

To Feed or Not to Feed Back

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Abstract

We study the communication over Finite State Channels (FSCs), where the encoder and the decoder can control the availability or the quality of the noise-free feedback. Specifically, the instantaneous feedback is a function of an action taken by the encoder, an action taken by the decoder, and the channel output. Encoder and decoder actions take values in finite alphabets, and may be subject to average cost constraints.

We prove capacity results for such a setting by constructing a sequence of achievable rates, using a simple scheme based on ‘code tree’ generation, that generates channel input symbols along with encoder and decoder actions. We prove that the limit of this sequence exists. For a given block length N and probability of error, ϵ , we give an upper bound on the maximum achievable rate. Our upper and lower bounds coincide and hence yield the capacity for the case where the probability of initial state is positive for all states. Further, for stationary *indecomposable* channels without intersymbol interference (ISI), the capacity is given as the limit of normalized directed information between the input and output sequence, maximized over an appropriate set of causally conditioned distributions. As an important special case, we consider the framework of ‘to feed or not to feed back’ where either the encoder or the decoder takes binary actions, which determine whether current channel output will be fed back to the encoder, with a constraint on the fraction of channel outputs that are fed back. As another special case of our framework, we characterize the capacity of ‘coding on the backward link’ in FSCs, i.e. when the decoder sends limited-rate instantaneous coded noise-free feedback on the backward link.

Index Terms

Actions, Causal Conditioning, Channel with States, Cost Constraints, Directed Information, Feedback Sampling, Indecomposable Channel, Intersymbol Interference, Sampled Feedback, Time-invariant Deterministic Feedback, To Feed or Not to Feed Back.

I. INTRODUCTION

Feedback plays a very important role in communication systems. Despite proving a pessimistic result in [1] that feedback does not increase the capacity of a memoryless channel, Shannon did foresee the important role of feedback, which he highlighted in the first Shannon Lecture. Indeed, even for memoryless channels, feedback has its merits, such as simple capacity achieving coding schemes and improved reliability, [2], [3]. Feedback is also known to increase the capacity for multiple-access channels, [4] and broadcast channels, [5],[6].

In his book [7], Gallager introduced finite state channels (FSCs) as an apt model for a very broad family of channels with memory. When no feedback is present and the channel is stationary and indecomposable without ISI, the capacity was shown by Gallager in [7] and by Blackwell, Breiman and Thomasian in [8] to be

$$C_{NF} = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{P(x^N)} I(X^N; Y^N). \quad (1)$$

For the case of no ISI, stationary and indecomposable finite state channels with time invariant deterministic feedback, the capacity was shown in [9] to be,

$$C_{FB} = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{Q(x^N \| z^{N-1})} I(X^N \rightarrow Y^N), \quad (2)$$

where $Q(x^N \| z^{N-1})$ is causal conditioning introduced by Kramer in [10], [11] and is defined as,

$$Q(x^N \| z^{N-1}) \triangleq \prod_{i=1}^N Q(x_i | x^{i-1}, z^{i-1}). \quad (3)$$

Here Z_i is a time-invariant deterministic function of the output Y_i . Subsequent work on FSCs included the characterization of the capacity of finite state multiple access channel in [12]. When the channels have memory, feedback can increase the capacity even for single user channels. One such example is the chemical channel introduced in 1961 by Blackwell in [13], also referred to as the ‘trapdoor channel’ by Ash in [14]. The capacity of this channel without feedback is a long-standing open problem with only bounds on it known, such as those established by Kobayashi *et al* in [15], [16]. With feedback, the capacity of the trapdoor channel was computed in [17] using dynamic programming approach and shown to be strictly higher than the capacity without feedback. For Gaussian channels with memory, Cover and Pombra in [18] showed feedback cannot

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increase the capacity of an additive white gaussian channel by more than half of a bit. Kim characterized the capacity of a wide class of stationary Gaussian channels with feedback in [19].

Directed information, denoted by $I(X^N \rightarrow Y^N)$, was introduced by Massey in [20], where he credits it to Marko [21]. It was further shown that directed information equals mutual information for memoryless channels iff there is no feedback by Massey and Massey in [22]. Directed information also appears in the work of Tatikonda *et al*, [23], [24], where there is generalization of work by Verdu and Han in [25] for the case of channels with feedback. Capacity of some Markovian Channels was computed using directed information by Yang *et al* in [26] and Chen and Berger in [27]. Tatikonda also formulated the problem of computing capacities of channels with feedback as a Markov Decision Process in [28]. Zero error capacity was also computed using dynamic programming in [29]. Recently, interpretations of directed information in gambling, portfolio theory and estimation have been characterized in [30], [31] and [32]. The capacity of the compound channel with feedback was computed in [33] using directed information. Directed information also appeared in rate distortion problems, such as source coding with feed-forward by Pradhan and Venkataramanan [34], and implicitly in the competitive prediction framework of [35].

In [36], the notion of *actions* in a source coding context was introduced. Their setting is a generalization of the Wyner-Ziv source coding with decoder side information problem in [37], where now the decoder can take actions based on the index obtained from the encoder to affect the formation or availability of side information. In [38], the channel coding dual is studied where the transmitter takes actions that affect the formation of channel states. This framework captures various new coding scenarios which include two stage recording on a memory with defects, motivated by similar problems in magnetic recording and computer memories. Kittichokechai *et al* in [39] studied a variant of the problem in [36] and [38], where encoder and decoder both have action dependent partial side information. However, in the source coding formulation of [36], attention was restricted to the case where the actions are taken by the decoder while in the channel coding scenario of [38] and [39], actions were taken only by the encoder. Recently, in [40], the channel coding setting in [38] and [39] was generalized, to accommodate the case where both the encoder and the decoder take channel probing actions, with associated costs, to maximize the rate of reliable communication. This was referred to as the ‘*Probing Capacity*’.

In this paper, we introduce the notion of actions in acquisition of noise-free feedback or its deterministic function for FSCs. The main contribution of this paper is in characterizing the cost-capacity trade-off when the feedback observed by the encoder is a deterministic function of an action taken by the encoder, an action taken by the decoder, and the channel output, when actions are required to satisfy an average cost constraint. More precisely, the encoder observes ‘*sampled*’ feedback $Z_i = f(A_{e,i}, A_{d,i}, Y_i)$, where $f(\cdot)$ is a deterministic function, Y_i is the channel output, $A_{e,i} = A_{e,i}(M, Z^{i-1})$ is the action taken by the encoder as a function of the message and the past sampled feedback, and $A_{d,i}$ is the action taken by the decoder, where we study two scenarios: one where that action is strictly causal in the channel output, i.e., $A_{d,i} = A_{d,i}(Y^{i-1})$, and one where it can depend also on the present channel output, i.e., $A_{d,i} = A_{d,i}(Y^i)$. The problem is motivated by practical applications where acquisition of the feedback may be costly, and either or both the encoder and decoder influence whether and what from the channel output is to be fed back.

The key technique in our achievability result lies in generating both actions and input symbol code trees, as described in Section IV. With this achievability, we find most of the proof follows that in [9], except for some cases where care has to be taken to properly handle cost constraints. This is because the presence of cost constraints results in breaking down of some properties that were used in [9] such as sub-additivity. The main contribution of our paper is in obtaining a multi-letter characterization of the capacity for our communication scenario, involving maximization over directed information. Such a characterization is of importance not only for its theoretical significance, but also because it admits a dynamic optimization formulation that can lead to analytic closed form capacity expressions for specific channels, similarly as in [17].

A special case of our framework is when only the encoder or the decoder is the one taking actions. Under this setting, we motivate and compute a special case of *to feed or not to feed back*, i.e., where actions are binary corresponding to observing the channel output or not observing it, the cost constraint corresponding to the fraction of channel output observations allowed, and the channel states evolve as a markov chain independent of the channel input process. When only the encoder takes actions, we derive a single letter lower bound on this capacity and show that it is strictly better than the rate achieved by a naive time sharing scheme between capacity at zero cost (corresponding to Gallager’s capacity for FSCs in [7]) and unit cost (corresponding to the complete noise-free feedback capacity of [9]). For the same FSC, we also derive a single letter lower bound on the capacity when only the decoder takes binary actions. A special case of the framework when only decoder takes actions is that of coding on the backward link, where the decoder sends a symbol from the action alphabet based on the channel outputs observed so far, thus operating at an instantaneous rate which is log the cardinality of said alphabet. The capacity for this case is characterized in single letter form for some Markovian Channels.

The rest of paper is organized as follows. Section II describes the channel model and formulates the problem studied in this paper. The main results of this paper are outlined in Section III. Section IV is dedicated to capacity-achieving coding schemes, while converse results are proved in Section V. Section VI characterizes the capacity for stationary, indecomposable, finite state channels without intersymbol interference (ISI). Section VII generalizes the framework from decoder taking actions strictly causally dependent on the channel output (i.e. $A_{d,i} = A_{d,i}(Y^{i-1})$) to the case when decoder can also use the current output to generate its actions (i.e. $A_{d,i} = A_{d,i}(Y^i)$). As special cases, Section VIII-A outlines the capacity results when actions are

taken only by the encoder while the case when only decoder takes actions is discussed in Section VIII-B. Section IX presents single letter lower bounds for a specific example of *to feed or not to feed back* (i.e. when actions are binary) for Markovian channels when only one of the two, encoder or decoder, takes the actions. Section X establishes that coding on the backward link for FSCs is a special case of our general framework, and computes the capacity for an example of a Markovian channel. The paper is summarized and concluded in Section XI.

II. CHANNEL MODEL AND PROBLEM FORMULATION

We begin by introducing the notation used throughout this paper. Let upper case, lower case, and calligraphic letters denote, respectively, random variables, specific or deterministic values they may assume, and their alphabets. For two jointly distributed random variables, X and Y , let P_X , P_{XY} and $P_{X|Y}$ respectively denote the marginal of X , joint distribution of (X, Y) and conditional distribution of X given Y . X_m^n is a shorthand for $n - m + 1$ tuple $\{X_m, X_{m+1}, \dots, X_{n-1}, X_n\}$. X^n will also denote X_1^n . When $i \leq 0$, X^i denotes null string as it is also for X_j^i , when $i \geq j$. $X^{n \setminus i}$ denotes $\{X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\}$. The cardinality of an alphabet \mathcal{X} is denoted by $|\mathcal{X}|$. We impose the assumption of finiteness of cardinality on all alphabets, unless otherwise indicated.

We use the *Causal Conditioning* notation $(\cdot \parallel \cdot)$ as introduced by Kramer in [10] and [11] :

$$P(y^N \parallel x^N) \triangleq \prod_{i=1}^N P(y_i | x^i, y^{i-1}). \quad (4)$$

We also use the following notation as introduced in [9] :

$$P(y^N \parallel x^{N-1}) \triangleq \prod_{i=1}^N P(y_i | x^{i-1}, y^{i-1}). \quad (5)$$

Note that both *causal conditioning*, $P(y^N \parallel x^N)$ and $P(y^N \parallel x^{N-1})$ are *distributions* on Y^n for a fixed x^N , as they are non negative for all x^N, y^N and they sum to unity, i.e.,

$$\sum_{y^N} P(y^N \parallel x^N) = \sum_{y^N} P(y^N \parallel x^{N-1}) = 1. \quad (6)$$

The directed information $I(X^N \rightarrow Y^N)$, as defined by Massey in [20], is given by,

$$I(X^N \rightarrow Y^N) = \sum_{i=1}^N I(X^i; Y_i | Y^{i-1}) = \mathbb{E} \left[\log \frac{P(Y^N \parallel X^N)}{P(Y^N)} \right], \quad (7)$$

where \mathbb{E} stands for expectation. Naturally, the directed information conditioned on a random object S , $I(X^N \rightarrow Y^N | S)$, is defined as,

$$I(X^N \rightarrow Y^N | S) \triangleq \sum_{i=1}^N I(X^i; Y_i | Y^{i-1}, S). \quad (8)$$

We model discrete time channels with memory as Finite State Channels (FSCs) introduced by Gallager in his book [7], as an apt class of models for channels with memory, e.g. channels with ISI, etc. The channel input symbols take values in the finite alphabet \mathcal{X} and output denoted by Y takes values in finite alphabet \mathcal{Y} . The state takes values in a finite alphabet \mathcal{S} . The stationary channel is characterized by the conditional probability law $P(y_i, s_i | x_i, s_{i-1})$ satisfying,

$$P(y_i, s_i | x^i, s^{i-1}, y^{i-1}, m) = P(y_i, s_i | x_i, s_{i-1}), \quad (9)$$

and by the probability of the initial state $P(s_0)$. More precisely, without loss of generality, we can make the following assumption on our channel model,

$$P(y_i, s_i | x^i, s^{i-1}, y^{i-1}, a_e^i, a_d^i, m) = P(y_i, s_i | x_i, s_{i-1}), \quad (10)$$

where $a_{e,i} \in \mathcal{A}_e$ and $a_{d,i} \in \mathcal{A}_d$ are the encoder and decoder actions respectively as will be explained later. Messages $M \in \mathcal{M}$ are assumed to be independent of initial state, s_0 . The FSC is without intersymbol interference (ISI) if

$$P(s_i | s_{i-1}, x_i) = P(s_i | s_{i-1}), \quad (11)$$

i.e., the evolution of the channel states is independent of the channel input process. The basic framework in this paper is the setting depicted in Fig. 1. The communication system has the following building blocks :

- *Encoder Feedback Logic* : Generates encoder actions, $\{A_{e,i}\}_{i=1}^N$, using the function $f_{A_{e,i}} : \mathcal{M} \times \mathcal{Z}^{i-1} \rightarrow \mathcal{A}_e$ i.e., $A_{e,i} = f_{A_{e,i}}(M, Z^{i-1})$, where $Z_i \in \mathcal{Z}$ is the sampled feedback component.

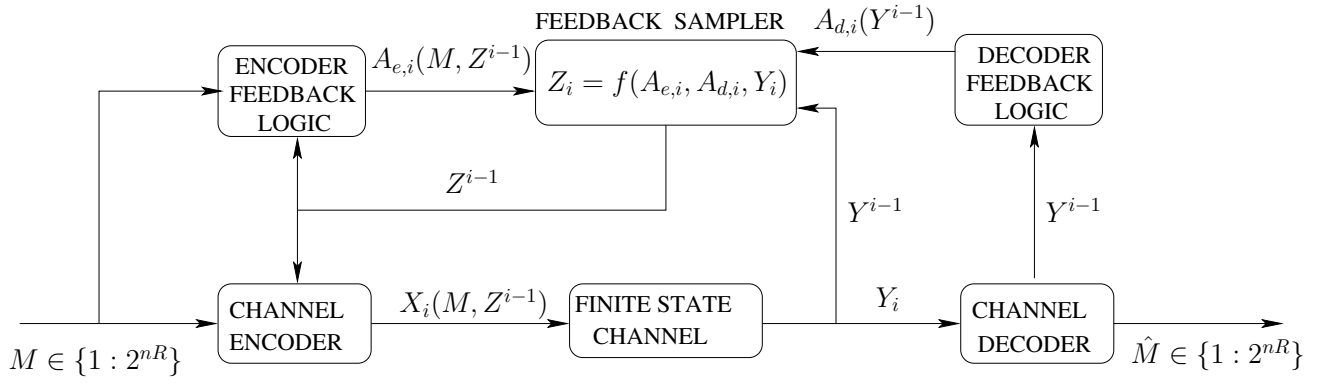


Fig. 1. Modeling **Feedback Sampling** for the acquisition of feedback in Finite State Channels (FSCs).

