

A CATEGORIFICATION OF THE STABLE $SU(2)$ WITTEN-RESHETIKHIN-TURAEV INVARIANT OF LINKS IN $\mathbb{S}^2 \times \mathbb{S}^1$

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ABSTRACT. The WRT invariant of a link L in $\mathbb{S}^2 \times \mathbb{S}^1$ at sufficiently high values of the level r can be expressed as an evaluation of a special polynomial invariant of L at $q = \exp(\pi i/r)$. We categorify this polynomial invariant by associating to L a bi-graded homology whose graded Euler characteristic is equal to this polynomial.

If L is presented as a circular closure of a tangle τ in $\mathbb{S}^2 \times \mathbb{S}^1$, then the homology of L is defined as the Hochschild homology of the H_n -bimodule associated to τ in [Kho00]. This homology can also be expressed as a stable limit of the Khovanov homology of the circular closure of τ in \mathbb{S}^3 through a torus braid with high twist.

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1. INTRODUCTION

1.1. The stable WRT invariant of links in a 3-sphere with handles. Let $Z(O) \in \mathbb{Q}[[q, q^{-1}]]$ be a Laurent series of q which is a topological invariant of an object O (e.g. a link in a 3-manifold). A general idea of a (weak) categorification, as presented in [Kho00], is to associate to O a $\mathbb{Z} \oplus \mathbb{Z}$ -graded vector space $\mathcal{H}(O) = \bigoplus_{i,j \in \mathbb{Z}} \mathcal{H}_{i,j}(O)$, the first degree being of homological nature, such that its graded Euler characteristic is equal to $Z(O)$:

$$Z(O) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}_{i,j}(O). \quad (1.1)$$

A full categorification extends this assignment to a functor from the category of link cobordisms to the category of homogeneous maps between bigraded vector spaces.

Let L be a framed link in a 3-manifold M . The $SU(2)$ Witten-Reshetikhin-Turaev (WRT) invariant $Z_r(L, M)$ is a \mathbb{C} -valued topological invariant of the pair (L, M) which depends on an integer number $r \geq 2$. We assume that the components of L are ‘colored’ by the fundamental representation of $SU(2)$ and if L is an empty link, then we omit it from notations.

Generally, the dependence of $Z_r(L, M)$ on r seems random and the non-polynomial nature of $Z_r(L, M)$ presents a challenge for anyone trying to categorify it. However, there is a special class of 3-manifolds (3-spheres with handles) for which $Z_r(L, M)$ is almost polynomial. Let \mathbb{S}_k^3 denote a 3-sphere \mathbb{S}^3 from which k 3-balls are cut. Let $\mathbb{S}_{g|k}^3$ denote oriented manifold constructed by gluing $2g$ spherical boundary components of $\mathbb{S}_{|2g+k}^3$ pairwise. A *3-sphere with g handles* \mathbb{S}_g^3 is defined as $\mathbb{S}_{g|0}^3$: $\mathbb{S}_g^3 = \mathbb{S}_{g|0}^3$. Alternatively, \mathbb{S}_g^3 can be constructed by gluing together two handlebodies of genus g through the identity isomorphism of their boundaries.

A link L in \mathbb{S}_g^3 can be constructed by taking a (n_1, \dots, n_{2g}) -tangle τ in $\mathbb{S}_{|2g}^3$ with pairwise matching valences $n_i = n_{i+g}$ and applying the pairwise gluing to the boundary components of $\mathbb{S}_{|2g}^3$ and to the tangle endpoints which reside there. If at least one of the valences is odd, then the WRT invariant associated with the resulting link is zero for any r , hence we will always assume that all valences are even. For a $(2n_1, \dots, 2n_{2g})$ -tangle τ in $\mathbb{S}_{|2g}^3$ we define the critical level:

$$r_{\text{cr}}(\tau, \mathbb{S}_{|2g}^3) = \max(n_1, \dots, n_g) + 2.$$

We define the critical level $r_{\text{cr}}(L, \mathbb{S}_g^3)$ of a link $L \subset \mathbb{S}_g^3$ as the minimum of $r_{\text{cr}}(\tau, \mathbb{S}_{|2g}^3)$ over all the presentation of L as the closure of a tangle.

The following theorem is easy to prove:

Theorem 1.1. *For a link L in a 3-sphere with g handles S_g^3 there exists a unique rational function $J(L, S_g^3) \in \mathbb{Q}(q)$ such that if $r \geq r_{\text{cr}}(L, S_g^3)$, then the modified WRT invariant*

$$\tilde{Z}_r(L, S_g^3) = \frac{Z_r(L, S_g^3)}{Z_r(S_g^3)}$$

is equal to the evaluation of $J(L, S_g^3)$ at $q = \exp(i\pi/r)$:

$$\tilde{Z}_r(L, S_g^3) = J(L, S_g^3)|_{q=\exp(i\pi/r)}. \tag{1.2}$$

We call $J(L, S_g^3)$ the stable WRT invariant of $L \subset S_g^3$. We expect that Laurent series expansion of $J(L, S_g^3)$ at $q = 0$ can be categorified in the sense of eq.(1.1).

If $g = 0$, then S_g^3 is a 3-sphere S^3 . In this case $r_{\text{cr}}(L, S^3) = 2$ and $J(L, S^3)$ is the Jones polynomial whose categorification was constructed in [Kho00].

In this paper we consider the case of $g = 1$, that is, we study $S_1^3 = S^2 \times S^1$. It turns out that the corresponding stable WRT invariant $J(L, S^2 \times S^1)$ is again a Laurent polynomial, and we construct its categorification.

1.2. A 3-dimensional oriented restricted TQFT. The stable WRT invariant $J(L, S_g^3)$ comes from a stable restricted topological WRT theory. Let us recall the definition of a 3-dimensional oriented restricted TQFT (a generalization to n dimensions is obvious).

Let \mathbb{F} be a field with an involution

$$\bar{\cdot} : \mathbb{F} \longrightarrow \mathbb{F}. \tag{1.3}$$

Usually, $\mathbb{F} = \mathbb{C}$, $\bar{\cdot}$ being the complex conjugation.

Let \mathcal{S} be the set of admissible connected boundaries: its elements are some closed oriented connected 2-manifolds with marked points. \mathcal{S} always includes the empty surface \emptyset .

Let M be a 3-dimensional manifold with a boundary consisting of k connected components from \mathcal{S} , each having n_i ($i = 1, \dots, k$) marked points. An *identification* of the boundary of M is a choice of a diffeomorphism between every connected component of ∂M and its ‘standard copy’ in \mathcal{S} .

A \mathbf{n} -tangle in M , where $\mathbf{n} = (n_1, \dots, n_k)$, is an embedding of a disjoint union of segments and circles in M such that the endpoints of segments are mapped one-to-one to the marked points on the boundary ∂M . Our tangles are assumed to be framed. Let $\tilde{\mathfrak{T}}(M)$ be the set of all \mathbf{n} -tangles in M distinguished up to boundary fixing ambient isotopy.

A diffeomorphism $M \rightarrow M$ is called *relative* if it acts trivially on the boundary of M . Let Map_M denote the mapping class group of relative homeomorphisms of M . This group

acts on $\tilde{\mathfrak{T}}(M)$, and we refer to the elements of the quotient $\mathfrak{T}(M) = \tilde{\mathfrak{T}}(M)/\text{Map}_M$ as *rough tangles*, that is, rough tangles are tangles distinguished up to relative homeomorphisms.

Let \mathcal{M} be the set of admissible manifolds: its elements are some oriented 3-manifolds with independently oriented boundary components, each boundary component being admissible. The set \mathcal{M} must include the manifolds $S \times \mathbb{I}$ for all $S \in \mathcal{S}$ and $\mathbb{I} = [0, 1]$. Also \mathcal{M} must be closed with respect to two operations. The first operation is the disjoint union of admissible manifolds. The second operation is the gluing of two diffeomorphic connected components of ∂M , provided that they have opposite relative orientations with respect to the orientation of M .

