Zero dissipation limit to rarefaction wave with vacuum for 1-D compressible Navier-Stokes equations

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Abstract

It is well-known that one-dimensional isentropic gas dynamics has two elementary waves, i.e., shock wave and rarefaction wave. Among the two waves, only the rarefaction wave can be connected with vacuum. Given a rarefaction wave with one-side vacuum state to the compressible Euler equations, we can construct a sequence of solutions to one-dimensional compressible isentropic Navier-Stokes equations which converge to the above rarefaction wave with vacuum as the viscosity tends to zero. Moreover, the uniform convergence rate is obtained. The proof consists of a scaling argument and elementary energy analysis based on the underlying rarefaction wave structures.

Keywords: compressible Navier-Stokes equations, zero dissipation limit, rarefaction wave, vacuum.

1 Introduction and main result

In this paper, we investigate the zero dissipation limit of the one-dimensional compressible isentropic Navier-Stokes equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, & x \in \mathbf{R}, t > 0, \\ (\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = \epsilon\, u_{xx}, \end{cases} \tag{1.1}$$

where $\rho(t,x) \geq 0$, $u(t,x)$ and p represent the density, the velocity and the pressure of the gas, respectively and $\epsilon > 0$ is the viscosity coefficient. Here we assume that the viscosity coefficient ϵ is a positive constant and the pressure p is given by the $\gamma-$law:

$$p(\rho) = \frac{\rho^\gamma}{\gamma}$$

with $\gamma > 1$ being the gas constant.

Formally, as ϵ tends to zero, the limit system of the compressible Navier-Stokes equations (1.1) is the following inviscid Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0. \end{cases} \tag{1.2}$$

The Euler system (1.2) is a strictly hyperbolic one for $\rho > 0$ whose characteristic fields are both genuinely nonlinear, that is, in the equivalent system

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0,$$

the Jacobi matrix

$$\begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix}$$

has two distinct eigenvalues

$$\lambda_1(\rho,u) = u - \sqrt{p'(\rho)}, \qquad \lambda_2(\rho,u) = u + \sqrt{p'(\rho)}$$

with corresponding right eigenvectors

$$r_i(\rho,u) = (1, (-1)^i \frac{\sqrt{p'(\rho)}}{\rho})^t, \qquad i = 1,2,$$

such that

$$r_i(\rho,u) \cdot \nabla_{\rho,u} \lambda_i(\rho,u) = (-1)^i \frac{\rho p''(\rho) + 2p'(\rho)}{2\rho\sqrt{p'(\rho)}} \neq 0, \quad i = 1,2.$$

We can define the $i-$Riemann invariant $(i = 1,2)$ by

$$\Sigma_i(\rho,u) = u + (-1)^{i+1} \int^\rho \frac{\sqrt{p'(s)}}{s} ds$$

such that

$$\nabla_{(\rho,u)} \Sigma_i(\rho,u) \cdot r_i(\rho,u) \equiv 0, \qquad \forall \rho > 0, u.$$

The study of the limiting process of viscous flows when the viscosity tends to zero, is one of the important problems in the theory of the compressible fluid. When the solution of the inviscid flow is smooth, the zero dissipation limit can be solved by classical scaling method. However, the inviscid compressible flow contains singularities such as shock and the vacuum in general. Therefore, how to justify the zero dissipation limit to the Euler equations with basic wave patterns and/or the vacuum is a natural and difficult problem.

There have been many results on the zero dissipation limit of the compressible fluid with basic wave patterns without vacuums. For the system of the hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx},$$

Goodman-Xin [4] verified the viscous limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. For the compressible isentropic Navier-Stokes equations (1.1), Hoff-Liu [5] first proved the vanishing viscosity limit for piecewise constant shock even with initial layer. Later Xin [15] obtained the zero dissipation limit for rarefaction waves without vacuum for both rarefaction wave data and well-prepared smooth data. Then Wang [13] generalized the result of Goodmann-Xin [4] to the isentropic Navier-Stokes equations (1.1). For the full Navier-Stokes equations where the conservation of the energy is also involved, there are also many results on the zero dissipation limit to the corresponding full Euler system with basic wave patterns without vacuums. We refer to Jiang-Ni-Sun [8] and Xin-Zeng [16] for the rarefaction wave, Wang [14] for the shock wave, Ma [10] for the contact discontinuity and Huang-Wang-Yang [7] for the superposition of two rarefaction waves and a contact discontinuity case.

More recently, Chen-Perepelitsa [2] proved the vanishing viscosity to the compressible Euler equations for the compressible Navier-Stokes equations (1.1) by compensated compactness method for the general case if the far field of the initial values of Euler system (1.2) has no vacuums. Note that this result is quite universal since the initial values of the Euler system can contain vacuum states in the interior domain.

Now we turn back to the case of the basic wave patterns with vacuum state. As pointed out by Liu-Smoller [9], among the two nonlinear waves, i.e., shock and rarefaction waves, to the one-dimensional compressible isentropic Euler equations (1.2), only the rarefaction wave can be connected with vacuum. However, to our knowledge, so far there is no any results on the zero dissipation limit of the system (1.1) in the case when the Euler system (1.2) contain the rarefaction wave connected with the vacuum. In this paper, we investigate this fundamental problem and want to obtain the decay rate with respect to the viscosity ϵ. First we give a description of the rarefaction wave connected with the vacuum to the compressible Euler equations (1.2), see also the references [9] and [12]. For definiteness, we consider the 2-rarefaction wave. If we consider the compressible Euler system (1.2) with the Riemann initial data

$$\begin{cases} \rho(0,x)=0, & x<0, \\ (\rho,u)(0,x)=(\rho_+,u_+), & x>0, \end{cases} \tag{1.3}$$

where the left side is the vacuum state and $\rho_+>0, u_+$ are prescribed constants on the right state, then the Riemann problem (1.2), (1.3) admits a $2-$rarefaction wave connected with the vacuum on the left side. By the fact that along the $2-$rarefaction wave curve, $2-$Riemann invariant $\Sigma_2(\rho,u)$ is constant in (x,t), we can get the velocity $u_-=\Sigma_2(\rho_+,u_+)$ being the speed of the fluid coming into the 2-rarefaction wave from the vacuum. This $2-$rarefaction wave connecting the vacuum $\rho=0$ to (ρ_+,u_+) is the self-similar solution $(\rho^{r_2},u^{r_2})(\xi)$, $(\xi=\frac{x}{t})$ of (1.2) defined by

$$\rho^{r_2}(\xi)=0, \ \text{ if } \ \xi<\lambda_2(0,u_-)=u_-,$$

$$\lambda_2(\rho^{r_2}(\xi),u^{r_2}(\xi))=\begin{cases} \xi, & \text{if } \ u_-\le\xi\le\lambda_2(\rho_+,u_+), \\ \lambda_2(\rho_+,u_+), & \text{if } \ \xi>\lambda_2(\rho_+,u_+), \end{cases} \tag{1.4}$$

$$\Sigma_2(\rho^{r_2}(\xi),u^{r_2}(\xi))=\Sigma_2(0,u_-)=\Sigma_2(\rho_+,u_+). \tag{1.5}$$

