On the continuous dependence of the minimal solution of constrained backward stochastic differential equations

Helin Wu[1]
School of Mathematics, Shandong University, Jinan 250100, China
E-mail: wuhewlin@gmail.com

Abstract

It is well-known that solutions of backward differential equations are continuously dependent on the terminal value. Since the increasing part of the minimal solution of a constrained backward differential equation (shortly CBSDE) varies against terminal value, the continuous dependence property of terminal value is not obvious for it. In this paper, we obtain a result about this problem under some mild assumptions. The main tool used here is the penalization method to get the minimal solution of a CBSDE and the property of convex functional that it is continuous when it is lower semi-continuous. The comparison theorem of the minimal solution of CBSDE plays a crucial role in our proof.

Keywords:CBSDE, continuous dependence, convex functional, minimal solution

1 Introduction

It is well-known that g-solution and g-supersolution are both continuously dependent on the terminal value, see El Karoui etc.[3] and Pardoux and Peng[5] for references. g-solution and g-supersolution are often used to price or hedge claims in mathematical finance. In practice, some constraints may put on the portfolio and wealth process, and the backward stochastic differential equation subjecting to some constraints comes up to our consideration. For such investigation on constrained backward stochastic equation, we refer to Cvitanic.J, Karatzas.I and Soner.H.M[1]. In their paper, the constraint is $z(t) \in K$ for some convex set K. Under the assumption that there exist at least one solution to such constrained BSDE, a penalization method can be used to get the minimal solution to this CBSDE. Similarly, Peng[6] pointed out that the smallest g-supersolution, which is denoted by $\mathcal{E}_t^{g,\phi}(\xi)$, can be obtained by penalization method, i.e, it can be approximated by an increasing sequence of g-supersolutions suppose that for the square integrable terminal value ξ in $L_T^2(R)$, there exist at least one g-supersolution (y_t, z_t, C_t) satisfying the constraint equation $\phi(t, y_t, z_t) = 0$. In these case, it is obvious that the associated increasing part C_t of the minimal solution of CBSDE varies against terminal value, the usual priori estimation of g-solution or g-supersolution does not work for the continuous dependence property. In this paper, we mainly investigate such a problem in the framework of Peng[6] when both g and ϕ are convex. The case $z \in K$ can be concluded in this case taking $\phi(z) = d(z, K)$, the distance function from z to the convex set K. It is obvious convex in z.

This paper is organized as follows: In section 2, we state the framework in Peng[6] and some propositions on the smallest g-supersolution, namely the minimal solution of CBSDE with constraint as $\phi(t, y_t, z_t) = 0$.

Under the assumption that when the generator g and constraint function ϕ are both convex, we obtain the continuous dependence property of $\mathcal{E}_t^{g,\phi}(\xi)(\ 0 \le t \le T)$ in terms of $\xi \in L_T^2(R)$ in section 3.

2 BSDE and the minimal solution of CBSDE

Given a probability space $(\Omega, \mathcal{F}, P)$ and R^d-valued Brownian motion $W(t)$, we consider a sequence $\{(\mathcal{F}_t); t \in [0, T]\}$ of filtrations generated by Brownian motion $W(t)$ and $\mathcal{P}$ is the σ-field of predictable sets of $\Omega \times [0, T]$. We use $L_T^2(R^d)$ to denote the space of all F_T-measurable random variables $\xi : \Omega \to R^d$ for which

$$\| \xi \|^2 = E[|\xi|^2] < +\infty.$$

and use $H_T^2(R^d)$ to denote the space of predictable process $\varphi : \Omega \times [0, T] \to R^d$ for which

$$\| \varphi \|^2 = E[\int_0^T |\varphi|^2] < +\infty.$$

+ Corresponding author.