To an admissible boundary $S \in \mathcal{S}$ the TQFT associates a Hilbert space $\mathcal{H}(S)$ over \mathbb{F} with three isomorphisms

$$\diamond, \nabla : \mathcal{H}(S) \longrightarrow \mathcal{H}^\vee(S), \quad \bar{\cdot} : \mathcal{H}(S) \longrightarrow \mathcal{H}(S). \quad (1.4)$$

Here \mathcal{H}^\vee is the dual of \mathcal{H} and the maps ∇ and $\bar{\cdot}$ are ‘anti-linear’: they involve the involution (1.3) of the field \mathbb{F} . The maps (1.4) should satisfy the relations

$$(\bar{v})^\diamond = v^\nabla, \quad (\bar{v})^\nabla = v^\diamond, \quad \bar{\bar{v}} = v. \quad (1.5)$$

for all $v \in \mathcal{H}(S)$. If any two of three isomorphisms (1.4) are defined, then the third one is determined by either of the first two relations (1.5). Slightly abusing notations, we will use the same notations \diamond and ∇ for the inverses of the isomorphisms (1.4); in each case it will be clear whether a direct or an inverse isomorphism is used.

To a disjoint union of admissible boundaries TQFT associates a tensor product of Hilbert spaces over \mathbb{F} :

$$\mathcal{H}(S_1 \sqcup \cdots \sqcup S_k) = \mathcal{H}(S_1) \otimes \cdots \otimes \mathcal{H}(S_k).$$

For the empty boundary $\mathcal{H}(\emptyset) = \mathbb{F}$.

Let M be an admissible manifold. The relative orientation of M and connected components of its boundary separates ∂M into two disjoint components: the ‘in’ boundary and the ‘out’ boundary: $\partial M = (\partial M)_{\text{in}} \sqcup (\partial M)_{\text{out}}$. Define

$$\mathcal{H}(\partial M) = \mathcal{H}(\partial M)_{\text{out}} \otimes \mathcal{H}^\vee(\partial M)_{\text{in}}.$$

For every identification of ∂M the TQFT provides a state map, which maps rough tangles τ in M to elements of the Hilbert space of its boundary:

$$\langle - \rangle : \mathfrak{T}(M) \longrightarrow \mathcal{H}(\partial M), \quad (\tau, M) \mapsto \langle \tau, M \rangle. \quad (1.6)$$

The state map should satisfy the following axioms:

Change of orientation of the boundary component: A change in the orientation of a boundary component of M results in the application of the \diamond map to the corresponding factor in the Hilbert space $\mathcal{H}(\partial M)$.

Change of orientation of M : If M' is the manifold M with reversed orientation, then $\langle \tau, M' \rangle = \langle \tau, M \rangle^\nabla \in \mathcal{H}^\vee(\partial M)$ (since we did not change the orientation of the boundary components, the ‘in’ boundary component of M becomes the ‘out’ boundary component of M' and vice versa).

Disjoint union: For two admissible pairs (τ_1, M_1) and (τ_2, M_2) , the state map of their disjoint union is $\langle \tau_1 \sqcup \tau_2, M_1 \sqcup M_2 \rangle = \langle \tau_1, M_1 \rangle \otimes \langle \tau_2, M_2 \rangle$.

Gluing: Suppose that the same admissible boundary component S appears in the ‘in’ and in the ‘out’ parts of M . Let M' be the manifold constructed by gluing these components together (according to their identifications with the ‘standard copy’). Then $\langle \tau, M' \rangle$ is the result of canonical pairing \perp applied to the tensor product $\mathcal{H}(S) \otimes \mathcal{H}^\vee(S)$ within the Hilbert space $\mathcal{H}(\partial M)$.

It follows from the change of orientation axioms and relations (1.5) that if M' is the manifold M in which the orientation of M and of its boundary are reversed simultaneously, then $\langle \tau, M' \rangle = \overline{\langle \tau, M \rangle}$.

In the stable restricted topological WRT theory the base field \mathbb{F} is $\mathbb{Q}(q)$ – the field of rational functions of q with rational coefficients. Admissible boundaries are 2-spheres with $2n$ marked points, $n \geq 0$ and admissible 3-manifolds are $S_{g|k}^3$ and their disjoint unions. In this paper we will consider a more restricted (‘toy’) version of this theory. Namely, the admissible 3-manifolds are only $S_0^3 = S^3$ (a 3-sphere), $S_{0|1}^3 = \mathbb{B}^3$ (a 3-ball), $S_{0|2}^3 = S^2 \times \mathbb{I}$, $S_1^3 = S^2 \times S^1$ and their disjoint unions.

We will show in subsections 3.3.4 that $J(L, S^3) = \langle L, S^3 \rangle$ is equal to the Jones polynomial of L and we will prove in subsection A.2 that the invariant $J(L, S^2 \times S^1) = \langle L, S^2 \times S^1 \rangle$ is a Laurent polynomial of q and satisfies the property (1.2), which in this case takes the form

$$Z_r(L, S^2 \times S^1) = \langle L, S^2 \times S^1 \rangle|_{q=\exp(i\pi/r)} \tag{1.7}$$

for $r \geq r_{\text{cr}}(L, S^2 \times S^1)$, because $Z_r(S^2 \times S^1) = 1$.

1.3. A \mathbb{Z} -graded weak algebraic categorification. In this paper we will construct a weak algebraic categorification of the toy stable restricted topological WRT theory.

Let us recall basic facts about a \mathbb{Z} -graded version of a weak algebraic categorical 3-dimensional TQFT. We use the same set of admissible boundaries \mathcal{S} and the set of admissible 3-manifolds \mathcal{M} as in previous subsection. To an admissible boundary $S \in \mathcal{S}$ an algebraic categorical TQFT associates a \mathbb{Z} -graded algebra $\mathcal{A}(S)$ and an additive category

$\mathcal{C}(S) = \mathbf{D}^b(\mathcal{A}(S))$, that is, the bounded derived category of \mathbb{Z} -graded $\mathcal{A}(S)$ -modules. Each algebra has a canonical involution

$$\diamond: \mathcal{A}(S) \longrightarrow \mathcal{A}^{\text{op}}(S) \quad (1.8)$$

which preserves \mathbb{Z} -grading. There are two equivalence functors

$$\diamond, \nabla: \mathbf{D}^b(\mathcal{A}(S)) \longrightarrow \mathbf{D}^b(\mathcal{A}^{\text{op}}(S)). \quad (1.9)$$

The functor \diamond is covariant: it turns a complex of $\mathcal{A}(S)$ -modules into a complex of $\mathcal{A}^{\text{op}}(S)$ -modules with the help of the involution (1.8) and shifts its homological and \mathbb{Z} gradings by an amount depending on S . The functor ∇ is contravariant: it maps objects to their duals and then shifts their degree by an amount depending on S . The contravariant functor

$$-: \mathbf{D}^b(\mathcal{A}(S)) \longrightarrow \mathbf{D}^b(\mathcal{A}(S))$$

is the composition of \diamond with the inverse of ∇ or the other way around. In other words, $-$ turns a complex of $\mathcal{A}(S)$ -modules into the dual complex of $\mathcal{A}^{\text{op}}(S)$ -modules, then replaces $\mathcal{A}^{\text{op}}(S)$ by $\mathcal{A}(S)$ with the help of the involution (1.8) and finally performs a degree shift which depends on S . Three functors satisfy the relations (1.5), where this time v is an object of $\mathbf{D}^b(\mathcal{A}(S))$.

