ON THE QUANTUM CLUSTER ALGEBRAS OF FINITE TYPES

MING DING

Abstract. We extend the definition of the quantum analogue of Caldero-Chapoton map defined by [12][11]. When Q is a quiver of finite type, we prove that the algebra $\mathcal{AH}_{|k|}(Q)$ generated by all generalized cluster variables(see Definition 2.1) is exactly the quantum cluster algebra $\mathcal{EH}_{|k|}(Q)$.

1. Introduction

Quantum cluster algebras were introduced by A. Berenstein and A. Zelevinsky [3] to study the canonical basis. When $q = 1$, the quantum cluster algebras are exactly the corresponding cluster algebras which were introduced and studied by S.Fomin and A. Zelevinsky in a series of papers [8][9][1]. The quantum analogue of the Caldero-Chapoton formula [4] was defined by D. Rupel [12] and the author conjectured that cluster variables could be expressed in terms of the quantum analogue of the Caldero-Chapoton formula and proved it for cluster variables in finite types as well as in almost acyclic clusters. Later this conjecture has been proved for acyclic equally valued quivers in [11].

The cluster category is introduced for its combinatorial similarities with cluster algebras. Different from the case in cluster algebras, for any objects M, N in cluster category associated to quantum cluster algebra, it does not generally hold that $X_N X_M = |k|^{\pm\frac{1}{2}n_{N\oplus M}} X_{N\oplus M}$ for any $n_{N\oplus M} \in \mathbb{Z}$. Thus the natural problem is to ask if $X_{N\oplus M}$ is in the corresponding quantum cluster algebra. Hence it becomes interesting to study the relation between the algebra generated by all generalized cluster variables(see Definition 2.1) and the corresponding quantum cluster algebra. In the case of cluster algebras, both of them are equal for finite and affine types [5][7]. The aim of this article is to prove that for any quiver Q of finite type, the algebra $\mathcal{AH}_{|k|}(Q)$ generated by all generalized cluster variables is still the quantum cluster algebra $\mathcal{EH}_{|k|}(Q)$.

2. Preliminaries and main result

2.1. **Definition of quantum cluster algebra.** Let L be a lattice of rank m and $\Lambda : L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. Set a formal variable q and the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the *based quantum torus* associated

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to the pair (L,Λ) to be the $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathcal{T}$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^e : e \in L\}$ and the multiplication

$$X^eX^f = q^{\Lambda(e,f)/2}X^{e+f}.$$

It is known that $\mathcal{T}$ is contained in its skew-field of fractions $\mathcal{F}$. A *toric frame* in $\mathcal{F}$ is a map $M : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ given by

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of $\mathcal{F}$ and $\eta : \mathbb{Z}^m \to L$ is an isomorphism of lattices. By the definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the based quantum torus $\mathcal{T}_M := \varphi(\mathcal{T})$ and satisfy the following relations:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2}M(\mathbf{c}+\mathbf{d}),\ M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})}M(\mathbf{d})M(\mathbf{c}),$$
$$M(\mathbf{0}) = 1,\ M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism η. Let Λ_M be the skew-symmetric $m\times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1,\ldots,e_m\}$ is the standard basis of $\mathbb{Z}^m$. Given a toric frame M, let $X_i = M(e_i)$. Then we have

$$\mathcal{T}_M = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1},\ldots,X_m^{\pm 1} : X_iX_j = q^{\lambda_{ij}}X_jX_i\rangle.$$

An easy computation shows that:

$$M(\mathbf{c}) = q^{\frac{1}{2}\sum_{i<j}c_ic_j\lambda_{ji}}X_1^{c_1}X_2^{c_2}\cdots X_m^{c_m} =: X^{(\mathbf{c})} \quad (\mathbf{c}\in\mathbb{Z}^m).$$

Let Λ be an $m\times m$ skew-symmetric matrix and $\tilde{B}$ an $m\times n$ matrix with $n \le m$. We call the pair $(\Lambda,\tilde{B})$ *compatible* if $\tilde{B}^T\Lambda = (D|0)$ is an $n\times m$ matrix with $D = diag(d_1,\cdots,d_n)$ where $d_i\in\mathbb{N}$ for $1\le i\le n$. The pair $(M,\tilde{B})$ is called a *quantum seed* if the pair $(\Lambda_M,\tilde{B})$ is compatible. Define the $m\times m$ matrix $E = (e_{ij})$ as follows