Thus we can define the momentum of 2-rarefaction wave by

$$m^{r_2}(\xi)=\begin{cases} \rho^{r_2}u^{r_2} & \text{if } \ \rho_2^r>0, \\ 0 & \text{if } \ \rho_2^r=0. \end{cases} \tag{1.6}$$

In the present paper, we want to construct a sequence of solutions $(\rho^\epsilon,m^\epsilon)(x,t)$ to the compressible Navier-Stokes equations (1.1) which converge to the 2-rarefaction wave $(\rho_2^r,m_2^r)(x/t)$ defined above as ϵ tends to zero. We will ignore the effects of initial layers by choosing the well-prepared initial data depending on the viscosity for the Navier-Stokes equations .

The main novelty and difficulty of the paper is how to control the degeneracies caused by the vacuum states. To overcome this difficulty, we will first cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve. More precisely, for any $\mu>0$ to be determined, the cut-off rarefaction wave will connect the state $(\rho,u)=(\mu,u_\mu)$ and (ρ_+,u_+) where u_μ can be obtained uniquely by the definition of the 2-rarefaction wave curve.

Then we will construct an approximate rarefaction wave to this cut-off rarefaction wave through the Burgers equation. Finally, the desired solution sequences to the compressible Navier-Stokes equations (1.1) could be constructed around the approximated rarefaction wave. The uniform estimates to the perturbation of the solution sequences around the approximate rarefaction wave can be got by the following two observations. One is the fact that the viscosity ϵ can control the degeneracies caused by the vacuum in some terms by choosing suitably $\mu = \mu(\epsilon)$. In fact, we choose $\mu = \epsilon^{\frac{1}{6}}|\ln\epsilon|$ in the present paper. The other observation is that we can carry out the energy estimates under the a priori assumption that the perturbation is suitably small in $H^1(\mathbf{R})$ norm with some decay rate with respect to ϵ as ϵ tends to zero. See (3.8) in the below for the details. Note that this a priori assumption is natural but is first used in studying zero dissipation limit to our knowledge. With these two observations, we can close the a priori assumption and obtain the desired results.

Now we state our main result as follows.

Theorem 1.1. *Let* $(\rho^{r_2}, m^{r_2})(x/t)$ *be the 2-rarefaction wave defined by* (1.4)-(1.6) *with one-side vacuum state and* $1 < \gamma \leq 2$. *Then there exists a small positive constant* ϵ_0 *such that for any* $\epsilon \in (0, \epsilon_0)$, *we can construct a global smooth solution* $(\rho^\epsilon, m^\epsilon = \rho^\epsilon u^\epsilon)(x,t)$ *to the compressible Navier-Stokes equation (1.1) satisfying*

(1)

$$\begin{aligned}(\rho^\epsilon - \rho^{r_2}, m^\epsilon - m^{r_2}), (\rho^\epsilon, m^\epsilon)_x &\in C^0(0,+\infty; L^2(\mathbf{R})),\\ m^\epsilon_{xx} &\in L^2(0,+\infty; L^2(\mathbf{R})).\end{aligned}$$

(2) As viscosity $\epsilon \to 0$, $(\rho^\epsilon, m^\epsilon)(x,t)$ *converges to* $(\rho^{r_2}, m^{r_2})(x/t)$ *pointwisely except the original point* $(0,0)$. *Furthermore, for any given positive constant* h, *there exists a constant* $C_h > 0$, *independent of* ϵ, *such that*

$$\begin{aligned}&\sup_{t\geq h}\|\rho^\epsilon(\cdot,t) - \rho^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h\epsilon^{1/6}|\ln\epsilon|,\\ &\sup_{t\geq h}\|m^\epsilon(\cdot,t) - m^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h\epsilon^{1/8}|\ln\epsilon|^{-1/2}.\end{aligned} \tag{1.7}$$

Remark 1.2. *Similar results to Theorem 1.1 holds for the case* $\gamma > 2$. *However, some terms like* $\bar{\rho}^{\gamma-2}$, $\rho^{\gamma-3}$ *in (3.14) below will be degenerate on the vacuum states when* $\gamma > 2$. *Consequently, the convergence rate with respect to* ϵ *may be slower than that in Theorem 1.1 and may depend on* γ. *Here we just consider the case* $1 < \gamma \leq 2$ *for simplicity. The case* $\gamma > 2$ *will be shown in the forthcoming paper.*

Remark 1.3. *From the proof of Theorem 1.1 we can see that our main result also holds for a one-dimensional compressible Navier-Stokes equation with density-dependent viscosity, which reads*

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + \rho^\gamma\big)_x = \epsilon\big(\rho^\alpha u_x\big)_x, \end{cases} \tag{1.8}$$

with $\alpha > 0$. For example, to the viscous Saint-Venant system for laminar shallow water derived from incompressible Navier-Stokes system with a moving free surface, see [1, 3], the parameter in (1.8) *is taken as $\alpha = 1, \gamma = 2$. In this situation, since viscosity vanishes at vacuum, the convergence rate may become slower than that in our main Theorem 1.1.*

Remark 1.4. *Our result and method can also be generalized to the full compressible Navier-Stokes equations with heat-conductivity and the Boltzmann equation with slab symmetry. This is left to the forthcoming paper.*

Remark 1.5. *It is also interesting to study the zero dissipation limit of compressible Navier-Stokes equations (1.1) in the case when the Euler system (1.2) has two rarefaction waves with the vacuum states in the middle. However, it is nontrivial to cut off these rarefaction waves with vacuum along corresponding rarefaction wave curves. In fact, the wave structure containing two rarefaction waves with the medium vacuum is destroyed in the cut-off precess, which is quite different from the single rarefaction wave case considered in the present paper.*

The rest of the paper is organized as follows. In section 2, we construct a smooth 2-rarefaction wave which approximates the cut-off rarefaction wave based on the inviscid Burgers equation. And the proof the Theorem 1.1 is given in Section 3.

Throughout this paper, $H^l(\mathbf{R}), l = 0, 1, 2, ...$, denotes the l-th order Sobolev space with its norm

$$\|f\|_l = (\sum_{j=0}^{l} \|\partial_y^j f\|^2)^{\frac{1}{2}}, \quad \text{and } \|\cdot\| := \|\cdot\|_{L^2(dy)},$$

while $L^2(dz)$ means the L^2 integral over $\mathbf{R}$ with respect to the Lebesgue measure dz, and $z = x$ or y. For simplicity, we also write C as generic positive constants which are independent of time t and viscosity ϵ unless otherwise stated.