To a disjoint union of admissible boundaries an algebraic categorical TQFT associates a tensor product of algebras

$$\mathcal{A}(S_1 \sqcup \cdots \sqcup S_k) = \mathcal{A}(S_1) \otimes \cdots \otimes \mathcal{A}(S_k).$$

and, consequently, the category

$$\mathcal{C}(S_1 \sqcup \cdots \sqcup S_k) = \mathbf{D}^b(\mathcal{A}(S_1) \otimes \cdots \otimes \mathcal{A}(S_k)). \quad (1.10)$$

For an empty boundary $\mathcal{A}(\emptyset) = \mathbb{Q}$ and $\mathcal{C}(\emptyset)$ is the category of bounded complexes of $\mathbb{Z} \oplus \mathbb{Z}$ -graded vector spaces over \mathbb{Q} (the first grading is homological and the second is related to \mathbb{Z} -grading of algebras). Equivalently, this is a category of $\mathbb{Z} \oplus \mathbb{Z}$ -graded vector spaces over \mathbb{Q} (both categories are related by taking homology).

Define

$$\mathcal{A}(\partial M) = \mathcal{A}(\partial M)_{\text{out}} \otimes \mathcal{A}^{\text{op}}(\partial M)_{\text{in}}, \quad \mathcal{C}(\partial M) = \mathbf{D}^b(\mathcal{A}(\partial M)).$$

For every identification of the boundary of an admissible manifold M the algebraic categorical TQFT provides an object map

$$\langle\langle - \rangle\rangle: \mathfrak{T}(M) \longrightarrow \mathcal{C}(\partial M), \quad (\tau, M) \mapsto \langle\langle \tau, M \rangle\rangle. \quad (1.11)$$

If $\partial M = \emptyset$, so that τ is a link L in M , we take the homology of the complex $\langle\langle L, M \rangle\rangle$ and refer to it as the stable homology of the link $L \subset M$:

$$H_{\bullet}^{\text{st}}(L, M) = H_{\bullet}(\langle\langle L, M \rangle\rangle).$$

The object map should satisfy the following axioms:

Change of orientation of M : If M' is the manifold M with reversed orientation, then $\langle\langle \tau, M' \rangle\rangle = \langle\langle \tau, M \rangle\rangle^{\nabla} \in \mathcal{D}^b(\mathcal{A}^{\text{op}}(\partial M))$.

Disjoint union: For two admissible pairs (τ_1, M_1) and (τ_2, M_2) , the object map of their disjoint union is $\langle\langle \tau_1 \sqcup \tau_2, M_1 \sqcup M_2 \rangle\rangle = \langle\langle \tau_1, M_1 \rangle\rangle \otimes \langle\langle \tau_2, M_2 \rangle\rangle$.

Change of orientation of a boundary component: A change in the orientation of a boundary component of M results in the application of the \diamond functor to the corresponding factor in the category of eq.(1.11).

Gluing: Suppose that the same admissible boundary component S appears in the ‘in’ and in the ‘out’ parts of M . Let M' be the manifold constructed by gluing these components together. Then $\langle\langle \tau, M' \rangle\rangle$ is the result of derived tensor product $\overset{\text{L}}{\otimes}$ applied to the category $\mathcal{D}^b(\mathcal{A}(S) \otimes \mathcal{A}^{\text{op}}(S))$ within the category $\mathcal{D}^b(\mathcal{A}(\partial M))$.

An application of Grothendieck’s K_0 -functor to the elements of a weak algebraic categorical TQFT construction produces an ordinary TQFT described in the previous subsection. This time \mathbb{F} is the field $\mathbb{Q}(q)$ of rational functions of q and $\bar{q} = q^{-1}$. The Grothendieck group $K_0(\mathcal{D}^b(\mathcal{A}(S)))$ is a module over $\mathbb{Z}[q, q^{-1}]$, the multiplication by q corresponding to the translation $[1]_q$ of \mathbb{Z} -grading. A Hilbert space $\mathcal{H}(S)$ is the Grothendieck group $K_0^{\mathbb{Q}}(\mathcal{D}^b(\mathcal{A}(S)))$ defined as an extension to $\mathbb{Q}(q)$:

$$\mathcal{H}(S) = K_0^{\mathbb{Q}}(\mathcal{D}^b(\mathcal{A}(S))) = K_0(\mathcal{D}^b(\mathcal{A}(S))) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q). \quad (1.12)$$

There is a canonical isomorphism $\mathcal{H}^{\vee}(S) = K_0(\mathcal{D}^b(\mathcal{A}^{\text{op}}(S)))$, because for two objects $A \in \mathcal{D}^b(\mathcal{A}(S))$ and $B \in \mathcal{D}^b(\mathcal{A}^{\text{op}}(S))$ there is a pairing $K_0(B) \lrcorner K_0(A) = \chi(\text{Tor}(B, A))$, where χ is the Euler characteristic. Hence K_0 reduces the functors (1.9) to the corresponding involutions (1.4).

Overall, for every admissible manifold M there should be a commutative triangle

$$\begin{array}{ccc} & \mathfrak{T}(M) & \\ \langle\langle - \rangle\rangle \swarrow & & \searrow \langle - \rangle \\ \mathcal{C}(\partial M) & \xrightarrow{K_0} & \mathcal{H}(\partial M) \end{array} \quad (1.13)$$

and if a manifold M' is constructed by gluing together two matching boundary components of M , then there should be a commutative prism

$$\begin{array}{ccccc}
 \mathfrak{T}(M) & \xrightarrow{\text{gluing}} & \mathfrak{T}(M') & & \\
 \downarrow \langle - \rangle & \searrow \langle\langle - \rangle\rangle & \downarrow \langle - \rangle & \searrow \langle\langle - \rangle\rangle & \\
 \mathcal{C}(\partial M) & \xrightarrow{\mathbb{L} \otimes} & \mathcal{C}(\partial M') & & \\
 \swarrow K_0 & & \swarrow K_0 & & \\
 \mathcal{H}(\partial M) & \xrightarrow{\mathbb{L}} & \mathcal{H}(\partial M') & &
 \end{array} \tag{1.14}$$

1.4. A brief account of the results and the plan of the paper. We construct a weak categorification of the stable toy WRT theory. Our construction is based on \mathbb{Z} -graded algebras H_n introduced in [Kho02]. In the same paper, a complex $\langle\langle \tau \rangle\rangle_{\mathbb{K}}$ of $H_n \otimes H_m^{\text{op}}$ -modules is associated to a diagram of a $(2m, 2n)$ -tangle τ .

In this paper we associate the algebra H_n and a derived category of \mathbb{Z} -graded H_n -modules $\mathbf{D}^b(H_n)$ to a 2-sphere with $2n$ marked points. A tangle τ in a 3-ball can be considered as a $(0, 2n)$ -tangle, hence we associate to it the same complex $\langle\langle \tau \rangle\rangle_{\mathbb{K}}$ this time considered as an object of $\mathbf{D}^b(H_n)$. A link L in \mathbb{S}^3 can be constructed by gluing two tangles in 3-balls along the common boundary. As explained in [Kho02], the H_n -modules in $\langle\langle \tau \rangle\rangle_{\mathbb{K}}$ are projective, hence the gluing rule implies that stable homology of $L \subset \mathbb{S}^3$ coincides with the homology introduced in [Kho00].

The boundary category of the manifold $\mathbb{S}^2 \times \mathbb{I}$ with $2m$ marked points on the ‘in’ boundary and $2n$ marked points on the ‘out’ boundary is $\mathbf{D}^b(H_n \otimes H_m^{\text{op}})$. If a tangle σ in $\mathbb{S}^2 \times \mathbb{I}$ is represented by a $(2m, 2n)$ -tangle τ then we set again $\langle\langle \sigma \rangle\rangle = \langle\langle \tau \rangle\rangle_{\mathbb{K}}$. We prove that if two different $(2m, 2n)$ -tangles τ_1 and τ_2 represent the same tangle in $\mathbb{S}^2 \times \mathbb{I}$, then the complexes $\langle\langle \tau_1 \rangle\rangle_{\mathbb{K}}$ and $\langle\langle \tau_2 \rangle\rangle_{\mathbb{K}}$ are quasi-isomorphic, that is, although they may be homotopically inequivalent, they represent isomorphic objects in derived category $\mathbf{D}^b(H_n \otimes H_m^{\text{op}})$.

