathsf{Trees}_{\mathrm{h}}(1, m)$ define $\mathrm{spine}(T) \subset E_i(T)$ to be the set of internal edges of T which lie on the path from the root vertex to the unique vertex with input edge of color e.Define sub collections G_k^s, G_k^+, G_k of $\mathsf{SC}_\mathsf{d}^{\mathrm{h}}(1, -)$ for each $k \geq 0$. First, given a tree $T \in \mathsf{Trees}_{\mathrm{h}}(1, m)$ define

$$\begin{aligned} W^-(T) &= \{(\epsilon_i) \in [0,\infty]^{E_i(T)} \mid \epsilon_i = 0 \text{ for some } \epsilon_i \in E_i(T)\} \\ W^s(T) &= \{(\epsilon_i) \in [0,\infty]^{E_i(T)} \mid \epsilon_i = 0 \text{ or } \infty \text{ for some } \epsilon_i \notin \mathrm{spine}(T) \\ &\qquad \text{OR } \epsilon_i = 0 \text{ for some } \epsilon_i \in \mathrm{spine}(T)\} \\ W^+(T) &= \{(\epsilon_i) \in [0,\infty]^{E_i(T)} \mid \epsilon_i = 0 \text{ or } \infty \text{ for some } \epsilon_i \in E_i(T)\} \\ W(T) &= [0,\infty]^{E_i(T)} \end{aligned}$$

For $T \in \mathsf{Trees}_{\mathrm{h}}(n, m)$, we have

$$SC_d(T) = \prod_{v \in V(T)} SC_d^{\mathrm{h}}(\mathrm{in}(v)).$$

This is the set of all vertex labels of T. Recall that $SC_d^{\mathsf{id}}(T) \subset SC_d(T)$ is the set of all vertex labels of T where at least one vertex is labeled with the identity. Define

$$\begin{aligned} (W \times SC_d)^-(T) &= W^-(T) \times SC_d(T) \cup_{W^-(T) \times SC_d^{\mathsf{id}}(T)} W(T) \times SC_d^{\mathsf{id}}(T) \\ (W \times SC_d)^s(T) &= W^s(T) \times SC_d(T) \cup_{W^s(T) \times SC_d^{\mathsf{id}}(T)} W(T) \times SC_d^{\mathsf{id}}(T) \\ (W \times SC_d)^+(T) &= W^+(T) \times SC_d(T) \cup_{W^+(T) \times SC_d^{\mathsf{id}}(T)} W(T) \times SC_d^{\mathsf{id}}(T) \end{aligned}$$

Since the inclusions $W^-(T) \subset W^s(T) \subset W^+(T) \subset W(T)$ are $\mathrm{Aut}(T)$-equivariant cofibrations and $SC_d^{\mathsf{id}}(T) \subset SC_d(T)$ is an $\mathrm{Aut}(T)$-cofibration, an equivariant version of the pushout-product axiom [**BM06**], lemma 2.5.2, gives us $\mathrm{Aut}(T)$-cofibrations

$$(W \times SC_d)^-(T) \to (W \times SC_d)^s(T) \to (W \times SC_d)^+(T) \to W(T) \times SC_d(T).$$

Define $G_0(m)$ to be

$$\coprod_{[T]\in\pi_0(\mathsf{Trees}^0_{\mathrm{h}}(1,m))} (W(T)\times SC_d(T))\times_{\mathrm{Aut}(T)} S_m,$$

and inductively define $G_k(m)\to G^s_{k+1}(m)\to G^+_{k+1}(m)\to G_{k+1}(m)$ by a succession of pushouts

$$\begin{array}{ccc}
\coprod_{[T]\in\pi_0(\mathsf{Trees}^{k+1}_{\mathrm{h}}(1,m))} (W\times SC_d)^-(T)\times_{\mathrm{Aut}(T)} S_m & \longrightarrow & G_k(m)\\
\downarrow & & \downarrow\\
\coprod_{[T]\in\pi_0(\mathsf{Trees}^{k+1}_{\mathrm{h}}(1,m))} (W\times SC_d)^s(T)\times_{\mathrm{Aut}(T)} S_m & \longrightarrow & G^s_{k+1}(m)\\
\downarrow & & \downarrow\\
\coprod_{[T]\in\pi_0(\mathsf{Trees}^{k+1}_{\mathrm{h}}(1,m))} (W\times SC_d)^+(T)\times_{\mathrm{Aut}(T)} S_m & \longrightarrow & G^+_{k+1}(m)\\
\downarrow & & \downarrow\\
\coprod_{[T]\in\pi_0(\mathsf{Trees}^{k+1}_{\mathrm{h}}(1,m))} (W(T)\times SC_d(T))\times_{\mathrm{Aut}(T)} S_m & \longrightarrow & G_{k+1}(m)
\end{array} \tag{7.3}$$

where the top horizontal map collapses edges labeled 0 and vertices labeled with the identity. The vertical maps on the left are S_m-cofibrations by lemma (induction and restriction functors from group homomorphism). By the same lemma, and the fact that $-\otimes A^{\otimes m}$ preserves cofibrations (since A is cofibrant in $\mathcal{C}$), we know that

$$\coprod_m G^*_k(1,m)\otimes_{S_m} A^{\otimes m}\to\coprod_m G^\bullet_{k'}(1,m)\otimes_{S_m} A^{\otimes m}$$

is a cofibration in $\mathcal{C}$ for any choice of $k,k',*,\bullet$ giving one of the vertical maps on the right in (7.3). Now set

$$A^{sc}_0=\coprod_m G_0(1,m)\otimes_{S_m} A^{\otimes m},$$

and inductively define $A^{sc}_k\to A^{sc+}_k\to A^{sc}_{k+1}$ by the pushouts

$$\begin{array}{ccc}
\coprod_m G^s_k(1,m)\otimes_{S_m} A^{\otimes m} & \longrightarrow & A^{sc}_k\\
\downarrow & & \downarrow\\
\coprod_m G^+_k(1,m)\otimes_{S_m} A^{\otimes m} & \longrightarrow & A^{sc+}_k\\
\downarrow & & \downarrow\\
\coprod_m G_{k+1}(1,m)\otimes_{S_m} A^{\otimes m} & \longrightarrow & A^{sc}_{k+1}
\end{array}$$

where the top horizontal arrow collapses edges labeled 0, and collapses edges labeled ∞ which are not on the spine, using the $\mathsf{E_{d-1}}$ algebra structure on A. Since the arrows on the left are cofibrations, so are the arrows on the right. Heuristically, A^{sc}_k is given by all trees in A^{sc} with at most k internal edges. There is a relation on A^{sc} which allows us to collapse any infinite edge which separates the tree into a composition of an element of $\mathsf{SC}^{\mathrm{h}}_{\mathsf{d}}(1,-)$ and an element of $\mathsf{E_{d-1}}$. We replace the $\mathsf{E_{d-1}}$ part of the tree by the appropriate label in A given by the $\mathsf{E_{d-1}}$ structure on A. In other words, we can collapse any infinite edge of the tree which does not live on its spine. Also A^{sc+}_k is given by all trees and A^{sc} with at most $k+1$ internal edges where at least one of those internal edges is labeled by ∞. The non-spine infinite edges are also collapsed in A^{sc+}_k. It should be clear that A^{sc}_k and A^{sc+}_k, $k\geq 0$ satisfy conditions (7.1). ☐