- *Decoder Feedback Logic* : Generates decoder actions, $\{A_{d,i}\}_{i=1}^N$, using the function $f_{A_{d,i}} : \mathcal{Y}^{i-1} \rightarrow \mathcal{A}_d$ i.e., $A_{d,i} = f_{A_{d,i}}(Y^{i-1})$, where $Y_i \in \mathcal{Y}$ is the channel output.
- *Feedback Sampler* : Generates *sampled* feedback, $Z_i = f(A_{e,i}, A_{d,i}, Y_i)$, where f is a deterministic function.
- *Channel Encoder* : Constructs channel input symbol, $X_i(M, Z^{i-1})$, using the encoding function, $f_{e,i} : \mathcal{M} \times \mathcal{Z}^{i-1} \rightarrow \mathcal{X}$.
- *Channel Decoder* : Generate the best estimate of the message given the channel output, $\hat{M}(Y^N)$, using the decoding function, $f_d : \mathcal{Y}^N \rightarrow \mathcal{M}$.

We are interested in characterizing the maximal rate of reliable communication under the average cost constraint,

$$\mathbb{E} [\Lambda(A_e^N, A_d^N)] = \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \Lambda(A_{e,i}, A_{d,i}) \right] \leq \Gamma, \quad (12)$$

where $\Lambda(\cdot, \cdot)$ is a given cost function satisfying $\max_{a_e \in \mathcal{A}_e, a_d \in \mathcal{A}_d} \Lambda(a_e, a_d) = \Lambda_{\max} < \infty$.

The joint probability distribution induced by a given scheme,

$$\begin{aligned} & P_{M, A_e^N, A_d^N, Z^N, X^N, S_0^N, Y^N, \hat{M}}(m, a_e^N, a_d^N, z^N, x^N, s_0^N, y^N, \hat{m}) \\ &= \frac{1}{|\mathcal{M}|} P_S(s_0) \prod_{i=1}^n \mathbf{1}_{\{a_{d,i} = f_{A_{d,i}}(y^{i-1})\}} \mathbf{1}_{\{a_{e,i} = f_{A_{e,i}}(m, z^{i-1})\}} \\ & \times \prod_{i=1}^n \mathbf{1}_{\{x_i = f_{e,i}(m, z^{i-1})\}} P(y_i, s_i | x_i, s_{i-1}) \mathbf{1}_{\{z_i = f(a_{e,i}, a_{d,i}, y_i)\}} \times \mathbf{1}_{\{\hat{m} = f_d(y^N)\}}. \end{aligned} \quad (13)$$

Definition 1: A rate R is said to be *achievable* if there exists a sequence of block codes $(N, \lceil 2^{NR} \rceil)$ satisfying (12) such that the maximal probability of error,

$$\max_{m \in \{1, \dots, \lceil 2^{NR} \rceil\}} \Pr(\hat{m} \neq m | \text{message } m \text{ was sent}),$$

vanishes as $N \rightarrow \infty$. The capacity of such a system is denoted by C which is the supremum of all achievable rates.

III. MAIN RESULTS

Let s_0 denote the initial state. We define $\underline{C}_N(\Gamma)$ and $\bar{C}_N(\Gamma)$ as,

$$\underline{C}_N(\Gamma) \triangleq \frac{1}{N} \max_{s_0} \min I(X^N \rightarrow Y^N | s_0) \quad (14)$$

$$\bar{C}_N(\Gamma) \triangleq \frac{1}{N} \max_{s_0} \max I(X^N \rightarrow Y^N | s_0). \quad (15)$$

Here **max** denotes maximization over the joint probability distribution,

$$P(s_0, x^N, a_e^N, a_d^N, y^N, z^N) = P(s_0) Q(x^N, a_e^N \| z^{N-1}) Q(a_d^N \| y^{N-1}) P(y^N \| x^N, s_0) \prod_{i=1}^N \mathbf{1}_{\{z_i = f(a_{e,i}, a_{d,i}, y_i)\}}, \quad (16)$$

such that $E[\Lambda(A_e^N, A_d^N)] \leq \Gamma$, where

$$Q(x^N, a_e^N \parallel z^{N-1}) = \prod_{i=1}^N Q(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, z^{i-1}) \quad (17)$$

$$Q(a_d^N \parallel y^{N-1}) = \prod_{i=1}^N Q(a_{d,i} | a_d^{i-1}, y^{i-1}) \quad (18)$$

$$\begin{aligned} I(X^N \rightarrow Y^N | s_0) &= \sum_{i=1}^N I(X^i; Y_i | Y^{i-1}, s_0) \\ &= E \left[\log \frac{P(Y^N \parallel X^N, s_0)}{P(Y^N | s_0)} \right] \end{aligned} \quad (19)$$

$$P(Y^N \parallel X^N, s_0) = \prod_{i=1}^N P(y_i | x^i, y^{i-1}, s_0). \quad (20)$$

As z^N is a deterministic function of (a_e^N, a_d^N, y^N) , from now on we will consider maximization over the joint probability distribution,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N, s_0), \quad (21)$$

where z_i will stand for $f(a_{e,i}, a_{d,i}, y_i)$ unless otherwise stated. Note that effectively maximization in definition of $\underline{C}_N(\Gamma)$ and $\overline{C}_N(\Gamma)$ is over $Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})$ as $P(s_0)$ is fixed and $P(y^N \parallel x^N, s_0)$ (and likewise $P(y^N \parallel x^N)$) is a characteristic of the channel given by (Lemma 6 of [9]),

$$P(y^N \parallel x^N, s_0) = \sum_{s_1^N} \prod_{i=1}^N P(y_i, s_i | x_i, s_{i-1}) \quad (22)$$

$$P(y^N \parallel x^N) = \sum_{s_0} P(s_0)P(y^N \parallel x^N, s_0) = \sum_{s_0^N} P(s_0) \left(\prod_{i=1}^N P(y_i, s_i | x_i, s_{i-1}) \right). \quad (23)$$

Our main results are as follows,

- **Achievable Rate** : For a communication abstraction as in Fig. 1, any rate R is achievable such that,

$$R < \lim_{N \rightarrow \infty} \underline{C}_N(\Gamma) = \sup_N \left[\underline{C}_N(\Gamma) - \frac{\log |S|}{N} \right]. \quad (24)$$

- **Converse** : Consider a coding scheme with rate R which achieves reliable communication over the FSC with feedback sampling as in Fig. 1. This implies the existence of $(N, \lceil 2^{NR} \rceil)$ codes such that the probability of error P_e^N goes to zero as $N \rightarrow \infty$. For such a scheme given $\epsilon > 0$, \exists block length N_0 such that for all block lengths $N > N_0$ we have

$$R \leq \overline{C}_N(\Gamma) + \epsilon. \quad (25)$$

- **Capacity** : In the following cases we characterize the capacity exactly,

1) For an FSC where the probability of the initial state is positive for all $s_0 \in \mathcal{S}$, the capacity is evaluated exactly,

$$C(\Gamma) = \lim_{N \rightarrow \infty} \underline{C}_N(\Gamma). \quad (26)$$

2) For stationary ‘*indecomposable*’ channels without ISI with feedback sampling as in Fig. 1, the capacity is,

$$C(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max I(X^N \rightarrow Y^N), \quad (27)$$

where **max** denotes maximization over the joint probability distribution,

$$P(x^N, a_e^N, a_d^N, y^N) = Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N), \quad (28)$$

such that $E[\Lambda(A_e^N, A_d^N)] \leq \Gamma$.

IV. ACHIEVABILITY

We begin this section by proving that the limit of the sequence $\underline{C}_N(\Gamma)$ exists. We then explain the encoding and decoding scheme followed by analysis of probability of error, showing that any rate R is achievable such that, $R < \underline{C}(\Gamma) = \lim_{N \rightarrow \infty} \underline{C}_N(\Gamma)$. Encoding uses random code-tree generation while decoding uses maximum likelihood decoding as in [7].

A. Existence of $\underline{C}(\Gamma)$

By the following theorem, we prove the existence of the limit of the sequence $\underline{C}_N(\Gamma)$.

Theorem 1: For a finite state channel with $|\mathcal{S}|$ states, $\lim_{N \rightarrow \infty} \underline{C}_N(\Gamma)$ exists and,

$$\lim_{N \rightarrow \infty} \underline{C}_N(\Gamma) = \sup_N \left[\underline{C}_N(\Gamma) - \frac{\log |\mathcal{S}|}{N} \right]. \quad (29)$$

Proof: Let $N = n + l$, $n, l \in \mathbf{Z}^+$. Note that from the Section IV-B we will show that we achieve \underline{C}_N by using random coding with distribution of form $Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})$ satisfying the cost constraints. Let us assume that $\underline{C}_n(\Gamma)$ and $\underline{C}_l(\Gamma)$ are achieved by $Q(x^n, a_e^n \| z^{n-1})Q(a_d^n \| y^{n-1})$ and $Q(x^l, a_e^l \| z^{l-1})Q(a_d^l \| y^{l-1})$ respectively. Consider

$$Q(x^N, a_e^N \| z^{N-1}) = Q(x^n, a_e^n \| z^{n-1})Q(x^l, a_e^l \| z^{l-1}) \quad (30)$$

$$Q(a_d^N \| y^{N-1}) = Q(a_d^n \| y^{n-1})Q(a_d^l \| y^{l-1}). \quad (31)$$

Therefore

$$\mathbb{E} [\Lambda(A_e^N, A_d^N)] = \frac{n}{N} \mathbb{E} [\Lambda(A_e^n, A_d^n)] + \frac{l}{N} \mathbb{E} [\Lambda(A_e^{n+l}, A_d^{n+l})] \quad (32)$$

$$\leq \frac{n\Gamma + l\Gamma}{N} = \Gamma. \quad (33)$$

Hence $Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})$ (which is a distribution) satisfies the cost requirements, but it may not be capacity achieving for blocklength N so,

$$N\underline{C}_N(\Gamma) \geq \min_{s_0} I(X^N \rightarrow Y^N | s_0). \quad (34)$$

We now follow the steps as in Proof of Theorem 8 in [9] to arrive at

$$N \left[\underline{C}_N(\Gamma) - \frac{\log |\mathcal{S}|}{N} \right] \geq n \left[\underline{C}_n(\Gamma) - \frac{\log |\mathcal{S}|}{n} \right] + l \left[\underline{C}_l(\Gamma) - \frac{\log |\mathcal{S}|}{l} \right]. \quad (35)$$

Hence the sequence, $\underline{C}_n(\Gamma)$ is super additive for all $n \in \mathbf{Z}^+$. The theorem is finally proved using the convergence of super additive sequences, as is done in Theorem 4.6.1 [7]. \blacksquare

B. Encoding Scheme

Encoding is based on generating separate code trees which is described below. These are then revealed to the encoder and the decoder.

- *Encoder Code-Tree* : 2^{NR} **code-trees** are generated as follows, the i^{th} encoder action and channel input symbol is generated using a probability mass function which depends on previous encoder action and channel input symbols and on the past sampled feedback sequence, i.e. $Q(x^i, a_e^i | x^{i-1}, a_e^{i-1}, z^{i-1})$.
- *Decoder Action Code-Tree* : We generate a single code tree at random, where the vertex represents decoder action symbol, $a_{d,i}$ generated with distribution $Q(a_{d,i} | a_d^{i-1}, y^{i-1})$. Thus the present decoder action depend on the past actions as well as the past channel output.

Note that $\{Q(x^i, a_e^i | x^{i-1}, a_e^{i-1}, z^{i-1})\}_{i=1}^N$ and $\{Q(a_{d,i} | a_d^{i-1}, y^{i-1})\}_{i=1}^N$ correspond to the joint distribution on $(X^N, A_e^N, A_d^N, S^N, Y^N)$ such that constraint $\mathbb{E} [\Lambda(A_e^N, A_d^N)] \leq \Gamma$ is satisfied.

Fig. 2 illustrates the *Encoder Code-Tree* for a specific example. The setting in the right in the figure is the illustration of the setting of *to feed or not to feed back*, when the output alphabet is binary and,

$$Z_i = f(A_{e,i}, A_{d,i}, Y_i) = * \text{ if } A_{e,i} \neq A_{d,i}. \quad (36)$$

$$Z_i = f(A_{e,i}, A_{d,i}, Y_i) = Y_i \text{ if } A_{e,i} = A_{d,i}, \quad (37)$$

where $*$ stands for erasure or no feedback. Knowing past channel outputs, decoder uses *Decoder Action Code-Tree* to figure out the decoder action symbol. Using the decoder action symbol $a_{d,i}$, along with encoder actions, $a_{e,i}$ and channel output y_i , feedback sampler produces sampled feedback as $z_i = f(a_{e,i}, a_{d,i}, y_i)$. In this way, given a message m , and the complete sampled feedback sequence z^{N-1} thus obtained, there is a particular (x^N, a_e^N) which can be found from the collection of *encoder code trees*. The encoder thus sends the corresponding x^N through the channel. *Note* that our coding scheme is similar in spirit to the code tree generation scheme as in [9]. However, here we generate both the cost constrained encoder actions and channel input symbols in one tree while decoder actions are generated in another tree.