The backward stochastic differential equation (shortly BSDE) driven by $g(t, y, z)$ is given by

$$-dy_t = g(t, y_t, z_t)dt - z_t^* dW(t) \tag{2.1}$$

where $y_t \in R$ and $W(t) \in R^d$. Suppose that $\xi \in L_T^2(R)$, $g(., 0, 0) \in H_T^2(R)$ and g is uniformly Lipschitz; i.e, there exists $M > 0$ such that $dP \otimes dt$ a.s.

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq M(|y_1 - y_2| + |z_1 - z_2|), \quad \forall (y_1, z_1), (y_2, z_2)$$

Pardoux and Peng [5] proved the existence of adapted solution $(y(t), z(t))$ of such BSDE. We call (g, ξ) standard parameters for the BSDE.

In this paper, we assume $g(\omega, t, y, z)$ are both convex in(y, z).

Dfinition 2.1. *(super-solution) A super-solution of a BSDE associated with the standard parameters (g, ξ) is a vector process (y_t, z_t, C_t) satisfying*

$$-dy_t = g(t, y_t, z_t)dt + dC_t - z_t^* dW(t), \quad y_T = \xi, \tag{2.2}$$

or being equivalent to

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s, \tag{2.2'}$$

where $(C_t, t \in [0, T])$ is an increasing, adapted, right-continuous process with $C_0 = 0$ and z_t^* is the transpose of z_t. When $C_t \equiv 0$, we call (y_t, z_t) a g-solution.

In this paper, we consider g-supersolutions (y_t, z_t, C_t) satisfying the constraint

$$\phi(t, y_t, z_t) = 0, \tag{2.3}$$

where $\phi(t, y, z) : [0, T] \times R \times R^d \to R^+$. In such case, we give the following definition,

Dfinition 2.2. *(The smallest g-supersolution or the minimal solution) A g-supersolution (y_t, z_t, C_t) is said to be the smallest g-supersolution or the minimal solution, given $y_T = \xi$, subjecting to the constraint* (2.3) *if for any other g-supersolution (y'_t, z'_t, C'_t) satisfies* (2.3) *with $y'_T = \xi$, we have $y_t \leq y'_t$ a.e., a.s., the smallest g-supersolution is denoted by $\mathcal{E}_t^{g,\phi}(\xi)$.*

For any $\xi \in L_T^2(R)$, we denote $\mathcal{H}^\phi(\xi)$ as the set of g-supersolutions (y_t, z_t, C_t) subjecting to (2.3) with $y_T = \xi$. When $\mathcal{H}^\phi(\xi)$ is not empty, Peng[6] proved that the smallest g-supersolution exists for $\xi \in L_T^2(R)$. In this paper, for simplicity, we first consider the continuous dependence property of $\mathcal{E}_t^{g,\phi}(\xi)$ at $t = 0$.

The convexity of $\mathcal{E}_0^{g,\phi}(\xi)$ can be easily deduced from the same propositions of g-solutions or g-supersolutions when both g and ϕ are convex.

Proposition 2.1. *Let $\phi(t, y, z)$ be a function: $[0, T] \times R \times R^d \to R^+$. If $\phi(t, y, z)$ is uniformly Lipschitz and convex in (y, z), then under the assumption that $g(t, y, z)$ is also uniformly Lipschitz and convex in (y, z) and $g(., 0, 0), \phi(., 0, 0) \in H_T^2(R)$, we have*

$$\mathcal{E}_t^{g,\phi}(a\xi + (1 - a)\eta) \leq a\mathcal{E}_t^{g,\phi}(\xi) + (1 - a)\mathcal{E}_t^{g,\phi}(\eta) \quad \forall t \in [0, T]$$

for any $\xi, \eta \in L_T^2(R)$ and $a \in [0, 1]$.

