Ergodic Description of STIT Tessellations

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Abstract

Let $(Y_t : t > 0)$ be the STIT tessellation process. We show that for all polytopes W with nonempty interior and all $a > 1$, the renormalized random sequence $(a^n Y_{a^n} : n \in \mathbb{Z})$ induced in W, is a finitary factor of a Bernoulli shift. As a corollary we get that the renormalized continuous time process $(a^t Y_{a^t} : t \in \mathbb{R})$ induced in W is a Bernoulli flow.

1 Introduction and main results

1.1 Introduction

Let $Y = (Y_t : t > 0)$ be the STIT tessellation process, which is a Markov process taking values in the space of tessellations of the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$. The process Y is spatially stationary (that is its law is invariant under translations of the space) and on every polytope with nonempty interior W (called a window) the induced tessellation process, which is denoted $Y \wedge W = (Y_t \wedge W : t > 0)$, is a pure jump process. The process Y was firstly constructed in [8] and in Subsection 1.4 we give a brief construction and recall some of its main properties.

Our results are stated in Subsection 1.6. In Theorems 1.1 and 1.2 we show that if $a > 1$ then the renormalized process $\mathcal{Z} = (\mathcal{Z}_t := a^t Y_{a^t} : t \in \mathbb{R})$ is a stationary (in time) Markov process and its restriction to a window $\mathcal{Z} \wedge W = (\mathcal{Z}_t \wedge W : t \in \mathbb{R})$ is mixing. In Theorem 1.3 we give an ergodic description of the discrete process $\mathcal{Z}^d \wedge W = (\mathcal{Z}_n \wedge W = a^n Y_{a^n} \wedge W : n \in \mathbb{Z})$ on a window W, where $\mathbb{Z}$ is the set of integers. There we show that $\mathcal{Z}^d \wedge W$ it is a finitary factor of a (generalized) Bernoulli shift with null anticipating length. We conclude in Corollaries 1.4 and 1.5 that $\mathcal{Z}^d \wedge W$ is isomorphic to a Bernoulli shift of infinite entropy and that $\mathcal{Z} \wedge W$ is isomorphic to a Bernoulli flow of infinite entropy defined on a Lebesgue probability space.

The proofs of these results are done in Section 2. We use strongly the fact that we are restricting the renormalized process to a window, indeed our main technical result, Lemma 2.1, gives the probability that in a nested sequence of decreasing windows the tessellation is reduced to the boundaries of the windows.

We need some background on Lebesgue probability spaces and on some elements on ergodic theory which are respectively given in Subsection 1.2.1 and Section 1.5.

1.2 Notation and some measurability facts

1.2.1 Notation and product spaces

For a set $\mathcal{X}$ we denote by $\mathcal{B}(\mathcal{X})$ a $\sigma-$field on $\mathcal{X}$ and the couple $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is called a measurable space. If $\mathcal{X}' \in \mathcal{B}(\mathcal{X})$ then we will always endow $\mathcal{X}'$ with the trace (or induced) $\sigma-$field $B(\mathcal{X}') = \{B \cap \mathcal{X}' : B \in B(\mathcal{X})\}$. When ν is a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, we will denote by $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu)$ the completed probability space, where completed means that we have added to $\mathcal{B}(\mathcal{X})$ all the negligible sets with respect to ν. We will always consider completed probability spaces, even if we do not explicit it. Sometimes the completed $\sigma-$field with respect to ν is denoted by $\mathcal{B}(\mathcal{X})_\nu$ but often (as we do here) it is not written to avoid overburden notation.

Let $(\mathcal{X}_i, \mathcal{B}(\mathcal{X}_i))$, $i \in L$, be a collection of measurable spaces. The (Cartesian) product space $\prod_{i \in L} \mathcal{X}_i$ will be endowed with the product σ-field $\otimes_{i \in L} \mathcal{B}(\mathcal{X}_i)$, which is the smallest $\sigma-$field containing the family of cylinders. We recall that a cylinder is a set of the form $C = \prod_{i \in J} A_i$ with $A_i \in \mathcal{B}(\mathcal{X}_i)$ and J a finite subset of L. We call $(\prod_{i \in L} \mathcal{X}_i, \otimes_{i \in L} \mathcal{B}(\mathcal{X}_i))$ the product measurable space.

Let $(\mathcal{X}_i, \mathcal{B}(\mathcal{X}_i), \nu_i)$, $i \in L$, be a family of probability spaces. The product measure $\otimes_{i \in L} \nu_i$ is such that on each cylinder $C = \prod_{i \in J} A_i$ it takes the value $(\otimes_{i \in L} \nu_i)(C) = \prod_{i \in J} \nu_i(A_i)$. We call $(\prod_{i \in L} \mathcal{X}_i, \otimes_{i \in L} \mathcal{B}(\mathcal{X}_i), \otimes_{i \in L} \nu_i)$ the product probability space.

When $(\mathcal{X}_i, \mathcal{B}(\mathcal{X}_i)) = (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ for all $i \in L$, instead of $\prod_{i \in L} \mathcal{X}_i$ and $\otimes_{i \in L} \mathcal{B}(\mathcal{X}_i)$ we simply put $\mathcal{X}^L$ and $\mathcal{B}(\mathcal{X})^{\otimes L}$. And if $\nu_i = \nu$ for all $i \in L$, the product probability measure $\otimes_{i \in L} \nu_i$ is simply written as $\nu^{\otimes L}$.

Let $(\mathcal{X}, d)$ be a metric space. It is called a Polish space if it is a complete separable metric space. For instance if $(\mathcal{X}, d)$ is a compact metric space then it is a Polish space. Let $(\mathcal{X}_i, d_i)$, $i \in L$, be a family of Polish spaces. When L is countable, the product space $\mathcal{X} = \prod_{i\in L} \mathcal{X}_i$ can be endowed with a metric $d_{\mathcal{X}}$ such that it is a Polish space and the topology generated by $d_{\mathcal{X}}$ is the product topology. It suffices to give $d_{\mathcal{X}}$ when $L = \mathbb{N}$, being $\mathbb{N} = \{1, 2, ..\}$ the set of positive integers. It is easily checked that the metric $d_{\mathcal{X}}(x, y) = \sum_{i\in\mathbb{N}} 2^{-i} \min(d_i(x_i, y_i), 1)$ for $x = (x_i : i \in \mathbb{N})$, $y = (y_i : i \in \mathbb{N}) \in \mathcal{X}$ does the job.

When $\mathcal{X}$ is a topological space we will reserve the notation $\mathcal{B}(X)$ to the Borel $\sigma-$field unless the contrary is explicitly specified. Let $(\mathcal{X}_i, \mathcal{B}(\mathcal{X}_i))$, $i \in L$, be a family of Polish spaces endowed with their Borel $\sigma-$fields. Consider the product space $\mathcal{X} = \prod_{i\in L} \mathcal{X}_i$. Then, on the space $\mathcal{X}$ we can consider both the Borel $\sigma-$field and the product $\sigma-$field. When L is countable both $\sigma-$fields coincide, that is $\otimes_{i\in L}\mathcal{B}(\mathcal{X}_i) = \mathcal{B}(\mathcal{X})$. This is not the case when L is non-countable, in this case the $\sigma-$fields are different. In fact the singletons belong to the Borel $\sigma-$field but not to the product $\sigma-$field. In the case L is non-countable, we will denote by $\widehat{\mathcal{B}}(\mathcal{X})$ the product $\sigma-$field to distinguish it from the Borel $\sigma-$field.

Let $\mathcal{X}$ be a topological space. For a set $B \subseteq \mathcal{X}$ we denote by $\text{int}(B)$ its interior, by $\overline{B}$ its closure and by $\partial B = \overline{B} \setminus \text{int}(B)$ its boundary.

1.2.2 Measurability facts

We recall that a Lebesgue probability space (or a standard probability space) is a probability space isomorphic to the unit interval endowed with a probability measure which is a convex combination of the Lebesgue measure and a pure atomic measure ('pure atomic' means that the measure is concentrated on points). Equivalent definitions and properties on these spaces can be found in Appendix 1 in [1], Appendix A in [13], Chapter 3 in [3] and [2]. In particular in Theorem $2-3$ in [2] it is shown that if $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is a Polish space endowed with its Borel $\sigma-$field and ν is a probability measure on it, then the completed probability space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu)$ is Lebesgue. Hence, if $\mathcal{X}' \in \mathcal{B}(\mathcal{X})$ is a Borel set of a Polish space and ν' is a probability measure on $(\mathcal{X}', \mathcal{B}(\mathcal{X}'))$ the complete probability space $(\mathcal{X}', \mathcal{B}(\mathcal{X}'), \nu')$ is Lebesgue. Let L be a countable set and let $(\mathcal{X}_i, \mathcal{B}(\mathcal{X}_i), \nu_i)$, $i \in L$, be a countable family of Lebesgue probability spaces, then the product probability space $(\prod_{i\in L} \mathcal{X}_i, \otimes_{i\in L}\mathcal{B}(\mathcal{X}_i), \otimes_{i\in L}\nu_i)$ is also Lebesgue.

Let us introduce the Skorohod topology. Let $\mathbb{R}_+ = [0, \infty)$. Let $(\mathcal{X}, d)$ be a metric space. We denote by $D_{\mathcal{X}}(\mathbb{R}_+)$ the space of càdlag trajectories taking values in $\mathcal{X}$ with time in $\mathbb{R}_+$. We recall that càdlag means that the trajectories are right continuous and have left limits. The space $D_{\mathcal{X}}(\mathbb{R}_+)$ is endowed with the Skorohod topology (see [4] chapter 3), which is metrizable (see Corollary 5.5 in Chapter 3 in [4]). Let $d_{\text{Sk}}^{\mathcal{X}}$ be a metric generating the Skorohod topology. When $(\mathcal{X}, d)$ is a separable space we get that $(D_{\mathcal{X}}(\mathbb{R}_+), d_{\text{Sk}}^{\mathcal{X}})$ is also a separable space (see Theorem 5.6 in Chapter 3 in [4]). We denote by $\mathcal{B}(D_{\mathcal{X}})$ the Borel $\sigma-$field associated to $(D_{\mathcal{X}}(\mathbb{R}_+), d_{\text{Sk}}^{\mathcal{X}})$. From Proposition 7.1 in [4] we get that the class of cylinders in $D_{\mathcal{X}}(\mathbb{R}_+)$ is a semi-algebra generating $\mathcal{B}(D_{\mathcal{X}})$. We will also need the following straightforward extension to processes with time in $\mathbb{R}$. Let $D_{\mathcal{X}}(\mathbb{R})$ be the space of càdlag trajectories with time in $\mathbb{R}$ taking values in $\mathcal{X}$. The Skorohod topology, the metric, the associated Borel $\sigma-$field and all the previous notions are analogously defined. We point out that the results previously formulated also hold, in particular the family of cylinders is a generating semi-algebra. We continue to denote the metric and the associated Borel $\sigma-$field by $d_{\text{Sk}}^{\mathcal{X}}$ and $\mathcal{B}(D_{\mathcal{X}})$ respectively, because we want to avoid overburden notation and because there will be no confusion from the context.

1.3 The space of tessellations

We will consider tessellations on $\mathbb{R}^\ell$, with $\ell \geq 1$.

A polytope is the compact convex hull of a finite point set, and we will always assume that it has nonempty interior. A locally finite covering of polytopes is a countable family of polytopes whose union is $\mathbb{R}^\ell$ and all bounded sets can only intersect a finite number of them. These polytopes will be called cells.