By the above code tree generation, we have in our achievability scheme,

$$P(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, a_d^{i-1}, y^{i-1}, s_0^i) = P(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, a_d^{i-1}, y^{i-1}, z^{i-1}, s_0^i) \quad (38)$$

$$= Q(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, z^{i-1}), \quad (39)$$

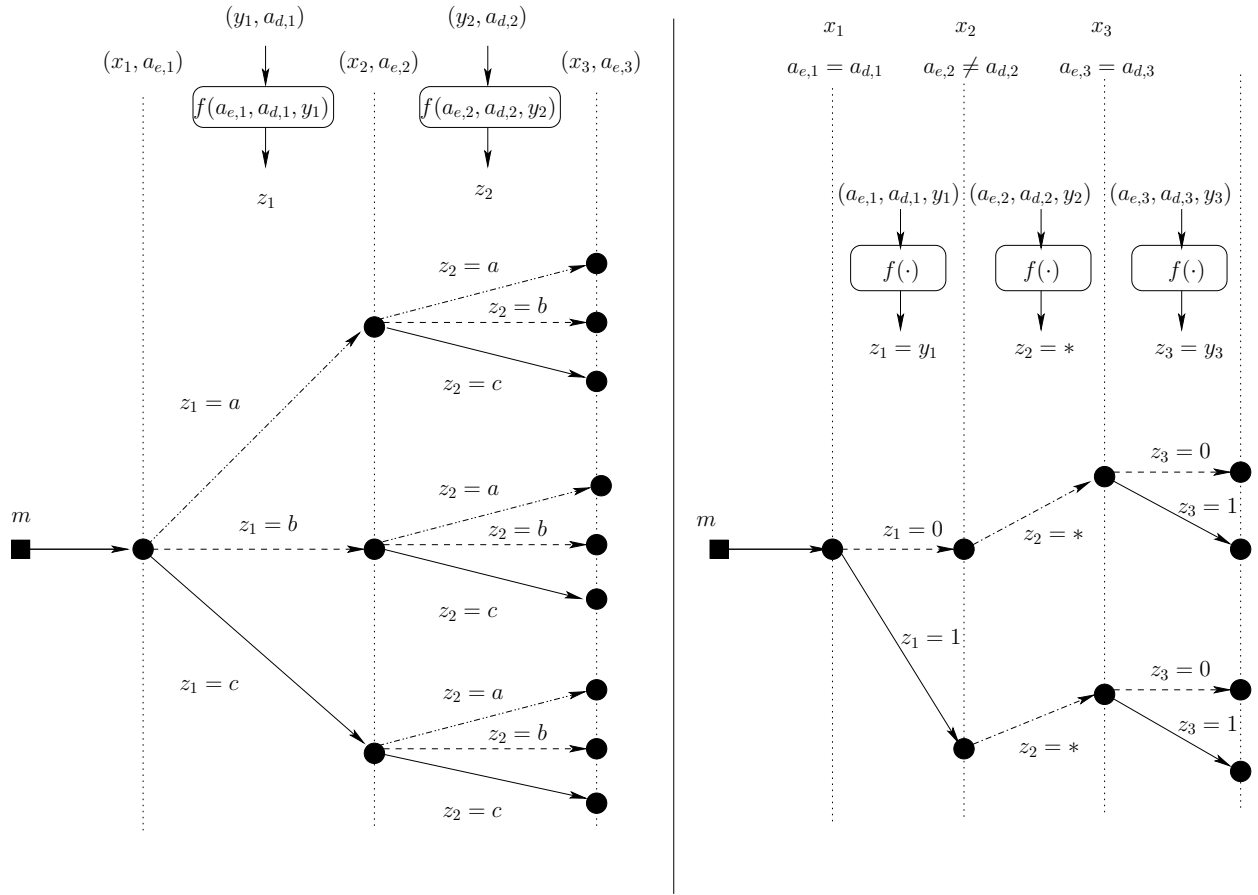


Fig. 2. This figure illustrates *Encoder Code-Trees* in our coding scheme. The left hand side figure depicts a general setting where $\mathcal{Z} = \{a, b, c\}$, and $z_i = f(a_{e,i}, a_{d,i}, y_i)$. The tree is shown for $N = 3$. The right hand side shows a specific example where $a_{d,i} = 0 \forall i$ and output is binary. Actions of encoder, $a_{e,i} \in \{0, 1\}$ and $z_i = f(a_{e,i}, a_{d,i}, y_i) = y_i$ if $a_{e,i} = a_{d,i}$ or $a_{e,i} = 0$, else it is erasure($*$). Hence some portion of the tree collapses as by knowing $a_{e,i}$ we know the possible values of z_i , for e.g. $a_{e,i} = 1$ implies $z_i = *$ and $a_{e,i} = 0$ implies, $z_i = 0$ or 1 .

where first equality follows from the fact, $z_i = f(a_{e,i}, a_{d,i}, y_i)$, while the second equality is due to our coding scheme where the i^{th} input and encoder action symbol only depend on past input symbols, actions and sampled feedback. Similarly since i^{th} decoder action only depends on past decoder actions and channel output, we have,

$$P(a_{d,i} | a_d^{i-1}, y^{i-1}, x^i, a_e^i, s_0^i) = Q(a_{d,i} | a_d^{i-1}, y^{i-1}). \quad (40)$$

Lemma 1: The joint probability distribution on $(s_0, x^N, a_e^N, a_d^N, y^N)$, by the achievability scheme described above is,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N, s_0). \quad (41)$$

Proof: Using Property 1 in Appendix A, we have,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0)Q(x^N, a_e^N, a_d^N \parallel y^{N-1}, s_0)P(y^N \parallel x^N, a_e^N, a_d^N, s_0). \quad (42)$$

From definition of causal conditioning and using Eq. (39) and (40) we have,

$$Q(x^N, a_e^N, a_d^N \parallel y^{N-1}, s_0) = Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1}). \quad (43)$$

Now again using Eq. (39) and (40) and the channel model assumption in Eq. (10) consider,

$$P(s_0^N, x^N, a_e^N, a_d^N, y^N) = P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1}) \prod_{i=1}^N P(y_i, s_i | x_i, s_{i-1}), \quad (44)$$

Summing over, s_1^N and using the characterization of $P(y^N \parallel x^N, s_0)$ in Section III as,

$$P(y^N \parallel x^N, s_0) = \sum_{s_1^N} \prod_{i=1}^N P(y_i, s_i | x_i, s_{i-1}), \quad (45)$$

we obtain,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = \sum_{s_1^N} P(s_0^N, x^N, a_e^N, a_d^N, y^N) \quad (46)$$

$$= P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N, s_0). \quad (47)$$

■

Corollary 1: From the steps in previous lemma it immediately implies,

$$P(y^N \parallel x^N, a_e^N, a_d^N, s_0) = P(y^N \parallel x^N, s_0). \quad (48)$$

Note that likewise it can be also shown as in Eq. (48) that,

$$P(y^N \parallel x^N, a_e^N, a_d^N) = P(y^N \parallel x^N), \quad (49)$$

which we will use in next section on decoding.

C. Decoding

The decoder performs ML decoding, i.e. it chooses the message m for which $P(y^N|m)$ is maximized.

$$P(y^N|m) = \prod_{i=1}^N P(y_i|y^{i-1}, m) \quad (50)$$

$$\stackrel{(a)}{=} \prod_{i=1}^N P(y_i|y^{i-1}, a_d^i(y^{i-1}), m, x^i(m, z^{i-1}), a_e^i(m, z^{i-1})) \quad (51)$$

$$\stackrel{(b)}{=} \prod_{i=1}^N P(y_i|y^{i-1}, a_d^i(y^{i-1}), x^i(m, z^{i-1}), a_e^i(m, z^{i-1})) \quad (52)$$

$$= P(y^N \parallel x^N, a_e^N, a_d^N) \quad (53)$$

$$\stackrel{(c)}{=} P(y^N \parallel x^N), \quad (54)$$

where (a) follows from the fact that knowing m and y^{i-1} , we know (x^i, a_e^i, a_d^i) . This can be iteratively shown. Given m we know $(x_1(m), a_1(m))$. We also know $a_{d,1}$. Given $y_1, z_1 = f(a_{e,1}, a_{d,1}, y_1)$. Hence now we know, $(x_2(m, z_1), a_{e,2}(m, z_1), a_{d,2}(y_1))$. Iteratively we can conclude that for a given message m and true feedback sequence, y^{i-1} , we can construct (x^i, a_e^i, a_d^i) knowing the codebooks. (b) follows from the assumption on channel model in Eq. (10) and (c) follows from Eq. (49). Hence ML decoding to construct message estimate, \hat{m} can also be done by maximizing causal conditioning, i.e.,

$$\hat{m} = \underset{m}{\operatorname{argmax}} P(y^N|m) = \underset{x^N}{\operatorname{argmax}} P(y^N \parallel x^N). \quad (55)$$

D. Calculation of Probability of Error

We will see in this section that most of the proofs are similar to that in [9] with $Q(x^N, a_e^N \parallel z^{N-1})$ being replaced with $Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})$. This is justifiable from our coding scheme that uses a distribution which is causal conditioning, $Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})$ and the optimal decoding which is finding

$$\underset{m}{\operatorname{argmax}} P(y^N|m) = \underset{x^N}{\operatorname{argmax}} P(y^N \parallel x^N). \quad (56)$$

Also from Lemma 1 we have,

$$P(x^N, a_e^N, a_d^N, y^N) = \sum_{s_0} P(s_0, x^N, a_e^N, a_d^N, y^N) \quad (57)$$

$$= Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1}) \sum_{s_0} P(s_0)P(y^N \parallel x^N, s_0) \quad (58)$$

$$= Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N), \quad (59)$$

where last equality follows from the characterization of $P(y^N \parallel x^N)$ in Section III. Due to factorization in Eq. (59) similar to the one in [9] as

$$P(x^N, y^N) = Q(x^N \parallel z^{N-1})P(y^N \parallel x^N), \quad (60)$$

we have parallelism in the proofs.

Note that from now on we will not state the condition of cost constraints, i.e., $E[\Lambda(A_e^N, A_d^N)] \leq \Gamma$ explicitly in maximizing distribution. The distribution $Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})$ will be assumed to be the one satisfying cost constraints. Let $P_{e,m}$ denote the probability of error of ML decoding when message m was sent. Given message m , Y_m^c denotes the set of outputs that cause error in decoding m , i.e.,

$$P_{e,m} = \sum_{y^N \in Y_m^c} P(y^N | m). \quad (61)$$

Theorem 2: Let M denote the total number of messages used in transmission and $E(P_{e,m})$ denote the average probability of error over these ensemble of codes. Then for any ρ , $0 < \rho \leq 1$,

$$E(P_{e,m}) \leq (M-1)^\rho \sum_{y^N} \left[\sum_{x^N, a_e^N, a_d^N} Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})P(y^N \| x^N)^{\frac{1}{1+\rho}} \right]^{1+\rho}. \quad (62)$$

Proof: Refer to Appendix B. ■

Let $P_{e,m}(s_0)$ denote the probability of error given the initial state of FSC was s_0 and the message m was sent.

Theorem 3: Consider FSC with feedback sampling (Fig. 1) having $|\mathcal{S}|$ states. For any positive integer N and any positive rate R , $\exists (N, M)$ code for which for all messages $m \in \{1, \dots, \lceil 2^{NR} \rceil\}$, all initial states s_0 and all ρ , $0 < \rho \leq 1$,

$$P_{e,m}(s_0) \leq 4 |\mathcal{S}| 2^{-N[-\rho R + F_N(\rho)]}, \quad (63)$$

where

$$F_N(\rho) = -\frac{\rho \log |\mathcal{S}|}{N} + \max_{Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})} \left[\min_{s_0} E_{o,N}(\rho, Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1}), s_0) \right], \quad (64)$$

and

$$\begin{aligned} & E_{o,N}(\rho, Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1}), s_0) \\ &= -\frac{1}{N} \log \sum_{y^N} \left[\sum_{x^N, a_e^N, a_d^N} Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})P(y^N \| x^N, s_0)^{\frac{1}{1+\rho}} \right]^{1+\rho}. \end{aligned} \quad (65)$$

Proof: Proof is following the steps in proof of Theorem 10 in [9] once we have obtained the bound on $E(P_{e,m})$ [Eq. (62)] in Theorem 2. ■

Theorem 4: $E_{o,N}(\rho, Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1}), s_0)$ has the following properties,

$$E_{o,N}(\rho, Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1}), s_0) \geq 0 \quad (66)$$

$$\frac{1}{N} I(X^N \rightarrow Y^N | s_0) \geq \frac{\partial E_{o,N}(\rho, Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1}), s_0)}{\partial \rho} > 0 \quad (67)$$

$$\frac{\partial^2 E_{o,N}(\rho, Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1}), s_0)}{\partial \rho^2} > 0, \quad (68)$$

where equality in Eq. (66) holds when $\rho = 0$, and equality holds on the left side of Eq. (67) when $\rho = 0$.

Proof: Omitted as it is similar to proof of Theorem 11 in [9] with $Q(x^N \| z^{N-1})$ replaced by $Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})$. ■

Lemma 2: We have the following results for the convergence of $F_N(\rho)$,

$$\lim_{N \rightarrow \infty} F_N(\rho) = F_\infty(\rho) = \sup_N F_N(\rho), \quad (69)$$

for $0 \leq \rho \leq 1$. The convergence of $F_N(\rho)$ is uniform in ρ and $F_\infty(\rho)$ is uniformly continuous for $\rho \in [0, 1]$.

Proof: Omitted. Proof similar to Lemma 13 in [9]. ■

Theorem 5: For any FSC with feedback logic let,

$$E_r(R) = \max_{0 \leq \rho \leq 1} [F_\infty(\rho) - \rho R]. \quad (70)$$

Then for any $\epsilon > 0$, $\exists N(\epsilon)$ such that for $N \geq N(\epsilon)$, \exists an (N, M) code such that for all m , $1 \leq m \leq M = \lceil 2^{NR} \rceil$, and all initial states,

$$P_{e,m}(s_0) \leq 2^{-N[E_r(R) - \epsilon]}. \quad (71)$$

Proof: Proof is similar to Theorem 14 in [9] using above Theorems 1, 2, 3, 4 and Lemma 2 to conclude that for every s_0 , there exists a ρ^* such that $F_\infty(\rho^*) - \rho^* R > 0$, for all $R < \underline{C}(\Gamma)$. ■

V. CONVERSE

In this section, we will first prove some converse results. Later in this section, we will show that for FSCs where probability of initial state is positive for all $s_0 \in \mathcal{S}$, the achievable rate and the upper bound coincide and hence the capacity is given by $\underline{C}(\Gamma)$.