Proof According to Peng[6], the solutions $y_t^m(\xi)$ of

$$y_t^m(\xi) = \xi + \int_t^T g(y_s^m(\xi), z_s^m, s)ds + A_T^m - A_t^m - \int_t^T z_s^m dW_s.$$

is an increasing sequence and converges to $\mathcal{E}_t^{g,\phi}(\xi)$, where

$$A_t^m := m \int_0^t \phi(y_s^m, z_s^m, s)ds.$$

For any fixed m, by the convexity of g and ϕ, $y_t^m(\xi)$ is a convex in ξ, that is

$$y_t^m(a\xi + (1 - a)\eta) \leq ay_t^m(\xi) + (1 - a)y_t^m(\eta),$$

taking limit as $m \to \infty$, we get the required result. $\square$

By the same method of penalization, we can get the comparison theorem of $\mathcal{E}_t^{g,\phi}(\xi)$.

Proposition 2.2. *Let $\phi(t, y, z)$ be a function: $[0, T] \times R \times R^d \to R^+$, then under the same assumptions as above proposition, we have*

$$\mathcal{E}_t^{g,\phi}(\xi) \leq \mathcal{E}_t^{g,\phi}(\eta)$$

for any $\xi, \eta \in L_T^2(R)$ when $P(\eta \geq \xi) = 1$.

3 Continuous dependence of the minimal solution with terminal value

Under the same assumptions as last section, for any $k \in R$, we define the k-level set of $\mathcal{E}_0^{g,\phi}(\xi)$ as

$$A_k \triangleq \{\xi \in L_T^2(R) | \mathcal{E}_0^{g,\phi}(\xi) \leq k\}.$$

For such sets, we have the following result.

Lemma 3.1. *For any $k \in R$, the set A_k is closed in $L_T^2(R)$-norm.*

Proof Suppose a sequence $\{\xi_n, n = 1, 2 \cdots\} \subset A_k$ converges under norm to some $\xi \in L_T^2(R)$. For any ξ_n, we take $y_0^m(\xi_n)$ as in proposition 2.1. Since $y_0^m(\xi_n)$ converges increasingly to $\mathcal{E}_0^{g,\phi}(\xi_n) \leq k$ as $m \to \infty$, $y_0^m(\xi_n) \leq k$ holds for any n and m.

For any fixed m, take $g_m = g + m\phi$, by the continuous dependence property of g_m-solution, we have $y_0^m(\xi_n) \to y_0^m(\xi)$ as $n \to \infty$ and $y_0^m(\xi) \leq k$ is obtained for any m. Again, for the fixed $\xi \in L_T^2(R)$, $y_0^m(\xi) \to \mathcal{E}_0^{g,\phi}(\xi)$ as $m \to \infty$. Thus one has $\mathcal{E}_0^{g,\phi}(\xi) \leq k$, this means A_k is closed under norm in $L_T^2(R)$. $\square$

Let $\varphi(\xi) = \mathcal{E}_0^{g,\phi}(\xi)$, under our assumption, by proposition 2.1, it is a convex functional on its domain of definition, which is obviously convex in $L_T^2(R)$.

By lemma 3.1, $\varphi(\xi) = \mathcal{E}_0^{g,\phi}(\xi)$ is lower semi-continuous on its domain of definition, but it is well-known that a convex functional is continuous if and only if it is lower semi-continuous, then we have the following theorem.

Theorem 3.1. *Suppose the generator function $g(t, y, z)$ and constraint function $\phi(t, y, z)$ are both convex and uniformly Lipschitz in (y, z), $g(., 0, 0)\phi(., 0, 0) \in H_T^2(R)$, then $\mathcal{E}_0^{g,\phi}(\xi)$ is a continuous functional in its domain of definition.*

Next we want to investigate the continuous dependence property of $\mathcal{E}_t^{g,\phi}(\xi)$ when $t \neq 0$. For this aim, we suppose $\{\xi_n, n = 1, 2 \cdots\}$ is a sequence in the domain of definition of $\mathcal{E}_t^{g,\phi}(\xi)$.

Theorem 3.2. *Suppose the generator function $g(t, y, z)$ and constraint function $\phi(t, y, z)$ are both convex and uniformly Lipschitz in (y, z), $g(., 0, 0), \phi(., 0, 0) \in H_T^2(R)$.Then for random variables $\xi_n, \xi \in L_T^2(R)$, we have*

$$E|\mathcal{E}_t^{g,\phi}(\xi_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2 \to 0$$

if $E|\xi_n - \xi|^2 \to 0$ as $n \to \infty$.