A tessellation T is a locally finite covering of polytopes with disjoint interiors. We denote by $\mathbb{T}$ the space of tessellations of $\mathbb{R}^\ell$. We define the boundary of a tessellation as the union of the boundaries of its cells. That is, for $T \in \mathbb{T}$ we define $\partial T := \bigcup_{C \in T} \partial C$

Let $\mathbb{F}$ be the family of closed sets of $\mathbb{R}^\ell$ endowed with the Fell topology $\mathcal{T}$, for definition and properties see [15], Subsections 12.2 and 12.3. We denote by $\mathbb{F}' = \mathbb{F} \setminus \{\emptyset\}$ the class of nonempty closed sets. We have that $(\mathbb{F}, \mathcal{T})$ is a compact Hausdorff space with a countable base, so it is metrizable and d denotes a metric on $\mathbb{F}$ whose topology is $\mathcal{T}$. Since $(\mathbb{F}, d_{\mathbb{F}})$ is a compact metric space, it is a Polish space (see Subsection 1.2.1). The set $\mathbb{F}'$ is an open set in $\mathcal{T}$. Let $\mathcal{T}'$ be the restriction of $\mathcal{T}$ to $\mathbb{F}'$, then $(\mathbb{F}', \mathcal{T}')$ is a locally compact Hausdorff space with a countable base.

Let us denote by $\mathbb{F}(\mathbb{F}')$ the family of closed sets of $\mathbb{F}'$. We endow it with the Fell topology and denote by $\mathcal{B}(\mathbb{F}(\mathbb{F}'))$ the associated Borel $\sigma-$field. Each tessellation $T \in \mathbb{T}$, as a countable collection of polytopes is an element of

$\mathbb{F}(\mathbb{F}')$, so $\mathbb{T} \subset \mathbb{F}(\mathbb{F}')$. Furthermore in Lemma 10.1.2. in [15] it was shown that $\mathbb{T} \in \mathcal{B}(\mathbb{F}(\mathbb{F}'))$.

We will often enumerate the family of countable cells of a tessellation $T \in \mathbb{T}$ in a prescribed and measurable form as $T = \{C(T)^l : l = 1, \ldots\}$. For a tessellation T such that the origin 0 is in the interior of its cell, the first cell $C(T)^1$ in the enumeration will be the one containing 0.

Let $W \subset \mathbb{R}^\ell$ be a fixed polytope with nonempty interior, we call it a window. As before, $\mathbb{F}_W$ denotes the set of closed subsets of W and we endow it with the Fell topology, and we put $\mathbb{F}'_W = \mathbb{F}_W \setminus \{\emptyset\}$. A tessellation R in W is the collection of all the cells of a locally finite countable covering of W by polytopes with disjoint interiors. We denote by $\mathbb{T}_W$ the space of tessellations of W. Since W is compact the locally finiteness property implies that every $R \in \mathbb{T}_W$ is constituted by a finite set of cells, and we will denote by $|R|$ the number of the cells. Each $R \in \mathbb{T}_W$ is an element of $\mathbb{F}(\mathbb{F}'_W)$. As before we can endow $\mathbb{F}(\mathbb{F}'_W)$ with the Fell topology which is metrizable and we denote by $d_{\mathbb{F}_W}$ a metric generating this topology. The space $(\mathbb{F}(\mathbb{F}'_W), d_{\mathbb{F}_W})$ is a compact metric space, we denote by $\mathcal{B}(\mathbb{F}(\mathbb{F}'_W))$ its Borel $\sigma-$field. We have $\mathbb{T}_W \in \mathcal{B}(\mathbb{F}(\mathbb{F}'_W))$, in fact the proof of Lemma 10.1.2. in [15] also works in this case. As before we also define the boundary of a tessellation $R \in \mathbb{T}_W$ by the union of the boundaries of its cells, $\partial R := \bigcup_{C \in R} \partial C$. The trivial tessellation R in $\mathbb{T}_W$ has a unique cell which is $R = \{W\}$, and so its boundary coincides with the boundary of the window $\partial R = \partial W$.

The tessellations in $\mathbb{T}_W$ can be also seen as induced from a tessellation in $\mathbb{T}$. In fact each $T \in \mathbb{T}$ induces a tessellation $T \wedge W$ in $\mathbb{T}_W$ given by the family of cells $T \wedge W = \{C \cap W : C \in T,\ \text{int}(C \cap W) \neq \emptyset\}$ (note that this set is finite by the locally finiteness property). Observe that $T \wedge W = \{W\}$ is the trivial tessellation when $W \subseteq C$ for some cell $C \in T$. When the windows W, W' are such that $W \subseteq W'$, every $Q \in \mathbb{T}_{W'}$ defines in the same way as before the tessellation $Q \wedge W \in \mathbb{T}_W$. In this case $Q \wedge W = \{W\}$ if $W \subseteq C$ for some cell $C \in Q$.

For $a \in \mathbb{R}$ and $B \subseteq \mathbb{R}^\ell$, we put $aB = \{ax : x \in B\}$. Observe that if W is a window and $a \neq 0$ then aW is also a window. For $T \in \mathbb{T}$ and $a \in \mathbb{R} \setminus \{0\}$ the tessellation aT is given by the set of cells $aT = \{aC : C \in T\}$. Analogously for a window W and a tessellation $Q \in \mathbb{T}_W$, the tessellation $aQ \in \mathbb{T} \wedge aW$ is given by $aQ = (aC : C \in Q)$. If W is a window containing 0, $a > 1$, and $Q \in \mathbb{T}_W$, the tessellation aQ belongs to $\mathbb{T}_{aW}$ and $W \subset aW$, so we can take the restriction $aQ \wedge W \in \mathbb{T}_W$.

Since $\mathbb{F}(\mathbb{F}')$ is a compact metric space, for a probability measure ν defined on $(\mathbb{F}(\mathbb{F}'), \mathcal{B}(F(\mathbb{F}')))$, the completed probability space $((\mathbb{F}(\mathbb{F}'), \mathcal{B}(F(\mathbb{F}')), \nu)$

is Lebesgue, see Subsection 1.2.2. Analogously, for any probability measure ν_W defined on $(\mathbb{F}(\mathbb{F}'_W), \mathcal{B}(F(\mathbb{F}'_W)))$, the completed probability space $((\mathbb{F}(\mathbb{F}'_W), \mathcal{B}(F(\mathbb{F}'_W)), \nu_W)$ is Lebesgue. Since $\mathbb{T} \in \mathcal{B}(\mathbb{F}(\mathbb{F}'))$ its associated Borel σ-field is $\mathcal{B}(\mathbb{T}) = \{B \cap \mathbb{T} : B \in \mathcal{B}(\mathbb{F}(\mathbb{F}'))\}$ and for any probability measure ν defined on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ the completed probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)$ is Lebesgue. Analogously for $\mathbb{T}_W$. We have $\mathcal{B}(\mathbb{T}_W) = \{B \cap \mathbb{T} : B \in \mathcal{B}(\mathbb{F}(\mathbb{F}'_W))\}$ and for any probability measure ν_W defined on $(\mathbb{T}_W, \mathcal{B}(\mathbb{T}_W))$ the completed probability space $(\mathbb{T}_W, \mathcal{B}(\mathbb{F}_W), \nu_W)$ is Lebesgue. Also for any countable set L the product probability spaces $(\mathbb{T}^L, \mathcal{B}(\mathbb{T}))^{\otimes L}, \nu^{\otimes L})$ and $(\mathbb{T}_W^L, (\mathcal{B}(\mathbb{T}_W))^{\otimes L}, \nu_W^{\otimes L})$ are Lebesgue.

1.4 The STIT tessellation process

Let us construct $Y = (Y_t : t > 0)$ a STIT tessellation process (see [8], [7]), which is a Markov processes where each marginal Y_t takes values in $\mathbb{T}$. The law of Y only depends on a (non-zero) σ-finite and translation invariant measure Λ on the space of hyperplanes $\mathcal{H}$ in $\mathbb{R}^\ell$. It is assumed that the support set of Λ satisfies that there is no line in $\mathbb{R}^\ell$ such that all hyperplanes of the support are parallel to it (in order to obtain a.s. bounded cells in the constructed tessellation). For all sets $W \subseteq \mathbb{R}^\ell$ put

$$[W] = \{H \in \mathcal{H} : H \cap W \neq \emptyset\}.$$

The assumptions imply $0 < \Lambda([W]) < \infty$ for every window W. The translation invariance of Λ yields (see e.g. [15], Theorem 4.4.1.)

$$\Lambda([cW]) = c\,\Lambda([W]) \quad \text{for all } c > 0. \tag{1}$$

Denote by $\Lambda_{[W]}$ the restriction of Λ to $[W]$ and by $\Lambda_W = \Lambda([W])^{-1}\Lambda_{[W]}$ the normalized probability measure.

Let us first construct $Y \wedge W = (Y_t \wedge W : t \geq 0)$ for a window W. We note that even if for $t = 0$ the object Y_0 does not exist as a tessellation of the whole $\mathbb{R}^\ell$ we define $Y_0 \wedge W = \{W\}$ the trivial tessellation for the window W. Let us take two independent families of independent random variables $D = (d_{n,m} : n, m \in \mathbb{N})$ and $\tau = (\tau_{n,m} : n, m \in \mathbb{N})$, where each $d_{n,m}$ has distribution Λ_W and each $\tau_{n,m}$ is exponentially distributed with parameter 1. We define a sequence of increasing random times $(S_n : n \geq 0)$ and a sequence of random tessellations $(Y_{S_n} \wedge W : n \geq 0)$ with, $S_0 = 0$ and $Y_0 \wedge W = \{W\}$. The process $Y \wedge W$ will satisfy

$$Y_t \wedge W = Y_{S_n} \wedge W, \;\; t \in [S_n, S_{n+1}). \tag{2}$$

The definition of $(S_n : n \geq 0)$ and $(Y_{S_n} \wedge W : n \geq 0)$ is done by an inductive procedure. Let $\{C_t^1, ..., C_t^{l_t}\}$ be the cells of $Y_{S_n} \wedge W$, we put

$$S_{n+1} = S_n + \tau(Y_{S_n} \wedge W) \text{ where } \tau(Y_{S_n} \wedge W) = \min\{\tau_{n,l}/\Lambda([C_t^l]) : l = 1, ..., l_t\}.$$

Let l_0 be such that $\tau_{n,l_0}/\Lambda([C_t^{l_0}]) = \tau(Y_{S_n} \wedge W)$ (it is a.s. uniquely defined). We denote by m the first index such that $d_{n+1,m} \in [C_t^{l_0}]$. The variable $d_{n+1,m}$ is distributed as $\Lambda_{C_t^{l_0}}$. The tessellation $Y_{S_{n+1}} \wedge W$ is defined as the one whose cells are $\{C_t^l : l \neq l_0\} \cup \{C_1', C_2'\}$ where C_1', C_2' is the partition of $C_t^{l_0}$ by the hyperplane $d_{n+1,m}$.

In particular, since S_1 is exponentially distributed with parameter $\Lambda([W])$,

$$\mathbb{P}(\partial(Y_t \wedge W) \cap \text{int} W = \emptyset) = \mathbb{P}(Y_t \wedge W = \{W\}) = \mathbb{P}(Y_t \wedge W = Y_0 \wedge W) = e^{-\Lambda([W])}. \tag{3}$$

The process $Y \wedge W$ is a Markov process. Also, this construction yields a law that is consistent with respect to W, that is if W and W' are windows and $W \subseteq W'$, then $(Y \wedge W') \wedge W \sim Y \wedge W$, where $\sim$ denotes the identity of distributions. A proof of consistency showing the existence of the law of the process Y was given in [8].

Since Λ is translation invariant, without loss of generality we can always use a window W with the origin 0 in its interior and we can also assume that $\mathbb{P}-$a.e. at all times the origin belongs to the interior of the its cell. This cell is called the $0-$cell.

From (2) it follows that for every window W the process $Y \wedge W$ is a pure jump Markov process with càdlag trajectories, so its trajectories take values in the space $D_{\mathbb{T}_W}(\mathbb{R}_+)$. Recall that $D_{\mathbb{T}_W}(\mathbb{R}_+)$ is endowed with the Skorohod topology generated by the metric $d_{\text{Sk}}^{\mathbb{T}_W}$ (see Subsection 1.2.2). Since $(\mathbb{T}_W, d_{\mathbb{F}_W})$ is a separable space, $(D_{\mathbb{T}_W}(\mathbb{R}_+), d_{\text{Sk}}^{\mathbb{T}_W})$ is also separable. $\mathcal{B}(D_{\mathbb{T}_W})$ denotes the Borel $\sigma-$field associated to $(D_{\mathbb{T}_W}(\mathbb{R}_+), d_{\text{Sk}}^{\mathbb{T}_W})$. As before $D_{\mathbb{T}_W}(\mathbb{R})$ is the space of càdlag trajectories taking values in $\mathbb{T}_W$ with time in $\mathbb{R}$. The respective metric and Borel $\sigma-$field continue to be written by $d_{\text{Sk}}^{\mathbb{T}_W}$ and $\mathcal{B}(D_{\mathbb{T}_W})$.