In analogy with the previous proo, we will define a sequence of categories

$$\cdots \to \mathsf{Op}_K^{k+1} \to \mathsf{Op}_K^k \to \cdots \to \mathsf{Op}_K^0 \to \mathsf{Op}_K^{-1} = \mathsf{Op}_K^{\mathrm{h1}},$$

where $\lim_k \mathsf{Op}_K^k = \mathsf{Op}_K^{\mathrm{h}}$.

We can think of Op_K as a category of symmetric monoidal functors from a category of rooted forests to topological spaces. This notion is due to Costello in [**Cos07**], and is expanded upon in [**Get07**]. Without going into the details of the category of rooted forests, we simply note that every tree T whose edges are colored by K gives us a map

$$\prod_{v \in V(T)} \mathcal{O}(\mathrm{in}(v), \mathrm{out}(v)) \to \mathcal{O}(\mathrm{in}(T), \mathrm{out}(T)) \tag{7.4}$$

where $V(T)$ is the set of vertices of T; $\mathrm{in}(v)$ is the K-colored set of incoming edges to the vertex v; $\mathrm{out}(v)$ is the K-colored singleton set containing the outgoing edge from v; similarly for $\mathrm{in}(T)$ and $\mathrm{out}(T)$. This is another way to look at the composition maps in the operad $\mathcal{O}$. These maps must be compatible with grafting of trees. In Op_K we allow all trees whose edges are colored by K. In $\mathsf{Op}_K^{\mathrm{h}}$ we only allow trees where every vertex has root color $\mathrm{h} \in K$. In $\mathsf{Op}_K^{\mathrm{h1}}$ we further restrict to trees where every vertex has at most one incoming edge of color e. Interpolating between the former two situations, in Op_K^k we only allow trees T satisfying the following condition

> Every vertex of T has root color h. Also, T has $\leq k$ internal edges or every vertex of T has at most one incoming edge of color e. (7.5)

The structure maps (7.4) for operads $\mathcal{O}$ in Op_K^k must be compatible with grafting of trees so long as all trees involved satisfy (7.5).

Proof of Theorem 5.1 part 1: $F(\mathsf{SC}_\mathsf{d}^{\mathrm{h1}}) \to \mathsf{SC}_\mathsf{d}^{\mathrm{h}}$ is a cofibration.. Define a sequence of collections $F_k \subset F_k^+ \subset F_{k+1}$ for $k \geq -1$, where $F_{-1} = F(\mathsf{SC}_\mathsf{d}^{\mathrm{h1}})$ and $\cup_k F_k = \mathsf{SC}_\mathsf{d}^{\mathrm{h}}$. This filtration will have the following properties.

- F_{-1} is a cofibrant collection.
- F_k lives in Op_K^k.
- F_k^+ is the free extension of F_k to an operad in Op_K^{k+1}.
- Each $F_k^+ \to F_{k+1}$ is a cofibration of collections.
- Every decomposable tree in F_{k+1} lives in F_k^+.
- A morphism of collections $\mathsf{SC}_\mathsf{d}^{\mathrm{h}} \to \mathcal{O}$, where $\mathcal{O}$ is an operad, is a morphism of operads if and only if each restriction $F_k \to \mathcal{O}$ is a morphism in Op_K^k.

Let $F_k \subset \mathsf{SC}_\mathsf{d}^{\mathrm{h}}$ be the sub collection given by edge and vertex-labeled trees T satisfying the following condition.

> In an edge-labelled tree T, either every finite subtree has ≤ 1 input edge of color e or the whole tree T has $\leq k$ internal edges. (7.6)

The finite subtrees of T are those given by breaking apart all infinite internal edges of T. It should be clear that a map of collections $\mathsf{SC}_\mathsf{d}^{\mathrm{h}} \to \mathcal{O}$ is a map of operads if and only if each $F_k \to \mathcal{O}$ is a morphism in Op_K^k. In addition, define F_k^+ to be the sub collection of $\mathsf{SC}_\mathsf{d}^{\mathrm{h}}$ given by trees T satisfying the following condition.

> In an edge-labelled tree T, either every finite subtree has ≤ 1 input edge of color e or the whole tree T has $\leq k+1$ internal edges. If T has exactly $k+1$ internal edges, then one edge is labeled ∞. (7.7)

First of all, it is clear that every decomposable tree in F_{k+1} lives in F_k^+. Also, we have $F_k \in \mathsf{Op}_K^k$, and $F_k^+ \in \mathsf{Op}_K^{k+1}$ is the image of F_k under the left adjoint to the forgetful functor $\mathsf{Op}_K^{k+1} \to \mathsf{Op}_K^k$.

To see that $F_k^+ \to F_{k+1}$ is a cofibration of collections we mimic the argument from the proof that A^{sc} is cofibrant. Also see the proof in [**BM06**] that W gives a cofibrant replacement functor. We will not be as detailed as in the previous proof, we will try to give more of the intuition. There is a pushout square

$$\begin{array}{ccc} \coprod_{[T]} \mathrm{Labels}^+(T) \times_{\mathrm{Aut}(T)} S_n \times S_m & \longrightarrow & F_k^+(n,m) \\ \downarrow & & \downarrow \\ \coprod_{[T]} \mathrm{Labels}(T) \times_{\mathrm{Aut}(T)} S_n \times S_m & \longrightarrow & F_k(n,m) \end{array} \tag{7.8}$$

where $\mathrm{Labels}^+(T)$ is the space of all labellings of the edges of T by $[0,\infty]$ and the vertices of T by $\mathsf{SC}_\mathsf{d}^\mathrm{h}$ such that T satisfies (7.7) or T has one vertex labeled with the identity of $\mathsf{SC}_\mathsf{d}^\mathrm{h}$. $\mathrm{Labels}(T)$ is the space of all edge and vertex labels of the T with no conditions. The coproducts on the left hand side run over all equivalence classes of trees with (n,m) input edges, $\leq k+1$ internal edges. One uses an equivariant version of the pushout-product axiom [**BM06**], lemma 2.5.2, to show that the vertical arrow on the left in the diagram above is and $S_n \times S_m$-cofibration. We conclude that the map of collections $F_k^+ \to F_{k+1}$ is a cofibration.