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j\neq k;\\ -1 & \text{if } i=j=k;\\ max(0,-b_{ik}) & \text{if } i\neq j=k.\end{cases}$$

For $n,k\in\mathbb{Z}$, $k\ge 0$, denote $\begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{(q^n-q^{-n})\cdots(q^{n-r+1}-q^{-n+r-1})}{(q^r-q^{-r})\cdots(q-q^{-1})}$. Let $\mathbf{c} = (c_1,\ldots,c_m)\in\mathbb{Z}^m$ with $c_k\ge 0$. Define the toric frame $M' : \mathbb{Z}^m\to\mathcal{F}\setminus\{0\}$ as follows:

$$M'(\mathbf{c}) = \sum_{p=0}^{c_k}\begin{bmatrix} c_k\\ p\end{bmatrix}_{q^{d_k/2}}M(E\mathbf{c}+p\mathbf{b}^k),\quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1}. \tag{2.1}$$

where the vector $\mathbf{b}^k\in\mathbb{Z}^m$ is the $k-$th column of $\tilde{B}$. Then the quantum seed $(M',\tilde{B}')$ is defined to be the mutation of $(M,\tilde{B})$ in direction k. Two quantum seeds are called mutation-equivalent if they can be obtained from each other by a sequence of mutations. Let $\mathcal{C} = \{M'(e_i) : i\in[1,n]\}$ where $(M',\tilde{B}')$ is mutation-equivalent to $(M,\tilde{B})$. The elements of $\mathcal{C}$ are called *cluster variables*. Let $\mathbb{P} = \{M(e_i) : i\in[n+1,m]\}$ and the elements of $\mathbb{P}$ are called *coefficients*. Denote by $\mathbb{ZP}$ the ring of laurent polynomials generated by $\{q^{\pm\frac{1}{2}}\} \cup \mathbb{P}$. Then the *quantum cluster algebra* $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is defined by the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{C}$.

2.2. **The quantum Caldero-Chapoton map and main result.** Let k be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers and $\widetilde{Q}$ an acyclic valued quiver with vertex set $\{1, \ldots, m\}$. Denote the subset $\{n+1, \ldots, m\}$ by C. The full subquiver Q on the vertices $1, \ldots, n$ is called the *principal part* of $\widetilde{Q}$.

Let $\widetilde{B}$ be the $m \times n$ matrix associated to the quiver $\widetilde{Q}$ whose entry in position (i, j) given by

$$b_{ij} = |\{\text{arrows}\, i \longrightarrow j\}| - |\{\text{arrows}\, j \longrightarrow i\}|$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Denote by $\widetilde{I}$ the left $m \times n$ submatrix of the identity matrix of size $m \times m$. Assume that there exists some antisymmetric $m \times m$ integer matrix Λ such that

$$\Lambda(-\widetilde{B}) = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{2.2}$$

where I_n is the identity matrix of size $n \times n$. Let $\widetilde{R} = \widetilde{R}_{\widetilde{Q}}$ be the $m \times n$ matrix with its entry in position (i, j) given by

$$\widetilde{r}_{ij} := \dim_k \mathrm{Ext}^1_{k\widetilde{Q}}(S_j, S_i) = |\{\text{arrows}\, j \longrightarrow i\}|.$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Set $\widetilde{R}^{tr} = \widetilde{R}_{\widetilde{Q}^{op}}$. Denote the principal parts of the matrices $\widetilde{B}$ and $\widetilde{R}$ by B and R respectively. Note that $\widetilde{B} = \widetilde{R}^{tr} - \widetilde{R}$ and $B = R^{tr} - R$.