By technical reasons it is useful to consider the closure $\overline{\mathbb{T}_W}$ of $\mathbb{T}_W$ in $\mathbb{F}(\mathbb{F}'_W)$. The space $D_{\overline{\mathbb{T}_W}}(\mathbb{R}_+)$ can be also endowed with the Skorohod topology which is generated by a metric $d_{\text{Sk}}^{\overline{\mathbb{T}_W}}$. Since $(\overline{\mathbb{T}_W}, d_{\text{Sk}}^{\overline{\mathbb{T}_W}})$ is a Polish space, from Theorem 5.6 in Chapter 3 in [4] we get that $(D_{\overline{\mathbb{T}_W}}(\mathbb{R}_+), d_{\text{Sk}}^{\overline{\mathbb{T}_W}})$ is also a Polish space. $\mathcal{B}(D_{\overline{\mathbb{T}_W}})$ denotes its Borel $\sigma-$field. Hence, for a window W we can also consider that the trajectories of the Markov process $Y \wedge W$ take values in the Polish space $(D_{\overline{\mathbb{T}_W}}(\mathbb{R}_+), d_{\text{Sk}}^{\overline{\mathbb{T}_W}})$. Also the extension $D_{\overline{\mathbb{T}_W}}(\mathbb{R})$ to processes with times in $\mathbb{R}$ is needed, all the previous definitions and results hold and we also denote by $\mathcal{B}(D_{\overline{\mathbb{T}_W}})$ the associated Borel $\sigma-$field.

1.4.1 Independent increments relation

It is useful to supply an independence relation on the increments of the Markov process Y which is written in terms of the following operation. For $T \in \mathbb{T}$ and $\vec{R} = (R^m : m \in \mathbb{N}) \in \mathbb{T}^{\mathbb{N}}$, we define the tessellation $T \boxplus \vec{R}$, referred to as the iteration of T and $\vec{R}$, by its set of cells

$$T \boxplus \vec{R} = (C(T)^k \cap C(R^k)^l : k=1,...; l=1,...; \mathrm{int}(C(T)^k \cap C(R^k)^l) \neq \emptyset).$$

So, we restrict R^k to the cell $C(T)^k$ and this is done for all $k = 1, \ldots$. The same definition holds when the tessellation and the sequence of tessellations are restricted to some window.

To state the independence relation of the increments of Y, we fix a copy of the random process Y and let $\vec{Y}' = (Y'^m : m \in \mathbb{N})$ be a sequence of independent copies of Y, all of them being also independent of Y. In particular $Y'^m \sim Y$. For a fixed time $s > 0$, we set $\vec{Y}'_s = (Y'^m_s : m \in \mathbb{N})$. Then, from the construction of Y it is straightforward to see that the following property holds

$$Y_{t+s} \sim Y_t \boxplus \vec{Y}'_s \text{ for all } t, s > 0 \,. \tag{4}$$

This relation was first stated in Lemma 2 in [8]. Moreover the construction done in the proof of this Lemma 2 also shows the following relation. Let $\vec{Y}'^{(i)}$, $i = 1, \ldots, j$, be a sequence of j independent copies of $\vec{Y}'$, which are also independent of Y. Then, for all $0 < s_1 < ... < s_j$ and all $t > 0$ we have

$$(Y_{t+s_1}, ..., Y_{t+s_j}) \sim (Y_t \boxplus \vec{Y}'^{(1)}_{s_1}, ..., (((Y_t \boxplus \vec{Y}'^{(1)}_{s_1}) \boxplus ....) \boxplus \vec{Y}'^{(j)}_{s_j - s_{j-1}})). \tag{5}$$

1.5 Elements of ergodic theory

A dynamical system $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ is such that $(\Omega, \mathcal{B}(\Omega), \mu)$ is a probability space and $\psi : \Omega \to \Omega$ is a measure-preserving measurable transformation, that is $\mu(\psi^{-1}(B)) = \mu(B) \ \ \forall B \in \mathcal{B}(\Omega)$. When $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ and $(\Omega', \mathcal{B}(\Omega'), \mu', \psi')$ are two dynamical systems, the measurable map $\varphi : \Omega \to \Omega'$ is called a factor map if it satisfies $\varphi \circ \psi = \psi' \circ \varphi \ \ \mu-$a.e. and $\mu(\varphi^{-1}(B')) = \mu'(B') \ \ \forall B' \in \mathcal{B}(\Omega')$. If a factor map φ is one-to-one $\mu-$a.e., onto $\mu'-$a.e. and φ^{-1} is also measurable, then it is called an isomorphism and the dynamical systems $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ and $(\Omega', \mathcal{B}(\Omega'), \mu', \psi')$ are called isomorphic. When $(\Omega, \mathcal{B}(\Omega), \mu)$ and $(\Omega', \mathcal{B}(\Omega'), \mu')$ are Lebesgue probability spaces, the measurability condition on φ^{-1} is not explicitly needed in the isomorphism requirements because it is implied by the other ones.

The dynamical system $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ is ergodic if $\mu(\psi^{-1}B \Delta B) = 0$ implies $\mu(B)\mu(B^c) = 0$ (where as usual we set $A \Delta B = (A \setminus B) \cup (B \setminus A)$). It is mixing if $\lim_{n\to\infty} \mu(\psi^{-n}A \cap B) = \mu(A)\mu(B)$ for all $A,\ B \in \mathcal{B}(\Omega)$. Mixing implies ergodicity. To avoid overburden notation the dynamical system $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ is usually denoted by (Ω, μ, ψ).

Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ be a measurable space (i.e. $\mathcal{S}$ is endowed with a $\sigma-$field $\mathcal{B}(\mathcal{S})$). Let $L = \mathbb{N}$ or $L = \mathbb{Z}$. The shift transformation $\sigma_{\mathcal{S}} : \mathcal{S}^L \to \mathcal{S}^L$ defined by $\sigma_{\mathcal{S}}(x_n : n \in L) = (x_{n+1} : n \in L)$ is a measurable transformation. If the probability measure μ defined on $(\mathcal{S}^L, \mathcal{B}(\mathcal{S}^L))$ is preserved by $\sigma_{\mathcal{S}}$ then $(\mathcal{S}^L, \mathcal{B}(\mathcal{S}^L), \mu, \sigma_{\mathcal{S}})$ (or simply $(\mathcal{S}^L, \mu, \sigma_{\mathcal{S}})$) is a dynamical system called a shift system (or simply a shift). When $L = \mathbb{N}$ it is called a one-sided shift, and if $L = \mathbb{Z}$ then it is called a two-sided shift. An example of a two-sided shift is given by a stationary random sequence $\mathcal{Y}^d = (\mathcal{Y}_n : n \in \mathbb{Z})$ with state space $\mathcal{S}$. Indeed, if $\mu^{\mathcal{Y}^d}$ is the distribution of $\mathcal{Y}^d$ on $\mathcal{S}^{\mathbb{Z}}$, the stationary property of $\mathcal{Y}^d$ means that $\mu^{\mathcal{Y}^d}$ is $\sigma_{\mathcal{S}}-$invariant and so $(\mathcal{S}^L, \mu^{\mathcal{Y}^d}, \sigma_{\mathcal{S}})$ is a shift system.

Let us recall the Bernoulli property. Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}), \nu_{\mathcal{S}})$ be a probability space and $L = \mathbb{N}$ or $L = \mathbb{Z}$. Let $(\mathcal{S}^L, \mathcal{B}(\mathcal{S}^L), \nu_{\mathcal{S}}^{\otimes L})$ be the product probability space. The shift action $\sigma_{\mathcal{S}}$ preserves the product probability measure $\nu_{\mathcal{S}}^{\otimes L}$ and $(\mathcal{S}^L, \nu_{\mathcal{S}}^{\otimes L}, \sigma_{\mathcal{S}})$ is called a Bernoulli shift, it is two-sided when $L = \mathbb{Z}$ and one- sided when $L = \mathbb{N}$. In notation of [13] Part I, Section 9, $(\mathcal{S}^{\mathbb{Z}}, \nu_{\mathcal{S}}^{\otimes \mathbb{Z}}, \sigma_{\mathcal{S}})$ is called a generalized two-sided Bernoulli shift (the name generalized is because $\mathcal{S}$ is not necessarily a countable set). A Bernoulli shift is mixing (so ergodic). Let us assume that $(\mathcal{S}, \mathcal{B}(\mathcal{S}), \nu_{\mathcal{S}})$ is a Lebesgue probability space. Then the entropy $h(\sigma_{\mathcal{S}}, \nu_{\mathcal{S}}^{\otimes L})$ of the Bernoulli shift satisfies $h(\sigma_{\mathcal{S}}, \nu_{\mathcal{S}}^{\otimes L}) = H(\nu_{\mathcal{S}})$, where $H(\nu_{\mathcal{S}}) = \infty$ if $\nu_{\mathcal{S}}$ has a non-atomic part and

$$H(\nu_{\mathcal{S}}) = - \sum_{A \in \mathcal{A}(\nu_{\mathcal{S}})} \nu_{\mathcal{S}}(A) \log(\nu_{\mathcal{S}}(A))$$

when $\nu_{\mathcal{S}}$ is purely atomic and where $\mathcal{A}(\nu_{\mathcal{S}})$ denotes the set of its atoms (singletons of positive $\nu_{\mathcal{S}}-$measure). The Ornstein isomorphism theorem (see [10] and [11]) states that two-sided Bernoulli shifts (defined on Lebesgue probability spaces) having the same entropy are isomorphic.

Let us introduce what a finitary factor is. If $(\mathcal{S}^{\mathbb{Z}}, \nu_{\mathcal{S}}^{\otimes \mathbb{Z}}, \sigma_{\mathcal{S}})$ and $(\mathcal{S}'^{\mathbb{Z}}, \nu_{\mathcal{S}'}^{\otimes \mathbb{Z}}, \sigma_{\mathcal{S}'})$ are two two-sided Bernoulli shifts, the measurable map $\varphi : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}'^{\mathbb{Z}}$ is a finitary factor map if it is a factor map and $\nu_{\mathcal{S}}^{\otimes \mathbb{Z}}-$a.e. in $x = (x_n : n \in \mathbb{Z}) \in \mathcal{S}^{\mathbb{Z}}$ the coordinate $(\varphi(x))_n$ only depends on a finite sequence of values $(x_m : m \in [n - M(x), n + M'(x)])$. From $(\varphi(x))_n = (\sigma_{\mathcal{S}'}^n \circ \varphi(x))_0 = (\varphi \circ \sigma_{\mathcal{S}}^n(x))_0$ we get that the finitary property can be stated as: $\nu_{\mathcal{S}}^{\otimes \mathbb{Z}}-$a.e. in $x = (x_n : n \in \mathbb{Z}) \in \mathcal{S}^{\mathbb{Z}}$ the $0-$th coordinate $(\varphi(x))_0$ only depends on a finite sequence $(x_m : m \in [-M(x), M'(x)])$. We call $M(x)$ and $M'(x)$ the memory length and the anticipation length (for x) respectively. A finitary isomorphism can be defined in an analogous way. We note that when the state spaces $\mathcal{S}$ and $\mathcal{S}'$ are finite, a finitary factor is a.e. continuous (that is in a set of full measure the factor map is continuous when the product spaces are endowed with the product topologies). In [5] and [6] there was introduced a method to construct a finitary isomorphism between two Bernoulli shifts of the same entropy with finite state spaces $\mathcal{S}$ and $\mathcal{S}'$. In [14] the finitary relation is studied for topological Markov chains with finite state spaces.