Theorem 6: Consider a coding scheme with rate R which achieves reliable communication over the FSC with feedback sampling as in Fig. 1 meeting the average cost constraints, Eq. (12). For such a scheme given any $\epsilon_N > 0$, \exists block length N_0 such that for all block lengths $N > N_0$ we have

$$R \leq \bar{C}_N(\Gamma) + \epsilon_N. \quad (72)$$

Proof: Let a message m is chosen uniformly with probability 2^{-NR} .

$$NR = H(M) \quad (73)$$

$$= I(M; Y^N) + H(M|Y^N) \quad (74)$$

$$\stackrel{(a)}{\leq} I(M; Y^N) + 1 + P_e^{(N)}NR \quad (75)$$

$$\stackrel{(b)}{=} \sum_{i=1}^N H(Y_i|Y^{i-1}) - H(Y_i|Y^{i-1}, X^i, A_e^i, A_d^i, M) + 1 + P_e^{(N)}NR \quad (76)$$

$$\stackrel{(c)}{=} \sum_{i=1}^N H(Y_i|Y^{i-1}) - H(Y_i|Y^{i-1}, X^i, A_e^i, A_d^i) + 1 + P_e^{(N)}NR \quad (77)$$

$$\stackrel{(d)}{=} \sum_{i=1}^N H(Y_i|Y^{i-1}) - H(Y_i|Y^{i-1}, X^i) + 1 + P_e^{(N)}NR \quad (78)$$

$$= \sum_{i=1}^N I(X^i; Y_i|Y^{i-1}) + 1 + P_e^{(N)}NR \quad (79)$$

$$= I(X^N \rightarrow Y^N) + 1 + P_e^{(N)}NR \quad (80)$$

$$\stackrel{(e)}{\leq} I(X^N \rightarrow Y^N|S_0) + \log |\mathcal{S}| + 1 + P_e^{(N)}NR \quad (81)$$

$$\leq \max_{s_0} I(X^N \rightarrow Y^N|s_0) + \log |\mathcal{S}| + 1 + P_e^{(N)}NR \quad (82)$$

$$\stackrel{(f)}{\leq} \max_{s_0} \max I(X^N \rightarrow Y^N|s_0) + \log |\mathcal{S}| + 1 + P_e^{(N)}NR \quad (83)$$

$$= N\bar{C}_N(\Gamma) + \log |\mathcal{S}| + 1 + P_e^{(N)}NR, \quad (84)$$

where

- (a) follows from Fano's inequality.
- (b) follow from similar arguments as in IV-C.
- (c) follows from assumption of channel model as in Eq. (10).
- (d) follows from proof of MC1 in Appendix C.
- (e) follows from Property 3 of Appendix A.
- (f) has its first maximization over the joint probability distribution

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0)Q(x^N, a_e^N \| z^{N-1})Q(a_d^N \| y^{N-1})P(y^N \| x^N, s_0), \quad (85)$$

which satisfy the expected cost constraints, $E[\Lambda(A_e^N, A_d^N)] \leq \Gamma$ and Eq. (85) follows from Lemma 5 in Appendix C.

Hence we have for sufficiently large N for any given $\epsilon_N > 0$,

$$R \leq \bar{C}_N(\Gamma) + \epsilon_N. \quad (86)$$

Note that unlike in [9], limit of $\bar{C}_N(\Gamma)$ may not exist because sub-additivity (like the one in Theorem 16 in [9]) breaks due to the presence of cost constraints. Hence for a general FSC, we have the above converse result for a give blocklength N and probability of error, ϵ_N . However if for the FSC, the probability of initial state is positive for all states, then we have the exact capacity as shown by the following theorem.

Theorem 7: Consider an FSC with feedback logic where all the initial states $\in \mathcal{S}$ have positive probability. The capacity is $\underline{C}(\Gamma)$. ■

Proof: The proof is similar to Theorem 17 in [9] with change in equalities in (c), (d) and (e) below. Let $P_e^N(s_0)$ denote the probability of error when the initial state is s_0 . Since every initial state $s_0 \in \mathcal{S}$ can occur with non zero probability, this implies that there exists a sequence of block codes $(N, \lfloor 2^{NR} \rfloor)$ with $P_e^N(s_0) \rightarrow 0, \forall s_0 \in \mathcal{S}$. Hence we have,

$$NR = H(M) \tag{87}$$

$$\stackrel{(a)}{=} H(M|s_0) \tag{88}$$

$$= I(M; Y^N|s_0) + H(M|Y^N, s_0) \tag{89}$$

$$\stackrel{(b)}{\leq} I(M; Y^N|s_0) + 1 + P_e^{(N)}(s_0)NR \tag{90}$$

$$\stackrel{(c)}{=} \sum_{i=1}^N H(Y_i|Y^{i-1}, s_0) - H(Y_i|Y^{i-1}, X^i, A_e^i, A_d^i, M, s_0) + 1 + P_e^{(N)}(s_0)NR \tag{91}$$

$$\stackrel{(d)}{=} \sum_{i=1}^N H(Y_i|Y^{i-1}, s_0) - H(Y_i|Y^{i-1}, X^i, A_e^i, A_d^i, s_0) + 1 + P_e^{(N)}(s_0)NR \tag{92}$$

$$\stackrel{(e)}{=} \sum_{i=1}^N H(Y_i|Y^{i-1}, s_0) - H(Y_i|Y^{i-1}, X^i, s_0) + 1 + P_e^{(N)}(s_0)NR \tag{93}$$

$$= \sum_{i=1}^N I(X^i; Y_i|Y^{i-1}, s_0) + 1 + P_e^{(N)}(s_0)NR \tag{94}$$

$$= I(X^N \rightarrow Y^N|s_0) + 1 + P_e^{(N)}NR \tag{95}$$

$$\stackrel{(f)}{\leq} \min_{s_0} \left[I(X^N \rightarrow Y^N|s_0) + 1 + P_e^{(N)}(s_0)NR \right], \tag{96}$$

where

- (a) follows from the fact that message M is independent of initial state s_0 .
- (b) follows from Fano's inequality.
- (c) follow from similar arguments as in IV-C.
- (d) follows from the assumption of channel model as in Eq. (10).
- (e) follows from proof of MC1 in Appendix C.
- (f) follows from the fact that Eq. (95) is true for all $s_0 \in \mathcal{S}$.

Hence since we have $P_e^N(s_0) \rightarrow 0, \forall s_0 \in \mathcal{S}$, we have,

$$R \leq \lim_{N \rightarrow \infty} \frac{1}{N} \max_{s_0} \min I(X^N \rightarrow Y^N), \tag{97}$$

where due to Lemma 5 in Appendix C the maximization is over the joint probability distribution,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N, s_0), \tag{98}$$

which satisfy the expected cost constraints, $E[\Lambda(A_e^N, A_d^N)] \leq \Gamma$. This implies from the achievability result of Section IV that capacity is,

$$C(\Gamma) = \underline{C}(\Gamma). \tag{99}$$

■

VI. CAPACITY FOR STATIONARY INDECOMPOSABLE FSC WITHOUT ISI

We assume now that state transition is a separate markov chain and does not depend on input, i.e., $P(y_i, s_i|s_{i-1}, x_i) = P(s_i|s_{i-1})P(y_i|s_i, s_{i-1}, x_i)$. Such a channel is said to have no ISI. We further assume this channel is *indecomposable* as the definition given below,

Definition 2: An FSC without ISI is said to be indecomposable if, for every $\epsilon > 0, \exists N_0$ such that $\forall N > N_0$

$$|P(s_N|s_0) - P(s_N|s'_0)| \leq \epsilon \forall s_N, s_0, s'_0. \tag{100}$$

A necessary and sufficient condition for a no ISI, FSC to be indecomposable [c.f. Theorem 4.6.3, [7]] is that there exists a choice for the n^{th} state, say s_n , such that,

$$q(s_n|s_0) > 0, \forall s_0 \in \mathcal{S}. \tag{101}$$

Furthermore, if the channel is indecomposable, n above can always be taken less than $2^{|\mathcal{S}|^2}$. This condition [Theorem 6.3.2, [7]] also implies existence of a unique steady-state stationary distribution $\pi(s)$, i.e.,

$$\lim_{N \rightarrow \infty} P(S_N = s | s_0) = \pi(s). \quad (102)$$

The channel is stationary if $P(s_0) = \pi(s_0)$.

Theorem 8: For a stationary and indecomposable FSC without ISI and with communication abstraction as in Fig. 1, the capacity of the channel is given by,

$$C(\Gamma) = \lim_{N \rightarrow \infty} C^N(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max I(X^N \rightarrow Y^N), \quad (103)$$

where **max** denotes maximization over $Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})$ such that $E[\Lambda(A_e^N, A_d^N)] \leq \Gamma$.

Proof: The proof is similar to proof of Theorem 18 in [9] with $Q(x^N \parallel z^{N-1})$ replaced by $Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})$. ■

VII. CAUSAL ACTION ENCODING AT DECODER

In this section we generalize the framework in Fig. 1, where now decoder actions also depend on the current channel output, i.e., $A_{d,i} = f_{A_{d,i}}(Y^i)$. The setting is depicted in Fig. 3. Note that the capacity in this generalized setting can be strictly better than that in Fig. 1. To get an intuition for it, one can consider a markovian channel, i.e., an FSC for which,

$$P(Y_i, S_i | X_i, S_{i-1}) = P(Y_i | X_i, S_{i-1})P(S_i | S_{i-1}). \quad (104)$$

The decoder knows the states along with the output on the fly, feeds back the effective output, $Y_{FB,i} = (Y_i, S_i)$ to the feedback sampler and the feedback sampling function is specialized to $f(A_{e,i}, A_{d,i}, Y_{FB,i}) = A_{d,i}$, $A_{d,i} = f_{A_{d,i}}(Y^i, S^i)$. Further $|\mathcal{A}| = |\mathcal{S}|$ and there are no cost constraints. We will see later in Section X that this is the setting of coding on the backward link in FSCs with no constraints on active feedback symbols. As will be shown in Section X that the capacity of this system is the same as that when encoder and decoder both have state information and it is achieved by setting $A_{d,i} = S_i$. Here we are able to do better because $X_i(M, A^{i-1})$ can be generated using S_{i-1} on which the channel output depends ($P(Y_i | X_i, S_{i-1})$). Thus, it is easy to see that under such a framework for the setting in Fig. 1, i.e., when $A_{d,i} = f_{A_{d,i}}(Y^{i-1}, S^{i-1})$, capacity can be comparatively strictly less, as channel input can at most depend on state upto S^{i-2} and has no information about the state S_{i-1} which determines the channel output.

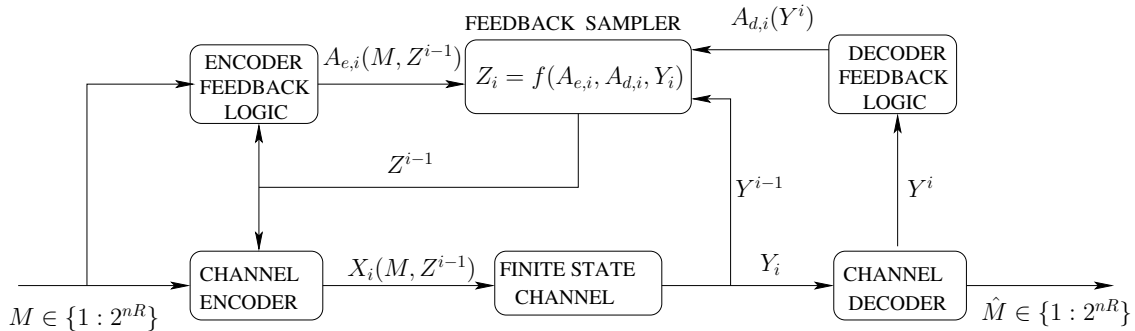


Fig. 3. Modeling **Feedback Sampling** for the acquisition of feedback in Finite State Channels (FSCs) when decoder can use the **current channel output** also to generate actions.

Theorem 9: Consider the system in Fig. 3. We have the following results paralleling those in Section III (for the setting of Fig. 1).

Let s_0 denotes the initial state. We define $\underline{C}_{N,causal}(\Gamma)$ and $\overline{C}_{N,causal}(\Gamma)$ as (where causal indicates that decoder actions can also depend on current channel output),

$$\underline{C}_{N,causal}(\Gamma) \triangleq \frac{1}{N} \max \min_{s_0} I(X^N \rightarrow Y^N | s_0) \quad (105)$$

$$\overline{C}_{N,causal}(\Gamma) \triangleq \frac{1}{N} \max \max_{s_0} I(X^N \rightarrow Y^N | s_0). \quad (106)$$

Here **max** denotes maximization over the joint probability distribution,

$$P(s_0, x^N, a_e^N, \phi_d^N, y^N, z^N) = P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(\phi_d^N \parallel y^{N-1})P(y^N \parallel x^N, s_0) \prod_{i=1}^N \mathbf{1}_{\{z_i = f(a_{e,i}, \phi_{d,i} | y_i, y_i)\}}, \quad (107)$$

such that $\mathbb{E}[\Lambda(A_e^N, \Phi_d^N |_{Y^N})] \leq \Gamma$. Here $\phi_d^N, \phi_d^N |_{y^N}$ are particular realizations of random variables $\Phi_d^N, \Phi_d^N |_{Y^N}$ and

$$\phi_{d,i} |_{y_i} = f_{A_{d,i}, y_i}(y^{i-1}) \in \mathcal{A}_d \quad (108)$$

$$\phi_{d,i} = \{f_{A_{d,i}, y}(y^{i-1}), y \in \mathcal{Y}\} \in \mathcal{A}_d^{|\mathcal{Y}|} \quad (109)$$

$$\phi_d^N |_{y^N} = \{\phi_{d,i} |_{y_i}\}_{i=1}^N. \quad (110)$$

With slight abuse of notation $\phi_{d,i} |_{y}$ for each $y \in \mathcal{Y}$ denotes a function from \mathcal{Y}^{i-1} to \mathcal{A}_d and $\phi_{d,i}$ can be treated as a vector of functions $\{\phi_{d,i} |_{y}\}_{y \in \mathcal{Y}}$. Note that $A_{d,i} = \phi_{d,i} |_{y_i}$ and hence $\{\phi_d^N |_{y^N}\}$ denotes the decoder action sequence.