2 Rarefaction waves

Since there is no exact rarefaction wave profile for the Navier-Stokes equations (1.1), the following approximate rarefaction wave profile satisfying the Euler equations was motivated by Matsumura-Nishihara [11] and Xin [15]. For the completeness of the presentation, we include its definition and the properties listed in Lemma 2.1. However, Lemma 2.1 is a little different from [15] as stated after the Lemma.

Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} w_t + ww_x = 0, \\ w(x,0) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases} \tag{2.1}$$

If $w_- < w_+$, then the Riemann problem (2.1) admits a rarefaction wave solution $w^r(x,t) = w^r(\frac{x}{t})$ given by

$$w^r(\frac{x}{t}) = \begin{cases} w_-, & \frac{x}{t} \le w_-, \\ \frac{x}{t}, & w_- \le \frac{x}{t} \le w_+, \\ w_+, & \frac{x}{t} \ge w_+. \end{cases} \tag{2.2}$$

As in [15], the approximate rarefaction wave to the compressible Navier-Stokes equations (1.1) can be constructed by the solution of the Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_\delta(x) = w(\frac{x}{\delta}) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\delta}, \end{cases} \tag{2.3}$$

where $\delta > 0$ is a small parameter to be determined. In fact, we choose $\delta = \epsilon^{\frac{1}{6}}$ in (3.9) in the following. Note that the solution $w^r_\delta(t,x)$ of the problem (2.3) is given by

$$w^r_\delta(t,x) = w_\delta(x_0(t,x)), \qquad x = x_0(t,x) + w_\delta(x_0(t,x))t.$$

And $w^r_\delta(t,x)$ has the following properties:

Lemma 2.1. *The problem* (2.3) *has a unique smooth global solution* $w^r_\delta(x,t)$ *for each* $\delta > 0$ *such that*

(1) $w_- < w^r_\delta(x,t) < w_+$, $\partial_x w^r_\delta(x,t) > 0$, *for* $x \in \mathbf{R}$, $t \ge 0$, $\delta > 0$.

(2) The following estimates hold for all $t>0$, $\delta>0$ *and* $p\in[1,\infty]$*:*

$$\|\partial_x w_\delta^r(\cdot,t)\|_{L^p} \le C(w_+-w_-)^{1/p}(\delta+t)^{-1+1/p}, \tag{2.4}$$

$$\|\partial_x^2 w_\delta^r(\cdot,t)\|_{L^p} \le C(\delta+t)^{-1}\delta^{-1+1/p}, \tag{2.5}$$

$$\left|\frac{\partial^2 w_\delta^r(x,t)}{\partial x^2}\right| \le \frac{4}{\delta}\frac{\partial w_\delta^r(x,t)}{\partial x}. \tag{2.6}$$

(3) There exist a constant $\delta_0\in(0,1)$ *such that for* $\delta\in(0,\delta_0], t>0$,

$$\|w_\delta^r(\cdot,t)-w^r(\frac{\cdot}{t})\|_{L^\infty} \le C\delta t^{-1}(\ln(1+t)+|\ln\delta|).$$

The proof of Lemma 2.1 can be found in [15]. However, the description of Lemma 2.1 is equivalent to but a little different from [15]. Take the estimation (2.4) as an example, which is described by

$$\|\partial_x w_\delta^r(\cdot,t)\|_{L^p} \le C\min\{(w_+-w_-)\delta^{-1+1/p}, (w_+-w_-)^{1/p}t^{-1+1/p}\}, \tag*{(2.4)$'$}$$

in Xin's paper [15]. In fact, two estimations (2.4) and (2.4)$'$ are equivalent for fixed wave strength w_+-w_-. However, the advantage of Lemma 2.1 is that the energy estimate can be carried out for all the time since there is no singularity to the approximate rarefaction wave even at $t=0$. While in Xin's paper [15], the energy estimate must be done in two time-scalings, that is, finite-time and large-time, due to the singularity of the estimations of the approximate rarefaction wave at $t=0$.

As mentioned in the introduction, we will cut off the 2-rarefaction wave with vacuum along the wave curve in order to overcome the difficulty caused by the vacuum,. More precisely, for any small $\mu>0$ to be determined, we can get a state $(\rho,u)=(\mu,u_\mu)$ belonging to the 2-rarefaction wave curve. From the fact that 2-Riemann invariant $\Sigma_2(\rho,u)$ is constant along the 2-rarefaction wave curve, we can compute explicitly that $u_\mu=\Sigma_2(\rho_+,u_+)+\frac{2}{\gamma-1}\mu^{\frac{\gamma-1}{2}}$. Now we can get a new 2-rarefaction wave $(\rho_\mu^{r_2},u_\mu^{r_2})(\xi)$, $(\xi=x/t)$ connecting the state (μ,u_μ) and the state (ρ_+,u_+) which can be expressed by

$$\lambda_2(\rho_\mu^{r_2},u_\mu^{r_2})(\xi)=\begin{cases}\lambda_2(\mu,u_\mu), & \xi<\lambda_2(\mu,u_\mu),\\ \xi, & \lambda_2(\mu,u_\mu)\le\xi\le\lambda_2(\rho_+,u_+),\\ \lambda_2(\rho_+,u_+), & \xi>\lambda_2(\rho_+,u_+).\end{cases} \tag{2.7}$$

$$\Sigma_2(\rho_\mu^{r_2},u_\mu^{r_2})=\Sigma_2(\mu,u_\mu)=\Sigma_2(\rho_+,u_+). \tag{2.8}$$

It is easy to show that the cut-off 2-rarefaction wave $(\rho_\mu^{r_2},u_\mu^{r_2})(x/t)$ approaches 2-rarefaction wave with vacuum $(\rho^{r_2},u^{r_2})(x/t)$ as μ tends to zero. So we have

Lemma 2.2. *There exist a constant $\mu_0 \in (0,1)$ such that for $\mu \in (0,\mu_0], t > 0$,*

$$\|(\rho_\mu^{r_2}, m_\mu^{r_2})(\cdot/t) - (\rho^{r_2}, m^{r_2})(\cdot/t)\|_{L^\infty} \leq C\mu.$$

The proof is directly from the explicit solution formula, so we omit it for brevity.