A flow (or continuous time dynamical system) $(\Omega, \mathcal{B}(\Omega), \mu, (\psi^t : t \in \mathbb{R}))$ is such that $(\Omega, \mathcal{B}(\Omega), \mu)$ is a probability space and $\psi^t : \Omega \to \Omega$ is a measure-preserving measurable transformation for all $t \in \mathbb{R}$. All the previous notions can be extended from dynamical systems to flows, in particular ergodicity, mixing and isomorphism of flows. The shift flows are defined with respect to the shift transformations $\sigma^t(x_s : s \in \mathbb{R}) = (x_{s+t} : s \in \mathbb{R})$ for $t \in \mathbb{R}$. An example of a shift flow is given by a stationary random process $\mathcal{Y} = (\mathcal{Y}_t : t \in \mathbb{R})$ with state space $\mathcal{S}$. If $\mu^{\mathcal{Y}}$ is the distribution of $\mathcal{Y}$ on the product measurable space $(\mathcal{S}^{\mathbb{R}}, \widehat{\mathcal{B}}(\mathcal{S}^{\mathbb{R}}))$ then the stationary property of $\mathcal{Y}$ means that $\mu^{\mathcal{Y}}$ is $\sigma^t_{\mathcal{S}}-$invariant for all $t \in \mathbb{R}$ and so $(\mathcal{S}^{\mathbb{R}}, \widehat{\mathcal{B}}(\mathcal{S}^{\mathbb{R}}), \mu^{\mathcal{Y}}, (\sigma^t_{\mathcal{S}} : t \in \mathbb{R}))$ is a shift flow. In the case $(\mathcal{S}, d_{\mathcal{S}})$ is a metric space and the stationary random process $\mathcal{Y}$ has càdlag trajectories, let $\mu^{\mathcal{Y}}$ be the distribution of $\mathcal{Y}$ on $D_{\mathcal{S}}(\mathbb{R})$. The stationary property of $\mathcal{Y}$ means that $\mu^{\mathcal{Y}}$ is $\sigma^t_{\mathcal{S}}-$invariant for $t \in \mathbb{R}$ and so $(D_{\mathcal{S}}(\mathbb{R}), \mathcal{B}(D_{\mathcal{S}}), \mu^{\mathcal{Y}}, (\sigma^t_{\mathcal{S}} : t \in \mathbb{R}))$ is a shift flow.

A Bernoulli flow is defined in Section 12, part 2 in [13] as a flow $(\Omega, \mathcal{B}(\Omega), \mu, (\psi^t : t \in \mathbb{R}))$ such that $(\Omega, \mathcal{B}(\Omega), \mu, \psi^1)$ is isomorphic to a Bernoulli shift. The entropy of the flow is defined to be the entropy of (Ω, μ, ψ^1). The isomorphism theorem for Bernoulli flows, Theorem 4 in Section 12, part 2 in [13], states that if $(\Omega, \mathcal{B}(\Omega), \mu, (\psi^t : t \in \mathbb{R}))$ and $(\Omega', \mathcal{B}(\Omega'), \mu', (\psi'^t : t \in \mathbb{R}))$ are two Bernoulli flows with the same entropy and such that its completed probability spaces $(\Omega, \mathcal{B}(\Omega), \mu)$ and $(\Omega', \mathcal{B}(\Omega'), \mu')$ are Lebesgue, then the two flows isomorphic.

1.6 Renormalized stationary tessellation process and main results

Fix $a > 1$ and define the process $\mathcal{Z} = (\mathcal{Z}_s : s \in \mathbb{R})$ by

$$\mathcal{Z}_s = a^s Y_{a^s}\,, \quad s \in \mathbb{R}\,.$$

Note that $\mathcal{Z}_0 = Y_1$. In this context the $t-$shift transformation $\sigma^t_{\mathbb{T}}$ can be expressed as

$$\sigma^t_{\mathbb{T}} \circ \mathcal{Z} = ((\sigma^t \circ \mathcal{Z})_s : s \in \mathbb{R}) \text{ with } (\sigma^t_{\mathbb{T}} \circ \mathcal{Z})_s = \mathcal{Z}_{s+t}.$$

For any window W we set $\mathcal{Z} \wedge W = (\mathcal{Z}_s \wedge W : s \in \mathbb{R})$, and the shift $\sigma_{\mathbb{T}}^t$ has an analogous expression as above.

We now state our main results whose proofs will be given in the next Section 2.

In the following result, the trajectories of the process $\mathcal{Z}$ take values in the product space $\mathbb{T}^{\mathbb{R}}$, which is endowed with the product $\sigma-$field $\widehat{\mathcal{B}}(\mathbb{T}^{\mathbb{R}})$. We denote by $\mu^{\mathcal{Z}}$ the law induced by $\mathcal{Z}$ on $(T^{\mathbb{R}}, \widehat{\mathcal{B}}(\mathbb{T}^{\mathbb{R}}))$.

Theorem 1.1. *$\mathcal{Z}$ is a stationary Markov process, this means that for all $t \in \mathbb{R}$ the equality in distribution $\mathcal{Z} \sim \sigma_{\mathbb{T}}^t \circ \mathcal{Z}$ is verified.*
Hence $(T^{\mathbb{R}}, \widehat{\mathcal{B}}(\mathbb{T}^{\mathbb{R}}), \mu^{\mathcal{Z}}, (\sigma^t : t \in \mathbb{R}))$ is a shift flow.

When W is a window the process $\mathcal{Z} \wedge W = (\mathcal{Z}_s \wedge W : s \in \mathbb{R})$ inherits from $Y \wedge W$ the property of having càdlag trajectories. Since the trajectories of $Y \wedge W$ take values in $D_{\mathbb{T}_W}(\mathbb{R}_+)$ then the trajectories $\mathcal{Z} \wedge W$ take values in $D_{\mathbb{T}_W}(\mathbb{R})$. We recall that $(D_{\mathbb{T}_W}(\mathbb{R}), d_{\text{Sk}}^{\mathbb{T}_W})$ is a separable space. By $\mathcal{B}(D_{\mathbb{T}_W})$ we denoted the Borel $\sigma-$field associated to $(D_{\mathbb{T}_W}(\mathbb{R}), d_{\text{Sk}}^{\mathbb{T}_W})$ and by $\mu_W^{\mathcal{Z}}$ we denote the law induced by $\mathcal{Z} \wedge W$ on $(D_{\mathbb{T}_W}(\mathbb{R}), \mathcal{B}(D_{\mathbb{T}_W}))$.

Theorem 1.2. *For any window W, the process $\mathcal{Z} \wedge W$ is a stationary Markov process that is mixing in time:*

$$\lim_{t\to\infty} \mathbb{P}(\mathcal{Z} \wedge W \in \widehat{A}, \sigma_{\mathbb{T}}^t \circ \mathcal{Z} \wedge W \in \widehat{B}) = \mathbb{P}(\mathcal{Z} \wedge W \in \widehat{A})\mathbb{P}(\mathcal{Z} \wedge W \in \widehat{B}) \quad (6)$$

for all events $\widehat{A}$, $\widehat{B}$ in the Borel $\sigma-$field $\mathcal{B}(D_{\mathbb{T}_W})$.
Hence, $(D_{\mathbb{T}_W}(\mathbb{R}), \mathcal{B}(D_{\mathbb{T}_W}), \mu_W^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^t : t \in \mathbb{R}))$ is a mixing shift flow.

Let $\mathcal{Z}^d = (\mathcal{Z}_n : n \in \mathbb{Z})$ be the restriction of $\mathcal{Z}$ to integer times and let $\mu^{\mathcal{Z}^d}$ be the law of $\mathcal{Z}^d$ on $\mathbb{T}^{\mathbb{Z}}$. Theorem 1.1 implies that $\mathcal{Z}^d$ is stationary in time. As we pointed out in Section 1.5, the stationary property can be stated by saying that $\mu^{\mathcal{Z}^d}$ is preserved by the shift transformation $\sigma_{\mathbb{T}}$.

Let W be a window. The random sequence $\mathcal{Z}^d \wedge W = (\mathcal{Z}_n \wedge W : n \in \mathbb{Z})$ is also stationary, so the law of $\mathcal{Z}^d \wedge W$ on $\mathbb{T}_W^{\mathbb{Z}}$, which is denoted by $\mu_W^{\mathcal{Z}^d}$, is $\sigma_{\mathbb{T}_W}-$invariant. We will give an ergodic description of the two-sided shift $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$. We recall that Theorem 1.2 states that this shift is mixing, so it is ergodic.

Let ξ be the law of $Y_1 = Z_0$, so $\xi(B) = \mathbb{P}(Y_1 \in B) = \mathbb{P}(Z_0 \in B)$ for $B \in \mathcal{B}(\mathbb{T})$. Let us denote by ξ_W the law of $Y_1 \wedge W = Z_0 \wedge W$, so

$$\forall B \in \mathcal{B}(\mathbb{T}_W): \quad \xi_W(B) = \mathbb{P}(Y_1 \wedge W \in B) = \mathbb{P}(Z_0 \wedge W \in B). \quad (7)$$

In the sequel we fix

$$\varrho = \xi_W^{\otimes \mathbb{N}}. \tag{8}$$

The following ergodic property is verified.

Theorem 1.3. *Let W be a window. Then the shift system $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$ is a factor of the Bernoulli shift $((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}, \varrho^{\otimes \mathbb{Z}}, \sigma_{\mathbb{T}_W^{\mathbb{N}}})$, that is $\exists\, \varphi : (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}} \to \mathbb{T}_W^{\mathbb{Z}}$ (a factor map) measurable and defined $\varrho^{\otimes \mathbb{Z}}-$a.e., which satisfies,*

$$\sigma_{\mathbb{T}_W} \circ \varphi = \varphi \circ \sigma_{\mathbb{T}_W^{\mathbb{N}}} \quad \varrho^{\otimes \mathbb{Z}} - a.e., \tag{9}$$

and

$$\varrho^{\otimes \mathbb{Z}} \circ \varphi^{-1} = \mu_W^{\mathcal{Z}^d}. \tag{10}$$

Moreover, the factor map is finitary with null anticipation, that is for all $m \in \mathbb{Z}$, $\varrho^{\otimes \mathbb{Z}}-$a.e. in $\mathbf{R} = (\vec{R}_n : n \in \mathbb{Z}) \in (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}$, the coordinate $\varphi(\mathbf{R})_m$ of the image point depends only on the finite set of coordinates $(\vec{R}_n : n \in [-N, m])$ of the point $\mathbf{R}$. (The memory length N depends on $\mathbf{R}$).

A consequence is the following result.

Corollary 1.4. *Let W be a window. Then $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$ is isomorphic to a Bernoulli shift of infinite entropy.*

Let us give the steps for its proof. In Subsection 2.4 we will show that the shift $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$ has infinite entropy. On the other hand $((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}, \mathcal{B}((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}), \varrho^{\otimes \mathbb{Z}})$ is a Lebesgue probability space. Then the proof that $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$ is isomorphic to a Bernoulli shift follows from Theorem 6, page 54 in [13] (see also [12]) because there it was shown that a factor of a Bernoulli shift defined on a Lebesgue probability space is isomorphic to a Bernoulli shift.

Corollary 1.5. *Let W be a window. Then $\mathcal{Z} \wedge W$ is a Bernoulli flow of infinite entropy that is isomorphic to any other Bernoulli flow of infinite entropy defined on a Lebesgue probability space.*

The proof is as follows. We have that $(D_{\overline{\mathbb{T}_W}}(\mathbb{R}), \mathcal{B}(D_{\overline{\mathbb{T}_W}}), \mu^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^t : t \in \mathbb{R}))$ is a flow and from Corollary 1.4 it follows that $(D_{\overline{\mathbb{T}_W}}(\mathbb{R}), \mathcal{B}(D_{\overline{\mathbb{T}_W}}), \mu^{\mathcal{Z}}, \sigma_{\mathbb{T}}^1)$ is a Bernoulli shift of infinite entropy. Since $(D_{\overline{\mathbb{T}_W}}(\mathbb{R}), \mathcal{B}(D_{\overline{\mathbb{T}_W}}), \mu^{\mathcal{Z}})$ is a Lebesgue probability space, Theorem 4 in Section 12, part 2 in [13] gives the result.