1) **Achievable Rate** : For a communication abstraction as in Fig. 3, any rate R is achievable such that,

$$R < \lim_{N \rightarrow \infty} \underline{C}_{N,causal}(\Gamma) = \sup_N \left[\underline{C}_{N,causal}(\Gamma) - \frac{\log |S|}{N} \right]. \quad (111)$$

2) **Converse** : Consider a coding scheme with rate R which achieves reliable communication over the FSC with feedback sampling as in Fig. 3. This implies the existence of $(N, \lceil 2^{NR} \rceil)$ codes such that the probability of error P_e^N goes to zero as $N \rightarrow \infty$. For such a scheme given $\epsilon > 0$, \exists block length N_0 such that for all block lengths $N > N_0$ we have

$$R \leq \overline{C}_{N,causal}(\Gamma) + \epsilon. \quad (112)$$

3) **Capacity** : In the following cases we characterize the capacity exactly,

a) For an FSC where the probability of the initial state is positive for all $s_0 \in \mathcal{S}$, the capacity is evaluated exactly,

$$C_{causal}(\Gamma) = \lim_{N \rightarrow \infty} \underline{C}_{N,causal}(\Gamma). \quad (113)$$

b) For stationary ‘indecomposable’ channels without ISI with feedback sampling as in Fig. 3, the capacity is,

$$C_{causal}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max I(X^N \rightarrow Y^N), \quad (114)$$

where maximization is over the joint probability distribution,

$$P(x^N, a_e^N, \phi_d^N, y^N, z^N) = P(s_0)Q(x^N, a_e^N \parallel z^{N-1})Q(\phi_d^N \parallel y^{N-1})P(y^N \parallel x^N) \prod_{i=1}^N \mathbf{1}_{\{z_i = f(a_{e,i}, \phi_{d,i} |_{y_i}, y_i)\}},$$

such that $\mathbb{E}[\Lambda(A_e^N, \Phi_d^N |_{Y^N})] \leq \Gamma$

Proof: The proof is straightforward as it uses the similar results as stated in Section III for the framework in Fig. 1 where decoder actions do not depend on current channel output. The argument is as follows. Notice that the setting in Fig. 3 where decoder takes actions $A_{d,i}(Y^i)$ and the sampling function is $Z_i = f(A_{e,i}, A_{d,i}, Y_i)$ is equivalent to the setting in Fig. 1 where decoder takes actions $\tilde{A}_{d,i}(Y^{i-1}) \in \tilde{\mathcal{A}}_d = \mathcal{A}_d^{|\mathcal{Y}|}$, or $\tilde{A}_{d,i} = \Phi_{d,i}$ as defined in the Theorem above and feedback sampling function is,

$$Z_i = g(A_{e,i}, \tilde{A}_{d,i}, Y_i) = f(A_{e,i}, \Phi_{d,i} |_{Y_i}, Y_i). \quad (115)$$

More precisely operationally, the generalized setting when decoder takes action which also depend on current output is equivalent when decoder takes an action vector, depending only on past channel output, for each of the possible Y 's, and the feedback sampling function uses the current output to extract out the corresponding action from this action vector to generate sampled feedback. Hence all the above results are derived from those in Section III by the transformation, $A_d \rightarrow \Phi_d, f(\cdot) \rightarrow g(\cdot)$. The cost constraints hence are equivalent to $\mathbb{E}[\Lambda(A_e^N, \Phi_d^N |_{Y^N})] \leq \Gamma$. ■

Note 1: Note that we started by solving a seemingly more restrictive case, i.e., the setting in Fig. 1 where decoder actions depend on the channel output strictly causally. In this section, we applied our results for the setting of Fig. 1 to characterize fundamental limits for the setting in Fig. 3, where decoder actions can depend also on the current channel output, by showing that the latter setting can be embedded in the former via an appropriate extension of the decoder action alphabet. Thus, the setting of Fig. 1 is in fact *more* general than that of Fig. 3. Interestingly, in the other direction, it does not appear that the results for the seemingly more restrictive setting of Fig. 1 can be deduced from those for the setting of Fig. 3.

VIII. SPECIAL CASES

A. Feedback Logic At Encoder

The basic framework in this subsection is the setting in Fig. 4. Here only the encoder takes actions to govern feedback sampling.

Theorem 10: For no ISI, stationary and indecomposable FSC with encoder *Feedback Logic* as in Fig. 4, the capacity is given by,

$$C_{enc}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{Q(x^N, a^N \parallel z^{N-1}), \mathbb{E}[\Lambda(A^N)] \leq \Gamma} I(X^N \rightarrow Y^N). \quad (116)$$

Proof: Specialize Theorem 8 as $A_d = \phi$, $A_e = A$ and $\Lambda(A_e, A_d) = \Lambda(A_e) = \Lambda(A)$. ■

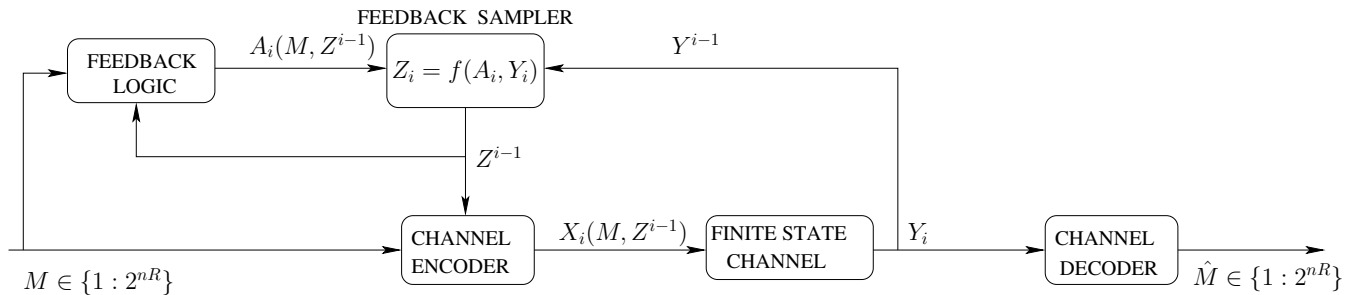


Fig. 4. Modeling **Feedback Sampling** for the acquisition of feedback in Finite State Channels (FSCs) with only *Encoder Feedback Logic*.

B. Feedback Logic At Decoder

In the previous sub-section, we characterized the fundamental limit for the communication system as depicted in Fig. 4 where encoder took actions to govern acquisition of feedback from decoder. However in some practical systems, it is the receiver (or decoder) which estimates the channel state perfectly and then decides to send it to the transmitter (or encoder) through noise-free feedback, [41]. To model such a system where sending noise free feedback from receiver to transmitter is costly and is governed by actions taken by the decoder, we consider the communication abstraction as in Fig. 5.

Theorem 11: For no ISI, stationary and indecomposable FSC with decoder *Feedback Logic* as in Fig. 5, the capacity is given by,

$$C_{dec}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{Q(x^N \| z^{N-1}) Q(a^N \| y^{N-1}), \mathbb{E}[\Lambda(A^N)] \leq \Gamma} I(X^N \rightarrow Y^N). \quad (117)$$

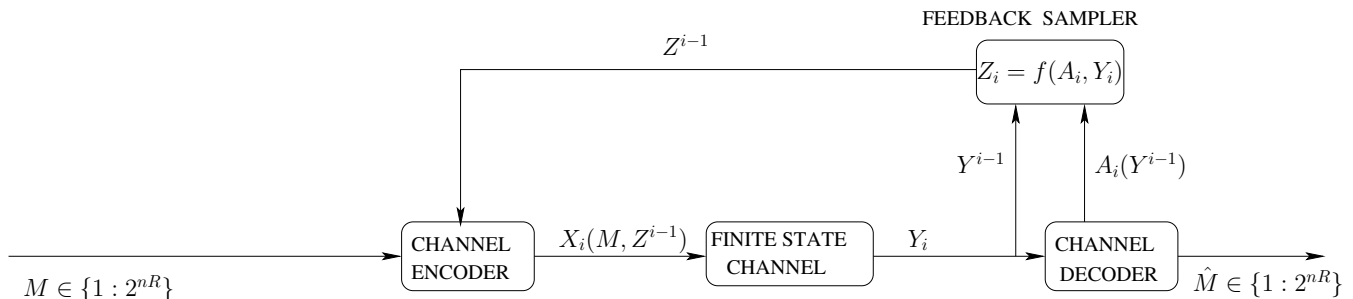


Fig. 5. Modeling **Feedback Sampling** for the acquisition of feedback in Finite State Channels (FSCs) with only *Decoder Feedback Logic*.

Proof: Specialize Theorem 8 as $A_e = \phi$, $A_d = A$ and $\Lambda(A_e, A_d) = \Lambda(A_d) = \Lambda(A)$.

Note 2: Also note that if in Fig. 5, decoder can use current channel output to generate actions, we can do the appropriate transformation in Theorem 9 to arrive at,

$$C_{dec,causal}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{Q(x^N \| z^{N-1}) Q(\phi_d^N \| y^{N-1}), \mathbb{E}[\Lambda(\Phi_d^N |_{Y^N})] \leq \Gamma} I(X^N \rightarrow Y^N). \quad (118)$$

IX. NUMERICAL EXAMPLE 1 : TO FEED OR NOT TO FEED BACK

A. Encoder Actions

Consider the setting as depicted in Fig. 6. Now the actions are binary, i.e., $\mathcal{A} = \{0, 1\}$. In this setting, action sequence determine *to feed or not to feed back* a deterministic function of the past channel output, i.e.,

$$\begin{aligned} Z_i &= f(A_i, Y_i) = g(Y_i), \text{ if } A_i = 1 \\ Z_i &= f(A_i, Y_i) = *, \text{ if } A_i = 0, \end{aligned} \quad (119)$$

where $*$ stands for erasure or no information about feedback. As a specific example for such a setting consider the communication system involving Markovian channel as in Fig. 7, which is essentially a no ISI, stationary, indecomposable FSC. Let the stationary distribution be given by $\pi(s)$, $\forall s \in \mathcal{S}$. The feedback from the decoder at time i consists of tuple

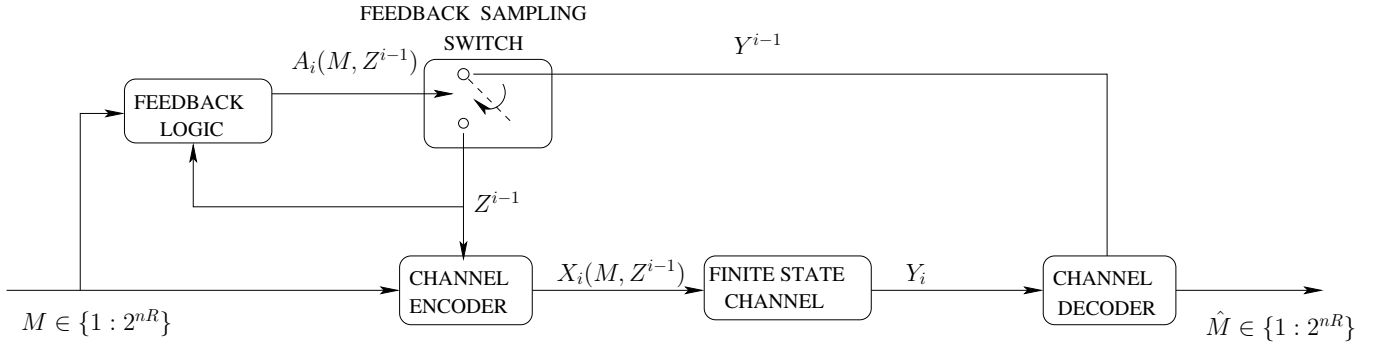


Fig. 6. Modeling logic ‘to feed or not to feed back’ in Finite State Channels (FSCs) with encoder taking actions.

$Y_{FB,i} = (Y_i, S_i)$ and observed or *sampled* feedback $Z_i = g(Y_{FB,i}) = S_i$, when $A_i = 1$. The cost function, $\Lambda(a) = a$, $a \in \mathcal{A}$ and the cost constraint is $\Gamma \in [0, 1]$. Using Theorem 10, the capacity of such a system is given by,

$$C_{enc}(\Gamma) = \lim_{N \rightarrow \infty} C_{enc}^N(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{Q(x^N, a^N \| z^{N-1}), \mathbb{E}[\Lambda(A^N)] \leq \Gamma} I(X^N \rightarrow (Y^N, S^N)). \quad (120)$$

In the following, we give single letter lower bound on $C_{enc}(\Gamma)$.

Theorem 12: The capacity of the system in Fig. 7 with encoder feedback logic is lower bounded as,

$$C_{enc}(\Gamma) \geq C_{enc,lower}(\Gamma) = \max I(X; Y|S), \quad (121)$$

where maximization is over joint probability distribution,

$$P_{S,A,Z,X,Y}(s, a, z, x, y) = \pi_S(s) P_A(a) \mathbf{1}_{\{z=f(a,s)\}} P_{X|Z,A}(x|z, a) P_{Y|X,S}(y|x, s). \quad (122)$$

and $\mathbb{E}[\Lambda(A)] \leq \Gamma$.

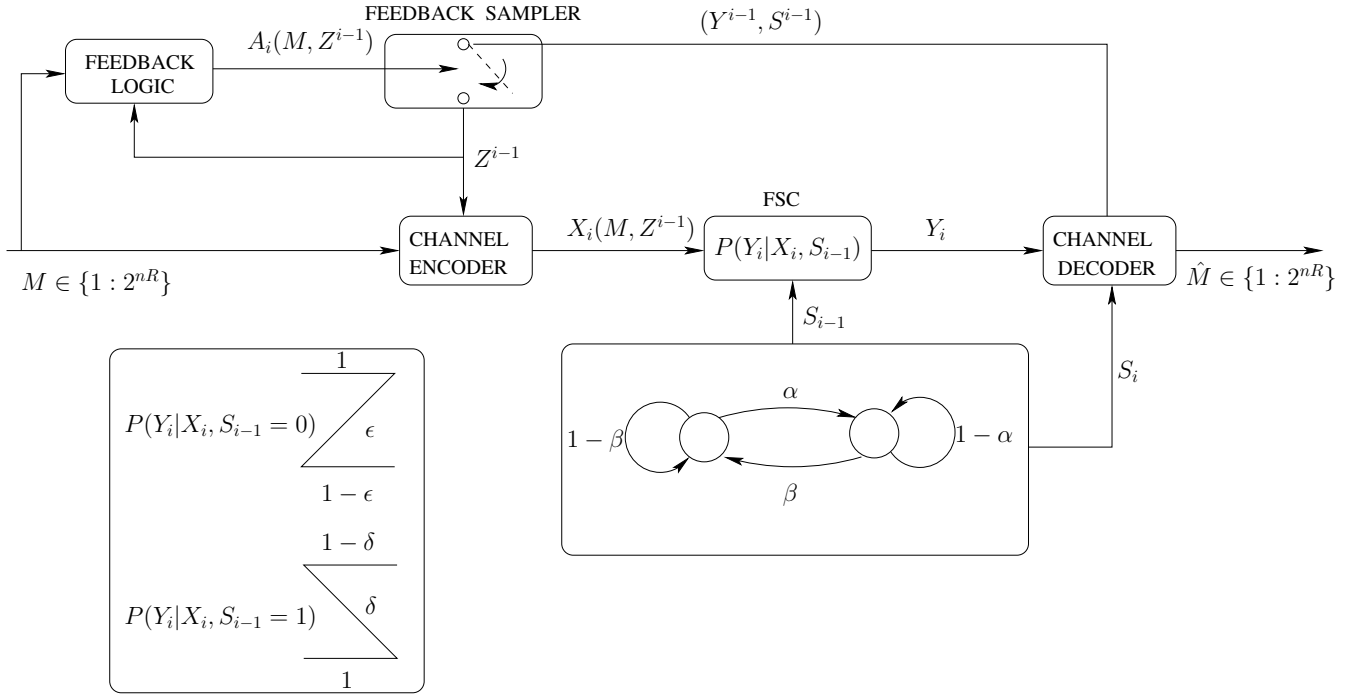


Fig. 7. To feed or not to feed back when encoder takes actions and decoder knows the state. States are stationary and evolve as a markov process.