Let $\mathcal{C}_{\widetilde{Q}}$ be the cluster category of $k\widetilde{Q}$, i.e., the orbit category of the derived category $\mathcal{D}^b(\widetilde{Q})$ by the functor $F = \tau \circ [-1]$(see [2]). Let I_i be the indecomposable injective $k\widetilde{Q}$ module for $1 \leq i \leq m$. Then the indecomposable $k\widetilde{Q}$-modules and $I_i[-1]$ for $1 \leq i \leq m$ exhaust all indecomposable objects of the cluster category $\mathcal{C}_{\widetilde{Q}}$. Each object M in $\mathcal{C}_{\widetilde{Q}}$ can be uniquely decomposed as:

$$M = M_0 \oplus I_M[-1]$$

where M_0 is a module and I_M is an injective module. The cluster category $\mathcal{C}_{\widetilde{Q}}$ is a 2-Calabi-Yau category, i.e, for any objects $M, N \in \mathcal{C}_{\widetilde{Q}}$:

$$\mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M, N) \cong \mathrm{D}\mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N, M)$$

where $\mathrm{D} = \mathrm{Hom}_k(-, k)$ is the standard duality.

The Euler form on $k\widetilde{Q}$-modules M and N is given by

$$\langle M, N \rangle = \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N).$$

Note that the Euler form only depends on the dimension vectors of M and N.

The quantum Caldero-Chapoton map of an acyclic quiver $\widetilde{Q}$ has been defined in [12][11]. For our purpose, we need to extend these definitions to the following map

$$X_? : \mathrm{obj}\mathcal{C}_{\widetilde{Q}} \longrightarrow \mathcal{T}$$

defined by the following rule: If M is a kQ-module and I is an injective $k\widetilde{Q}$-module, then

$$X_{M\oplus I[-1]}=\sum_{\underline{e}}|\mathrm{Gr}_{\underline{e}}M|q^{-\frac{1}{2}\langle \underline{e},\underline{m}-\underline{e}-\underline{i}\rangle}X^{-\widetilde{B}\underline{e}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI},$$

where $\underline{\dim}I=\underline{i},\underline{\dim}M=\underline{m}$ and $\mathrm{Gr}_{\underline{e}}M$ denotes the set of all submodules V of M with $\underline{\dim}V=\underline{e}$. We note that

$$X_{P[1]}=X_{\tau P}=X^{\underline{\dim}P/radP}=X^{\underline{\dim}\mathrm{soc}I}=X_{I[-1]}=X_{\tau^{-1}I}.$$

for any projective $k\widetilde{Q}$-module P and injective $k\widetilde{Q}$-module I with $\mathrm{soc}I=P/\mathrm{rad}P$. In the following, we denote by the corresponding underlined small letter $\underline{x}$ the dimension vector of a kQ-module X and view $\underline{x}$ as a column vector in $\mathbb{Z}^n$.

Definition 2.1. X_L is called *the generalized cluster variable*, if L is kQ-module or $L=M\oplus I[-1]\in\mathcal{C}_{\widetilde{Q}}$ satisfying that M is a kQ-module and I is an injective $k\widetilde{Q}$-module.

Denote by $\mathcal{AH}_{|k|}(Q)$ is the $\mathbb{ZP}$-algebra generated by all the generalized cluster variables and by $\mathcal{EH}_{|k|}(Q)$ is the corresponding quantum cluster algebra, i.e, the $\mathbb{ZP}$-algebra generated by all the cluster variables. The main result of this article is the following theorem:

Theorem 2.2. *For any quiver Q of finite type, we have $\mathcal{EH}_{|k|}(Q)=\mathcal{AH}_{|k|}(Q)$.*

Remark 2.3. It is an open problem that Theorem 2.2 holds for affine types because it is difficult to check whether the generalized cluster variables associated to regular modules in homogeneous tubes of degree at least two are in quantum cluster algebra.

3. Proof of the main theorem 2.2

In this section, we fix a quiver Q of finite type with n vertices. Firstly, we recall some notations. For any $k\widetilde{Q}-$modules M,N and E, denote by ε_{MN}^{E} the cardinality of the set $\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)_E$ which is the subset of $\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)$ consisting of those equivalence classes of short exact sequences with middle term isomorphic to M ([10, Section 4]). Let F_{AB}^{M} be the number of submodules U of M such that U is isomorphic to B and M/U is isomorphic to A. Then by definition, we have

$$|\mathrm{Gr}_{\underline{e}}(M)|=\sum_{A,B;\underline{\dim}B=\underline{e}}F_{AB}^{M}.$$

Denote by $[M,N]^1=\dim_k\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)$ and $[M,N]=\dim_k\mathrm{Hom}_{k\widetilde{Q}}(M,N)$. The following Theorem 3.1 proved in [6] and Proposition 3.2 give the explicit relations between X_NX_M and $X_{N\oplus M}$.