Observe that all the results in relation with $\mathcal{Z}^d$ are also true when instead of the discrete time process $(\mathcal{Z}_n \wedge W : n \in \mathbb{Z})$ we consider $(\mathcal{Z}_{hn} \wedge W : n \in \mathbb{Z})$ for a fixed positive real h. In fact this last process corresponds to the former one when a^h is used instead of a.

2 Proof of the Main Results

We recall that without loss of generality we can assume that window W contains the origin in its interior, $0 \in \text{int}(W)$. Also we can assume that 0 belongs to the interior of $0-$cell during all the tessellation process Y.

2.1 Proof of Theorem 1.1

Let us first note that since the space $(\mathbb{F}, d_{\mathbb{F}})$ is a Polish space and $\mathbb{T}$ is a Borel subset of $\mathbb{F}$, a probability measure on the product space $(\mathbb{T}^{\mathbb{R}}, \widehat{\mathcal{B}}(\mathbb{T})^{\otimes \mathbb{R}})$ is defined by the finite dimensional distributions verifying the consistency property (see [9], Corollary Section III.3).

Hence, to show that $\mathcal{Z} = (\mathcal{Z}_s : s \in \mathbb{R})$ is stationary it suffices to prove that the finite dimensional distributions are stationary. So, we must show that

$$\begin{aligned} &\forall t > 0\,,\ \forall s_1 < ... < s_n\ \forall B_1, ..., B_n \in \mathcal{B}(\mathbb{T}) : \\ &\mathbb{P}(Z_{s_i+t} \in B_i : i = 1, ..., n) = \mathbb{P}(Z_{s_i} \in B_i : i = 1, ..., n)\,. \end{aligned} \tag{11}$$

Let us do it. Since $(Y_t : t > 0)$ is a Markov process, so is $(\mathcal{Z}_s : s \in \mathbb{R})$. On the other hand it was shown in Lemma 5 in [8] that

$$tY_t \sim Y_1 \ \text{ for all } t > 0\,, \tag{12}$$

and hence all 1-dimensional distributions of $(\mathcal{Z}_s : s \in \mathbb{R})$ are identical. Therefore the proof of (11) will be finished once we show that the transition probabilities from $\mathcal{Z}_s$ to $\mathcal{Z}_{s+t}$ depend only on the time difference $t > 0$. Now, from (12) and (4) we get that for all $z \in \mathbb{T}$ and all measurable $B \in \mathcal{B}(\mathbb{T})$ it is satisfied,

$$\begin{aligned} \mathbb{P}\left(\mathcal{Z}_{s+t} \in B \,|\, \mathcal{Z}_s = z\right) &= \mathbb{P}\left(a^{s+t} Y_{a^{s+t}} \in B \,|\, Y_{a^s} = a^{-s} z\right) \\ &= \mathbb{P}\left(z \boxplus a^s \vec{Y}'_{a^{s+t}-a^s} \in a^{-t} B\right) = \mathbb{P}\left(z \boxplus \vec{Y}'_{a^t-1} \in a^{-t} B\right). \end{aligned}$$

So the stationary property holds.

2.2 Proof of Theorem 1.2

The process $\mathcal{Z} \wedge W$ takes values in the separable metric space $D_{\mathbb{T}_W}(\mathbb{R})$ which is endowed with its Borel $\sigma-$field $\mathcal{B}(D_{\mathbb{T}_W})$. As we pointed out in Section 1.4, since $\mathbb{T}_W$ is separable then the class of cylinders in $D_{\mathbb{T}_W}(\mathbb{R})$ is a semi-algebra generating $\mathcal{B}(D_{\mathbb{T}_W})$. From the Carathéodory theorem on exterior measures we get that for all sets $E \in \mathcal{B}(D_{\mathbb{T}_W})$ and all $\epsilon > 0$ exists $E' \in \mathcal{B}(D_{\mathbb{T}_W})$ which is a disjoint union of a finite number of cylinders such that $\mathbb{P}(E\Delta E') < \epsilon$. Therefore it suffices to show the stationary and the mixing property for the cylinders in $D_{\mathbb{T}_W}(\mathbb{R})$. Hence, the above proof of Theorem 1.1 also shows that $\mathcal{Z}\wedge W$ is a stationary Markov process when considered in $D_{\mathbb{T}_W}(\mathbb{R})$.

Let us prove (6). From the above discussion it suffices to prove it for cylinders all $\widehat{A}$ and $\widehat{B}$ such that $\mathbb{P}(\mathcal{Z}\wedge W\in\widehat{A})>0$ and $\mathbb{P}(\mathcal{Z}\wedge W\in\widehat{B})>0$. So let

$$\widehat{A}=\{\mathcal{Z}_{s_1}\in A_1,\dots,\mathcal{Z}_{s_j}\in A_j\} \text{ and } \widehat{B}=\{\mathcal{Z}_{u_1}\in B_1,\dots,\mathcal{Z}_{u_l}\in B_l\}$$

be such that $s_1<...<s_j$, $u_1<...<u_l$ in $\mathbb{R}$, $A_1,...,A_j,B_1,...,B_l\in\mathcal{B}(\mathbb{T}\wedge W)$ and $\mathbb{P}(\mathcal{Z}_0\wedge W\in A_p)>0$, $\mathbb{P}(\mathcal{Z}_0\wedge W\in B_q)>0$ for $p=1,...,j$, $q=1,...,l$. Note that by time invariance property shown in Theorem 1.1 and since in (6) time $t\to\infty$, we can assume $s_j=u_1=0$.

First, let us show (6) in the case $j=l=1$. It suffices to show that for all $A,B\in\mathcal{B}(\mathbb{T}\wedge W)$ which satisfy $\mathbb{P}(\mathcal{Z}_0\wedge W\in A)>0$, $\mathbb{P}(\mathcal{Z}_0\wedge W\in B)>0$, it is fulfilled

$$\lim_{t\to\infty}\mathbb{P}(\mathcal{Z}_0\wedge W\in A,\mathcal{Z}_t\wedge W\in B)=\mathbb{P}(\mathcal{Z}_0\wedge W\in A)\mathbb{P}(\mathcal{Z}_0\wedge W\in B)\,. \quad (13)$$

Let us consider the events $\{\partial Y_1\cap\,\mathrm{int}(a^{-t}W)=\emptyset\}$ with $t\in\mathbb{R}$. This family of sets increases with t and when $t\to\infty$ it converges to the set $\bigcup_{m\in\mathbb{N}}\{\partial Y_1\cap\mathrm{int}(a^{-m}W)=\emptyset\}=\{\partial Y_1\cap\{0\}=\emptyset\}$. So

$$\lim_{t\to\infty}\mathbb{P}(\partial Y_1\cap\,\mathrm{int}(a^{-t}W)=\emptyset)=\mathbb{P}(\partial Y_1\cap\{0\}=\emptyset)=1. \quad (14)$$

For $t>0$

$$\begin{aligned}&\mathbb{P}(\mathcal{Z}_0\wedge W\in A,\mathcal{Z}_t\wedge W\in B)=\mathbb{P}(Y_1\wedge W\in A,a^tY_{a^t}\wedge W\in B)\\&=\mathbb{P}(Y_1\wedge W\in A,\partial Y_1\cap\mathrm{int}(a^{-t}W)\neq\emptyset,a^tY_{a^t}\wedge W\in B)\\&\quad+\mathbb{P}(Y_1\wedge W\in A,\partial Y_1\cap\mathrm{int}(a^{-t}W)=\emptyset,a^tY_{a^t}\wedge W\in B).\end{aligned} \quad (15)$$

From (14) the first item converges to 0 for $t\to\infty$, in fact

$$\begin{aligned}&\lim_{t\to\infty}\mathbb{P}(Y_1\wedge W\in A,\partial Y_1\cap\mathrm{int}(a^{-t}W)\neq\emptyset,a^tY_{a^t}\wedge W\in B)\\&\leq\lim_{t\to\infty}\mathbb{P}(\partial Y_1\cap\mathrm{int}(a^{-t}W)\neq\emptyset)=0.\end{aligned} \quad (16)$$

Let us now turn to the analysis of the second item. The assumption $\mathbb{P}(\mathcal{Z}_0\wedge W\in A)>0$ and (14) imply for sufficiently large $t>0$ that $\mathbb{P}(Y_1\wedge W\in A,\partial Y_1\cap\mathrm{int}(a^{-t}W)=\emptyset)>0$. Thus

$$\begin{aligned}&\mathbb{P}(Y_1\wedge W\in A,\partial Y_1\cap\mathrm{int}(a^{-t}W)=\emptyset,a^tY_{a^t}\wedge W\in B)\\&=\mathbb{P}(a^tY_{1+(a^t-1)}\wedge W\in B\mid Y_1\wedge W\in A,\partial(a^tY_1)\cap\mathrm{int}(W)=\emptyset)\times\\&\quad\times\mathbb{P}(Y_1\wedge W\in A,\partial Y_1\cap\mathrm{int}(a^{-t}W)=\emptyset).\end{aligned}$$

Conditioned on $\partial(a^t Y_1) \cap \mathrm{int}(W) = \emptyset$, the Markov property and the consistency of the construction (described in Subsection 1.4) yield that $a^t Y_{1+(a^t-1)} \wedge W$ is distributed as $a^t Y_{a^t-1} \wedge W$, that is

$$\mathbb{P}(a^t Y_{1+(a^t-1)} \wedge W \in B \mid Y_1 \wedge W \in A, \partial(a^t Y_1) \cap \mathrm{int}(W) = \emptyset) = \mathbb{P}(a^t Y_{a^t-1} \wedge W \in B).$$

Hence

$$\begin{aligned} &\mathbb{P}(Y_1 \wedge W \in A, \partial Y_1 \cap \mathrm{int}(a^{-t} W) = \emptyset, a^t Y_{a^t} \wedge W \in B) \\ &= \mathbb{P}(Y_1 \wedge W \in A, \partial Y_1 \cap \mathrm{int}(a^{-t} W) = \emptyset)\ \mathbb{P}(a^t Y_{a^t-1} \wedge W \in B). \quad (17) \end{aligned}$$

Note that (14) also yields

$$\lim_{t\to\infty} \mathbb{P}(Y_1 \wedge W \in A, \partial Y_1 \cap \mathrm{int}(a^{-t} W) = \emptyset) = \mathbb{P}(Y_1 \wedge W \in A) = \mathbb{P}(\mathcal{Z}_0 \wedge W \in A). \quad (18)$$

Therefore, from the relations (15),(16), (17) and (18), we get that the result will be proven once we show

$$\lim_{t\to\infty} \mathbb{P}(a^t Y_{a^t-1} \wedge W \in B) = \mathbb{P}(Y_1 \wedge W \in B). \quad (19)$$

From (12) we have $\mathbb{P}(a^t Y_{a^t-1} \wedge W \in B) = \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B)$ and so it suffices to show,

$$\lim_{t\to\infty} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B) = \mathbb{P}(Y_1 \wedge W \in B). \quad (20)$$

For $k \in \mathbb{N}$ and $t > 0$ define the events

$$D_{k,t} = \{\partial Y'^{m}_{a^{-t}} \cap \mathrm{int}(W) = \emptyset \ \forall m \in \{1, ..., k\}\} \quad (21)$$

with $\vec{Y}'$ as introduced in Subsection 1.4.1. Notice that for any fixed k the events are monotonically increasing in t because due to the construction of the process the sets $Y'^{m}_{a^{-t}}$ are decreasing in t. Moreover, from (14) we get

$$\lim_{t\to\infty} \mathbb{P}(D_{k,t}) = \lim_{t\to\infty} \mathbb{P}(\partial Y_1 \cap \mathrm{int}(a^{-t} W) = \emptyset)^k = 1. \quad (22)$$

Further, recall that $|Y \cap W|$ denotes the number of cells of $Y \cap W$. We have the following decomposition,

$$\begin{aligned} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B) &= \sum_{k\in\mathbb{N}} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B, |Y_{1-a^{-t}} \cap W| = k, D^c_{k,t}) \\ &+ \sum_{k\in\mathbb{N}} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B, |Y_{1-a^{-t}} \cap W| = k, D_{k,t}). \quad (23) \end{aligned}$$