Now the approximate rarefaction wave $(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x,t)$ of the cut-off 2-rarefaction wave $(\rho_\mu^{r_2}, u_\mu^{r_2})(\frac{x}{t})$ to compressible Navier-Stokes equations (1.2) can be defined by

$$\begin{cases} w_+ = \lambda_2(\rho_+, u_+), \quad w_- = \lambda_2(\mu, u_\mu), \\ w_\delta^r(t,x) = \lambda_2(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(t,x), \\ \Sigma_2(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x,t) = \Sigma_2(\rho_+, u_+) = \Sigma_2(\mu, u_\mu), \end{cases} \tag{2.9}$$

where w_δ^r is defined in (2.3). From then on, we will omit the subscription of $(\bar{\rho}_{\delta,\mu}, \bar{u}_{\delta,\mu})(x,t)$ as $(\bar{\rho}, \bar{u})(x,t)$ for simplicity. Then the approximate cut-off 2-rarefaction wave $(\bar{\rho}, \bar{u})$ defined above satisfies

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\ (\bar{\rho}\bar{u})_t + \big(\bar{\rho}\bar{u}^2 + p(\bar{\rho})\big)_x = 0, \end{cases} \tag{2.10}$$

and the properties listed in the following Lemma.

Lemma 2.3. *The approximate cut-off 2-rarefaction wave $(\bar{\rho}, \bar{u})$ defined in (2.9) satisfies the following properties:*

(i) $\bar{u}_x(x,t) = \frac{2}{\gamma+1}(w_\delta^r)_x > 0$, *for* $x \in \mathbf{R}$, $t \geq 0$;
$\bar{\rho}_x = \bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_x$, *and* $\bar{\rho}_{xx} = \bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_{xx} + \frac{3-\gamma}{2} \bar{\rho}^{2-\gamma} (\bar{u}_x)^2$.

(ii) The following estimates hold for all $t > 0$, $\delta > 0$ *and* $p \in [1,\infty]$*:*

$$\|\bar{u}_x(\cdot,t)\|_{L^p} \leq C(w_+ - w_-)^{1/p}(\delta + t)^{-1+1/p},$$

$$\|\bar{u}_{xx}(\cdot,t)\|_{L^p} \leq C(\delta + t)^{-1}\delta^{-1+1/p}.$$

(iii) There exist a constant $\delta_0 \in (0,1)$ such that for $\delta \in (0,\delta_0], t > 0$,

$$\|(\bar{\rho} - \rho_\mu^{r_2}, \bar{u} - u_\mu^{r_2})(\cdot,t)\|_{L^\infty} \leq C\delta t^{-1}(\ln(1+t) + |\ln\delta|).$$

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we regard the solution $(\rho^\epsilon, u^\epsilon)$ as the perturbation around the approximate rarefaction wave $(\bar{\rho}, \bar{u})$. Consider the Cauchy problem for (1.1) with smooth initial data

$$(\rho^\epsilon, u^\epsilon)(x, t = 0) = (\bar{\rho}, \bar{u})(x, 0). \tag{3.1}$$

Then we introduce the perturbation

$$(\phi, \psi)(y, \tau) = (\rho^\epsilon, u^\epsilon)(x, t) - (\bar{\rho}, \bar{u})(x, t), \tag{3.2}$$

where y, τ are the scaled variables as

$$y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}, \tag{3.3}$$

and $(\rho^\epsilon, u^\epsilon)$ is assumed to be the solution to the problem (1.1). For the simplicity of the notation, we will omit the superscription of $(\rho^\epsilon, u^\epsilon)$ as (ρ, u) from now on if there is no confusion of the notation. Substituting (3.2) and (3.3) into (1.1) and (3.1) and using the definition for $(\bar{\rho}, \bar{u})$, we obtain

$$\phi_\tau + \rho\psi_y + u\phi_y = -f, \tag{3.4}$$

$$\rho\psi_\tau + \rho u\psi_y + p'(\rho)\phi_y - \psi_{yy} = -g, \tag{3.5}$$

$$(\phi, \psi)(y, 0) = 0, \tag{3.6}$$

where

$$\begin{cases} f = \bar{u}_y\phi + \bar{\rho}_y\psi, \\ g = -\bar{u}_{yy} + \rho\psi\bar{u}_y + \bar{\rho}_y\Big[p'(\rho) - \dfrac{\rho}{\bar{\rho}}p'(\bar{\rho})\Big]. \end{cases} \tag{3.7}$$

We seek a global (in time) solution (ϕ, ψ) to the problem (3.4) – (3.6). To this end, we define the solution space for (3.4) – (3.6) by

$$\begin{aligned} X(0, \tau_1) = \Big\{(\phi, \psi)\Big|(\phi, \psi) \in C^0(0, \tau_1; H^1(\mathbf{R})), \quad \phi_y \in L^2(0, \tau_1; L^2(\mathbf{R})), \\ \psi_y \in L^2(0, \tau_1; H^1(\mathbf{R}))\Big\} \end{aligned}$$

with $0 < \tau_1 \leq +\infty$.

In what follows, we always carry out the analysis under the a priori assumptions

$$\sup_{0\leq\tau\leq\tau_1} \|\phi(\cdot, \tau)\|_1 \leq \epsilon^{1/6}, \quad \sup_{0\leq\tau\leq\tau_1} \|\psi(\cdot, \tau)\|_1 \leq \epsilon^{1/8}. \tag{3.8}$$

Take

$$\mu = \epsilon^{1/6} |\ln \epsilon|, \qquad \delta = \varepsilon^{1/6} \tag{3.9}$$

in the sequel. Then we have $\mu \geq 2\epsilon^{1/6}$ if $\epsilon \ll 1$. Under the a priori assumption (3.8), we can get

$$\frac{\bar{\rho}}{2} \leq \rho \leq \frac{3\bar{\rho}}{2}. \tag{3.10}$$

In fact,

$$\rho = \bar{\rho} + \phi \geq \bar{\rho} - \|\phi\|_{L^\infty} \geq \bar{\rho} - \epsilon^{1/6} \geq \bar{\rho} - \frac{1}{2}\mu \geq \frac{\bar{\rho}}{2}, \tag{3.11}$$

$$\rho = \bar{\rho} + \phi \leq \bar{\rho} + \|\phi\|_{L^\infty} \leq \bar{\rho} + \epsilon^{1/6} \leq \bar{\rho} + \frac{1}{2}\mu \leq \frac{3\bar{\rho}}{2}, \tag{3.12}$$

where we used Sobolev inequality $\|\phi\|_{L^\infty} \leq \sqrt{2}\|\phi\|^{1/2}\|\phi_y\|^{1/2}$. Moreover, we can obtain

$$C_1\bar{\rho}^{\gamma-2}\phi^2 \leq p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi \leq C_2\bar{\rho}^{\gamma-2}\phi^2. \tag{3.13}$$

under the a priori assumption (3.8) and the condition $1 \leq \gamma \leq 2$.