Now we make the lifting argument. Suppose $P \to Q$ is an acyclic fibration in Op_K^h. By induction on k we have a commutative diagram in Op_K^k.

$$\begin{array}{ccc} F_k^+ & \longrightarrow & P \\ \downarrow & & \downarrow \\ F_{k+1} & \longrightarrow & Q \end{array}$$

where the arrow on the right is an acyclic fibration, and the map on the left is a cofibration of collections. Thus, a priori, we only get a lift of collections. However, since every decomposable tree in the F_{k+1} lives in F_k^+ any such lift is a morphism in Op_K^k. This uniquely defines a map in Op_K^{k+1}, $F_{k+1}^+ \to P$ fitting into a commutative square

$$\begin{array}{ccc} F_{k+1}^+ & \longrightarrow & P \\ \downarrow & & \downarrow \\ F_{k+2} & \longrightarrow & Q \end{array}$$

enabling the next step of the induction. □

7.2. *Weak equivalence proof*

This section contains a proof of the second part of theorem 5.1, which states that the inclusion $F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1}) \to \mathsf{SC}_\mathsf{d}^\mathrm{h}$ is a weak equivalence of operads in Op_K^h. The idea of the proof is as follows. Consider the maps $p_1\colon F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1})(n,m) \to F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1})(n-1,m)$ and $p\colon \mathsf{SC}_\mathsf{d}^\mathrm{h}(n,m) \to \mathsf{SC}_\mathsf{d}^\mathrm{h}(n-1,m)$ given by forgetting the n^{th} disc. By induction, we can suppose $F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1})(n-1,m) \to \mathsf{SC}_\mathsf{d}^\mathrm{h}(n-1,m)$ is a weak equivalence. If we can then show that $p_1^{-1}(\alpha) \to p^{-1}(\alpha)$ is a weak equivalence for every $\alpha \in F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1})(n-1,m)$, then by the long exact sequence of homotopy groups for a fibration, we can conclude that $F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1})(n,m) \to \mathsf{SC}_\mathsf{d}^\mathrm{h}(n,m)$ is a weak equivalence as well.

For computational purposes, we will collapse the n^{th} disc of $\alpha \in \mathsf{SC}_\mathsf{d}^\mathrm{h}(n,m)$ to a point. Our goal in the next section is to make this precise.

7.3. *Defining $SC_d(k,l;n,m)$*

When we collapse the n^{th} disc of $\alpha \in SC_d^\mathrm{h}(n,m)$ to its center, we really should think of the result $\hat{\alpha}$ as living in a four-colored operad which we also denote by SC_d. We add the colors $\hat{\mathrm{e}}$ and $\hat{\mathrm{h}}$. Let $\widehat{K} = \{\hat{\mathrm{e}}, \hat{\mathrm{h}}, \mathrm{e}, \mathrm{h}\}$ be the set of colors for this new operad, which lives in the category

$\mathsf{Op}_{\widehat{K}}^{\mathrm{h}\hat{\mathrm{h}};\hat{\mathrm{e}}+\hat{\mathrm{h}}\leq 1}$. This category consists of $\widehat{K}$-colored operads $\mathcal{O}$ satisfying

$$\mathcal{O}^{\mathrm{e}}(k,l;n,m) = \mathcal{O}^{\hat{\mathrm{e}}}(k,l;n,m) = \emptyset$$

for all $(\hat{\mathrm{e}}, \hat{\mathrm{h}}; \mathrm{e}, \mathrm{h})$-colored sets $(k,l;n,m)$. This condition is denoted by the superscript $\mathrm{h}\hat{\mathrm{h}}$, denoting that these are the only colors allowed in any output edges. The superscript $\hat{\mathrm{e}} + \hat{\mathrm{h}} \leq 1$ denotes the condition

$$\mathcal{O}^{\mathrm{h}}(k,l;n,m) = \emptyset \text{ when } k+l \geq 2.$$

To define $SC_d^{\mathrm{h}}(k,l;n,m)$ we need the notion of the geometric realization of $\alpha \in SC_d^{\mathrm{h}}(n,m)$.

Definition 7.1. Given $\beta \in SC_d^{\mathrm{h}}(n,m)$, let $|\beta|$ be the geometric realization of β. This is the subset of $\mathbb{R}^d$ given by deleting the open discs and half-discs of β from the closed unit half-disc. More precisely, if $\bar{D}_+^d$ is the closed unit half-disc in $\mathbb{R}^d$, $\{(D_{\mathrm{e}}^d)_j\}_{j=1}^m$ are the open discs of β, and $\{(D_{\mathrm{h}}^d)_i\}_{i=1}^n$ are the open half-discs of β considered as open discs in $\mathbb{R}^d$ whose center lies in $\mathbb{R}^{d-1}$, then

$$|\beta| = \bar{D}_+^d - \left(\left(\bigcup_{i=1}^{n} (D_{\mathrm{h}}^d)_i \right) \cup \left(\bigcup_{j=1}^{m} (D_{\mathrm{e}}^d)_j \right) \right).$$

Now we can set

$$SC_d^{\mathrm{h}}(k,l;n,m) = \begin{cases} SC_d^{\mathrm{h}}(n,m) & k=l=0 \\ \{(\alpha,q) \mid \alpha \in SC_d^{\mathrm{h}}(n,m), q \in |\alpha|\} & k=1, l=0 \\ \{(\alpha,q) \mid \alpha \in SC_d^{\mathrm{h}}(n,m), q \in |\alpha| \cap \mathbb{R}^{d-1}\} & k=0, l=1 \end{cases}$$

and

$$SC_d^{\hat{\mathrm{h}}}(k,l;n,m) = \begin{cases} * & (k,l;n,m) = (1,0;0,0) \\ \emptyset & \text{else} \end{cases}$$

We think of the point $q \in |\alpha|$ as a collapsed disc and the point $q \in |\alpha| \cap \mathbb{R}^{d-1}$ as a collapsed half-disc. Composition in $SC_d^{\mathrm{h}}(*,*,*)$ takes place in the half-discs and collapsed half-discs only. The discs play no part in composition. However the collapsed half-discs and collapsed discs only play a part in composition when we plug a collapsed disc into a collapsed half-disc. The result is a collapsed disc which happens to live on the boundary of the geometric realization.