Proof: The joint distribution in maximization in Eq. (120) is

$$P(s_0^N, a^N, z^N, x^N, y^N) = P(s_0) \prod_{i=1}^N Q(x_i, a_i | x^{i-1}, a^{i-1}, z^{i-1}) P(y_i | x_i, s_{i-1}) P(s_i | s_{i-1}) \mathbf{1}_{\{z_i=f(a_i, s_i)\}}. \quad (123)$$

To derive the lower bound we consider the following special type of above distribution,

$$P'(s_0^N, a^N, z^N, x^N, y^N) = P(s_0) \prod_{i=1}^N Q(a_i) Q(x_i | z_{i-1}) P(y_i | x_i, s_{i-1}) P(s_i | s_{i-1}) \mathbf{1}_{\{z_i = f(a_i, s_i)\}}. \quad (124)$$

Note that right hand side of the above distribution can be factorized as,

$$P'(s_0^N, a^N, z^N, x^N, y^N) = \Phi_i(a_i, a_{i-1}, s_{i-1}, z_{i-1}, s_i, z_i, x_i, y_i) \Phi_{n \setminus i}(a^{i-2}, a_{i+1}^N, s_0^{n \setminus i}, z^{i-2}, z_{i+1}^N, x^{n \setminus i}, y^{n \setminus i}), \quad (125)$$

which proves the markov chain,

$$(Y_i, S_i) - S_{i-1} - (Y^{i-1}, S_0^{i-2}). \quad (126)$$

Hence we have,

$$C_{enc}^N \geq \max \frac{1}{N} I(X^N \rightarrow (Y^N, S^N)) \quad (127)$$

$$= \max \frac{1}{N} \sum_{i=1}^N I(X^i; Y_i, S_i | Y^{i-1}, S^{i-1}) \quad (128)$$

$$= \max \frac{1}{N} \sum_{i=1}^N H(Y_i, S_i | S_{i-1}, S^{i-2}, Y^{i-1}) - H(Y_i, S_i | X_i, S_{i-1}, X_{i-1}, S^{i-2}, Y^{i-1}) \quad (129)$$

$$\stackrel{(a)}{=} \max \frac{1}{N} \sum_{i=1}^N H(Y_i, S_i | S_{i-1}) - H(Y_i, S_i | X_i, S_{i-1}) \quad (130)$$

$$= \max \frac{1}{N} \sum_{i=1}^N I(X_i; Y_i, S_i | S_{i-1}) \quad (131)$$

$$\stackrel{(b)}{=} \max \frac{1}{N} \sum_{i=1}^N I(X_i; Y_i | S_{i-1}), \quad (132)$$

$$(133)$$

where

- (a) follows from Markov Chain (126) and from the channel model assumption [Eq. 10].
- (b) follows from the chain rule

$$I(X_i; Y_i, S_i | S_{i-1}) = I(X_i; Y_i | S_{i-1}) + I(X_i, S_i | S_{i-1}, Y_i), \quad (134)$$

and from the fact that $I(X_i, S_i | S_{i-1}, Y_i) = 0$, due the following state evolution,

$$P(S_i | S_{i-1}) = P(S_i | S_{i-1}, X_i, Y_i), \quad (135)$$

which follows from the assumption on FSC for example in Fig. 7 as,

$$P(Y_i, S_i | X_i, S_{i-1}) = P(Y_i | X_i, S_{i-1}) P(S_i | S_{i-1}). \quad (136)$$

The maximum in the above inequalities is taken over set of the distributions,

$$\mathcal{S}_1 = \left\{ \prod_{i=1}^N Q(a_i) Q(x_i | z_{i-1}) : \mathbb{E}[\Lambda(A^N)] \leq \Gamma \right\}. \quad (137)$$

Clearly

$$\mathcal{S}_2 = \left\{ \prod_{i=1}^N Q(a_i) Q(x_i | z_{i-1}), \mathbb{E}[\Lambda(A_i)] \leq \Gamma, \forall i \right\} \subseteq \mathcal{S}_1. \quad (138)$$

Now since the channel is stationary, it is invariant in time shift, hence $P(s_i) = \pi(s_i), \forall i$. Therefore we have the lower bound,

$$C_{enc, lower}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{\prod_{i=1}^N Q(a_i) Q(x_i | z_{i-1}) : \mathbb{E}[\Lambda(A_i)] \leq \Gamma} \sum_{i=1}^N I(X_i; Y_i | S_{i-1}) \quad (139)$$

$$\stackrel{(c)}{\leq} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \max_{\prod_{i=1}^N Q(a_i) Q(x_i | z_{i-1}) : \mathbb{E}[\Lambda(A_i)] \leq \Gamma} I(X_i; Y_i | S_{i-1}) \quad (140)$$

$$\stackrel{(d)}{=} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \max_{P(a_{i-1}, s_{i-1}, z_{i-1}, x_i, y_i)} I(X_i, Y_i | S_{i-1}), \quad (141)$$

where

- (c) follows from the identity, $\max_a [f(x) + g(x)] \leq \max_a f(x) + \max_a g(x)$.
- (d) has,

$$P(a_{i-1}, s_{i-1}, z_{i-1}, x_i, y_i) = \pi(s_{i-1})Q(a_{i-1})\mathbf{1}_{\{z_{i-1}=f(a_{i-1}, s_{i-1})\}}Q(x_i|z_{i-1})P(y_i|x_i, s_{i-1}), \quad (142)$$

such that $E[\Lambda(A_{i-1})] \leq \Gamma$. a_0 is assumed to be a constant.

The inequality (c) holds with equality iff,

$$P(a_{i-1}, s_{i-1}, z_{i-1}, x_i, y_i) = \underset{P(a,s,z,x,y): E[\Lambda(A)] \leq \Gamma}{\operatorname{argmax}} I(X; Y|S) \quad \forall i, \quad (143)$$

where

$$P(a, s, z, x, y) = \pi(s)Q(a)\mathbf{1}_{\{z=f(a,s)\}}Q(x|z)P(y|x, s). \quad (144)$$

Note that for our setting, $P(x|z, a) = P(x|z)$ as knowing z determines a . Therefore, we have

$$C_{enc, lower}(\Gamma) = \max I(X; Y|S). \quad (145)$$

with maximization over the joint distribution,

$$P_{S,A,Z,X,Y}(s, a, z, x, y) = \pi_S(s)P_A(a)\mathbf{1}_{\{z=f(a,s)\}}P_{X|Z,A}(x|z, a)P_{Y|X,S}(y|x, s), \quad (146)$$

such that $E[\Lambda(A)] \leq \Gamma$. ■

Note 3: Note that lower bound on capacity at zero cost is,

$$C_{enc, lower}(\Gamma = 0) = \max_{\pi_S P_X P_{Y|X,S}} I(X; Y|S). \quad (147)$$

This is indeed also the capacity at zero cost as derived below, Clearly

$$C_{enc}(\Gamma = 0) \geq C_{enc, lower}(\Gamma = 0). \quad (148)$$

Now

$$C_{enc}^N(\Gamma = 0) \stackrel{(a)}{=} \frac{1}{N} \max_{Q(x^N)} I(X^N; Y^N, S^N) \quad (149)$$

$$\stackrel{(b)}{=} \frac{1}{N} \max_{Q(x^N)} I(X^N; Y^N|S^N) \quad (150)$$

$$= \frac{1}{N} \max \sum_{i=1}^N H(Y_i|S_{i-1}, S^{n \setminus (i-1)}, Y^{i-1}) - H(Y_i|X_i, S_{i-1}, S^{n \setminus (i-1)}, X^{n \setminus i}, Y^{i-1}) \quad (151)$$

$$\stackrel{(c)}{\leq} \frac{1}{N} \max \sum_{i=1}^N H(Y_i|S_{i-1}) - H(Y_i|X_i, S_{i-1}) \quad (152)$$

$$\stackrel{(d)}{=} \frac{1}{N} \sum_{i=1}^N \max I(X_i; Y_i|S_{i-1}) \quad (153)$$

$$\stackrel{(e)}{=} \max_{\pi_S P_X P_{Y|X,S}} I(X; Y|S), \quad (154)$$

where

- (a) follows from the fact that $A_i = 0$ for all $1 \leq i \leq N$ and hence since there is no feedback mutual information is equal to directed information ([20]).
- (b) follows from the fact that X^N and S^N are independent.
- (c) follows from the fact that conditioning reduces entropy and from the channel model assumption for the FSC, i.e. Eq. (10).
- (d) follows from the identity, $\max_a [f(x) + g(x)] \leq \max_a f(x) + \max_a g(x)$.
- (e) follows from the fact that maximization in (e) is on the joint,

$$P(x^N, s_0^N, y^N) = P(x^N)P(s_0) \prod_{i=1}^N P(y_i, s_i|x_i, s_{i-1}). \quad (155)$$

Hence the joint on (X_i, Y_i, S_{i-1}) is equivalent to,

$$P(X_i, Y_i, S_{i-1}) = P(S_{i-1})P(X_i)P(Y_i|X_i, S_{i-1}) = \pi(S_{i-1})P(X_i)P(Y_i|X_i, S_{i-1}). \quad (156)$$

Hence combining Eq. (148) and (154) we establish equality,

Note 4: Just like capacity at zero cost, we can also show that lower bound on capacity at unit cost is indeed tight too with similar steps as above. Hence we have,

$$C_{enc}(\Gamma = 1) = \max_{\pi_S P_{X|S} P_{Y|X,S}} I(X; Y|S). \quad (157)$$

Note that this scenario of complete feedback from decoder with state information is similar to the scenario where encoder and decoder know the states. The capacity result for such a communication system was characterized in [42] for channels with memory and indeed it coincides with Eq. (157).

Note 5: It is interesting to observe that the lower bound on capacity is the *Probing Capacity* of the system considered in [40](as depicted in Fig. 8), where

- Channel is memoryless with state distribution that is i.i.d. as the stationary distribution π_S of the channel with memory considered here.
- Encoder takes message dependent actions that are binary and decide **to observe or not to observe** channel state.
- Decoder has complete state information.

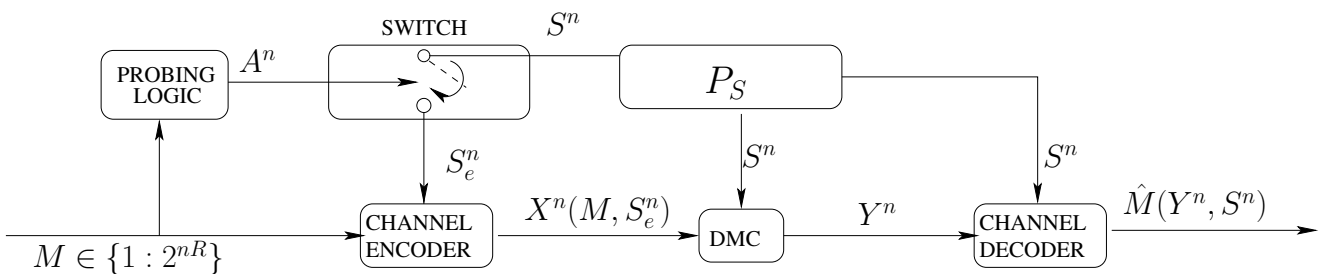


Fig. 8. The **Probing Capacity** setting in [40] where encoder takes message dependent actions to observe state, encodes using partial state information non-causally while decoder knows the complete channel state. Note that here the state is i.i.d.

B. Decoder Actions

Theorem 13: Consider again the system in Fig. 7 but now with decoder feedback logic (instead of encoder feedback logic) with decoder taking actions causally dependent on the channel output and state (for the sake of simplicity of notation here we denote this capacity by $C_{dec}(\Gamma)$ instead of $C_{dec,causal}(\Gamma)$). The capacity of such a system is lower bounded as,

$$C_{dec}(\Gamma) \geq C_{dec,lower}(\Gamma) = \max I(X; Y|S), \quad (158)$$

where maximization is over joint probability distribution,

$$P_{S,A,Z,X,Y}(s, a, z, x, y) = \pi_S(s) P_{A|S}(a|s) \mathbf{1}_{\{z=f(a,s)\}} P_{X|Z,A}(x|z, a) P_{Y|X,S}(y|x, s), \quad (159)$$

such that $E[\Lambda(A)] \leq \Gamma$.

Proof: Note the only difference in this lower bound as compared to $C_{enc,lower}$ is that in that there is $P_{A|S}$ instead of P_A in the distribution of maximization. The proof is similar to proof of Theorem 12, except the set of maximizing distributions taken here is,

$$\mathcal{S} = \left\{ \prod_{i=1}^N Q(a_i|s_i) Q(x_i|z_{i-1}), E[\Lambda(A_i)] \leq \Gamma \right\}. \quad (160)$$

Therefore,

$$C_{dec}(\Gamma) \geq C_{dec,lower}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{\mathcal{S}} I(X^N \rightarrow (Y^N, S^N)). \quad (161)$$

All the other steps follow as in proof of Theorem 12. ■

We evaluate the lower bounds for the example in Fig. 7, when $\alpha = \beta = \epsilon = \delta = 0.5$. The region is shown in Fig. 9. From the plot it is clear we can do much better than time sharing between capacity at zero and unit cost when either encoder or decoder takes actions.

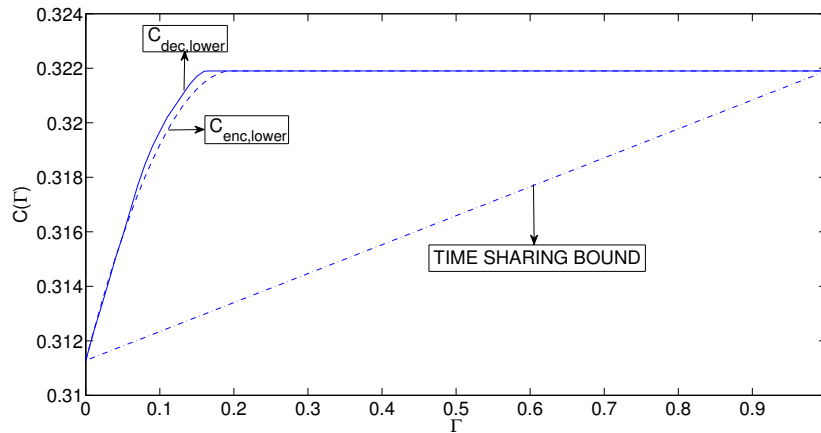


Fig. 9. Cost-capacity trade off for example in Fig. 7. $C_{enc,lower}$ is the lower bound on capacity with encoder feedback logic. If instead of encoder decoder decides (causally dependent on channel output and state) when encoder will sample feedback, $C_{dec,lower}$ is a lower bound on the capacity. The straight time represents time sharing scheme which is strictly sub-optimal.