Theorem 3.1. [6] *Let M and N be kQ-modules. Then*

$$q^{[M,N]^1}X_MX_N=q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}^{tr})\underline{m},(\widetilde{I}-\widetilde{R}^{tr})\underline{n})}\sum_{E}\varepsilon_{MN}^{E}X_E.$$

Let M be any $kQ-$module and I any injective $k\widetilde{Q}-$module. Define

$$\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'} := \{f : M \longrightarrow I | kerf \cong B, cokerf \cong I'\}.$$

The following result together with Theorem 3.1 is essentially important for us to prove Theorem 2.2.

Proposition 3.2. *With the above notations, we have*

$$q^{[M,I]}X_M X_{I[-1]} = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}^{tr})\underline{m},-\underline{\dim} socI)}\sum_{B,I'}|Hom_{k\widetilde{Q}}(M,I)_{BI'}|X_{B\oplus I'[-1]}.$$

Proof.

$$\begin{aligned}
& X_M X_{I[-1]} \\
= & \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} F^M_{GH} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}} X^{\underline{\dim} socI} \\
= & \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} F^M_{GH} q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m},\underline{\dim} socI)} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI} \\
= & q^{\frac{1}{2}\Lambda(-(\widetilde{I}-\widetilde{R}^{tr})\underline{m},\underline{\dim} socI)} \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{h},\underline{\dim} socI)} F^M_{GH} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI} \\
= & q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}^{tr})\underline{m},-\underline{\dim} socI)} \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} q^{-\frac{1}{2}[H,I]} F^M_{GH} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI}.
\end{aligned}$$

Here we use the fact that

$$\Lambda(-\widetilde{B}\underline{h},\underline{\dim} socI) = -\underline{h}^{tr}\widetilde{B}^{tr}\Lambda(\underline{\dim} socI) = -\underline{h}^{tr}(\underline{\dim} socI) = -[H,I].$$

Note that we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y & = & Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & I \longrightarrow & I' & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

And short exact sequences

$$0 \longrightarrow B \longrightarrow M \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow I \longrightarrow I' \longrightarrow 0.$$

It follows that

$$\sum_{B} F_{XY}^{B} F_{AB}^{M} = \sum_{G} F_{AX}^{G} F_{GY}^{M},\ |\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}| = \sum_{A} |\mathrm{Aut}(A)| F_{AB}^{M} F_{I'A}^{I}$$

and

$$\sum_{A,I',X} |\mathrm{Aut}(A)| F_{I'A}^{I} F_{AX}^{G} = \sum_{I',X} |\mathrm{Hom}_{k\widetilde{Q}}(G,I)_{XI'}| = q^{[G,I]} = q^{\langle G,I\rangle}.$$

By [10, Lemma 1], we have $(\widetilde{I}-\widetilde{R}^{tr})\underline{i} = \underline{\dim} socI$. Now we can compute the term

$$\begin{aligned}
& \sum_{B,I'} |\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}| X_{B\oplus I'[-1]} \\
= & \sum_{A,B,I',X,Y} |\mathrm{Aut}(A)| F_{AB}^{M} F_{I'A}^{I} q^{-\frac{1}{2}\langle Y, X-I'\rangle} F_{XY}^{B} X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}^{tr})\underline{b}+\underline{\dim} socI'} \\
= & \sum_{A,G,I',X,Y} q^{-\frac{1}{2}\langle Y, X-I'\rangle} |\mathrm{Aut}(A)| F_{I'A}^{I} F_{AX}^{G} F_{GY}^{M} X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}^{tr})\underline{b}+\underline{\dim} socI'} \\
= & \sum_{G,H} q^{\langle G,I\rangle} q^{-\frac{1}{2}\langle H, G-I\rangle} F_{GH}^{M} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI} \\
= & \sum_{G,H} q^{\langle M,I\rangle} q^{-\frac{1}{2}\langle H,I\rangle} q^{-\frac{1}{2}\langle H,G\rangle} F_{GH}^{M} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI} \\
= & q^{[M,I]} \sum_{G,H} q^{-\frac{1}{2}[H,I]} q^{-\frac{1}{2}\langle H,G\rangle} F_{GH}^{M} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI}.
\end{aligned}$$