Let $\mathsf{Op}_{\widehat{K}}^{\mathrm{h}\hat{\mathrm{h}};\hat{\mathrm{e}}+\hat{\mathrm{h}}+\mathrm{e}\leq 1}$ be the category consisting of operads $\mathcal{O} \in \mathsf{Op}_{\widehat{K}}^{\mathrm{h}\hat{\mathrm{h}};\hat{\mathrm{e}}+\hat{\mathrm{h}}\leq 1}$ which further satisfy the condition

$$\mathcal{O}^{\mathrm{h}}(k,l;n,m) = \emptyset \text{ if } k+l+n \geq 2.$$

We can consider $SC_d^{\mathrm{h}}(*,*)$ as a $\widehat{K}$ colored operad by setting

$$SC_d^{\mathrm{h}}(k,l;n,m) = SC_d^{\mathrm{h}}(k+n, l+m) \text{ and } SC_d^{\hat{h}}(1,0;0,0) = SC_d^{\mathrm{h}}(1,0).$$

Let $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(*,*;*,*)$ denote the W construction applied to the four-colored version of $\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}$. Consider the commutative diagram

$$\begin{array}{ccc} F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}})(n+1,m) & \longrightarrow & \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n+1,m) \\ \sim\downarrow & & \downarrow\sim \\ F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}})(1,0;n,m) & \longrightarrow & \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(1,0;n,m) \\ p_1\downarrow & & \downarrow p \\ F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h}})(n,m) & \longrightarrow & \mathsf{SC}_{\mathsf{d}}^{\mathrm{h}}(n,m) \end{array}$$

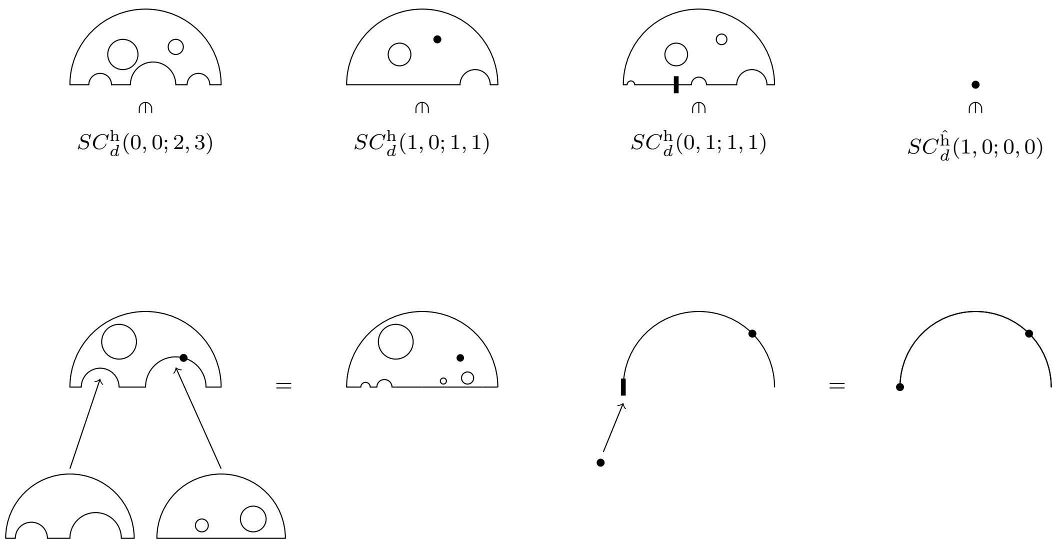


FIGURE 4. *The collapsed discs are denoted by dots and the collapsed half-discs by tick marks. Collapsed discs are color* ê *input edges and collapsed half-discs are color* $\hat{\mathrm{h}}$ *input edges. To keep the collapsed discs and half-discs from coinciding, we only allow one or the other in any composition. Composition in* $SC_d^{\mathrm{h}}(*,*;*,*)$ *takes place only in the half-discs and collapsed half discs. The only composition we can do in a collapsed half-disc is given by plugging in a collapsed disc. The result is a collapsed disc replacing the collapsed half-disc.*

The maps p_1 and p delete the collapsed disc and, if necessary, a left over collapsed half-disc. By induction on n we assume the bottom horizontal arrow is an equivalence. We will show that for each $\alpha \in F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}})(n,m)$ the inclusion $p_1^{-1}(\alpha) \to p^{-1}(\alpha)$ is an equivalence. Then by the long exact sequence of homotopy groups we conclude that the middle horizontal arrow is an equivalence. Clearly the top two vertical arrows are equivalences since they given by collapsing the n^{th} disc, so we conclude that the top horizontal arrow is also an equivalence.

7.4. *Computing* $p^{-1}(\alpha)$ *and* $p_1^{-1}(\alpha)$.

We have shown that the proof rests on the following lemma. This section is dedicated to the proof of this lemma.

LEMMA 7.2. *Fix* $\alpha \in F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}1})(n,m)$. *The inclusion of the fiber* $p_1^{-1}(\alpha)$ *into the fiber* $p^{-1}(\alpha)$ *is a weak equivalence.*

The proof for this theorem will use some observations about the W construction. We will compute $p^{-1}(\alpha)$ and $p_1^{-1}(\alpha)$ as homotopy colimits of functors F and F_1. We can more easily examine these homotopy colimits and see by inspection that they are equivalent.

Define $\mathsf{Trees}(k,l;n,m)$ to be the category whose objects are trees with $(k,l;n,m)$ input edges of color $(\hat{\mathrm{e}},\hat{\mathrm{h}};\mathrm{e},\mathrm{h})$, and whose morphisms are given by collapsing internal edges or inserting a vertex along an edge. The category $\mathsf{Trees}(n,m)$ is defined in a similar way. There is a functor $p : \mathsf{Trees}(1,0;n,m) \to \mathsf{Trees}(n,m)$, which deletes the single input edge of color ê and the subsequent edge of color $\hat{\mathrm{h}}$, if it is there.

Given $\alpha \in F(\mathsf{SC}_\mathsf{d}^{\mathrm{h}})(n,m)$, let $T_\alpha \in \mathsf{Trees}(n,m)$ be the underlying tree of α after we delete all edges labeled 0 and all vertices labeled with the identity. Let the category $\mathsf{Trees}_{T_\alpha/}$ be the under-category of T_α with respect to the functor p. The objects of this under-category are pairs $(T, T_\alpha \to p(T))$ where T is a tree in $\mathsf{Trees}(1,0;n,m)$ and $T_\alpha \to p(T)$ is a morphism in $\mathsf{Trees}(n,m)$.