Let us analyze the first sum in (23). From (1) we get

$$\mathbb{P}(D_{k,t}) = e^{-a^{-t}\Lambda([W])k}.$$

Then, by independence between Y and $\{D_{k,t} : k \in \mathbb{N}\}$ and by using that $|Y_s \cap W|$ increases with s we obtain,

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \mathbb{P}(|Y_{1-a^{-t}} \cap W| = k, D_{k,t}^c) = \sum_{k\in\mathbb{N}} \mathbb{P}(|Y_{1-a^{-t}} \cap W| = k)\, \mathbb{P}(D_{k,t}^c) \\
&\leq \sum_{k\in\mathbb{N}} \mathbb{P}(|Y_{1-a^{-t}} \cap W| = k) \left(1 - e^{-a^{-t}\Lambda([W])k}\right) \\
&= \mathbb{E}\left(1 - e^{-a^{-t}\Lambda([W])|Y_{1-a^{-t}}\cap W|}\right) \leq \mathbb{E}\left(1 - e^{-a^{-t}\Lambda([W])|Y_1\cap W|}\right).
\end{aligned}$$

Since the term $\left(1 - e^{-a^{-t}\Lambda([W])\,|Y_1\cap W|}\right)$ is dominated by 1 and it decreases with t, the Monotone Convergence Theorem gives

$$\lim_{t\to\infty} \mathbb{E}\left(1 - e^{-a^{-t}\Lambda([W])\,|Y_1\cap W|}\right) = \mathbb{E}\left(\lim_{t\to\infty}\left(1 - e^{-a^{-t}\Lambda([W])\,|Y_1\cap W|}\right)\right) = 0.$$

We have shown that

$$\lim_{t\to\infty} \sum_{k\in\mathbb{N}} \mathbb{P}(|Y_{1-a^{-t}} \cap W| = k,\ D_{k,t}^c) = 0\,. \tag{24}$$

Then,

$$\lim_{t\to\infty} \sum_{k\in\mathbb{N}} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k,\ D_{k,t}^c) = 0\,.$$

Let us turn to the second term in (23). With an appropriate measurable numbering of the cells of $Y_{1-a^{-t}}$, we get the inclusion of events

$$\{|Y_{1-a^{-t}} \cap W| = k\} \cap D_{k,t} \subseteq \{Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}} = Y_{1-a^{-t}}\},$$

and so,

$$\begin{aligned}
&\{|Y_{1-a^{-t}} \cap W| = k\} \cap D_{k,t} \\
&= \{Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}} = Y_{1-a^{-t}}\} \cap \{|Y_{1-a^{-t}} \cap W| = k\} \cap D_{k,t}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\mathbb{P}(Y_{1-a^{-t}} \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k,\ D_{k,t}) \\
&= \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B, |Y_{1-a^{-t}} \cap W| = k,\ D_{k,t}).
\end{aligned}$$

We have

$$
\begin{aligned}
&\mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k,\ D_{k,t}) \\
&= \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k) \\
&\quad - \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B, |Y_{1-a^{-t}} \cap W| = k, D^c_{k,t}).
\end{aligned}
$$

By summing this equality over $k \in \mathbb{N}$ and by using that the family of events $(|Y_{1-a^{-t}} \cap W| = k : k \in \mathbb{N})$ is disjoint and covers the whole space we obtain,

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}} \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k,\ D_{k,t}) \\
&= \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B) \\
&\quad - \sum_{k\in\mathbb{N}} \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k,\ D^c_{k,t}).
\end{aligned}
$$

Since $Y_1 \sim (Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}})$, also

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}} \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B, |Y_{1-a^{-t}} \cap W| = k, D_{k,t}) \\
&= \mathbb{P}(Y_1 \wedge W \in B) \\
&\quad - \sum_{k\in\mathbb{N}} \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k, D^c_{k,t}).
\end{aligned}
$$

From (24) we get,

$$
\lim_{t\to\infty} \sum_{k\in\mathbb{N}} \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B,\ |Y_{1-a^{-t}} \cap W| = k,\ D^c_{k,t}) = 0\,.
$$

Hence

$$
\lim_{t\to\infty} \sum_{k\in\mathbb{N}} \mathbb{P}((Y_{1-a^{-t}} \boxplus \vec{Y}'_{a^{-t}}) \wedge W \in B, |Y_{1-a^{-t}} \cap W| = k, D_{k,t}) = \mathbb{P}(Y_1 \wedge W \in B).
$$

Thus we have shown $\lim\limits_{t\to\infty} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B) = \mathbb{P}(Y_1 \wedge W \in B)$ and (19) is verified. The proof of Theorem 1.1 in the case $j = l = 1$ is complete.

Let us now show the general case $j > 0$, $l > 0$, $s_1 < ... < s_j = 0$ and $u_1 = 0 < ... < u_l$ in $\mathbb{R}$, and $A_1, \dots, A_j, B_1, \dots, B_l \in \mathcal{B}(\mathbb{T}_W)$. Since the proof is entirely similar to the case $j = l = 1$ we will only give the main steps. We put

$$
\mathcal{Z}[s_1, \dots, 0] \wedge W \in [A_1, \dots, A_j] := \{\mathcal{Z}_{s_1} \wedge W \in A_1, \dots, \mathcal{Z}_0 \wedge W \in A_j\}.
$$

The same notation is used for Y. We must prove

$$\lim_{t\to\infty} \mathbb{P}(\mathcal{Z}[s_1,\dots,0]\wedge W\in[A_1,\dots,A_j],\mathcal{Z}[t,\dots,t+u_l]\wedge W\in[B_1,\dots,B_l]) \\ = \mathbb{P}(\mathcal{Z}[s_1,\dots,0]\wedge W\in[A_1,\dots,A_j])\mathbb{P}(\mathcal{Z}[0,\dots,u_l]\wedge W\in[B_1,\dots,B_l])\,.$$

Let $t>0$. We have

$$\{\mathcal{Z}[s_1,..,0]\wedge W\in[A_1,..,A_j],\mathcal{Z}[t,..,t+u_l]\wedge W\in[B_1,..,B_l]\} \\ =\{Y[a^{s_1},..,1]\wedge W\in[a^{-s_1}A_1,..,A_j],a^tY[a^t,..,a^{t+u_l}]\wedge W\in[B_1,..,a^{-u_l}B_l]\}.$$

By using (14) and the same arguments as those we used from (15) to (17) we get,

$$\lim_{t\to\infty}\mathbb{P}(\mathcal{Z}[s_1,..,0]\wedge W\in[A_1,..,A_j],\mathcal{Z}[t,..,t+u_l]\wedge W\in[B_1,..,B_l]) \\ =\lim_{t\to\infty}\mathbb{P}(Y[a^{s_1},\dots,1]\wedge W\in[a^{-s_1}A_1,\dots,A_j],\partial Y_1\cap \operatorname{int}(a^{-t}W)=\emptyset, \\ a^tY[a^t,\dots,a^{t+u_l}]\wedge W\in[B_1,\dots,a^{-u_l}B_l]) \\ =\mathbb{P}(Y[a^{s_1},\dots,1]\wedge W\in[a^{-s_1}A_1,\dots,A_j])\times \\ \times\lim_{t\to\infty}\mathbb{P}(a^tY[a^t-1,\dots,a^{t+u_l}-1]\wedge W\in[B_1,\dots,a^{-u_l}B_l]).$$

Since (12) implies

$$\mathbb{P}(a^tY[a^t-1,\dots,a^{t+u_l}-1]\wedge W\in[B_1,\dots,a^{-u_l}B_l]) \\ =\mathbb{P}(Y[1-a^{-t},\dots,a^{u_l}-a^{-t}]\wedge W\in[B_1,\dots,a^{-u_l}B_l]),$$

the result will be proven once we show

$$\lim_{t\to\infty}\mathbb{P}(Y[1-a^{-t},\dots,a^{u_l}-a^{-t}]\wedge W\in[B_1,\dots,a^{-u_l}B_l]) \\ =\mathbb{P}(Y[1,\dots,a^{u_l}]\wedge W\in[B_1,\dots,a^{-u_l}B_l]). \tag{25}$$

Now, let $\vec{Y}'^{(i)}$, $i=1,...,l$ be l independent copies of $\vec{Y}'$, which are also independent of Y. Even if $\boxplus$ is not associative for sequences of tessellations we use $Y_{v_0}\boxplus_{i=2}^{l}\vec{Y}'^{(i)}_{v_i}$ to mean $\left(..(Y_{v_0}\boxplus\vec{Y}'^{(2)}_{v_2})\boxplus..)\boxplus\vec{Y}'^{(l)}_{v_l}\right)$. From the construction in Lemma 2 in [8], see (5), we have,

$$\mathbb{P}(Y[1-a^{-t},a^{u_2}-a^{-t},\dots,a^{u_l}-a^{-t}]\wedge W\in[B_1,a^{-u_2}B_2,\dots,a^{-u_l}B_l], \\ |Y_{1-a^{-t}}\cap W|=k) \\ =\mathbb{P}(Y_{1-a^{-t}}\wedge W\in B_1,Y_{1-a^{-t}}\boxplus\vec{Y}'^{(2)}_{a^{u_2}-1}\wedge W\in a^{-u_2}B_2,\dots, \\ Y_{1-a^{-t}}\boxplus_{i=2}^{l}\vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}}\wedge W\in a^{-u_l}B_l,|Y_{1-a^{-t}}\cap W|=k)\,.$$

This yields

$$\begin{aligned}
&\mathbb{P}(Y[1-a^{-t}, a^{u_2}-a^{-t}, \dots, a^{u_l}-a^{-t}] \wedge W \in [B_1, a^{-u_2}B_2, \dots, a^{-u_l}B_l]) \\
&= \sum_{k\in\mathbb{N}} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B_1, Y_{1-a^{-t}} \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_2}B_2, \dots, \\
&\qquad Y_{1-a^{-t}} \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l, |Y_{1-a^{-t}} \cap W| = k)\,.
\end{aligned}$$

Now we use the definition of $D_{k,t}$ done in (21) with $\vec{Y}' = \vec{Y}'^{(1)}$. From the equality of events

$$\begin{aligned}
&(Y_{1-a^{-t}} \wedge W \in B_1, Y_{1-a^{-t}} \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_2}B_2, \dots, \\
&\qquad Y_{1-a^{-t}} \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l, |Y_{1-a^{-t}} \cap W| = k, D_{k,t}) \\
&= (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}} \wedge W \in B_1, (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_l}B_2, \dots, \\
&\qquad (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l, |Y_{1-a^{-t}} \cap W| = k, D_{k,t})
\end{aligned}$$

and by using twice (24) we get

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}(Y[1-a^{-t}, a^{u_2}-a^{-t}, \dots, a^{u_l}-a^{-t}] \wedge W \in [B_1, a^{-u_2}B_2, .., a^{-u_l}B_l]) \\
&= \lim_{t\to\infty} \sum_{k\in\mathbb{N}} \mathbb{P}(Y_{1-a^{-t}} \wedge W \in B_1, Y_{1-a^{-t}} \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_2}B_2, .., \\
&\qquad Y_{1-a^{-t}} \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l, |Y_{1-a^{-t}} \cap W| = k) \\
&= \lim_{t\to\infty} \sum_{k\in\mathbb{N}} \mathbb{P}(Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}} \wedge W \in B_1, (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_2}B_2, \\
&\qquad .., (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l, |Y_{1-a^{-t}} \cap W| = k) \\
&= \lim_{t\to\infty} \mathbb{P}(Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}} \wedge W \in B_1, (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_2}B_2, \\
&\qquad .., (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l).
\end{aligned}$$

Finally from

$$\begin{aligned}
&\mathbb{P}(Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}} \wedge W \in B_1, (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus \vec{Y}'^{(2)}_{a^{u_2-1}} \wedge W \in a^{-u_2}B_2, \\
&\qquad \dots, (Y_{1-a^{-t}} \boxplus \vec{Y}'^{(1)}_{a^{-t}}) \boxplus_{i=2}^{l} \vec{Y}'^{(i)}_{a^{u_i}-a^{u_{i-1}}} \wedge W \in a^{-u_l}B_l) \\
&= \mathbb{P}(Y_1 \wedge W \in B_1, Y_{a^{u_2}} \wedge W \in a^{-u_2}B_2, Y_{a^{u_l}} \wedge W \in a^{-u_l}B_l),
\end{aligned}$$

the relation (25) follows. The proof of Theorem 1.2 is complete. □

2.3 Proof of Theorem 1.3

We will show some intermediate results -some of them having their own interest-, that will be needed in the proof of the Theorem.