If a link L in $\mathbb{S}^2 \times \mathbb{S}^1$ is presented as a closure of a $(2n, 2n)$ -tangle τ within $\mathbb{S}^2 \times \mathbb{S}^1$, then in accordance with the gluing rule we define the stable homology of L as the Hochschild homology of $\langle\langle \tau \rangle\rangle_{\mathbb{K}}$: $\text{H}^{\text{st}}(L; \mathbb{S}^2 \times \mathbb{S}^1) = \text{HH}_{\bullet}(\langle\langle \tau \rangle\rangle_{\mathbb{K}})$. We prove that this homology does not depend on the choice of a tangle τ which represents L .

Finally, we suggest a practical method of computing the stable homology $\text{H}^{\text{st}}(L; \mathbb{S}^2 \times \mathbb{S}^1)$. If $L \subset \mathbb{S}^2 \times \mathbb{S}^1$ can be constructed by a $\mathbb{S}^2 \times \mathbb{S}^1$ closure of a $(2n, 2n)$ -tangle τ , then it is known that the stable invariant $\langle L, \mathbb{S}^2 \times \mathbb{S}^1 \rangle$ can be approximated (up to the coefficients at high powers of q) by the Kauffman bracket of the \mathbb{S}^3 closure of τ through the torus braid with high twist number. We show that the same is true in categorified theory: Khovanov homology of the \mathbb{S}^3 closure of τ through a high twist torus braid approximates the stable homology

of L in $S^2 \times S^1$. In fact, a close relation between the Hochschild homology of H_n and the Khovanov homology of a high twist torus link was first observed by Jozef Przytycki [Prz10] in case of $n = 1$.

In Section 2 we review the definition and properties of Kauffman bracket and Temperley-Lieb algebra. We define the stable toy WRT theory and prove that it satisfies the axiom requirement. Then we review Khovanov homology, Bar-Natan’s universal categorification of the Temperley-Lieb algebra [BN05] and bi-module categorification of Temperley-Lieb algebra [Kho02].

In Section 6 we present main results of the paper: we define the categorified theory, explain how stable homology is related to Khovanov homology through torus braid closure and conjecture the structure of Hochschild homology and cohomology of algebras H_n .

In Section 7 we prove that the categorified theory is well-defined and satisfies the axioms. In Section 8 we prove that torus braid closures of tangles within S^3 approximate the stable homology of their closures within $S^2 \times S^1$. In Section A we review the properties of Jones-Wenzl projectors and prove the relation (1.7) between the complete and stable WRT invariants of links in $S^2 \times S^1$.

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2. TANGLES AND LINKS

2.1. Basic definition.

2.1.1. *Tangles.* All tangles and links in this paper are framed, and in pictures we assume blackboard framing. We use a notation

$$\text{twist} = \phi_1$$

for a framing twist.

For a 3-manifold M with a boundary we have defined the set of tangles $\tilde{\mathfrak{T}}(M)$, the relative mapping class group Map_M and the set of rough tangles $\mathfrak{T}(M) = \tilde{\mathfrak{T}}(M)/\text{Map}_M$ in subsection 1.2. For a 2-dimensional manifold Σ being either \mathbb{R}^2 or S^2 , let $\tilde{\mathfrak{T}}_{m,n}(\Sigma \times \mathbb{I}) \subset \tilde{\mathfrak{T}}(\Sigma \times \mathbb{I})$ denote the set of (m, n) -tangles, that is, tangles with m marked points on the ‘in’ boundary $\Sigma \times \{0\}$ and with n marked points on the ‘out’ boundary $\Sigma \times \{1\}$. We use a shortcut notation

$\tilde{\mathfrak{T}}_n(\Sigma \times \mathbb{I}) = \tilde{\mathfrak{T}}_{n,n}(\Sigma \times \mathbb{I})$ and we use similar notations for rough tangles with $\tilde{\mathfrak{T}}$ replaced by \mathfrak{T} . Note that $\text{Map}_{\mathbb{R}^2}$ is trivial, so $\mathfrak{T}(\mathbb{R}^2) = \tilde{\mathfrak{T}}(\mathbb{R}^2)$.

The set $\tilde{\mathfrak{T}}(\Sigma \times \mathbb{I})$ of all tangles in $\Sigma \times \mathbb{I}$ and its quotient $\mathfrak{T}(\Sigma \times \mathbb{I})$ have the structure of a set of morphisms: the multiplication corresponds to the composition of tangles $\tau_1 \circ \tau_2$. We use the notation $\tilde{\mathfrak{T}}^{\text{op}}(\Sigma \times \mathbb{I})$ to denote the same set of morphisms with the opposite order of composing elements. The set of morphisms $\tilde{\mathfrak{T}}(\Sigma \times \mathbb{I})$ is ‘quiver-like’, because the composition of tangles is meaningful only when the numbers of end-points match, otherwise we set it to zero; in other words, the set of morphisms $\tilde{\mathfrak{T}}_n(\Sigma \times \mathbb{I})$ stand at the vertices of the quiver and the sets $\tilde{\mathfrak{T}}_{m,n}(\Sigma \times \mathbb{I})$ stand at its oriented edges.

The braid group $\mathfrak{B}_n(\Sigma)$ is a subgroup of $\tilde{\mathfrak{T}}_n(\Sigma \times \mathbb{I})$ and, in particular, the Artin braid group \mathfrak{B}_n is a subgroup of $\mathfrak{T}(\mathbb{R}^2)$. We use the following notations for some important framed braids in \mathfrak{B}_n :

$$\begin{array}{c} \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n = \begin{array}{c} \diagup \quad \dots \quad \diagdown \\ \text{---} \quad \text{---} \quad \text{---} \\ \diagdown \quad \dots \quad \diagup \end{array} \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array}, \quad \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n = \begin{array}{c} \diagdown \quad \dots \quad \diagup \\ \text{---} \quad \text{---} \quad \text{---} \\ \diagup \quad \dots \quad \diagdown \end{array} \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \end{array} \quad (2.1)$$

$$\begin{array}{c} \begin{array}{c} \textcircled{2} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n = \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n \circ \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n = \begin{array}{c} \textcircled{2} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n \end{array}, \quad \begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n = \left(\begin{array}{c} \textcircled{1} \\ \vdots \\ \textcircled{n} \end{array} \text{---} n \right)^n, \quad (2.2)$$

the latter braid representing the full rotation of n strands accompanied by a framing shift.

2.1.2. *Tangles in $\mathbb{R}^2 \times \mathbb{I}$.* The most commonly used tangles are tangles in $\mathbb{R}^2 \times \mathbb{I}$ and we refer to them simply as tangles. Tangles may be considered as morphisms of the tangle category: an object of this category is a set of m linearly ordered points, an (m, n) -tangle τ is a morphism between m points and n points and a composition of morphisms is a composition of tangles. Thus the set of all tangles $\mathfrak{T} = \mathfrak{T}(\mathbb{R}^2)$ is the set of morphisms in this category.

The set of morphisms \mathfrak{T} has three special involutions

$$\diamond, \nabla: \mathfrak{T} \longrightarrow \mathfrak{T}^{\text{op}}, \quad \mathfrak{T}_{m,n} \longrightarrow \mathfrak{T}_{n,m}, \quad \bar{}: \mathfrak{T} \longrightarrow \mathfrak{T}, \quad \mathfrak{T}_{m,n} \longrightarrow \mathfrak{T}_{m,n} \quad (2.3)$$

satisfying the relations (1.5). The first two turn a (m, n) -tangle τ into a (n, m) -tangle. The flip \diamond rotates τ by 180° about an axis which lies in the blackboard plane and is perpendicular to the time-line. The duality ∇ performs the flip and then switches all crossings of the tangle

into the opposite ones. In other words, τ^∇ is the mirror image of τ with respect to the plane which is perpendicular to the time line. $\bar{\cdot}$ just switches all crossings of a tangle.