Define a functor $W\colon \mathsf{Trees}(1,0;n,m)^{op} \to \mathsf{Top}$ by assigning a length to every internal edge of a tree T. That is, $W(T) = [0,\infty]^{E_i(T)}$. The morphisms in $\mathsf{Trees}(1,0;n,m)^{op}$ are given by inserting edges and the deleting unary vertices. The label on an inserted edge is 0; the label on an edge from which a unary vertex was deleted is the sum of the labels of the edges which were on either side of the unary vertex. In the same way, there is a functor $W : \mathsf{Trees}(n,m)^{op} \to \mathsf{Top}$. There is a projection $W(T) \to W(p(T))$ since $p(T)$ has the same internal edges as T. Combine this with the image under W of the morphism $T_\alpha \to p(T)$ to get an arrow $W(p(T)) \to W(T)$. Define $W_\alpha(T, T_\alpha \to p(T)) = W_\alpha(\mathcal{T})$ to be the pullback

$$\begin{array}{ccc} W_\alpha(\mathcal{T}) & \longrightarrow & W(T) \\ \Big\downarrow & & \downarrow \\ & & W(p(T)) \\ & & \downarrow \\ * & \xrightarrow{\mathbf{t}_\alpha} & W(T_\alpha) \end{array}$$

where α is represented by $(\mathbf{t}_\alpha, \tilde{\alpha}) \in W(T_\alpha) \times SC_d(T_\alpha)$. This defines a functor

$$W_\alpha\colon (\mathsf{Trees}_{T_\alpha/})^{op} \to \mathsf{Top}.$$

Indeed, if $\mathcal{T} \to \mathcal{T}'$ is a morphism in $\mathsf{Trees}_{T_\alpha/}$ then the following diagrams commute.

$$\begin{array}{ccc} p(T) & \longrightarrow & p(T') \\ \nwarrow & & \nearrow \\ & T_\alpha & \end{array} \qquad \begin{array}{ccc} W(T) & \longleftarrow & W(T') \\ \downarrow & & \downarrow \\ W(p(T)) & \longleftarrow & W(p(T')) \end{array}$$

On the other hand SC_d defines a functor $\mathsf{Trees}_{T_\alpha/} \to \mathsf{Top}$. First, there is a map $SC_d(T) \to SC_d(p(T))$ for every $T \in \mathsf{Trees}(1,0;n,m)$ which deletes the marked point corresponding to the edge of color ê. Now define $\mathsf{SC}_\alpha(T)$ to be the pullback

$$\begin{array}{ccccc} \mathsf{SC}_\alpha(T) & & \longrightarrow & & SC_d(T) \\ \Big\downarrow & & & & \Big\downarrow \\ * & \xrightarrow{\tilde{\alpha}} & SC_d(T_\alpha) & \longrightarrow & SC_d(p(T)) \end{array}$$

It is easy to see that SC_α gives a functor $\mathsf{Trees}_{T_\alpha/} \to \mathsf{Top}$.

Let Trees_α be the subcategory of $\mathsf{Trees}_{T_\alpha/}$ where the morphism $T_\alpha \to p(T)$ is insertion of a single vertex which is made unary by deletion either of its ê or its ĥ incoming edge. Now restrict the functors W_α and SC_α to Trees_α.

LEMMA 7.3. *The fiber $p^{-1}(\alpha)$ is given by the coend*

$$W_\alpha \otimes_{\mathsf{Trees}_\alpha} \mathsf{SC}_\alpha.$$

Proof. Given any tree T the space $W_\alpha(T) \times \mathsf{SC}_\alpha(T)$ is the space of all edge and vertex labels of T which, after deleting the marked point corresponding to the ê input edge, give the representative $(\mathbf{t}_\alpha, \tilde{\alpha}) \in W(T) \times SC_d(T)$ of α up to a possible extra identity vertex. The relations that hold in the coend are exactly the relations in the W construction which allow us to delete edges labeled 0 and vertices labeled with the identity. □

The objects of the category Trees_α can be described in terms of the edges and vertices of T_α. Define graphs Γ_i for $i = 0, 1$ whose geometric realizations are homeomorphic to line segments. Let Γ_0 be the graph with one vertex and one edge of color $\hat{\mathrm{e}}$. Let Γ_1 be the graph with two vertices, one internal edge of color $\hat{\mathrm{h}}$ and one tail of color $\hat{\mathrm{e}}$. Let the leftmost vertex of Γ_i be denoted Γ_i^v. Figure 5 displays these graphs.

FIGURE 5. *The graphs* Γ_i. *Straight edges have color* $\hat{\mathrm{h}}$. *Squiggly edges have color* $\hat{\mathrm{e}}$. *These edges are gray because later we will draw* e *and* h *edges black.*

Each edge ϵ of T_α gives two objects ϵ_i for $i = 0, 1$. In addition each vertex v of T_α gives two objects v_i. The object ϵ_i is given by the tree T_ϵ^i where Γ_i has been grafted on to T_α by gluing Γ_i^v to the middle of ϵ. The object v_i is given by the tree T_v^i which is obtained from T_α by gluing Γ_i^v to v. There is a morphism $\zeta_i \to \zeta_k$ whenever $i \geq k$ for $\zeta = \epsilon$ or v. If such a morphism exists, it is unique. In addition, there is a unique morphism $\epsilon_i \to v_j$ if and only if $i \geq j$ and v is attached to ϵ. See figure 6 for an illustration.

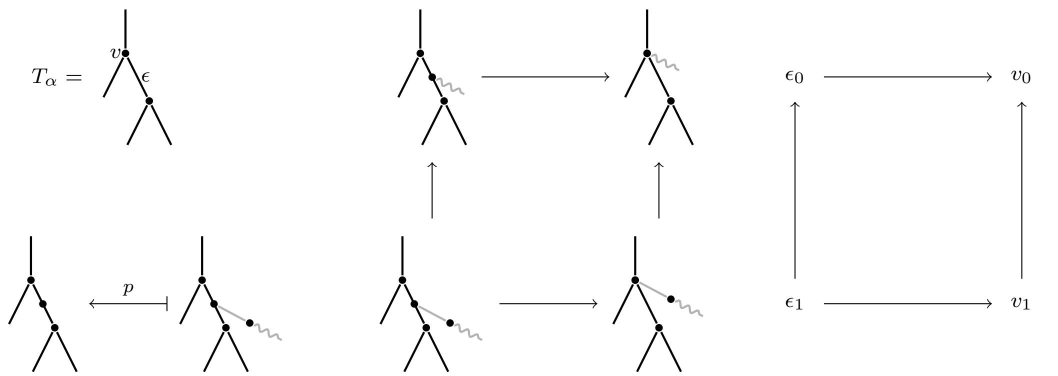


FIGURE 6. *The edge* ϵ *and vertex* v *of* T_α *give a commutative square in* Trees_α. *The graphs* Γ_0, Γ_1 *are in gray, attached to* T_α *drawn in black. The edge of color* $\hat{\mathrm{e}}$ *is drawn with a squiggly line. In addition, the image of* T_ϵ^1 *under the functor* p *is shown. This makes it clear that the map* $T_\alpha \to p(T_\epsilon^1)$ *is given by inserting a single vertex.*