Proof Let $\overline{\xi}_n = \xi_n \bigvee \xi$, $\underline{\xi}_n = \xi_n \bigwedge \xi$, by assumption, $\overline{\xi}_n$, $\underline{\xi}_n \in L_T^2(R)$ and they both converge to ξ in the space $L_T^2(R)$ with norm.

First, we consider $\alpha_n \triangleq E|\mathcal{E}_t^{g,\phi}(\overline{\xi}_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2$ and show that $\alpha_n \to 0$ as $n \to \infty$.

In fact, for any fixed $\xi \in L_T^2(R)$, by proposition 2.1, the functional $\varphi(\eta) \triangleq E|\mathcal{E}_t^{g,\phi}(\eta) - \mathcal{E}_t^{g,\phi}(\xi)|^2$ is convex on the set $K = \{\eta \in L_T^2(R) | \eta \geq \xi \quad a.s\}$. It is also easy to show by the same technique used in lemma 3.1 that $\varphi(\eta)$ is lower semi-continuous on K, thus $\alpha_n \to 0$ comes true.

Secondly, let $\beta_n \triangleq E|\mathcal{E}_t^{g,\phi}(\underline{\xi}_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2$ and we can prove $\beta_n \to 0$ as $n \to \infty$.

This time let us consider the functional $\varphi(\eta) \triangleq E|\mathcal{E}_t^{g,\phi}(\eta) - \mathcal{E}_t^{g,\phi}(\xi)|^2$ on the convex set $\tilde{K} = \{\eta \in L_T^2(R) | \eta \leq \xi \quad a.s\}$. For any sequence $\{\eta_n \in \tilde{K}, n = 1, 2, \cdots\}$ which converges to $\eta \in \tilde{K}$, we can show that $\varphi(\eta) \geq c$ if $\varphi(\eta_n) \geq c$ for all n. In fact, let $y_t^m(\eta_n)$ be the approximating sequence of $\mathcal{E}_t^{g,\phi}(\eta_n)$ as in proposition 2.1, that is $y_t^m(\eta_n) \leq \mathcal{E}_t^{g,\phi}(\eta_n)$ and converges increasingly to $\mathcal{E}_t^{g,\phi}(\eta_n)$ as $m \to \infty$. Since $\eta_n \leq \xi$ $a.s$, thus

$E|y_t^m(\eta_n)-\mathcal{E}_t^{g,\phi}(\xi)|^2 \geq E|\mathcal{E}_t^{g,\phi}(\eta_n)-\mathcal{E}_t^{g,\phi}(\xi)|^2 \geq c$ for any n and m. For any fixed m, by continuous dependence property of unconstrained BSDE, $E|y_t^m(\eta_n) - y_t^m(\eta)|^2 \to 0$ as $n \to \infty$, this gives $E|y_t^m(\eta) - \mathcal{E}_t^{g,\phi}(\xi)|^2 \geq c$ for any m, it is then easy to conclude that $E|\mathcal{E}_t^{g,\phi}(\eta) - \mathcal{E}_t^{g,\phi}(\xi)|^2 \geq c$ by monotone convergence theorem.

Now suppose on the contrary, if $\beta_n \nrightarrow 0$ as $n \to \infty$, then there is some subsequence of $\{\underline{\xi_n}, n = 1, 2, \cdots\}$ (for convenience, we still denote it as $\{\underline{\xi_n}, n = 1, 2, \cdots\}$), such that $\beta_n \triangleq E|\mathcal{E}_t^{g,\phi}(\underline{\xi}_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2 \geq \delta$ for some $\delta > 0$. But if we take $\underline{\xi}_n$ as η_n in the above argument, it will be a contradiction noting that $\underline{\xi}_n \to \xi \in \tilde{K}$.