X. NUMERICAL EXAMPLE 2 : CODING ON THE BACKWARD LINK IN FSC

Consider the setting depicted in Fig. 10. We allow *coding on the backward link*, i.e., decoder encodes the channel outputs causally ($A_i(Y^i) \in \mathcal{A}$) and sends it to the encoder. The encoder uses the acquired *active feedback* symbols to generate channel input symbols, i.e., $X_i(M, A^{i-1})$. For stationary indecomposable FSCs with active feedback we denote the capacity by C_{AF} . The setting in Fig. 10 is a very special case of the framework of decoder feedback logic considered in Note 2 at the end of Section VIII-B, when we let, $f(A_{d,i}, Y_i) = A_{d,i} = A_i$. Hence using the above conditions in the capacity expression mentioned in Note 2 at the end of Theorem 13 we have,

$$C_{AF} = \lim_{N \rightarrow \infty} \frac{1}{N} \max I(X^N \rightarrow Y^N), \quad (162)$$

where maximization is over joint distribution,

$$P(x^N, a^N, \phi_d^N, y^N) = Q(x^N \parallel a^{N-1})Q(\phi_d^N \parallel y^{N-1})P(y^N \parallel x^N) \prod_{i=1}^N \mathbf{1}_{\{a_i = \phi_{d,i} | y_i\}}, \quad (163)$$

such that $E[\Lambda(a^N)] = E[\Lambda(\Phi_d^N | Y^N)] \leq \Gamma$, where $\Phi_d^N | Y^N, \phi_d^N$ are as defined in Section VII.

Now consider an example under this setting where the channel evolution is markovian with binary states, i.e.

$$P(Y_i, S_i | X_i, S_{i-1}) = P(Y_i | X_i, S_{i-1})P(S_i | S_{i-1}), \quad (164)$$

and states are known to the decoder on the fly. Hence, decoder performs coding on backward link as, $A_i = A_i(Y^i, S^i) \in \mathcal{A}$. The markov chain is assumed to be stationary with distribution π_S and states take values in a finite alphabet \mathcal{S} . We consider following special cases when $|\mathcal{A}| \geq |\mathcal{S}|$:

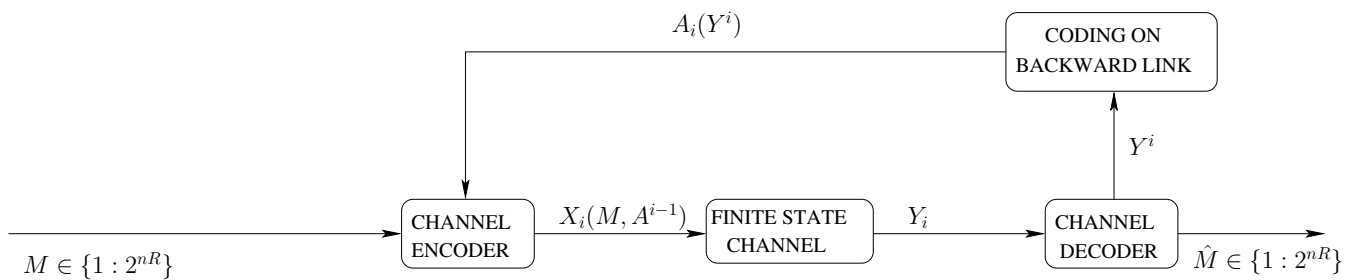


Fig. 10. Modeling **coding on backward link** in finite state channels (FSCs).

A. No Cost Constraints

Theorem 14: Under this setting, the capacity is given by,

$$C_{AF} = \max_{\pi_S P_{X|S} P_{Y|X,S}} I(X; Y|S). \quad (165)$$

Proof: Since the decoder knows the states, the effective output is the tuple $Y_{FB,i} = (Y_i, S_i)$. Achievability is straightforward. Actions basically communicate the state, $A_i = S_i$. Hence in this case, the setup is same as encoder and decoder knowing states, and by the notes at the end of Section VIII-A, we have,

$$C_{AF} \geq \max_{\pi_S P_{X|S} P_{Y|X,S}} I(X; Y|S). \quad (166)$$

Now consider for the converse,

$$C_{AF} = \lim_{N \rightarrow \infty} \max \frac{1}{N} I(X^N \rightarrow (Y^N, S^N)) \quad (167)$$

$$= \lim_{N \rightarrow \infty} \max \frac{1}{N} \sum_{i=1}^N H(Y_i, S_i | Y^{i-1}, S^{i-1}) - H(Y_i, S_i | X^i, Y^{i-1}, S^{i-1}) \quad (168)$$

$$\leq \lim_{N \rightarrow \infty} \max \frac{1}{N} \sum_{i=1}^N H(Y_i, S_i | S_{i-1}) - H(Y_i, S_i | X_i, S_{i-1}) \quad (169)$$

$$= \lim_{N \rightarrow \infty} \max \frac{1}{N} \sum_{i=1}^N I(X_i; Y_i, S_i | S_{i-1}) \quad (170)$$

$$\stackrel{(a)}{=} \lim_{N \rightarrow \infty} \max \frac{1}{N} \sum_{i=1}^N I(X_i; Y_i | S_{i-1}) \quad (171)$$

$$\stackrel{(b)}{\leq} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \max I(X_i; Y_i | S_{i-1}), \quad (172)$$

where (a) follows from the similar arguments used for Eq. (132) while (b) follows from the identity $\max_a [f(x) + g(x)] \leq \max_a f(x) + \max_a g(x)$. Note that maximization in (b) is over the joint probability distribution,

$$P(s_0, x^N, a^N, \phi_d^N, y^N) = Q(x^N \parallel a^{N-1}) Q(\phi_d^N \parallel y^{N-1}) P(y^N \parallel x^N, s_0) \prod_{i=1}^N \mathbf{1}_{\{a_i = \phi_{d,i} | y_i\}}, \quad (173)$$

but one can average out, $(a^N, \phi_d^N, s^{n \setminus i-1}, x^{n \setminus i}, y^{n \setminus i})$ so that for the i^{th} term, maximization is over the joint probability distribution,

$$P(s_{i-1}, x_i, y_i) = \pi_S(s_{i-1}) P(x_i | s_{i-1}) P(y_i | x_i, s_{i-1}), \quad (174)$$

since states are stationary and distributed as π_S . Hence we have,

$$C_{AF} \leq \max_{\pi_S P_{X|S} P_{Y|X,S}} I(X; Y|S). \quad (175)$$

Proof is completed using Eq. (166) and (175). ■

B. Cost constraint Γ

The condition of this subsection differs from those of the previous in cost constraints. The intuition here is to look for an achievability scheme which decides when to send or not send state information from decoder to encoder depending on cost constraints.

Theorem 15: For the system in Fig. 11, the capacity is lower bounded as,

$$C_{AF}(\Gamma) \geq C_{AF,lower}(\Gamma) = \max I(X; Y|S), \quad (176)$$

where maximization is over joint probability distribution,

$$P_{S,A,X,Y}(s, a, x, y) = \pi_S(s) P_{A|S}(a|s) P_{X|A}(x|a) P_{Y|X,S}(y|x, s), \quad (177)$$

where $E[\Lambda(A)] \leq \Gamma$.

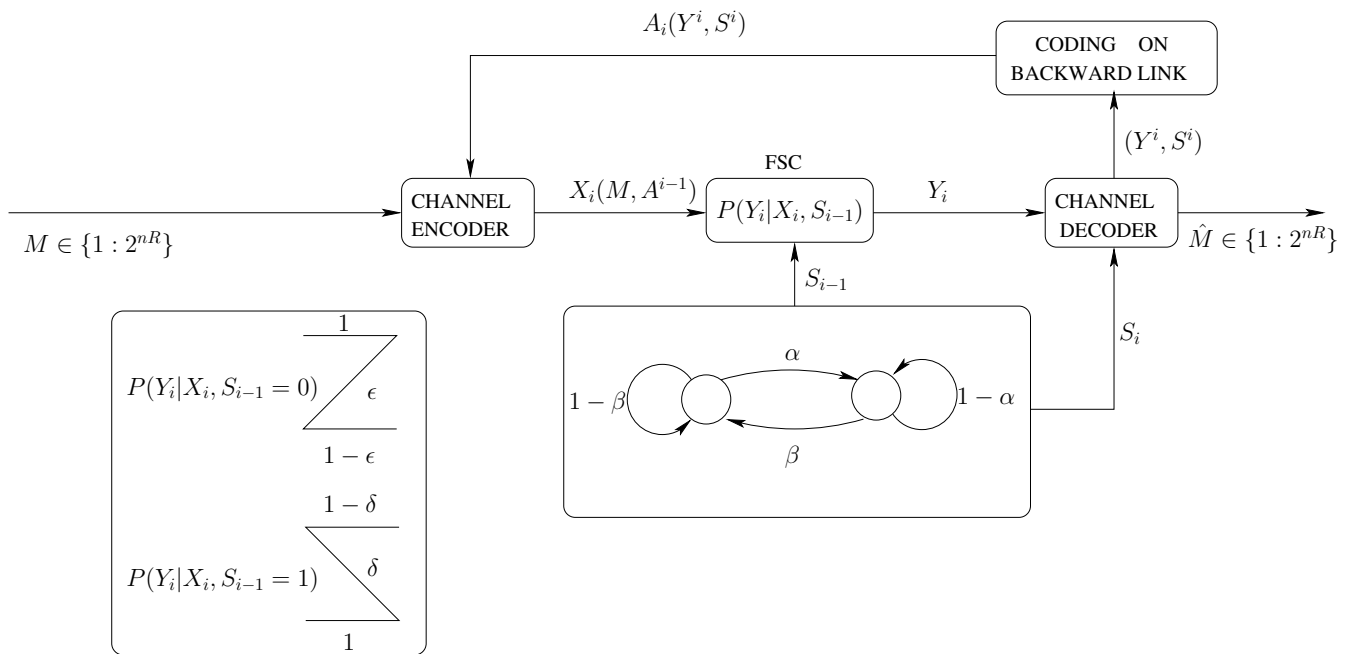


Fig. 11. Modeling coding on the backward link for markovian channel with binary states.

Proof: We outline only the sketch of the proof as it is similar to the proof of Theorem 12, except the set of maximizing distributions taken here is,

$$\mathcal{S} = \left\{ \prod_{i=1}^N Q(a_i|s_i)Q(x_i|a_{i-1}), \mathbb{E}[\Lambda(A_i)] \leq \Gamma \right\}. \quad (178)$$

Therefore,

$$C_{AF}(\Gamma) \geq C_{AF,lower}(\Gamma) = \lim_{N \rightarrow \infty} \frac{1}{N} \max_{\mathcal{S}} I(X^N \rightarrow (Y^N, S^N)). \quad (179)$$

All the other steps follow as in proof of Theorem 12. \blacksquare

We consider an example under this setting as depicted in Fig. 11. We assume \mathcal{A} is a binary alphabet with cost function, $\Lambda(a) = a$, $a \in \{0, 1\}$, hence this models the scenario of cost constrained *one-bit active feedback* in the given finite state channel. The plot for $\alpha = \beta = \delta = 0.5$ is shown in Fig. 12. Note that this bound is equal to the bound, $C_{dec,lower}$ in Theorem 13 for $f(a, s) = a$.

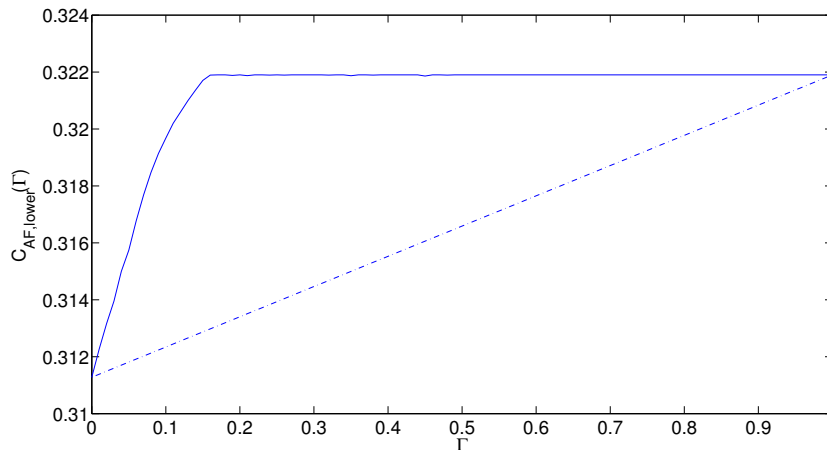


Fig. 12. Cost-capacity trade off for example in Fig. 11. $C_{AF,lower}$ is the lower bound on capacity. The straight line represents naive time sharing scheme.

XI. CONCLUSION

In this paper, we studied communication systems with finite state channels (FSCs), where the encoder and decoder adaptively decide what to feed back from the decoder to encoder to optimize for the rate of reliable communication, under an average cost constraint. For FSCs where probability of initial state is positive for all states or for stationary indecomposable FSCs, we have the exact characterization of the capacity. We also discuss the special case of *to feed or not to feed back* where either the encoder or the decoder takes binary actions that determine whether or not a deterministic function of channel output will be fed back to the encoder. As another special case, we characterize the capacity in case of *coding on the backward link* for FSCs. In case of markovian channels, with explicit computation we show our achievable schemes outperform naive time sharing schemes significantly.

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APPENDIX A SOME PROPERTIES OF CAUSAL CONDITIONING AND DIRECTED INFORMATION

Here we present some of the basic properties of causal conditioning and directed information. The proofs are omitted as being similar to the corresponding Lemmas in [9].

- *Property 1* :[Chain rule for causal conditioning]

$$P(x^N, a^N, y^N) = P(y^N \parallel x^N, a^N)P(x^N, a^N \parallel y^{N-1}). \quad (180)$$

Similarly,

$$P(x^N, a^N, y^N, s_0) = P(y^N \parallel x^N, a^N, s_0)P(x^N, a^N \parallel y^{N-1}, s_0). \quad (181)$$

- *Property 2* :

$P(x^N, a_e^N \parallel z^{N-1})$ uniquely determines $P(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, z^{i-1}) \forall 1 \leq i \leq N$ and all the arguments $(x^{i-1}, a_e^{i-1}, z^{i-1})$, for which $P(x^{i-1}, a_e^{i-1}, z^{i-1}) > 0$. Similar results holds for $P(a_d^N \parallel y^{N-1})$.