Since the proof for the local existence of the solution to (3.4) $-$ (3.6) is standard, we omit it for brevity. To prove Theorem 1.1, it is sufficient to prove the following a priori estimates.

Lemma 3.1. ***(A priori estimates)*** *Let $1 < \gamma \leq 2$ and $(\phi, \psi) \in X(0, \tau_1)$ be a solution to the problem (3.4) $-$ (3.6). Then under the a priori assumption (3.8), there exist positive constants ϵ_0 and C independent of ϵ, such that if $0 < \epsilon \leq \epsilon_0$, then*

$$\begin{aligned} &\int_{\mathbf{R}} \Big(\bar{\rho}\psi^2 + \bar{\rho}^{\gamma-2}\phi^2 + \phi_y^2 + \psi_y^2\Big)(\tau, y)dy \\ &+ \int_0^\tau \int_{\mathbf{R}} \Big[\psi_y^2 + \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2 + \bar{\rho}\bar{u}_y\psi^2 + \bar{\rho}^{\gamma-3}\phi_y^2 + \frac{\psi_{yy}^2}{\bar{\rho}}\Big]dyd\tau \leq C\epsilon^{1/3}|\ln \epsilon|^{-1/2}, \end{aligned} \tag{3.14}$$

Consequently,

$$\begin{aligned} \sup_{0\leq\tau\leq\tau_1} \|\phi(\cdot, \tau)\|_{L^\infty} \leq C\epsilon^{1/6}|\ln \epsilon|^{-1/4}, \\ \sup_{0\leq\tau\leq\tau_1} \|\psi(\cdot, \tau)\|_{L^\infty} \leq C\epsilon^{1/8}|\ln \epsilon|^{-1/2}. \end{aligned} \tag{3.15}$$

Proof of Lemma 3.1: The proof of Lemma 3.1 consists of the following steps.

Step 1. First, define

$$E := \Phi(\rho, \bar{\rho}) + \frac{\psi^2}{2}, \tag{3.16}$$

where

$$\Phi(\rho,\bar\rho):=\int_{\bar\rho}^{\rho}\frac{p(\xi)-p(\bar\rho)}{\xi^2}d\xi=\frac{1}{(\gamma-1)\rho}\big(p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi\big). \tag{3.17}$$

Direct computations yield

$$\begin{aligned}&\Big(\rho E\Big)_\tau+\Big[\rho uE-\psi_y\psi+\big(p(\rho)-p(\bar\rho)\big)\psi\Big]_y\\&+\psi_y^2+\bar u_y\Big(p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi\Big)+\psi^2\rho\bar u_y=\bar u_{yy}\psi.\end{aligned}$$

Then integrating the above equation over $\mathbf{R}^1\times[0,\tau]$ and using (3.10), (3.13) and (3.17) imply

$$\int_{\mathbf{R}}\Big(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\Big)dy+\int_0^\tau\int_{\mathbf{R}}\Big(\psi_y^2+\bar\rho^{\gamma-2}\bar u_y\phi^2+\bar\rho\bar u_y\psi^2\Big)dyd\tau\le C\int_0^\tau\int_{\mathbf{R}}|\bar u_{yy}\psi|dyd\tau. \tag{3.18}$$

By Sobolev inequality and Lemma 2.3, we can obtain

$$\begin{aligned}&\int_0^\tau\int_{\mathbf{R}}|\bar u_{yy}\psi|dyd\tau\le C\int_0^\tau\|\bar u_{yy}\|_{L^1}\|\psi\|^{1/2}\|\psi_y\|^{1/2}d\tau\\ \le\;& C\int_0^\tau\frac{1}{\tau+\delta/\epsilon}\|\psi\|^{1/2}\|\psi_y\|^{1/2}d\tau\\ \le\;&\frac18\int_0^\tau\|\psi_y\|^2d\tau+C\int_0^\tau(\frac{1}{\tau+\delta/\epsilon})^{4/3}\|\psi\|^{2/3}d\tau\\ \le\;&\frac18\int_0^\tau\|\psi_y\|^2d\tau+\frac18\sup_{[0,\tau]}\|\sqrt{\bar\rho}\psi\|^2+C\Big(\mu^{-1/3}\int_0^\infty(\frac{1}{\tau+\delta/\epsilon})^{4/3}d\tau\Big)^{3/2}\\ \le\;&\frac18\int_0^\tau\|\psi_y\|^2d\tau+\frac18\sup_{[0,\tau]}\|\sqrt{\bar\rho}\psi\|^2+C(\frac{\epsilon}{\mu\delta})^{1/2}.\end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) and recalling (3.9) yield

$$\int_{\mathbf{R}}\Big(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\Big)dy+\int_0^\tau\int_{\mathbf{R}}\Big[\psi_y^2+\bar\rho^{\gamma-2}\bar u_y\phi^2+\bar\rho\bar u_y\psi^2\Big]dyd\tau\le C\epsilon^{1/3}|\ln\epsilon|^{-1/2}. \tag{3.20}$$

Step 2. We make use of the idea in [6] with modifications to derive the estimation of ϕ_y. We first differentiate (3.4) with respect to y and then multiply the resulted equation by ϕ_y/ρ^3 to get

$$(\frac{\phi_y^2}{2\rho^3})_\tau+(\frac{u\phi_y^2}{2\rho^3})_y+\frac{\psi_{yy}\phi_y}{\rho^2}=-\frac{\phi_y}{\rho^3}(\bar u_{yy}\phi+\bar\rho_{yy}\psi+2\bar\rho_y\psi_y). \tag{3.21}$$

Multiplying (3.5) by ϕ_y/ρ^2 gives

$$
\begin{aligned}
&(\frac{\psi\phi_y}{\rho})_\tau-(\frac{\psi\phi_\tau}{\rho}+\bar\rho_y\frac{\psi^2}{\rho})_y-\psi_y^2+p'(\rho)\frac{\phi_y^2}{\rho^2}-\frac{\psi_{yy}\phi_y}{\rho^2}\\
&-\bar u_y\frac{\psi_y\phi}{\rho}+2\bar\rho_y\frac{\psi\psi_y}{\rho}+\bar\rho_{yy}\frac{\psi^2}{\rho}+\bar\rho_y\bar u_y\frac{\psi\phi}{\rho^2}-\bar\rho\bar u_y\frac{\psi\phi_y}{\rho^2}=-g\frac{\phi_y}{\rho^2}.
\end{aligned}
\tag{3.22}
$$