2.2. Temperley-Lieb tangles and crossingless matchings.

2.2.1. *Definitions.* A tangle is *planar* if it can be presented by a diagram without crossings. A planar tangle is called Temperley-Lieb (TL) if it is boundary connected, that is, if it does not contain disjoint circles. We denote TL tangles by the letter λ and $\mathfrak{T}^{\text{TL}} \subset \mathfrak{T}$ denotes the set of all TL tangles. Since TL tangles have no crossings, then $\bar{\lambda} = \lambda$ and the action of \diamond and ∇ coincide: $\lambda^\diamond = \lambda^\nabla$.

The TL $(0, 2n)$ -tangles are called *crossingless n -matchings*, and their set is denoted $\mathfrak{C}_n = \mathfrak{T}_{0,2n}^{\text{TL}}$. All crossingless matchings form the set $\mathfrak{C} = \bigcup_n \mathfrak{C}_n \subset \mathfrak{T}^{\text{TL}}$. We denote crossingless matchings by letters α and β .

2.2.2. *The structure of TL tangles.* Let $n \overset{i}{\cup}$ and $\overset{i}{\cap} n$, $1 \leq i \leq n-1$, denote the elementary cup and cap tangles:

$$n \overset{i}{\cup} = \left| \begin{array}{c} 1 \\ \vdots \\ \dots \\ \vdots \\ n \end{array} \right| \dots \left| \begin{array}{c} i \\ \vdots \\ \cup \\ \vdots \\ i+1 \end{array} \right| \dots \left| \begin{array}{c} n \\ \vdots \\ \dots \\ \vdots \\ n \end{array} \right|, \quad \overset{i}{\cap} n = \left| \begin{array}{c} 1 \\ \vdots \\ \dots \\ \vdots \\ n \end{array} \right| \dots \left| \begin{array}{c} i \\ \vdots \\ \cap \\ \vdots \\ i+1 \end{array} \right| \dots \left| \begin{array}{c} n \\ \vdots \\ \dots \\ \vdots \\ n \end{array} \right|.$$

For a positive integer $d \leq \frac{n}{2}$ let $\mathbf{I} = (i_1, \dots, i_d)$ be a sequence of positive integers such that $i_k < n - 2k + 2$ for all $k \in \{1, \dots, d\}$. A *cap-tangle* $\overset{\mathbf{I}}{\cap} n$ is a $(n, n - 2d)$ -tangle which can be presented as a composition of d tangles of the form $\overset{i}{\cap} m$:

$$\overset{\mathbf{I}}{\cap} n = \overset{i_d}{\cap} n-2d+2 \circ \dots \circ \overset{i_2}{\cap} n-2 \circ \overset{i_1}{\cap} n.$$

A *cup-tangle* $n \overset{\mathbf{I}}{\cup}$ is defined similarly:

$$n \overset{\mathbf{I}}{\cup} = n \overset{i_1}{\cup} \circ n-2 \overset{i_2}{\cup} \circ \dots \circ n-2d+2 \overset{i_d}{\cup}.$$

Let $\mathcal{I}_{n,t}$, where $t = n - 2d$, be the set of all sequences \mathbf{I} mentioned above with an additional condition that $i_{k+1} \geq i_k - 1$ for all $k \in \{2, \dots, d\}$.

The following proposition is obvious:

Proposition 2.1. *For every TL (m, n) -tangle λ there exists a number t_λ and a unique presentation*

$$\lambda = n \overset{\mathbf{I}}{\cup} \circ \overset{\mathbf{J}}{\cap} m, \quad \mathbf{I} \in \mathcal{I}_{n,t_\lambda}, \quad \mathbf{J} \in \mathcal{I}_{m,t_\lambda}, \quad t_\lambda = n - 2d_\lambda = m - 2d'_\lambda. \quad (2.4)$$

The number t_λ is called a *through degree*, it equals the number of strands that go through from the bottom to the top of the tangle. Obviously, $t_\lambda \leq m, n$ and the numbers $n - t_\lambda$, $m - t_\lambda$ are even.

If $t_\lambda = 0$ then the TL tangle λ is called *split*. A split $(2m, 2n)$ -tangle λ has a unique presentation $\lambda = \alpha \circ \beta^\nabla$, where α is a TL $(0, 2n)$ -tangle and β is a TL $(0, 2m)$ -tangle.

2.3. Tangles in admissible manifolds.

2.3.1. *Tangles in \mathbb{B}^3* . The relative mapping class group of \mathbb{B}^3 is trivial, and there is an obvious canonical bijective map

$$\mathfrak{T}_{0,n} \xrightarrow{=} \mathfrak{T}_n(\mathbb{B}^3) = \tilde{\mathfrak{T}}_n(\mathbb{B}^3). \quad (2.5)$$

2.3.2. *Tangles in $\mathbb{S}^2 \times \mathbb{I}$* . We call the tangles of $\tilde{\mathfrak{T}}(\mathbb{S}^2 \times \mathbb{I})$ *spherical* tangles and we call the tangles of the quotient $\mathfrak{T}(\mathbb{S}^2 \times \mathbb{I}) = \tilde{\mathfrak{T}}(\mathbb{S}^2 \times \mathbb{I}) / \text{Map}_{\mathbb{S}^2 \times \mathbb{I}}$ *rough* spherical tangles.

For a fixed product structure in $\mathbb{S}^2 \times \mathbb{I}$ there are obvious surjective homomorphisms (with respect to tangle composition) $\tilde{\mathfrak{s}}$ and \mathfrak{s} :

$$\begin{array}{ccc} & \mathfrak{T} & \\ \tilde{\mathfrak{s}} \swarrow & & \searrow \mathfrak{s} \\ \tilde{\mathfrak{T}}(\mathbb{S}^2 \times \mathbb{I}) & \xrightarrow{-/\text{Map}_{\mathbb{S}^2 \times \mathbb{I}}} & \mathfrak{T}(\mathbb{S}^2 \times \mathbb{I}) \end{array}$$

Theorem 2.2. *The kernel of the homomorphism $\mathfrak{T} \xrightarrow{\tilde{\mathfrak{s}}} \tilde{\mathfrak{T}}(\mathbb{S}^2 \times \mathbb{I})$ is a congruence on \mathfrak{T} generated by the braid $\frac{\overline{\cdot}}{\mathfrak{S}^2} \Big|_2^n$, that is, an equivalence ($\tau_1 \sim \tau_2 \Leftrightarrow \mathfrak{s}(\tau_1) \approx \mathfrak{s}(\tau_2)$) is the minimal equivalence which includes the relation $\frac{\overline{\cdot}}{\mathfrak{S}^2} \Big|_2^n \sim \overline{\cdot} \Big|_2^n$ and respects the tangle composition.*

The relative mapping class group of $\mathbb{S}^2 \times [0, 1]$ is \mathbb{Z}_2 . Let tw (twist) denote its generator. It acts on spherical tangles by composing them with the braid $\overset{1}{\times} \Big|_2^n$: if σ is a spherical (m, n) -tangle, then

$$tw(\sigma) = \mathfrak{s} \left(\overset{1}{\times} \Big|_2^n \right) \circ \sigma.$$

Hence we get the following extension of Theorem 2.2:

Theorem 2.3. *The kernel of the homomorphism $\mathfrak{T} \xrightarrow{\mathfrak{s}} \mathfrak{T}(\mathbb{S}^2 \times \mathbb{I}) = \tilde{\mathfrak{T}}(\mathbb{S}^2 \times \mathbb{I}) / \text{Map}_{\mathbb{S}^2 \times \mathbb{I}}$ is the congruence generated by the braids $\frac{\overline{\cdot}}{\mathfrak{S}^2} \Big|_2^n$ and $\overset{1}{\times} \Big|_2^n$.*

The homomorphisms $\tilde{\mathfrak{s}}$ and \mathfrak{s} transfer the involutions ∇ , \diamond and $\bar{\cdot}$ from \mathfrak{T} to $\tilde{\mathfrak{T}}(\mathbb{S}^2 \times \mathbb{I})$ and to $\mathfrak{T}(\mathbb{S}^2 \times \mathbb{I})$.