Computing W_α is simple,

$$W_\alpha(T_\epsilon^i) = \{(t_{\epsilon'}, t_{\epsilon''}, s) \in [0, \infty] \times W(\Gamma_i) \mid t_{\epsilon'} + t_{\epsilon''} = \mathbf{t}_\alpha(\epsilon)\},$$

where $\mathbf{t}_\alpha \in [0, \infty]^{E(T_\alpha)}$ is the edge-labelling for α where the external edges have length ∞. The parameters $t_{\epsilon'}$ and $t_{\epsilon''}$ measure how far along ϵ the vertex Γ_i^v is placed, while $s \in W(\Gamma_1)$ measures the length of the color $\hat{\mathrm{h}}$ edge. If ϵ' is the segment of ϵ closer to the root, then it has length $t_{\epsilon'}$. If ϵ'' is the segment of ϵ further from the root, it has length $t_{\epsilon''}$.

In addition, we have $W_\alpha(T_v^i) = \{*\}_{t_v} \times W(\Gamma_i)$. The labels t helps make it clear how W_α behaves on morphisms. For example $W_\alpha(T_v^i) \to W_\alpha(T_\epsilon^i)$ sends (t_v, s) to $(0, \mathbf{t}_\alpha(\epsilon), s)$ if ϵ is an input edge to v and sends (t_v, s) to $(\mathbf{t}_\alpha(\epsilon), 0, s)$ if ϵ is the output edge from v. Figure 7 shows how the parameter spaces $W_\alpha(T)$ vary with T.

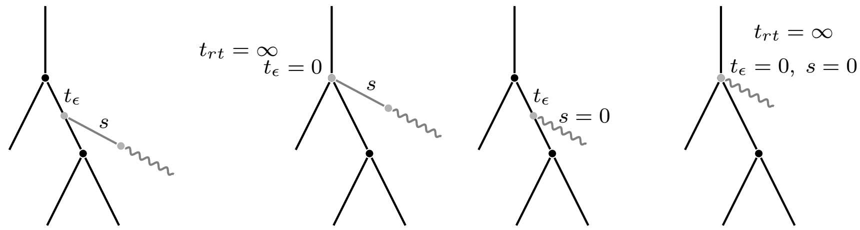


FIGURE 7. *The tree T^1_ϵ on the left is obtained from T_α by gluing Γ^v_1 to ϵ, splitting the edge into ϵ' nearer to the root and ϵ'' further from the root. It has two parameters, $t_{\epsilon'}$ and s. The root edge is denoted rt and the root vertex by v. The labelling where $t_{rt} = \infty$ coincides with the labelling where $t_{\epsilon'} = 0$.*

LEMMA 7.4. *For any functor $F\colon \mathsf{Trees}_\alpha \to \mathsf{Top}$ the coend $W_\alpha \otimes_{\mathsf{Trees}_\alpha} F$ is homeomorphic to the homotopy colimit of F over Trees_α.*

Proof. The category Trees_α is a poset. It is made up of a bunch of commutative diagrams just like the one below, one for each edge ϵ of T_α with adjacent vertices v and v'.

$$\begin{array}{ccccc} v'_1 & \longleftarrow & \epsilon_1 & \longrightarrow & v_1 \\ \downarrow & & \downarrow & & \downarrow \\ v'_0 & \longleftarrow & \epsilon_0 & \longrightarrow & v_0 \end{array} \tag{7.9}$$

By pasting such rectangles together along the vertices of T_α we obtain the whole category Trees_α. Since we took T_α to be a tree with no edges of length 0, we find that $W_\alpha(\epsilon_1)$ is homeomorphic to the square $[0, \infty]^2$. Every other object in the diagram is carried into a subspace of $W_\alpha(\epsilon_1)$.

$$W_\alpha(v'_1) = 0 \times [0, \infty] \quad W_\alpha(v_1) = \infty \times [0, \infty]$$
$$W_\alpha(v'_0) = (0, 0) \quad W_\alpha(\epsilon_0) = [0, \infty] \times 0 \quad W_\alpha(v_0) = (\infty, 0)$$

It is clear that W_α is naturally homeomorphic to the functor

$$N(x) = \left|\mathrm{Nerve}(\mathsf{Trees}_\alpha)_{x/}\right|$$

where $(\mathsf{Trees}_\alpha)_{x/}$ is the under category of x. ☐

In order to mimick these results for $p_1^{-1}(\alpha)$, we need an appropriate version of W which computes $F(\mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}})$ as a coend with SC_α. Recall that $\beta \in \mathsf{SC}^{\mathrm{h}}_{\mathsf{d}}$ lives in $F(\mathsf{SC}^{\mathrm{h}1}_{\mathsf{d}})$ if and only if, when we break apart the edges of length ∞, we are left with trees each living in $\mathsf{SC}^{\mathrm{h}}_{\mathsf{d}}(1, m)$ or $\mathsf{SC}^{\mathrm{h}}_{\mathsf{d}}(0, m)$. We say that the distance between any two discs must be ∞. Discs correspond to input edges of color e, so we are interested in the subset $W^1(T) \subset W(T)$ consisting of all edge-labellings such that every edge of color e is distance ∞ from every other edge of color e.

The distance $\mathrm{dist}(v, v')$ from a vertex v to a vertex v' is the sum

$$t(\epsilon_1) + \cdots + t(\epsilon_k)$$

where $\epsilon_1, \ldots, \epsilon_k$ is the shortest path in T from v to v' and $t(\epsilon_i)$ is the length of ϵ_i. This defines a distance function on the input edges of T as well by taking the distance between the corresponding vertices. In order to write $W^1_\alpha\colon \mathsf{Trees}_\alpha \to \mathsf{Top}$ we need to consider the distance of every edge and vertex of T_α from edges of color e. Call a vertex *far from* e if it is distance ∞ from any vertex touching an edge of color e. Call an input edge *far from* e if its vertex is far from e. Otherwise, we will say a vertex or input edge is close to e. Since internal edges are adjacent to two vertices they can be close to e on one end and far from e on the other.