- *Property 3* :

$$|I(X^N \rightarrow Y^N) - I(X^N \rightarrow Y^N | S)| \leq H(S) \leq \log |\mathcal{S}|.$$

APPENDIX B PROOF OF THEOREM 2

From our achievability scheme and Lemma 1 we have

$$P(x^N, a_e^N, a_d^N, y^N) = \sum_{s_0} P(s_0, x^N, a_e^N, a_d^N, y^N) \quad (182)$$

$$= Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1}) \sum_{s_0} P(s_0)P(y^N \parallel x^N, s_0) \quad (183)$$

$$= Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N), \quad (184)$$

where last equality follows from Eq. (23). Hence we have

$$E(P_{e,m}) = \sum_{y^N} \sum_{x^N, a_e^N, a_d^N} P(x^N, a_e^N, a_d^N, y^N)P(\text{error} | m, x^N, a_e^N, a_d^N, y^N) \quad (185)$$

$$= \sum_{y^N} \sum_{x^N, a_e^N, a_d^N} Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1})P(y^N \parallel x^N)P(\text{error} | m, x^N, a_e^N, a_d^N, y^N). \quad (186)$$

Let $A_{m'} = \{\text{event such that } P_{y^N | m'} > P_{y^N | m} \text{ for } m' \neq m\}$. Alternatively if for a message m , encoder generated x^N , $A_{m'} = \{\text{event such that } P(y^N \parallel x^N) > P(y^N \parallel x'^N) \text{ for } x'^N \neq x^N\}$

$$P(A_{m'} | m, x^N, a_e^N, a_d^N, y^N) = \sum_{x'^N, a_e'^N, a_d'^N} Q(x'^N, a_e'^N \parallel z^{N-1})Q(a_d'^N \parallel y^{N-1})\mathbf{1}_{\{P(y^N \parallel x'^N) > P(y^N \parallel x^N)\}} \quad (187)$$

$$P(A_{m'} | m, x^N, a_e^N, a_d^N, y^N) \leq \sum_{x'^N, a_e'^N, a_d'^N} Q(x'^N, a_e'^N \parallel z^{N-1})Q(a_d'^N \parallel y^{N-1}) \left[\frac{P(y^N \parallel x'^N)}{P(y^N \parallel x^N)} \right]^s \text{ any } s > 0. \quad (188)$$

Hence,

$$P(\text{error}|m, x^N, a_e^N, a_d^N, y^N) = P(\cup_{m' \neq m} A_{m'} | m, x^N, a_e^N, a_d^N, y^N) \quad (189)$$

$$\leq \min \left[\sum_{m' \neq m} P(A_{m'} | m, x^N, a_e^N, a_d^N, y^N), 1 \right] \quad (190)$$

$$\leq \left[\sum_{m' \neq m} P(A_{m'} | m, x^N, a_e^N, a_d^N, y^N) \right]^\rho, \text{ for any } 0 \leq \rho \leq 1 \quad (191)$$

$$\leq \left[(M-1) \sum_{x'^N, a_e'^N, a_d'^N} Q(x'^N, a_e'^N \| z^{N-1}) Q(a_d'^N \| y^{N-1}) \left(\frac{P(y^N \| x'^N)}{P(y^N \| x^N)} \right)^s \right]^\rho. \quad (192)$$

Now substituting (192) in (186) and using $s = \frac{1}{\rho+1}$ we obtain,

$$E_{P_{e,m}} \leq (M-1)^\rho \sum_{y^N} \left[\sum_{x^N, a_e^N, a_d^N} Q(x^N, a_e^N \| z^{N-1}) Q(a_d^N \| y^{N-1}) P(y^N \| x^N)^{\frac{1}{\rho+1}} \right]^{\rho+1}. \quad (193)$$

APPENDIX C PROOF OF SOME MARKOV CHAINS

If a given scheme satisfies joint probability distribution, as in Eq. (13), we have the following markov chains,

- MC1 $Y_i - (X^i, Y^{i-1}, S_0) - (M, A_e^i, A_d^i)$.
- MC2 $(X_i, A_{e,i}) - (X^{i-1}, A_e^{i-1}, Z^{i-1}) - (Y^{i-1}, A_d^{i-1}, S_0)$.
- MC3 $A_{d,i} - (Y^{i-1}, A_d^{i-1}) - (X^i, A_e^i, Z^{i-1}, S_0)$.

To prove MC1, consider again the joint probability distribution induced by a given scheme,

$$\begin{aligned} & P_{M, A_e^N, A_d^N, Z^N, X^N, S_0^N, Y^N, \hat{M}}(m, a_e^N, a_d^N, z^N, x^N, s_0^N, y^N, \hat{m}) \\ &= \frac{1}{|\mathcal{M}|} P_S(s_0) \prod_{i=1}^n \mathbf{1}_{\{a_{d,i} = f_{A_{d,i}}(y^{i-1})\}} \mathbf{1}_{\{a_{e,i} = f_{A_{e,i}}(m, z^{i-1})\}} \\ & \quad \times \prod_{i=1}^n \mathbf{1}_{\{x_i = f_{e,i}(m, z^{i-1})\}} P(y_i, s_i | x_i, s_{i-1}) \mathbf{1}_{\{z_i = f(a_{e,i}, a_{d,i}, y_i)\}} \times \mathbf{1}_{\{\hat{m} = f_d(y^n)\}}. \end{aligned} \quad (194)$$

Summing over $(\hat{M}, X_{i+1}^N, Z_i^N, S_{i+1}^N, Y_{i+1}^N, A_{e,i+1}^N, A_{d,i+1}^N)$ in Eq. (194) we obtain,

$$\begin{aligned} & P_{M, A_e^i, A_d^i, Z^{i-1}, X^i, S_0^i, Y^i}(m, a_e^i, a_d^i, z^{i-1}, x^i, s_0^i, y^i) \\ &= \frac{1}{|\mathcal{M}|} \prod_{j=1}^i \mathbf{1}_{\{a_{d,j} = f_{A_{d,j}}(y^{j-1})\}} \mathbf{1}_{\{a_{e,j} = f_{A_{e,j}}(m, z^{j-1})\}} \mathbf{1}_{\{x_j = f_{e,j}(m, z^{j-1})\}} \times \prod_{j=1}^{i-1} \mathbf{1}_{\{z_i = f(a_{e,i}, a_{d,i}, y_i)\}} \end{aligned} \quad (195)$$

$$\times \prod_{j=1}^i P(y_j, s_j | x_j, s_{j-1}) \times P_S(s_0) \quad (196)$$

$$= \Phi_1(M, A_e^i, A_d^i, Z^{i-1}, Y^{i-1}, X^i) \Phi_2(Y^i, S_0^i, X^i) \quad (197)$$

$$= \Phi_1'(M, A_e^i, A_d^i, Z^{i-1}, Y^{i-1}, X^i) \Phi_2(Y^i, S_0^i, X^i), \quad (198)$$

which implies markov chain $(M, A_e^i, A_d^i, Z^{i-1}) - (Y^{i-1}, X^i, S_0) - (Y_i, S^i)$ which implies MC1.

Lemma 3: If MC1 holds,

$$P(y^N \| x^N, a_e^N, a_d^N, s_0) = P(y^N \| x^N, s_0). \quad (199)$$

Proof: This follows by chain rule in expanding $P(y^N \| x^N, a_e^N, a_d^N, s_0)$,

$$P(y^N \| x^N, a_e^N, a_d^N, s_0) = \prod_{i=1}^N P(y_i | y^{i-1}, x^i, a_e^i, a_d^i, s_0) \quad (200)$$

$$\stackrel{(*)}{=} \prod_{i=1}^N P(y_i | y^{i-1}, x^i, s_0) \quad (201)$$

$$= P(y^N \| x^N, s_0), \quad (202)$$

where (*) follows from MC1. ■

To prove MC2, again summing over $(\hat{M}, X_{i+1}^N, Z_i^N, S_i^N, Y_i^N, A_{e,i+1}^N, A_{d,i}^N)$ in Eq. (194) we obtain,

$$P_{M, A_e^i, A_d^{i-1}, Z^{i-1}, X^i, S_0^{i-1}, Y^{i-1}}(m, a_e^i, a_d^{i-1}, z^{i-1}, x^i, s_0^{i-1}, y^{i-1}) \quad (203)$$

$$= \frac{1}{|\mathcal{M}|} \mathbf{1}_{\{x_i=f_{e,i}(m, z^{i-1})\}} \mathbf{1}_{\{a_{e,i}=f_{A_{e,i}}(m, z^{i-1})\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{a_{e,j}=f_{A_{e,j}}(m, z^{j-1})\}} \mathbf{1}_{\{x_j=f_{e,j}(m, z^{j-1})\}} \\ \times \prod_{j=1}^{i-1} \mathbf{1}_{\{a_{d,j}=f_{A_{d,j}}(y^{j-1})\}} P(y_j, s_j | x_j, s_{j-1}) \mathbf{1}_{\{z_i=f(a_{e,i}, a_{d,i}, y_i)\}} \times P_S(s_0) \quad (204)$$

$$= \Phi_1(M, X_i, A_{e,i}, X^{i-1}, A_e^{i-1}, Z^{i-1}) \Phi_2(X^{i-1}, A_e^{i-1}, Z^{i-1}, A_d^{i-1}, S_0^{i-1}, Y^{i-1}), \quad (205)$$

which implies the markov chain, $(M, X_i, A_{e,i}) - (X^{i-1}, A_e^{i-1}, Z^{i-1}) - (Y^{i-1}, S_0^{i-1}, A_d^{i-1})$ which implies MC2.

To prove MC3, we sum over $(\hat{M}, X_{i+1}^N, Z_i^N, S_i^N, Y_i^N, A_{e,i+1}^N, A_{d,i+1}^N)$ in Eq. (194) and obtain,

$$P_{M, A_e^i, A_d^i, Z^{i-1}, X^i, S_0^{i-1}, Y^{i-1}}(m, a_e^i, a_d^i, z^{i-1}, x^i, s_0^{i-1}, y^{i-1}) \\ = \frac{P_S(s_0)}{|\mathcal{M}|} \mathbf{1}_{\{x_i=f_{e,i}(m, z^{i-1}), a_{e,i}=f_{A_{e,i}}(m, z^{i-1})\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{a_{e,j}=f_{A_{e,j}}(m, z^{j-1}), x_j=f_{e,j}(m, z^{j-1}), z_i=f(a_{e,i}, a_{d,i}, y_i)\}} P(y_j, s_j | x_j, s_{j-1}) \\ \times \prod_{j=1}^{i-1} \mathbf{1}_{\{a_{d,j}=f_{A_{d,j}}(y^{j-1})\}} \quad (206)$$

$$= \Phi_1(M, S_0^{i-1}, X^i, A_e^i, Z^{i-1}, A_d^{i-1}, Y^{i-1}) \Phi_2(A_d^{i-1}, Y^{i-1}, A_{d,i}), \quad (207)$$

which implies markov chain, $A_{d,i} - (Y^{i-1}, A_d^{i-1}) - (X^i, A_e^i, Z^{i-1}, S_0^{i-1}, M)$, which implies MC3.

Lemma 4: If MC2 and MC3 holds, then,

$$Q(x^N, a_e^N, a_d^N \parallel y^{N-1}, s_0) = Q(x^N, a_e^N \parallel z^{N-1}) Q(a_d^N \parallel y^{N-1}). \quad (208)$$

Proof: Applying chain rule for causal conditioning,

$$Q(x^N, a_e^N, a_d^N \parallel y^{N-1}, s_0) = \prod_{i=1}^N Q(x_i, a_{e,i}, a_{d,i} | x^{i-1}, a_e^{i-1}, a_d^{i-1}, y^{i-1}, s_0) \quad (209)$$

$$= Q(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, a_d^{i-1}, y^{i-1}, s_0) Q(a_{d,i} | x^i, a_e^i, a_d^{i-1}, y^{i-1}, s_0) \quad (210)$$

$$\stackrel{(*)}{=} Q(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, a_d^{i-1}, y^{i-1}, z^{i-1}, s_0) Q(a_{d,i} | x^i, a_e^i, a_d^{i-1}, y^{i-1}, s_0) \quad (211)$$

$$\stackrel{(**)}{=} Q(x_i, a_{e,i} | x^{i-1}, a_e^{i-1}, z^{i-1}) Q(a_{d,i} | a_d^{i-1}, y^{i-1}) \quad (212)$$

$$= Q(x^N, a_e^N \parallel z^{N-1}) Q(a_d^N \parallel y^{N-1}) P(y^N \parallel x^N), \quad (213)$$

where (*) follows from the fact that, $z_i = f(a_{e,i}, a_{d,i}, y_i)$, while (**) follows from MC2 and MC3. ■

Lemma 5: If a given scheme satisfies joint as in Eq. (13) then,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0) Q(x^N, a_e^N \parallel z^{N-1}) Q(a_d^N \parallel y^{N-1}) P(y^N \parallel x^N, s_0) \quad (214)$$

$$P(x^N, a_e^N, a_d^N, y^N) = Q(x^N, a_e^N \parallel z^{N-1}) Q(a_d^N \parallel y^{N-1}) P(y^N \parallel x^N). \quad (215)$$

$$(216)$$

Proof: Using Property 1 in Appendix A, we have,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0) Q(x^N, a_e^N, a_d^N \parallel y^{N-1}, s_0) P(y^N \parallel x^N, a_e^N, a_d^N, s_0), \quad (217)$$

but already proved that MC1, MC2 and MC3 holds which implies by Lemmas 3 and 4 that,

$$P(y^N \parallel x^N, a_e^N, a_d^N, s_0) = P(y^N \parallel x^N, s_0) \quad (218)$$

$$Q(x^N, a_e^N, a_d^N \parallel y^{N-1}, s_0) = Q(x^N, a_e^N \parallel z^{N-1}) Q(a_d^N \parallel y^{N-1}) P(y^N \parallel x^N), \quad (219)$$

which implies,

$$P(s_0, x^N, a_e^N, a_d^N, y^N) = P(s_0) Q(x^N, a_e^N \parallel z^{N-1}) Q(a_d^N \parallel y^{N-1}) P(y^N \parallel x^N, s_0). \quad (220)$$

Summing over s_0 ,

$$P(x^N, a_e^N, a_d^N, y^N) = \sum_{s_0} P(s_0, x^N, a_e^N, a_d^N, y^N) \quad (221)$$

$$= Q(x^N, a_e^N \parallel z^{N-1})Q(a_d^N \parallel y^{N-1}) \sum_{s_0} P(s_0)P(y^N \parallel x^N, s_0), \quad (222)$$

where last equality follows from Eq. (23). ■