As announced we assume that the interior of the window W contains the origin 0.

Let $\mathbb{Z}_- = \{n \in \mathbb{Z} : n \le 0\}$. For a measurable space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ we shall use the one sided Bernoulli shift $\sigma_{\mathcal{S}} : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$, $(\sigma_{\mathcal{S}}(x))_n = x_{n+1} \ \forall\, n \in \mathbb{N}$, and the inverse shift

$$\sigma_{\mathcal{S}}^- : \mathcal{S}^{\mathbb{Z}_-} \to \mathcal{S}^{\mathbb{Z}_-}, \quad (\sigma_{\mathcal{S}}^-(x))_n = x_{n-1} \ \forall\, n \in \mathbb{Z}_- .$$

We will set $\sigma_{\mathcal{S}}^{-n} := (\sigma_{\mathcal{S}}^-)^n$ for $n \in \mathbb{N}$. For a probability measure ν on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, both one-sided Bernoulli shifts $(\mathcal{S}^{\mathbb{N}}, \nu^{\otimes \mathbb{N}}, \sigma_{\mathcal{S}})$ and $(\mathcal{S}^{\mathbb{Z}_-}, \nu^{\otimes \mathbb{Z}_-}, \sigma_{\mathcal{S}}^-)$ are canonically isomorphic. We recall that the Bernoulli shifts are mixing, so ergodic.

In the sequel we use the notion of a boundary ∂T of a tessellation T which was defined in Section 1.3 as the union of the boundaries of its cells.

Observe that from property (4) and definition of $\mathcal{Z}$ it follows that

$$\mathcal{Z}_{n+1} \sim a\mathcal{Z}_n \boxplus a^{n+1} \vec{Y}'_{a^{n+1}-a^n} .$$

Since $a^{n+1}\vec{Y}'_{a^{n+1}-a^n} = \frac{a}{a-1}(a^n(a-1)\vec{Y}'_{a^n(a-1)})$ we get from (12),

$$\mathcal{Z}_{n+1} \sim a\mathcal{Z}_n \boxplus \frac{a}{a-1} \vec{Y}'_1 . \tag{26}$$

Let $(\vec{Y}_1'^{(i)} : i \ge 0)$ be independent copies of $\vec{Y}_1'$. A simple recurrence on (26) and (5) give the formula

$$(\mathcal{Z}_{n+k} : k \in \mathbb{N}) \sim \left(a^k \mathcal{Z}_n \boxplus_{i=1}^k \frac{a^{k+1-i}}{a-1} \vec{Y}_1'^{(i)} : k \in \mathbb{N} \right) . \tag{27}$$

We recall $M \boxplus_{i=1}^k \vec{M}'^{(i)}$ is an abbreviation for $\left(\ldots \left(M \boxplus \vec{M}'^{(1)} \right) \boxplus \ldots \right) \boxplus \vec{M}'^{(k)}$, where M is a tessellation and $\vec{M}'^{(i)}$ a sequence of tessellations.

The following fact will be useful. We recall that ξ_W is the distribution of $Y_1 \wedge W$, see (7).

Lemma 2.1. *Let W be a window containing the origin 0 in its interior. Let $\vec{R}^- = (R^k : k \in \mathbb{Z}_-)$ be a random sequence of independent copies of $Y_1 \wedge W$, that is $\vec{R}^- \sim \xi_W^{\otimes \mathbb{Z}_-}$. Then, for $a > 1$ we have*

$$\forall\, k \in \mathbb{Z}_- : \quad \mathbb{P}(\partial R^k \cap \mathrm{int}(a^k W) = \emptyset) = \mathbb{P}(\partial R^0 \cap \mathrm{int}(W) = \emptyset)^{a^k} , \tag{28}$$

and

$$\mathbb{P}(\forall\, k \in \mathbb{Z}_- : \partial R^k \cap \mathrm{int}(a^k W) = \emptyset) = \mathbb{P}(\partial R^0 \cap \mathrm{int}(W) = \emptyset)^{\frac{a}{a-1}} > 0 . \tag{29}$$

Moreover

$$\mathbb{P}(\vec{R}^- : \exists (n_i \ge 1 : i \in \mathbb{N}) \nearrow, \forall i \in \mathbb{N}\, \forall k \in \mathbb{Z}_- : \partial R^{-n_i+k} \cap \mathrm{int}(a^k W) = \emptyset) = 1. \tag{30}$$

($\nearrow$ means strictly increasing; so the sequence (n_i) satisfies $\lim_{i\to\infty} n_i = \infty$).

Proof. The consistency of the STIT tessellations and (3) yield for all windows $W' \subseteq W$

$$\mathbb{P}(\partial Y_1 \cap \mathrm{int}(W') = \emptyset) = e^{-\Lambda([W'])} > 0.$$

Hence for all $k \in \mathbb{Z}_-$ we use (1) to get,

$$\mathbb{P}(\partial R^k \cap \mathrm{int}(a^k W) = \emptyset) = e^{-\Lambda([a^k W])} = e^{-a^k \Lambda([W])} = \mathbb{P}(\partial R^0 \cap \mathrm{int}(W) = \emptyset)^{a^k}$$

which shows (28).

Further, by monotonicity

$$\begin{aligned}
&\mathbb{P}\left(\forall k \in \mathbb{Z}_- : \partial R^k \cap \mathrm{int}(a^k W) = \emptyset\right) \\
&= \lim_{m\to-\infty} \mathbb{P}\left(\forall k \in \{m, ..., 0\} : \partial R^k \cap \mathrm{int}(a^k W) = \emptyset\right) \\
&= \lim_{m\to-\infty} \prod_{k=m}^{0} \exp(-a^k \Lambda([W])) = \lim_{m\to-\infty} \exp\left(-\Lambda([W]) \sum_{k=m}^{0} a^k\right) \\
&= \exp\left(-\Lambda([W]) \frac{a}{a-1}\right) = \mathbb{P}(\partial R^0 \cap \mathrm{int}(W) = \emptyset)^{\frac{a}{a-1}} > 0.
\end{aligned}$$

This proves (29).

Let us show (30). Consider the inverse Bernoulli shift $\left(\mathbb{T}_W^{\mathbb{Z}_-}, \xi_W^{\otimes \mathbb{Z}_-}, \sigma^-_{\mathbb{T}_W}\right)$ with $(\sigma^-(R'))^k = R'^{k-1}\ \forall\, k \in \mathbb{Z}_-$. Define

$$A^* = \{\vec{R}' \in \mathbb{T}_W^{\mathbb{Z}_-} \,:\, \partial R'^k \cap \mathrm{int}(a^k W) = \emptyset\ \forall k \in \mathbb{Z}_-\}\,.$$

By (29) we have $\xi_W^{\otimes \mathbb{Z}_-}(A^*) > 0$. Since Bernoulli shifts are ergodic the Birkhoff Ergodic Theorem gives

$$\lim_{N\to\infty} \frac{1}{N} \left(\sum_{k=0}^{N-1} \mathbf{1}_{A^*}(\sigma^{-k}_{\mathbb{T}_W}(\vec{R}'))\right) = \xi_W^{\otimes \mathbb{Z}_-}(A^*) > 0 \quad \xi_W^{\otimes \mathbb{Z}_-} - \text{a.e.}\,.$$

Therefore $\xi_W^{\otimes \mathbb{Z}_-}$−a.e. in $\vec{R}' \in \mathbb{T}_W^{\mathbb{Z}_-}$ there exists a strictly increasing sequence $(n_i \ge 1 : i \in \mathbb{N})$ such that $\{\sigma^{-n_i}_{\mathbb{T}_W}(\vec{R}') \in A\}$ for all $i \in \mathbb{N}$. This is exactly (30) because the distribution of $\vec{R}^- = (R^k : k \in \mathbb{Z}_-)$ is $\xi_W^{\otimes \mathbb{Z}_-}$. □

We will also use the following elementary result.

Lemma 2.2. *Let W be a window containing the origin 0 in its interior. Let T^0 and R^0 be two tessellations, $(\vec{Q}_n : n \in \mathbb{N})$ be a family of sequences of tessellations (so for each $n \in \mathbb{N}$, $\vec{Q}_n = (Q_n^m : m \in \mathbb{N}) \in \mathbb{T}^{\mathbb{N}}$ is a sequence of tessellations). Define the following sequences of tessellations in $\mathbb{T}_W$:*

$$\forall n \in \mathbb{N} : \; T^{n+1} = (aT^n \boxplus \frac{a}{a-1}\vec{Q}_{n+1}) \wedge W, \;\; R^{n+1} = (aR^n \boxplus \frac{a}{a-1}\vec{Q}_{n+1}) \wedge W. \quad (31)$$

Then

$$T^0 \wedge a^{-n}W = R^0 \wedge a^{-n}W \Rightarrow \;\; T^n \wedge W = R^n \wedge W \,. \quad (32)$$

Proof. By iterating (31) we find

$$T^n \wedge W = a^n(T^0 \wedge a^{-n}W) \boxplus \left(\boxplus_{i=1}^{n} \frac{a^{n+1-i}}{a-1}\vec{Q}_i \right) \wedge W \,.$$

and the result follows straightforward. □

Proof of Theorem 1.3. The last part of the Theorem (the fact that the factor map satisfies the finitary property) will be part of the construction of the factor map.

We recall the notation in (8), $\varrho = \xi_W^{\otimes \mathbb{N}}$. For the tessellation $T = \{C(T)^l : l = 1, ...\} \in \mathbb{T}_W$ (the number of cells is finite) we prescribe $C(T)^1$ to be the cell containing the origin 0. For $\vec{R} = (R^m : m \in \mathbb{N}) \in \mathbb{T}_W^{\mathbb{N}}$, the set of cells of the tessellation $T \boxplus \vec{R} \in \mathbb{T}_W$ is

$$\{C(T)^i \cap C(R_i)^j : j = 1, ...; i = 1, ...; \text{ with } \operatorname{int}(C(T)^i \cap C(R_i)^j) \neq \emptyset\}.$$

As noted in Subsection 1.3 for $b > 1$ and $T \in \mathbb{T}_W$, $b\,T \wedge W$ is also in $\mathbb{T}_W$. When $\vec{R} = (R^m : m \in \mathbb{N}) \in \mathbb{T}_W^{\mathbb{N}}$ we put $b\,\vec{R} \wedge W = (bR^m \wedge W : m \in \mathbb{N})$.

The factor map $\varphi : (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}} \to \mathbb{T}_W^{\mathbb{Z}}$ which must satisfy (9) and (10) is constructed in an iterative way: we will define a sequence of functions $(\varphi^N : N \geq 0)$ and will show that the function $\varphi = \lim_{N \to \infty} \varphi^N$ is pointwise $\varrho^{\otimes \mathbb{Z}}$−a.e. defined and fulfills the property of being a factor. Then, we start by defining φ^N.