2.3.3. *Links in S^3 and in $S^2 \times S^1$.* For a 3-manifold M let $\mathfrak{L}(M)$ denote the set of links in M up to ambient isotopy and let $\tilde{\mathfrak{L}}(M)$ be the set of links in M up to diffeomorphism: $\tilde{\mathfrak{L}}(M) = \mathfrak{L}(M)/\text{Map}_M$, where Map_M is the mapping class group of M . We use a shortcut notation $\mathfrak{L} = \mathfrak{L}(S^3) = \tilde{\mathfrak{L}}(S^3)$ for links in S^3 .

Let $(\tau; S^3)$ denote a circular closure of a (n, n) -tangle within S^3 and let $(\sigma; S^2 \times S^1)$ denote a circular closure of a spherical tangle within $S^2 \times S^1$. Sometimes we also use an abbreviated notation $(\tau; S^2 \times S^1) = (\mathfrak{s}(\tau); S^2 \times S^1)$. Thus we have two closure maps

$$\begin{array}{ccc} & \mathfrak{T}_{n,n} & \\ \begin{array}{c} \swarrow \\ (-; S^3) \end{array} & & \begin{array}{c} \searrow \\ (-; S^2 \times S^1) \end{array} \\ \mathfrak{L}(S^3) & & \mathfrak{L}(S^2 \times S^1) \end{array}$$

Both are trace-like and invariant under the involution \diamond :

$$(\tau_1 \circ \tau_2; *) = (\tau_2 \circ \tau_1; *), \tag{2.6}$$

$$(\tau^\diamond; *) = (\tau; *), \tag{2.7}$$

where $*$ stands for S^3 or $S^2 \times S^1$.

The quotients over the action of the mapping class groups combine into the following commutative diagram:

$$\begin{array}{ccccc} \mathfrak{T} & \xrightarrow{\tilde{\mathfrak{s}}} & \tilde{\mathfrak{T}}(S^2 \times \mathbb{I}) & \xrightarrow{(-; S^2 \times S^1)} & \tilde{\mathfrak{L}}(S^2 \times S^1) \\ & \searrow \mathfrak{s} & \downarrow & & \downarrow \\ & & \mathfrak{T}(S^2 \times \mathbb{I}) & \xrightarrow{(-; S^2 \times S^1)} & \mathfrak{L}(S^2 \times S^1) \end{array}$$

The mapping class group of $S^2 \times S^1$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ with generators tw (twist) and \diamond (flip), that is,

$$tw(\tau; S^2 \times S^1) = \left(\bigcirc \begin{array}{c} 1 \\ \vdots \\ n \end{array} \circ \tau; S^2 \times S^1 \right), \quad (\tau; S^2 \times S^1)^\diamond = (\tau^\diamond; S^2 \times S^1).$$

Theorem 2.4. *The equivalence relation $(\sigma_1 \sim \sigma_2 \Leftrightarrow (\sigma_1; S^2 \times S^1) \cong (\sigma_2; S^2 \times S^1))$ within $\tilde{\mathfrak{T}}(S^2 \times \mathbb{I})$ is generated by the relation $\sigma_1 \circ \sigma_2 \sim \sigma_2 \circ \sigma_1$. The same equivalence within $\mathfrak{T}(S^2 \times \mathbb{I})$ is generated by two relations: $\sigma_1 \circ \sigma_2 \sim \sigma_2 \circ \sigma_1$ and $\sigma_1^\diamond \sim \sigma_1$.*

3. QUANTUM INVARIANTS OF LINKS AND TANGLES

3.1. **The Temperley-Lieb algebra.** A detailed account of Temperley-Lieb algebras and Jones-Wenzl projectors can be found in the book by Lou Kauffman and Sostenes Lins [KL94]. Here we summarize relevant facts and set our notations.

3.1.1. *Definitions.* A Temperley-Lieb algebra TL is a ring generated, as a module, freely by TL tangles, that is, by the elements $\langle \lambda \rangle$, $\lambda \in \mathfrak{T}^{\text{TL}}$ over the ring $\mathbb{Z}[q, q^{-1}]$. The product in TL corresponds to the composition of tangles and is denoted by \circ . If the numbers of endpoints do not match, then the product is defined to be zero. Any disjoint circle appearing in the composition is replaced by the factor $-(q + q^{-1})$:

$$\langle \bigcirc \rangle = -(q + q^{-1}). \quad (3.1)$$

The Kauffman bracket relation

$$\langle \text{cross} \rangle = q^{\frac{1}{2}} \langle \text{cup} \rangle \langle \text{cap} \rangle + q^{-\frac{1}{2}} \langle \text{cup-cap} \rangle \quad (3.2)$$

associates a TL element

$$\langle \tau \rangle = \sum_{\lambda \in \mathfrak{T}^{\text{TL}}} a_{\lambda}(\tau) \langle \lambda \rangle, \quad a_{\lambda}(\tau) = \sum_{i \in \mathbb{Z}} a_{\lambda, i}(\tau) q^i \quad (3.3)$$

to any tangle τ . Note that the element $\langle \tau \rangle$ depends on the framing of τ :

$$\langle \text{framed unknot} \rangle = -q^{\frac{3}{2}} \langle \text{unknot} \rangle. \quad (3.4)$$

The homomorphisms (2.3) extend to the Temperley-Lieb algebra

$$\diamond, \nabla: \text{TL} \longrightarrow \text{TL}^{\text{op}}, \quad \text{TL}_{m,n} \longrightarrow \text{TL}_{n,m}, \quad \bar{}: \text{TL} \longrightarrow \text{TL}, \quad \text{TL}_{m,n} \longrightarrow \text{TL}_{m,n} \quad (3.5)$$

by their action on generating TL tangles, the involutions ∇ and $\bar{}$ being accompanied by the involution of the base ring $\bar{}: \mathbb{Z}[q, q^{-1}] \rightarrow \mathbb{Z}[q, q^{-1}]$, $\bar{q} = q^{-1}$. This extension is well-defined, because the involutions preserve the Kauffman bracket relation (3.2) as well as the unknot invariant formula (3.1).

3.1.2. *Subrings of the Temperley-Lieb algebra and their bimodules.* As a $\mathbb{Z}[q, q^{-1}]$ -module, TL contains submodules $\text{TL}_{m,n}$ ($n - m$ is even) generated by (m, n) -tangles. The ‘diagonal’ submodules $\text{TL}_{n,n}$, which we denote for brevity as TL_n , are, in fact, subrings of TL. Obviously, $\text{TL}_{m,n}$ is a module over the ring $\text{TL}_n \otimes \text{TL}_m^{\text{op}}$.

The empty tangle λ_{\emptyset} is the only TL $(0, 0)$ -tangle, hence the ring TL_0 is canonically isomorphic to $\mathbb{Z}[q, q^{-1}]$. A $(0, 0)$ -tangle L is just a link, and the corresponding Kauffman bracket equals its Jones polynomial: $\langle L \rangle = J(L, \mathbb{S}^3)$. The cyclic closure of an (n, n) -tangle in \mathbb{S}^3 produces a trace on the ring TL_n : $\text{TL}_n \xrightarrow{(-; \mathbb{S}^3)} \text{TL}_0 = \mathbb{Z}[q, q^{-1}]$.

Theorem 3.1. *There exists a unique element $P_{2n,0} \in \text{QTL}_{2n}^{\text{spl}}$ such that*

$$P_{2n,0} \circ \langle \alpha \rangle = \langle \alpha \rangle \quad (3.10)$$

for any $\alpha \in \mathfrak{C}_n$. This element is idempotent.