Adding (3.21) and (3.22) together, then integrating the resulted equation over $\mathbf{R}^1\times[0,\tau]$ imply

$$
\begin{aligned}
&\int_{\mathbf{R}}\Big(\frac{\phi_y^2}{2\rho^3}+\frac{\psi\phi_y}{\rho}\Big)dy+\int_0^\tau\int_{\mathbf{R}}p'(\rho)\frac{\phi_y^2}{\rho^2}dyd\tau\\
=\;&\int_0^\tau\int_{\mathbf{R}}\Big\{\psi_y^2+\bar u_y\frac{\psi_y\phi}{\rho}-2\bar\rho_y\frac{\psi\psi_y}{\rho}-\bar\rho_{yy}\frac{\psi^2}{\rho}-\bar\rho_y\bar u_y\frac{\psi\phi}{\rho^2}+\bar\rho\bar u_y\frac{\psi\phi_y}{\rho^2}\\
&-\frac{\phi_y}{\rho^3}\Big(\bar u_{yy}\phi+\bar\rho_{yy}\psi+2\bar\rho_y\psi_y\Big)-g\frac{\phi_y}{\rho^2}\Big\}dyd\tau.
\end{aligned}
\tag{3.23}
$$

Combination of (3.20) and (3.23) leads to

$$
\begin{aligned}
&\int_{\mathbf{R}}\Big(\frac{\phi_y^2}{\bar\rho^3}+\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\Big)dy+\int_0^\tau\int_{\mathbf{R}}\Big(\psi_y^2+\bar\rho^{\gamma-2}\bar u_y\phi^2+\bar\rho\bar u_y\psi^2+\bar\rho^{\gamma-3}\phi_y^2\Big)dyd\tau\\
&\le C\int_0^\tau\int_{\mathbf{R}}\Big\{|\bar u_y\frac{\psi_y\phi}{\bar\rho}|+|\bar\rho_y\frac{\psi\psi_y}{\bar\rho}|+|\bar\rho_{yy}\frac{\psi^2}{\bar\rho}|+|\bar\rho_y\bar u_y\frac{\psi\phi}{\bar\rho^2}|+|\bar u_y\frac{\psi\phi_y}{\bar\rho}|+|\bar\rho_y\frac{\phi_y}{\bar\rho^3}\psi_y|\\
&\qquad+|\frac{\bar u_{yy}}{\bar\rho^3}\phi_y\phi|+|\frac{\bar\rho_{yy}}{\bar\rho^3}\phi_y\psi|+|g\frac{\phi_y}{\bar\rho^2}|\Big\}dyd\tau+C\epsilon^{1/3}|\ln\epsilon|^{-1/2}\\
&:=\sum_{i=1}^9 I_i+C\epsilon^{1/3}|\ln\epsilon|^{-1/2}.
\end{aligned}
\tag{3.24}
$$

Then we estimate the terms on the right hand side of (3.24) one by one. By Cauchy's inequality, Lemma 2.3 and (3.10), we obtain

$$
\begin{aligned}
I_1\;&=\int_0^\tau\int_{\mathbf{R}}|\bar u_y\frac{\psi_y\phi}{\bar\rho}|dyd\tau\le\frac18\int_0^\tau\|\psi_y\|^2d\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma-2}\bar u_y\phi^2\frac{\bar u_y}{\bar\rho^\gamma}dyd\tau\\
&\le\frac18\int_0^\tau\|\psi_y\|^2d\tau+C\mu^{-\gamma}\delta^{-1}\epsilon\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma-2}\bar u_y\phi^2dyd\tau\\
&\le\frac18\int_0^\tau\|\psi_y\|^2d\tau+\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma-2}\bar u_y\phi^2dyd\tau
\end{aligned}
\tag{3.25}
$$

where we have used the fact that

$$
C\mu^{-\gamma}\delta^{-1}\epsilon\le C\mu^{-2}\delta^{-1}\epsilon=C\epsilon^{\frac12}|\ln\epsilon|^{-2}\le\frac18,\qquad \text{if }\ \epsilon\ll1.
$$

Note that

$$\bar{\rho}_y = \bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_y, \tag{3.26}$$

we can get

$$\begin{aligned} I_2 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_y \frac{\psi\psi_y}{\bar{\rho}}| dy d\tau \\ &\le \frac{1}{8}\int_0^\tau \|\psi_y\|^2 d\tau + C\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{2-\gamma}\bar{u}_y(\bar{\rho}\bar{u}_y\psi^2) dy d\tau \\ &\le \frac{1}{8}\int_0^\tau \|\psi_y\|^2 d\tau + C\delta^{-1}\epsilon \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau \\ &\le \frac{1}{8}\int_0^\tau \|\psi_y\|^2 d\tau + \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau \end{aligned} \tag{3.27}$$

if $\epsilon \ll 1$.

Recalling that $|\omega^r_{\delta xx}| \le C\dfrac{\omega^r_{\delta x}}{\delta}$ from Lemma 2.1 and the fact (i) in Lemma 2.3, we have

$$\begin{aligned} \bar{\rho}_{xx} &= \frac{2}{\gamma+1}\bar{\rho}^{\frac{3-\gamma}{2}}\omega^r_{\delta xx} + \frac{2(3-\gamma)}{(\gamma+1)^2}\bar{\rho}^{2-\gamma}(\omega^r_{\delta x})^2 \\ &\le C\Big(\bar{\rho}^{\frac{3-\gamma}{2}}\frac{\bar{u}_x}{\delta} + \bar{\rho}^{2-\gamma}\bar{u}_x^2\Big). \end{aligned} \tag{3.28}$$

Thus we can compute

$$\begin{aligned} I_3 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_{yy}\frac{\psi^2}{\bar{\rho}}| dy d\tau \\ &\le C\frac{\epsilon}{\delta}\int_0^\tau \int_{\mathbf{R}} |\bar{\rho}\bar{u}_y\psi^2\bar{\rho}^{-\frac{\gamma+1}{2}}| dy d\tau + C\frac{\epsilon}{\delta}\int_0^\tau \int_{\mathbf{R}} |\bar{\rho}\bar{u}_y\psi^2\bar{\rho}^{2-\gamma}| dy d\tau \\ &\le C\frac{\epsilon}{\delta\mu^{\frac{3}{2}}}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau \\ &\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau \qquad \text{if } \ \gamma \in (1,2] \ \text{ and } \ \varepsilon \ll 1. \end{aligned} \tag{3.29}$$

Similarly, we can get

$$I_8 = \int_0^\tau \int_{\mathbf{R}} |\frac{\bar{\rho}_{yy}}{\bar{\rho}^3}\phi_y\psi| dy d\tau \le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau. \tag{3.30}$$