Here we use the facts

$$\underline{i'} + \underline{a} = \underline{i},\ \underline{x} + \underline{a} = \underline{g} \Longrightarrow \underline{x} - \underline{i'} = \underline{g} - \underline{a}.$$

And

$$\begin{aligned}
& -\widetilde{B}\underline{y} - (\widetilde{I}-\widetilde{R}^{tr})\underline{b} + \underline{\dim} socI' \\
= & -\widetilde{B}\underline{h} - (\widetilde{I}-\widetilde{R}^{tr})(\underline{m} - \underline{i} - \underline{i'}) + \underline{\dim} socI' \\
= & -\widetilde{B}\underline{h} - (\widetilde{I}-\widetilde{R}^{tr})\underline{m} + (\widetilde{I}-\widetilde{R}^{tr})(\underline{i} - \underline{i'}) + \underline{\dim} socI' \\
= & -\widetilde{B}\underline{h} - (\widetilde{I}-\widetilde{R}^{tr})\underline{m} + (\widetilde{I}-\widetilde{R}^{tr})\underline{i} \\
= & -\widetilde{B}\underline{h} - (\widetilde{I}-\widetilde{R}^{tr})\underline{m} + \underline{\dim} socI.
\end{aligned}$$

This finishes the proof. □

Remark 3.3. Proposition 3.2 holds for any acyclic quiver.

The following lemma is well-known. Here we give a sketch of proof.

Lemma 3.4. *Let*

$$M \longrightarrow E \longrightarrow N \xrightarrow{\epsilon} M[1]$$

be a non-split triangle in $\mathcal{C}_{\widetilde{Q}}$. *Then*

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,E) < \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M\oplus N, M\oplus N).$$

Proof. For any object $L \in \mathcal{C}_{\widetilde{Q}}$, applying the functor $\mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(-,L)$ to the above non-split triangle gives rise to the exact sequence

$$0 \longrightarrow ker f_L \longrightarrow \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N,L) \xrightarrow{f_L} \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,L) \xrightarrow{g_L} \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M,L) \longrightarrow coker g_L \longrightarrow 0$$

Thus we have

$$\dim_k ker f_L + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,L) + \dim_k coker g_L = \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N,L) + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M,L)$$

Hence

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,N) \leq \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N,N) + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M,N)$$

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,E) \leq \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N,E) + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M,E).$$

Note that $0 \neq \epsilon \in ker f_M$, we have

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,M) < \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N,M) + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M,M)$$

Therefore

$$\begin{aligned} \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M\oplus N, M\oplus N) &> \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,N) + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,M) \\ &= \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(N,E) + \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(M,E) \\ &\geq \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,E). \end{aligned}$$

This proves our assertion. □

Proof of the Theorem 2.2: We only need to prove that for any generalized cluster variable $X_L \in \mathcal{AH}_{|k|}(Q)$, then $X_L \in \mathcal{EH}_{|k|}(Q)$.

Let $L \cong \bigoplus_{i=1}^{l} n_i L_i, n_i \in \mathbb{N}$ where $L_i (1 \leq i \leq l)$ are indecomposable objects in $\mathcal{C}_{\widetilde{Q}}$. Thus $X_{L_i} (1 \leq i \leq l)$ are in $\mathcal{EH}_{|k|}(Q)$. By Theorem 3.1, Proposition 3.2 and Lemma 3.4, we get that

$$X_{L_1}^{n_1} X_{L_2}^{n_2} \cdots X_{L_l}^{n_l} = q^{\frac{1}{2}n_L} X_L + \sum_{\dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(E,E) < \dim_k \mathrm{Ext}^1_{\mathcal{C}_{\widetilde{Q}}}(L,L)} f_{n_E}(q^{\pm\frac{1}{2}}) X_E$$

where $n_L \in \mathbb{Z}$ and $f_{n_E}(q^{\pm\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Thus by induction, we can prove that $X_L \in \mathcal{EH}_{|k|}(Q)$ which implies $\mathcal{EH}_{|k|}(Q) = \mathcal{AH}_{|k|}(Q)$.

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Institute for advanced study, Tsinghua University, Beijing 100084, P. R. China
E-mail address: m-ding04@mails.tsinghua.edu.cn (M.Ding)