Proof. The elements $\langle \beta \circ \alpha^\diamond \rangle$ form a basis of $\text{QTL}_{2n}^{\text{spl}}$, hence an element $P_{2n,0} \in \text{QTL}_{2n}^{\text{spl}}$ has a unique presentation $P_{2n,0} = \sum_{\alpha, \beta \in \mathfrak{C}_n} c_{\alpha\beta} \langle \beta \circ \alpha^\diamond \rangle$. The condition (3.10) determines the coefficients $c_{\alpha\beta}$:

$$P_{2n,0} = \sum_{\alpha, \beta \in \mathfrak{C}_n} B_{\alpha\beta}^{-1} \langle \beta \circ \alpha^\diamond \rangle, \quad (3.11)$$

where a symmetric matrix $(B_{\alpha\beta})_{\alpha, \beta \in \mathfrak{C}_n}$ is defined by the formula

$$B_{\alpha\beta} = \langle \alpha^\diamond \circ \beta \rangle = (-q - q^{-1})^{n_{\alpha\beta}}. \quad (3.12)$$

It is easy to verify that the element (3.11) is idempotent. \square

Let $\text{QTL}_{2m,2n}^{\text{spl}} \subset \text{QTL}^{\text{spl}}$ be the submodule generated by $(2m, 2n)$ -tangles. Obviously,

$$\text{QTL}_{2m,2n}^{\text{spl}} = \text{QTL}_{0,2n} \otimes_{\mathbb{Q}(q)} \text{QTL}_{2m,0} \quad (3.13)$$

and the homomorphism

$$\text{QTL}_{2n,2m} \xrightarrow{\hat{P}_{*,0}} \text{QTL}_{2m,2n}^{\text{spl}}, \quad \hat{P}_{*,0}(x) = P_{2n,0} \circ x = x \circ P_{2m,0}$$

is surjective.

3.3. A stable Witten-Reshetikhin-Turaev theory. Let us describe in more detail the stable toy WRT theory that we want to categorify.

3.3.1. Basic structure. As we mentioned at the end of subsections:ortqft, its base field \mathbb{F} is the field $\mathbb{Q}(q)$ of rational functions of q . The involution $\bar{\cdot}$ is defined by its action on q : $\bar{q} = q^{-1}$. Admissible boundaries are 2-spheres \mathbb{S}_n^2 with $2n$ marked points. Admissible manifolds are disjoint unions of four 3-manifolds: \mathbb{B}^3 , $\mathbb{S}^2 \times \mathbb{I}$, \mathbb{S}^3 and $\mathbb{S}^2 \times \mathbb{S}^1$. In fact, all admissible manifolds are generated by \mathbb{B}^3 and $\mathbb{S}^2 \times \mathbb{I}$ through disjoint union and gluing. Therefore the TQFT is defined by the choice of Hilbert spaces $\mathcal{H}(\mathbb{S}_n^2)$ and state maps for rough tangles in \mathbb{B}^3 and in $\mathbb{S}^2 \times \mathbb{I}$, because the other state maps are determined by the axioms.

3.3.2. Hilbert spaces. To a 2-sphere with $2n$ marked points the stable toy theory associates a $\mathbb{Q}(q)$ -module $\mathcal{H}(\mathbb{S}_n^2) = \text{QTL}_{0,2n}$, which, by definition, is a free $\mathbb{Q}(q)$ -module generated by (Kauffman brackets of) crossingless n -matchings. The module $\mathcal{H}^\vee(\mathbb{S}_n^2) = \text{QTL}_{2n,0}$ is canonically dual to $\mathcal{H}(\mathbb{S}_n^2)$, the pairing being defined by the Kauffman bracket of the composition of tangles: $\langle \alpha^\diamond \rangle \lrcorner \langle \beta \rangle = \langle \alpha^\diamond \circ \beta \rangle$. The involutions (1.4) are the corresponding involutions (3.5) restricted to $\text{QTL}_{2n,0}$.

3.3.3. *A state map for tangles in \mathbb{B}^3 and in $S^2 \times \mathbb{I}$.* If we orient the boundary of an oriented 3-ball \mathbb{B}^3 in the ‘out’ direction and put $2n$ marked points on it, then using the bijection (2.5) we set $\langle \tau, \mathbb{B}^3 \rangle = \langle \tau \rangle$.

We set the orientation of the boundary component $S^2 \times \{0\}$ as ‘in’, the orientation of the boundary component $S^2 \times \{1\}$ as ‘out’ and put $2m$ and $2n$ marked points on them. According to eq.(3.9),

$$\mathcal{H}(S_n^2) \otimes_{\mathbb{Q}(q)} \mathcal{H}^\vee(S_m^2) = \text{QTL}_{2m,2n}^{\text{spl}}, \tag{3.14}$$

hence the state map has the form

$$\langle -, S^2 \times \mathbb{I} \rangle : \mathfrak{T}(S^2 \times \mathbb{I}) \longrightarrow \text{QTL}^{\text{spl}}. \tag{3.15}$$

The gluing axiom implies that it must be a homomorphism and we define it by the following theorem:

Theorem 3.2. *There exists a unique homomorphism (3.15) such that the diagram*

$$\begin{array}{ccc} \mathfrak{T} & \xrightarrow{\mathfrak{s}} & \mathfrak{T}(S^2 \times \mathbb{I}) \\ \downarrow \langle - \rangle & & \downarrow \langle -, S^2 \times \mathbb{I} \rangle \\ \text{TL} & \xrightarrow{\hat{P}_{*,0}} & \text{QTL}^{\text{spl}} \end{array} \tag{3.16}$$

is commutative, that is, for any $\tau \in \mathfrak{T}_{2m,2n}$

$$\langle \mathfrak{s}(\tau), S^2 \times \mathbb{I} \rangle = P_{2n,0} \circ \langle \tau \rangle. \tag{3.17}$$

Proof. Since the homomorphism \mathfrak{s} is surjective, it is sufficient to show that $\ker \mathfrak{s}$ is ‘untangled’ by the composition of the Kauffman bracket $\langle - \rangle$ and the projector $\hat{P}_{*,0}$. According to Theorem 2.3, $\ker \mathfrak{s}$ is generated by the braids $\frac{\overline{\cdot}}{\ominus} \Big|_2^{2n}$ and $\overset{1}{\times} \circlearrowleft_{2n}$, so we have to show that $P_{2n,0} \circ \langle \tau \rangle = P_{2n,0}$ for τ being either of these $\ker \mathfrak{s}$ -generating braids. The latter relations follow from the formula (3.11) and from the isotopies $\alpha^\diamond \circ \tau \approx \alpha^\diamond$ which hold true for any TL $(0, 2n)$ -tangle α . □

3.3.4. *The invariant of links in S^3 and in $S^2 \times S^1$.* Since S^3 can be constructed by gluing together two 3-balls \mathbb{B}^3 , the gluing property dictates that the toy invariant of a link L in \mathbb{B}^3 is its Kauffman bracket: $\langle L, \mathbb{B}^3 \rangle = \langle L \rangle$.