By Lemma 2.3 and (3.26), we can obtain

$$
\begin{aligned}
I_4 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_y \bar{u}_y \frac{\psi\phi}{\bar{\rho}^2}| dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y \psi^2 dy d\tau + C\int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}^{\gamma-2}\phi^2 \frac{\bar{\rho}_y^2}{\bar{\rho}^{3+\gamma}} dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y \psi^2 dy d\tau + C\frac{\epsilon^2}{\mu^{3+\gamma}\delta^2}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2}\bar{u}_y \phi^2 dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y \psi^2 dy d\tau + \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2}\bar{u}_y \phi^2 dy d\tau
\end{aligned}
\tag{3.31}
$$

due to the fact that

$$
C\frac{\epsilon^2}{\mu^{3+\gamma}\delta^2} \le C\epsilon^{\frac{5}{6}}|\ln\epsilon|^{-5} \le \frac{1}{8}, \qquad \text{if } \gamma \in (1,2] \text{ and } \varepsilon \ll 1.
$$

Similarly, we can get

$$
\begin{aligned}
I_5 &= \int_0^\tau \int_{\mathbf{R}} |\bar{u}_y \frac{\psi\phi_y}{\bar{\rho}}| dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y \psi^2 \frac{\bar{u}_y}{\bar{\rho}^\gamma} dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\frac{\epsilon}{\delta\mu^\gamma}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y \psi^2 dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y \psi^2 dy d\tau.
\end{aligned}
\tag{3.32}
$$

By Lemma 2.3, the equality (3.26) and Cauchy's inequality, we have

$$
\begin{aligned}
I_6 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_y \frac{\phi_y}{\bar{\rho}^3}\psi_y| dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\int_0^\tau \int_{\mathbf{R}} \frac{\bar{u}_y^2}{\bar{\rho}^{2\gamma}}\psi_y^2 dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \|\rho^{(\gamma-3)/2}\phi_y\|^2 d\tau + C\frac{\epsilon^2}{\delta^2\mu^{2\gamma}}\int_0^\tau \int_{\mathbf{R}} \psi_y^2 dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \|\rho^{(\gamma-3)/2}\phi_y\|^2 d\tau + C\epsilon|\ln\epsilon|^{-4}\int_0^\tau \int_{\mathbf{R}} \psi_y^2 dy d\tau.
\end{aligned}
\tag{3.33}
$$

Similarly, we can calculate

$$
\begin{aligned}
I_7 &= \int_0^\tau \int_{\mathbf{R}} |\frac{\bar{u}_{yy}}{\bar{\rho}^3}\phi_y\phi| dyd\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\frac{\epsilon}{\delta}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2|\bar{u}_{yy}|\bar{\rho}^{-1-2\gamma} dyd\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\frac{\epsilon^3}{\delta^3\mu^{1+2\gamma}}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y\bar{\rho}^{\gamma-2}\phi^2 dyd\tau \\
&\le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\epsilon^{\frac{5}{3}}|\ln\epsilon|^{-5}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y\bar{\rho}^{\gamma-2}\phi^2 dyd\tau.
\end{aligned}
\tag{3.34}
$$

Finally, we have

$$
I_9 = \int_0^\tau \int_{\mathbf{R}} |g\frac{\phi_y}{\bar{\rho}^2}| dyd\tau \le \frac{1}{8}\int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2 d\tau + C\int_0^\tau \int_{\mathbf{R}} \frac{g^2}{\bar{\rho}^{1+\gamma}} dyd\tau. \tag{3.35}
$$

Recalling that (3.7), (3.10) and (3.26), we obtain

$$
\begin{aligned}
|g| &\le |\bar{u}_{yy}| + |\bar{\rho}\bar{u}_y\psi| + C|\bar{\rho}^{\gamma-2}\bar{\rho}_y\phi| \\
&\le |\bar{u}_{yy}| + |\bar{\rho}\bar{u}_y\psi| + C|\bar{\rho}^{\frac{\gamma-1}{2}}\bar{u}_y\phi|.
\end{aligned}
\tag{3.36}
$$

Thus the last term in (3.35) can be estimated by

$$
\begin{aligned}
& \Big|\int_0^\tau \int_{\mathbf{R}} \frac{g^2}{\bar{\rho}^{1+\gamma}} dyd\tau\Big| \\
\le\ & C\int_0^\tau \int_{\mathbf{R}} \frac{\bar{u}_{yy}^2}{\bar{\rho}^{1+\gamma}} dyd\tau + C\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{1-\gamma}\bar{u}_y^2\psi^2 dyd\tau + C\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{-2}\bar{u}_y^2\phi^2 dyd\tau \\
\le\ & C\frac{\epsilon^3}{\mu^{1+\gamma}}\int_0^\tau \|\bar{u}_{xx}\|_{L^2(dx)}^2 d\tau + C\frac{\epsilon}{\delta\mu^\gamma}\int_0^\tau \int_{\mathbf{R}} \left(\bar{\rho}\bar{u}_y\psi^2 + \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2\right) dyd\tau \\
\le\ & C\frac{\epsilon}{\delta\mu^3}\int_0^\tau (\tau+\frac{\delta}{\epsilon})^{-2} d\tau + C\frac{\epsilon}{\delta\mu^2}\int_0^\tau \int_{\mathbf{R}} \left(\bar{\rho}\bar{u}_y\psi^2 + \bar{u}_y\bar{\rho}^{\gamma-2}\phi^2\right) dyd\tau \\
\le\ & C\epsilon^{\frac{7}{6}}|\ln\epsilon|^{-3} + C\epsilon^{\frac{1}{2}}|\ln\epsilon|^{-2}\int_0^\tau \int_{\mathbf{R}} \left(\bar{\rho}\bar{u}_y\psi^2 + \bar{u}_y\bar{\rho}^{\gamma-2}\phi^2\right) dyd\tau \\
\le\ & C\epsilon^{\frac{7}{6}}|\ln\epsilon|^{-3} + \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \left(\bar{\rho}\bar{u}_y\psi^2 + \bar{u}_y\bar{\rho}^{\gamma-2}\phi^2\right) dyd\tau, \qquad \text{if } \epsilon \ll 1.
\end{aligned}
\tag{3.37}
$$

Combining (3.24)-(3.37), we obtain

$$
\begin{aligned}
& \int_{\mathbf{R}} \left(\bar{\rho}\psi^2 + \bar{\rho}^{\gamma-2}\phi^2 + \phi_y^2\right) dy + \int_0^\tau \int_{\mathbf{R}} \left(\psi_y^2 + \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2 + \bar{\rho}\bar{u}_y\psi^2 + \bar{\rho}^{\gamma-3}\phi_y^2\right) dyd\tau \\
\le\ & C\epsilon^{1/3}|\ln\epsilon|^{-1/2}.
\end{aligned}
\tag{3.38}
$$

Step 3. As the last step, we estimate $\sup\limits_{\tau}\|\psi_y\|$. For this, multiplying (3.5) by $-\psi_{yy}/\rho$ gives

$$(\frac{\psi_y^2}{2})_\tau-(\psi_y\psi_\tau+u\frac{\psi_y^2}{2})_y+u_y\frac{\psi_y^2}{2}-p'(\rho)\frac{\phi_y\psi_{yy}}{\rho}+\frac{\psi_{yy}^2}{\rho}=g\frac{\psi_{yy}}{\rho}. \tag{3.39}$$