Suppose ϵ is joined to v and v' with v closer to the root. We have the following possibilities for $W^1_\alpha(\epsilon_1)$.

List of Cases 7.1.

(i) If v and v' are far from e, then

$$W^1_\alpha(\epsilon_1) = \{(t_{\epsilon'}, t_{\epsilon''}, s) \in [0,\infty]^3 \mid t_{\epsilon'} + t_{\epsilon''} = \mathbf{t}_\alpha(\epsilon)\}.$$

In other words $W^1_\alpha(\epsilon_1) = W_\alpha(\epsilon_1)$.

(ii) If v is close to e and v' is far from e, then $\mathbf{t}_\alpha(\epsilon) = \infty$, so

$$W^1_\alpha(\epsilon_1) = \{(t_{\epsilon'}, t_{\epsilon''}, s) \in [0,\infty]^3 \mid t_{\epsilon'} + t_{\epsilon''} = \infty \text{ and } (t_{\epsilon'} = \infty \text{ or } s = \infty)\}.$$

(iii) If v is far from e and v' is close to e, then $\mathbf{t}_\alpha(\epsilon) = \infty$, so

$$W^1_\alpha(\epsilon_1) = \{(t_{\epsilon'}, t_{\epsilon''}, s) \in [0,\infty]^3 \mid t_{\epsilon'} + t_{\epsilon''} = \infty \text{ and } (t_{\epsilon''} = \infty \text{ or } s = \infty)\}.$$

(iv) If v and v' are close to e, then $\mathbf{t}_\alpha(\epsilon) = \infty$, and

$$W^1_\alpha(\epsilon_1) = \{(t_{\epsilon'}, t_{\epsilon''}, s) \in [0,\infty]^3 \mid t_{\epsilon'} + t_{\epsilon''} = \infty \text{ and } s = \infty\}.$$

The space $W^1_\alpha(\epsilon_0)$ is the subspace of $W^1_\alpha(\epsilon_1)$ where $s = 0$, which may be empty. Also, $W^1_\alpha(v_i)$ is the subspace of $W^1_\alpha(\epsilon_i)$ where $t_{\epsilon'} = 0$ and $s = 0$ if $i = 0$. The space $W^1_\alpha(v'_i)$ is the same except we have $t_{\epsilon''} = 0$.

Notice that $W^1_\alpha(v_0) = \emptyset$ if v is close to e and $W^1_\alpha(\epsilon_0) = \emptyset$ if v and v' are close to e. Thus we can compute the coend $W^1_\alpha \otimes_{\mathsf{Trees}_\alpha} \mathsf{SC}_\alpha$ over the smaller category Trees^1_α where we throw away the objects v_0 and ϵ_0 where v is close to e and ϵ is close to e at both ends. Using this category we have

Lemma 7.5. *For any functor $F\colon \mathsf{Trees}^1_\alpha \to \mathsf{Top}$ the coend $W^1_\alpha \otimes_{\mathsf{Trees}^1_\alpha} F$ is the homotopy colimit of F.*

Proof. We follow the proof of lemma 7.4. Let ϵ be an edge joined to v and v'. Following the four cases 7.1, the category Trees^1_α is one of the diagrams in figure 8 near the object ϵ_1. The left hand column lets us compute $N(x)$, the nerve of $(\mathsf{Trees}^1_\alpha)_{x/}$ for $x = \epsilon_i, v_i, v'_i$ in all four cases. It is clear that this agrees with the right hand side, which computes $W^1_\alpha(x)$ for all such x in all four cases. There is clearly a natural homeomorphism $N \to W^1_\alpha$. □

7.5. *Examining* SC_α

The following 4 cases encode the functor $\mathsf{SC}_\alpha\colon \mathsf{Trees}_\alpha \to \mathsf{Top}$ up to homotopy.

List of Cases 7.2.

Case 1: $\mathsf{SC}_\alpha(\epsilon_1)$ We get the equator of S^{d-1}_+, which is S^{d-2}. The vertex where Γ^v_1 is attached to ϵ_1 must be labeled by $(\beta, q) \in SC^{\hat{\mathrm{h}}}_d(0,1;0,1) \simeq S^{d-2}$. The other new vertex must be labelled by $\{\bullet\} = SC^{\hat{\mathrm{h}}}_d(1,0;0,0)$.

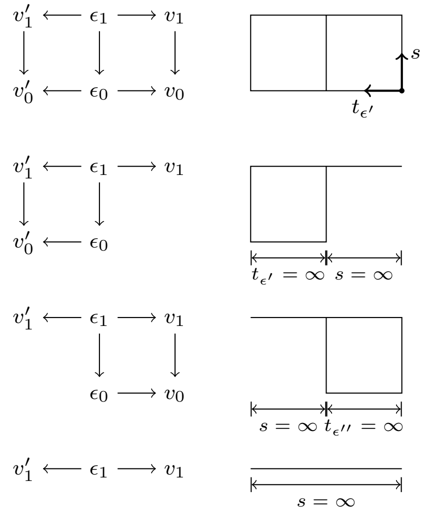
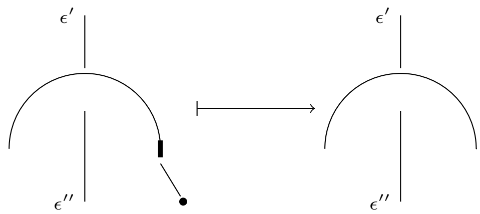
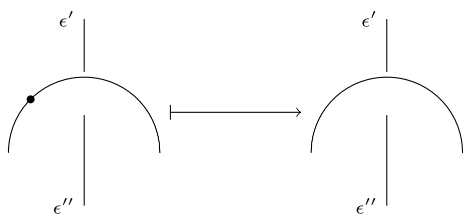


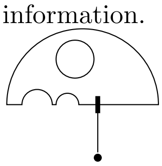
FIGURE 8. *The left hand column shows the category* Trees^1_α *near the object* ϵ_1. *The right hand column shows the spaces* $W^1_\alpha(\epsilon_1)$. *The rows represent the four possibilities for the vertices on either side of* α *to be near to or far from* e.