At last, by the comparison theorem of proposition 2.2, we have $\mathcal{E}_t^{g,\phi}(\underline{\xi}_n) \leq \mathcal{E}_t^{g,\phi}(\xi_n) \leq \mathcal{E}_t^{g,\phi}(\overline{\xi}_n)$ and

$$E|\mathcal{E}_t^{g,\phi}(\xi_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2 \leq \max\{E|\mathcal{E}_t^{g,\phi}(\overline{\xi}_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2, E|\mathcal{E}_t^{g,\phi}(\underline{\xi}_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2\},$$

thus complete our proof. □

From the proof of theorem 3.2, we have following corollary for general generator function and constraint function,

Corollary 3.1. *Suppose the generator function $g(t, y, z)$ and constraint function $\phi(t, y, z)$ are both uniformly Lipschitz in (y, z) and $g(., 0, 0), \phi(., 0, 0) \in H_T^2(R)$. If $\{\xi_n \in L_T^2(R),\ n = 1, 2, \cdots\}$ is a sequence which converges to $\xi \in L_T^2(R)$ with $\xi_n \leq \xi$ a.s for any n, then we have*

$$E|\mathcal{E}_t^{g,\phi}(\xi_n) - \mathcal{E}_t^{g,\phi}(\xi)|^2 \to 0 \quad \forall t \in [0, T].$$

The proof is just a restatement of the second part of the proof of theorem 3.2.

Remark 3.1. *El Karoui etc.[2] proved the existence of solution of reflected backward stochastic differential equation(shortly RBSDE) reflected by $S_t (0 \leq t \leq T)$, it is just the smallest g-supersolution or minimal solution with the constraint $\phi(t, y_t, z_t) = (y_t - S_t)^- = 0$. The domain of definition of the RBSDE is $\{\xi \in L_T^2(R)|\xi \geq S_T, a.s\}$, it is a convex closed set in $L_T^2(R)$. By priori estimation, the minimal solution is uniformly continuous with terminal value.*

Remark 3.2. *Note that in El Karoui etc.[2], the continuity of $\mathcal{E}_t^{g,\phi}(\xi)$ for any $t \in [0, T]$ is unform and the generator function need not to be convex. Furthermore, under the assumption of $\mathcal{E}_0^{g,\phi}(\xi) < \infty$ for all $\xi \in L_T^2(R)$, with the constraint function $\phi(t, y, z)$ being the distance function of $z/\sigma y$ from a convex closed set in R^n, Karatzas and Shreve[4] proved that $\mathcal{E}_0^{g,\phi}(\xi)$ can be represent as a supremum of a family of linear functional on $L_T^2(R)$, and then by the resonance theorem, $\mathcal{E}_0^{g,\phi}(\xi)$ is also uniformly continuous with ξ, although the constraint function $\phi(t, y, z)$ is not convex. Inspired by these facts, we conjecture more general continuous dependence theorem may exist.*

References

[1] Civitanic,J., Karatzas,I., Soner,H.M., 1998. Backward stochastic differential equation with constraints on the gains-process. The Annals of Probability, Vol.26, No.4, 1522-1551.

[2] El Karoui,N.,Kapoudjian,C.,Pardoux,E.,Peng,S.,Quenez,M.C., 1997. Reflected solution of backward SDE and related obstacle prolem for PDEs. Ann.Probab. 25(2), 702-737.

[3] El Karoui,N.,Peng,S.,Quenez,M.C., 1997. Backward stochastic differential equation in finance. Mathematical Finance, 7(1), 1-71.

[4] Karatzas,I., Shreve,S.E., 1998. Methods of mathematical finance. Springer Verlag New York, Inc.

[5] Pardoux,E.,Peng,S., 1990. Adapted solution of a backward stochastic differential equation. Systems and control letters 14, 55-62.

[6] Peng,S., 1999. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type. Probab.Theory Relat.Fields. 113, 473-499.