Let $\mathbf{R} = (\vec{R}_n : n \in \mathbb{Z}) \in (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}$, so each $\vec{R}_n = (R_n^m : m \in \mathbb{N})$ is a sequence of tessellations in the window W. We must define the image point $\varphi^N(\mathbf{R}) = (\varphi_n^N(\mathbf{R}) : n \in \mathbb{Z})$ in $\mathbb{T}_W^{\mathbb{Z}}$. We fix (recall $N \geq 0$),

$$\forall\, n \leq -N : \;\; \varphi_n^N(\mathbf{R}) = \{W\} \,,$$

and we define by recurrence,

$$\forall\, n \geq -N: \quad \varphi^N_{n+1}(\mathbf{R}) = \left(a\, \varphi^N_n(\mathbf{R}) \boxplus \frac{a}{a-1} \vec{R}_n \right) \wedge W\,. \tag{33}$$

We claim that $\varphi = \lim_{N\to\infty} \varphi^N$ is defined $\varrho^{\otimes\mathbb{Z}}-$a.e. In fact, from property (30) in Lemma 2.1, applied to the sequences $(R^1_n : n \in \mathbb{Z}_-)$, we get that $\varrho^{\otimes\mathbb{Z}}-$a.e. there exists a sequence $N_i \geq 1$, $N_i \to \infty$ (depending on $\mathbf{R}$) such that $\partial R^1_{k-N_i} \cap \mathrm{int}(a^k W) = \emptyset$ for all $k \in \mathbb{Z}_-$. Hence, from Lemma 2.2 we deduce that for all N_i

$$\forall N \geq N_i \;\forall n \geq -N_i: \quad \varphi^N_n(\mathbf{R}) = \varphi^{N_i}_n(\mathbf{R}).$$

Therefore

$$\forall N \geq N_i \;\forall n \geq -N_i: \quad \varphi_n(\mathbf{R}) = \varphi^N_n(\mathbf{R}) = \varphi^{N_i}_n(\mathbf{R}), \tag{34}$$

that is all the components $\varphi_n(\mathbf{R})$ for $n \geq -N_i$ are well-defined as $\varphi^{N_i}_n(\mathbf{R})$. Since the sequence $N_i \geq 1$ exists $\varrho^{\otimes\mathbb{Z}}-$a.e. the claim is verified, so φ is defined $\varrho^{\otimes\mathbb{Z}}-$a.e.

From the definition of φ^N we have

$$\sigma_{\mathbb{T}_W}(\varphi^{N+1}(\mathbf{R})) = (\varphi^N(\sigma_{\mathbb{T}^{\mathbb{N}}_W}(\mathbf{R})).$$

Then φ satisfies the commuting property (9). The equality (34) also shows that the factor map φ satisfies the finitary property stated in the Theorem.

Let us now turn to the proof of relation (10). We first note that since $\lim_{N\to\infty} \mathbb{P}(N_i \leq N) = 1$ for all N_i, from the above construction we obtain

$$\forall \epsilon > 0\; \forall k \in \mathbb{Z}\; \exists N(\epsilon, k): \;\; \mathbb{P}(\forall N \geq N(\epsilon, k)\, \forall n \geq k: \, \varphi^N_n = \varphi_n) > 1 - \epsilon. \tag{35}$$

We proved in Theorem 1.2 that $\mathcal{Z} \wedge W$ is mixing, then it is ergodic. Since $\mathbb{P}(\partial \mathcal{Z}_n \cap \mathrm{int}(W) = \emptyset) > 0$, the ergodic theorem applied to the ergodic stationary sequence $\mathcal{Z} \wedge W$ gives

$$\lim_{N\to\infty} \frac{1}{N} \left(\sum_{i=0}^{N-1} \mathbf{1}_{\{\partial \mathcal{Z}_i \cap \mathrm{int}(W) = \emptyset\}} \right) = \mathbb{P}(\partial \mathcal{Z}_n \cap \mathrm{int}(W) = \emptyset) \;>\; 0 \;\; \mathbb{P}-\text{a.e.}\,.$$

Then,

$$\mathbb{P}\left(\exists n_k \geq 0: \lim_{k\to\infty} n_k = \infty, \partial \mathcal{Z}_{n_k} \cap \mathrm{int}(W) = \emptyset \right) = 1\,.$$

Hence for all $\epsilon > 0$ there exists $K(\epsilon) > 0$ such that

$$\mathbb{P}\left(\exists n \in \{0, ..., K(\epsilon)\} : \partial \mathcal{Z}_n \cap \text{int}(W) = \emptyset\right) > 1 - \epsilon\,. \tag{36}$$

Consider $\mathcal{Z}_-^d = (\mathcal{Z}_n : n \leq 0)$. For each $M \geq 0$ we define the random sequence $V^M = (V_n^M : n \in \mathbb{Z})$ taking values in $\mathbb{T}_W^{\mathbb{Z}}$ by:

$$\forall\, n \leq -M : \;\; V_n^M = \mathcal{Z}_{n+M} \wedge W\,,$$

and by recurrence

$$\forall\, n \geq -M : \;\; V_{n+1}^M = \left(a\, V_n^M \boxplus \frac{a}{a-1} \vec{R}_n \right) \wedge W\,. \tag{37}$$

The sequence V^M depends on $\mathcal{Z}_-^d \wedge W$ and $\mathbf{R}$, if we need to explicit its dependence on $\mathbf{R}$ we put $V_n^M(\mathbf{R})$. We claim that $V^M \sim \mathcal{Z} \wedge W$. To show it first note that from the definition of V^M and by the time-stationarity of $\mathcal{Z} \wedge W$ we have

$$(V_n^M : n \leq -M) = (\mathcal{Z}_n \wedge W : n \leq 0) \sim (\mathcal{Z}_n \wedge W : n \leq -M). \tag{38}$$

Let us now define the shifted sequence $U^M(\mathbf{R}) = \sigma_{\mathbb{T}_W}^{-M} V^M(\mathbf{R})$ that satisfies

$$\forall\, n \in \mathbb{Z} : \; U_n^M(\mathbf{R}) = V_{n-M}^M(\mathbf{R})\,.$$

We have $(U_n^M : n \in \mathbb{Z}_-) = \mathcal{Z}_-^d \wedge W$ and by stationarity $U^M(\mathbf{R}) \sim \mathcal{Z} \wedge W$. From (36) we obtain

$$\forall\, M > 0 : \;\; \mathbb{P}\left(\exists n \in \{0, ..., K(\epsilon)\} : U_n^M(\mathbf{R}) = \{W\}\right) > 1 - \epsilon\,.$$

This is equivalent to

$$\mathbb{P}\left(\exists n \in \{-M, ..., K(\epsilon) - M\} : V_n^M(\mathbf{R}) = \{W\}\right) > 1 - \epsilon\,. \tag{39}$$

In analogy to (27) we obtain for all $M \geq 0$ and $l \geq 0$

$$V_{-M+l}^M(\mathbf{R}) = \left((a^l \mathcal{Z}_0 \wedge W) \boxplus_{i=1}^{l} \frac{a^{l+1-i}}{a-1} \vec{R}_{-M+i-1} \right) \wedge W,$$

$$\varphi_{-M+l}^M(\mathbf{R}) = \left((a^l W \wedge W) \boxplus_{i=1}^{l} \frac{a^{l+1-i}}{a-1} \vec{R}_{-M+i-1} \right) \wedge W.$$

Thus, if $V_{-M+l}^M(\mathbf{R}) = \{W\}$ for some $M,\; l \geq 0$ then necessarily $a^l \mathcal{Z}_0 \wedge W = \{W\}$ and hence also $\varphi_{-M+l}^M(\mathbf{R}) = \{W\}$. So, if $V_n^M(\mathbf{R}) = \{W\}$ for some

$n \in \{-M, ..., K(\epsilon) - M\}$ the iteration relations (33) and (37) allow to deduce $\varphi_n^M(\mathbf{R}) = V_n^M(\mathbf{R})$ for all $n \geq K(\epsilon) - M$. Therefore we find

$$\forall\, N \geq K(\epsilon): \quad \mathbb{P}\left(\forall n \geq K(\epsilon) - N : V_n^N(\mathbf{R}) = \varphi_n^N(\mathbf{R})\right) > 1 - \epsilon\,. \tag{40}$$

We can now state the proof of (10). Let us fix $k \in \mathbb{Z}$ and $l \geq 0$, it is sufficient to show that

$$\forall B_j \in \mathcal{B}(\mathbb{T}_W): \mathbb{P}(\varphi_{k+j}(\mathbf{R}) \in B_j : j = 0, ..., l) = \mathbb{P}(\mathcal{Z}_{k+j} \wedge W \in B_j : j = 0, ..., l).$$

Fix $M \geq 0$. Since $V^M \sim \mathcal{Z} \wedge W$ it suffices to prove that for all $\delta > 0$,

$$|\mathbb{P}(\varphi_{k+j}(\mathbf{R}) \in B_j : j = 0, ..., l) - \mathbb{P}(V_{k+j}^M(\mathbf{R}) \in B_j : j = 0, ..., l)| \leq \delta.$$

Therefore, it suffices to show that for any $\delta > 0$ we have,

$$\mathbb{P}\left(\exists j \in \{0, \ldots, l\} : \{\varphi_{k+j}(\mathbf{R}) \in B_j\} \Delta \{V_{k+j}^M(\mathbf{R}) \in B_j\}\right) \leq \delta.$$

Hence it suffices to prove that for any $\delta > 0$ it is satisfied,

$$\mathbb{P}(\exists j \in \{0, ..., l\}: \ \varphi_{k+j}(\mathbf{R}) \neq V_{k+j}(\mathbf{R})) \leq \delta. \tag{41}$$

To this purpose let us take $N(\delta/2, k)$ in (35) and $K(\delta/2)$ in (40), to obtain

$$\mathbb{P}(\forall N \geq \max(N(\delta/2, k), K(\delta/2), -k + K(\delta/2))\, \forall n \geq k: \ V_n^N = \varphi_n^N = \varphi_n) > 1 - \delta.$$

Then, (41) is verified and the proof of Theorem 1.3 is complete. □

2.4 Proof of Corollary 1.4

The only relation left to prove is that $h(\sigma_{\mathbb{T}_W}, \mu_W^{\mathcal{Z}^d}) = \infty$, where $h(\sigma_{\mathbb{T}_W}, \mu_W^{\mathcal{Z}^d})$ denotes the entropy of $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$. Recall that ξ_W is the law of $Y_1 \wedge W = \mathcal{Z}_0 \wedge W$. From the Markov property we have

$$h(\sigma_{\mathbb{T}_W}, \mu_W^{\mathcal{Z}^d}) = \int_{\mathbb{T}_W} H(\kappa_T)\, d\xi_W(T)\,,$$

where κ_T is the law of $\mathcal{Z}_1 \wedge W$ conditioned to $\mathcal{Z}_0 \wedge W = T$. We have $H(\kappa_T) = \infty$ when κ_T is not purely atomic and $H(\kappa_T) = -\sum_{a \in \mathcal{A}(\kappa_T)} \kappa_T(A) \log(\kappa_T(A))$ if κ_T is purely atomic and $\mathcal{A}(\kappa_T)$ is the set of its atoms. So, it suffices to show that

$$\xi_W(T \in \mathbb{T}_W : \kappa_T \text{ has a non-atomic part }) > 0\,.$$

We will show the stronger property: κ_T has a non-atomic part ξ_W-a.e.. First note that κ_T has an atom at $\{aT \wedge W\}$: $\kappa_T(\{aT \wedge W\}) > 0$. This is a consequence of the following facts: if $Y_a \wedge W = Y_1 \wedge W$ then $\mathcal{Z}_1 \wedge W = aY_a \wedge W = aY_1 \wedge W = a\mathcal{Z}_0 \wedge W$; and the construction of the process yields that $\mathbb{P}(Y_a \wedge W = Y_1 \wedge W) > 0$. Also from the construction of the process Y it follows that $\kappa_T(\{aT \wedge W\}) < 1$.

Assume that κ_T has an atom $T^0 \in \mathbb{T}_W$ different from the atom $\{aT \wedge W\}$. From the construction there is an hyperface r such that $aT \cup r \subseteq T^0$ and $r \subset H \in \mathcal{H}$, that is r is a part of an hyperplane H. The translation invariance and $\sigma-$finiteness of the hyperplane measure Λ implies that $\Lambda_W(\{H\}) = 0$ for all $H \in \mathcal{H}$. Consequently, the hyperface r in T^0 appears in the construction with probability 0. We conclude that $\{aT \wedge W\}$ is the unique atom of κ_T. Since $\kappa_T(\{aT \wedge W\}) < 1$, κ_T has a non-atomic part and so $H(\kappa_T) = \infty$ for all $T \in \mathbb{T}_W$. We conclude $h(\sigma_{\mathbb{T}_W}, \mu_W^{\mathcal{Z}^d}) = \infty$. $\square$

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