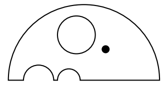
Case 2: $\mathsf{SC}_\alpha(\epsilon_0)$ Now we get the hemisphere S^{d-1}_+. This is similar to case 1, except that now we have no edge of color $\hat{\text{h}}$. The marked ê point lives in $|1_{SC_d}| \simeq S^{d-1}_+$. Note that the inclusion $S^{d-2} \to S^{d-1}_+$ corresponds to the inclusion $\mathsf{SC}_\alpha(\epsilon_1) \to \mathsf{SC}_\alpha(\epsilon_0)$.



Case 3: $\mathsf{SC}_\alpha(v_1)$ Here we get $|\alpha_v| \cap \mathbb{R}^{d-1}$ where $\alpha_v \in SC^{\text{h}}_d$ is the label on the vertex v of T_α. A point of this space is an $\hat{\text{h}}$ marked point of $|\alpha_v|$, which is attached by an edge to an ê point. Forgetting the ê point loses no information.



Case 4: $\mathsf{SC}_\alpha(v_0)$ Finally the last case gives us $|\alpha_v|$. Now we are allowed to put the ê marked point wherever we like in $|\alpha_v|$.



7.6. *Comparing $p^{-1}(\alpha)$ and $p_1^{-1}(\alpha)$*

The eight cases 7.2 define a functor $F\colon \mathsf{Trees}_\alpha \to \mathsf{Top}$ and a natural transformation $\mathsf{SC}_\alpha \to F$. Applying F to the rectangle (7.9) we have the commutative diagram below.

$$\begin{array}{ccccc} |\alpha_{v'}| \cap \mathbb{R}^{d-1} & \longleftarrow & S^{d-2} & \longrightarrow & |\alpha_v| \cap \mathbb{R}^{d-1} \\ \downarrow & & \downarrow & & \downarrow \\ |\alpha_{v'}| & \longleftarrow & S_+^{d-1} & \longrightarrow & |\alpha_v| \end{array}$$

The map $S_+^{d-1} \to |\alpha_v|$ is the inclusion of the boundary of the half-disc of α_v corresponding to the edge ϵ. If v' is further from the root than v then $S_+^{d-1} \to |\alpha_v|$ is the inclusion of the top hemisphere of the closed unit half-disc. The map $S^{d-2} \to S_+^{d-1}$ is the inclusion of the equator.

By lemma 7.3 and lemma 7.4 we have

$$p^{-1}(\alpha) \simeq \operatorname*{hocolim}_{\mathsf{Trees}_\alpha} \mathsf{SC}_\alpha.$$

Furthermore, the natural transformation $\mathsf{SC}_\alpha \to F$ gives an equivalence

$$\operatorname*{hocolim}_{\mathsf{Trees}_\alpha} \mathsf{SC}_\alpha \simeq \operatorname*{hocolim}_{\mathsf{Trees}_\alpha} F$$

since all spaces in sight are cofibrant and the transformation is an object-wise weak equivalence. We can now compute $p^{-1}(\alpha)$ using F as follows. For each edge ϵ of T_α there is a copy of $(S_+^{d-1})_\epsilon \times [0,1]$. Each vertex v of T_α such that α_v contains only one disc gives a copy of $|\alpha_v|$. If ϵ is attached to v at the end closer to the root, we glue $(S_+^{d-1})_\epsilon \times \{0\}$ to $\partial_\epsilon |\alpha_v|$, which is the boundary of the half-disc in $|\alpha_v|$ corresponding to ϵ. If ϵ is attached to v' at the end further from the root we glue $(S_+^{d-1})_\epsilon \times \{1\}$ to $\partial_{rt} |\alpha_{v'}|$, which is the upper hemisphere of the unit half-disc. An example of α and $p^{-1}(\alpha)$ is shown in figure 9.

It is easy now to do the same thing for $p_1^{-1}(\alpha)$. Just restrict F to Trees_α^1 then

$$p_1^{-1}(\alpha) \simeq \operatorname*{hocolim}_{\mathsf{Trees}_\alpha^1} \mathsf{SC}_\alpha \simeq \operatorname*{hocolim}_{\mathsf{Trees}_\alpha^1} F.$$

For each edge ϵ of T_α attached to the vertex v toward the root and v' away from the root we have a copy of

$$\begin{array}{ll} (S_+^{d-1})_\epsilon \times [0,1] & \text{if } v, v' \text{ are far from e;} \\ (S_+^{d-1})_\epsilon \times [0,1/2] \cup (S^{d-2})_\epsilon \times [1/2,1] & \text{if } v \text{ is far from e and } v' \text{ is close to e;} \\ (S^{d-2})_\epsilon \times [0,1/2] \cup (S_+^{d-1})_\epsilon \times [1/2,1] & \text{if } v \text{ is close to e and } v' \text{ is far from e;} \\ (S^{d-2})_\epsilon \times [0,1] & \text{if } v, v' \text{ are close to e.} \end{array}$$

If ϵ is external it gives a copy of $(S_+^{d-1})_\epsilon \times [0,1/2]$. If v is close to e then there is a copy of $|\alpha_v| \cap \mathbb{R}^{d-1}$. Gluing these pieces together we find that each fiber is equivalent to a wedge of $n-1$ spheres $(S^{d-1})^{\vee n-1}$. Figure 9 shows this for the two-dimensional case, with $\alpha \in F(\mathsf{SC}_\mathsf{d}^{\mathrm{h1}})(2,3)$.

We conclude that $p_1^{-1}(\alpha) \to p^{-1}(\alpha)$ is a weak equivalence, so the proof of lemma 7.2 is complete.

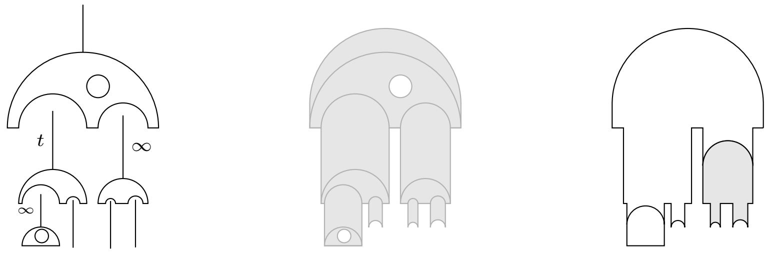


FIGURE 9. *On the left is* $\alpha \in F(\mathsf{SC}_{\mathsf{d}}^{\mathrm{h1}})(2,3)$. *In the middle is* $p^{-1}(\alpha)$*, and on the right is* $p_1^{-1}(\alpha)$. *Both* $p^{-1}(\alpha)$ *and* $p^{-1}(\alpha)$ *have the homotopy type of a wedge of spheres, one for each disc in* α.

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