Integrating the above equation over $\mathbf{R}^1\times[0,\tau]$ yields

$$\int_{\mathbf{R}}\frac{\psi_y^2}{2}dy+\int_0^\tau\int_{\mathbf{R}}\Big(\frac{\bar{u}_y\psi_y^2}{2}+\frac{\psi_{yy}^2}{\rho}\Big)dyd\tau=\int_0^\tau\int_{\mathbf{R}}\Big\{p'(\rho)\frac{\psi_{yy}\phi_y}{\rho}+g\frac{\psi_{yy}}{\rho}-\frac{\psi_y^3}{2}\Big\}dyd\tau. \tag{3.40}$$

Firstly, we have

$$\Big|\int_0^\tau\int_{\mathbf{R}}p'(\rho)\frac{\psi_{yy}\phi_y}{\rho}dyd\tau\Big|\le\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\psi_{yy}^2}{\rho}dyd\tau+C\int_0^\tau\|\bar{\rho}^{\frac{\gamma-3}{2}}\phi_y\|^2d\tau. \tag{3.41}$$

Then from (3.37) and (3.38), we obtain

$$\begin{aligned}\Big|\int_0^\tau\int_{\mathbf{R}}g\frac{\psi_{yy}}{\rho}dyd\tau\Big|\le&\ \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\psi_{yy}^2}{\rho}dyd\tau+C\Big|\int_0^\tau\int_{\mathbf{R}}\frac{g^2}{\bar{\rho}}dyd\tau\Big|\\ \le&\ \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\psi_{yy}^2}{\rho}dyd\tau+C\epsilon^{1/3}|\ln\epsilon|^{-1/2}.\end{aligned} \tag{3.42}$$

Furthermore, we can compute that

$$\begin{aligned}\Big|\int_0^\tau\int_{\mathbf{R}}\frac{\psi_y^3}{2}dyd\tau\Big|&\le C\int_0^\tau\|\psi_{yy}\|^{\frac{1}{2}}\|\psi_y\|^{\frac{5}{2}}d\tau\\ &\le\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\psi_{yy}^2}{\rho}dyd\tau+C\int_0^\tau\|\psi_y\|^{\frac{10}{3}}d\tau\\ &\le\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\psi_{yy}^2}{\rho}dyd\tau+C\sup_{[0,\tau_1]}\|\psi_y\|^{\frac{4}{3}}\int_0^\tau\|\psi_y\|^2d\tau\\ &\le\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\psi_{yy}^2}{\rho}dyd\tau+C\epsilon^{\frac{1}{6}}\int_0^\tau\|\psi_y\|^2d\tau.\end{aligned} \tag{3.43}$$

Combining (3.40)-(3.43) and using (3.10) and (3.38), we have

$$\int_{\mathbf{R}}\psi_y^2dy+\int_0^\tau\int_{\mathbf{R}}\Big(\bar{u}_y\psi_y^2+\frac{\psi_{yy}^2}{\bar{\rho}}\Big)dyd\tau\le C\epsilon^{1/3}|\ln\epsilon|^{-1/2}. \tag{3.44}$$

Therefore, (3.14) can be verified by combining (3.38) and (3.44). By using Sobolev inequality, we can get

$$\sup_{0\le\tau\le\tau_1}\|\phi(\cdot,\tau)\|_{L^\infty}\le\sqrt{2}\sup_{0\le\tau\le\tau_1}\|\phi(\cdot,\tau)\|^{1/2}\|\phi_y(\cdot,\tau)\|^{1/2}\le C\epsilon^{1/6}|\ln\epsilon|^{-1/4}, \tag{3.45}$$

and

$$
\begin{aligned}
\sup_{0\le\tau\le\tau_1}\|\psi(\cdot,\tau)\|_{L^\infty} &\le \sqrt{2}\sup_{0\le\tau\le\tau_1}\|\psi(\cdot,\tau)\|^{1/2}\|\psi_y(\cdot,\tau)\|^{1/2}\\
&\le \sqrt{2}\mu^{-\frac14}\sup_{0\le\tau\le\tau_1}\|\sqrt{\bar\rho}\psi(\cdot,\tau)\|^{1/2}\|\psi_y(\cdot,\tau)\|^{1/2} \qquad (3.46)\\
&\le C\epsilon^{1/8}|\ln\epsilon|^{-1/2}.
\end{aligned}
$$

So we verify the a priori assumption (3.8) if $\epsilon \ll 1$ and complete the proof of Lemma 3.1.

□

Proof of Theorem 1.1: Recall that $\mu = \epsilon^{1/6}|\ln\epsilon|,\ \ \delta = \epsilon^{1/6}$. For any given positive constant h, there exist a constant $C_h > 0$, independent of ϵ, such that

$$
\begin{aligned}
&\sup_{t\ge h}\|\rho(\cdot,t)-\rho^r(\frac{\cdot}{t})\|_{L^\infty}\\
\le\ & \sup_{t\ge h}\Big(\|\phi(\cdot,\tau)\|_{L^\infty}+\|\bar\rho(\cdot,t)-\rho^r_\mu(\frac{\cdot}{t})\|_{L^\infty}+\|\rho^r_\mu(\frac{\cdot}{t})-\rho^r(\frac{\cdot}{t})\|_{L^\infty}\Big)\\
\le\ & C_h\Big(\epsilon^{1/6}|\ln\epsilon|^{-1/4}+\delta|\ln\delta|+\mu\Big)\\
\le\ & C_h\epsilon^{1/6}|\ln\epsilon|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{t\ge h}\|m(\cdot,t)-m^r(\frac{\cdot}{t})\|_{L^\infty}\\
\le\ & C\sup_{t\ge h}\Big(\|\psi(\cdot,\tau)\|_{L^\infty}+\|\phi(\cdot,\tau)\|_{L^\infty}+\|\bar m(\cdot,t)-m^r_\mu(\frac{\cdot}{t})\|_{L^\infty}+\|m^r_\mu(\frac{\cdot}{t})-m^r(\frac{\cdot}{t})\|_{L^\infty}\Big)\\
\le\ & C_h\Big(\epsilon^{1/8}|\ln\epsilon|^{-1/2}+\epsilon^{1/6}|\ln\epsilon|^{-1/4}+\delta|\ln\delta|+\mu\Big)\\
\le\ & C_h\epsilon^{1/8}|\ln\epsilon|^{-1/2}.
\end{aligned}
$$

Thus the proof of Theorem 1.1 is complete. □

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