If the manifold $S^2 \times \mathbb{I}$ has $2n$ marked points on both boundary components, then the corresponding Hilbert module (3.14) becomes the module of endomorphisms:

$$\mathcal{H}(\partial(S^2 \times \mathbb{I})) = \text{QTL}_{2n}^{\text{spl}} = \text{End}_{\mathbb{Q}(q)}(\text{QTL}_{0,2n}). \tag{3.18}$$

Theorem 3.3. *A trace of an element $x \in \mathbb{QTL}_{2n,2n}^{\text{spl}}$, considered as an endomorphism of $\mathbb{QTL}_{0,2n}$, is equal to its circular closure within \mathbb{S}^2 :*

$$\text{Tr}_{\mathbb{QTL}_{0,2n}} x = \langle x; \mathbb{S}^3 \rangle. \quad (3.19)$$

Proof. It is sufficient to verify the formula for $x = \beta \circ \alpha^\diamond$, where α and β are TL $(0, 2n)$ -tangles. The only diagonal element in the matrix of x in the basis of TL tangles comes from the tangle β , and the corresponding matrix element is $\langle \alpha^\diamond \circ \beta \rangle$, which is equal to the Kauffman bracket of the closure of $\beta \circ \alpha^\nabla$ within \mathbb{S}^3 . \square

If a rough link $L \subset \mathbb{S}^2 \times \mathbb{S}^1$ is presented as a circular closure of a rough spherical tangle $\sigma \in \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I})$, then the gluing axiom of a TQFT says that the invariant of L must be equal to the trace of the state map of σ :

$$\langle \sigma, \mathbb{S}^2 \times \mathbb{S}^1 \rangle = \text{Tr}_{\mathbb{QTL}_{0,2n}} \langle \sigma, \mathbb{S}^2 \times \mathbb{I} \rangle. \quad (3.20)$$

The presentation $\sigma = \mathfrak{s}(\tau)$ together with eqs.(3.19) and (3.17), allows us to recast eq.(3.20) in the following equivalent form:

$$\langle \mathfrak{s}(\tau), \mathbb{S}^2 \times \mathbb{S}^1 \rangle = (\text{P}_{2n,0} \circ \langle \tau \rangle; \mathbb{S}^3). \quad (3.21)$$

In other words, the map

$$\langle -, \mathbb{S}^2 \times \mathbb{S}^1 \rangle : \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) \rightarrow \mathbb{Q}(q) \quad (3.22)$$

should provide the commutativity of the right square of the diagram

$$\begin{array}{ccccc} \mathfrak{T}_{2n,2n} & \xrightarrow{\mathfrak{s}} & \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I}) & \xrightarrow{(-; \mathbb{S}^2 \times \mathbb{S}^1)} & \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) \\ \downarrow \langle - \rangle & & \downarrow \langle -, \mathbb{S}^2 \times \mathbb{I} \rangle & & \downarrow \langle -, \mathbb{S}^2 \times \mathbb{S}^1 \rangle \\ \mathbb{QTL}_{2n,2n} & \xrightarrow{\text{P}_{2n,0} \circ -} & \mathbb{QTL}_{2n,2n}^{\text{spl}} & \xrightarrow{(-; \mathbb{S}^3)} & \mathbb{Q}(q) \end{array} \quad (3.23)$$

(the commutativity of the left square is a particular case of Theorem 3.2).

Theorem 3.4. *There exists a unique map (3.22) such that the right square of the diagram (3.23) is commutative*

Proof. Since the map $(-; \mathbb{S}^2 \times \mathbb{S}^1)$ is surjective, then according to Theorem 2.4, we have to check relations

$$(\langle \sigma_1 \circ \sigma_2, \mathbb{S}^2 \times \mathbb{I} \rangle; \mathbb{S}^3) = (\langle \sigma_2 \circ \sigma_1, \mathbb{S}^2 \times \mathbb{I} \rangle; \mathbb{S}^3), \quad (\langle \sigma^\diamond, \mathbb{S}^2 \times \mathbb{I} \rangle; \mathbb{S}^3) = (\langle \sigma, \mathbb{S}^2 \times \mathbb{I} \rangle; \mathbb{S}^3),$$

where σ , σ_1 and σ_2 are arbitrary spherical tangles. Present them as $\tilde{\mathfrak{f}}$ images of ordinary tangles, then eq.(3.17) reduces these relations to

$$(P_{2n,0} \circ \langle \tau_1 \rangle \circ \langle \tau_2 \rangle; \mathbb{S}^3) = (P_{2n,0} \circ \langle \tau_2 \rangle \circ \langle \tau_1 \rangle; \mathbb{S}^3), \quad (P_{2n,0} \circ \langle \tau^\diamond \rangle; \mathbb{S}^3) = (P_{2n,0} \circ \langle \tau \rangle; \mathbb{S}^3).$$

The first of this relations follows easily from the trace-like property (2.6) of the circular closure of a tangle within \mathbb{S}^3 and from the commutativity (3.8) of projectors, while the second one follows from the invariance of the projector (3.7) and of the circular closure (2.7) under the involution \diamond . \square

Theorem 3.5. *The link invariant $\langle L, S^2 \times S^1 \rangle$ defined by eq. (3.21) is a Laurent polynomial of q .*

Formula (3.21) indicates that relation (1.7) between the full and stable WRT invariants of links in $S^2 \times S^1$ follows from the following theorem:

Theorem 3.6. *For any $(2n, 2n)$ -tangle τ there is a relation*

$$Z_r(\tau, S^2 \times S^1) = (P_{2n,0} \circ \langle \tau \rangle; \mathbb{S}^3)|_{q=\exp(i\pi/r)} \tag{3.24}$$

if $r \geq n + 2$.

This theorem is well-known, but we will provide its proof in subsection A.2 for completeness.

4. CATEGORIFICATION

4.1. A triply graded categorification of the Jones polynomial. In [Kho00] M. Khovanov introduced a categorification of the Jones polynomial of links. To a diagram L of a link he associates a complex of graded vector spaces over \mathbb{Q}

$$\langle\langle L \rangle\rangle = (\cdots \rightarrow C_i \rightarrow C_{i+1} \rightarrow \cdots) \tag{4.1}$$

so that if two diagrams represent the same link then the corresponding complexes are homotopy equivalent, and the graded Euler characteristic of $\langle\langle L \rangle\rangle$ is equal to the Jones polynomial of L . As a complex of vector spaces, $\langle\langle L \rangle\rangle$ is homotopically equivalent to its homology known as Khovanov homology of the link L : $H_\bullet(\langle\langle L \rangle\rangle) = H_\bullet^{\text{Kh}}(L)$.

Thus, overall, the complex (4.1) has two gradings: the first one was the homological grading of the complex, the corresponding degree being equal to i , and the second grading was the grading related to powers of q . In this paper we adopt a slightly different convention which is convenient for working with framed links and tangles. It is inspired by matrix factorization categorification [KR08] and its advantage is that it is no longer necessary to assign orientation

to link strands in order to obtain the grading of the categorification complex (4.1) which would make it invariant under the second Reidemeister move.

To a framed link diagram L we associate a $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ -graded complex (4.1) with degrees \deg_h , \deg_q and \deg_2 . The first two gradings are of the same nature as in [Kho00] and, in particular, $\deg_h C_i = i$. The third grading is an inner grading of chain modules defined modulo 2 and of homological nature, that is, the homological parity of an element of $\langle\langle L \rangle\rangle$, which affects various sign factors, is the sum of \deg_h and \deg_2 . Both homological degrees are either integer or half-integer simultaneously, so the homological parity is integer and takes values in \mathbb{Z}_2 . The q -degree \deg_q may also take half-integer values.

Let $[m, l, n]$ denote the shift of three degrees by l , m and n units respectively¹. We use abbreviated notations

$$[l, m] = [l, m, 0], \quad [m]_{h,q} = [m, m], \quad [m]_q = [0, m, 0], \quad [m]_{q,2} = [0, m, m].$$

After the grading modification, the categorification formulas of [Kho00] take the following form: the module associated with an unknot is still $\mathbb{Z}[x]/(x^2)$ but with a different degree assignment:

$$\langle\langle \bigcirc \rangle\rangle = \mathbb{Z}[x]/(x^2) [0, -1, 1], \quad (4.2)$$

$$\deg_q 1 = 0, \quad \deg_q x = 2, \quad \deg_h 1 = \deg_h x = \deg_2 1 = \deg_2 x = 0, \quad (4.3)$$

and the categorification complex of a crossing is the same as in [Kho00] but with a different degree shift:

$$\langle\langle \text{crossing} \rangle\rangle = \left(\langle\langle \text{left} \rangle\rangle \langle\langle \text{right} \rangle\rangle \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \xrightarrow{f} \langle\langle \text{right} \rangle\rangle \langle\langle \text{left} \rangle\rangle \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] \right), \quad (4.4)$$

where f is either a multiplication or a comultiplication of the ring $\mathbb{Z}[x]/(x^2)$ depending on how the arcs in the *r.h.s.* are closed into circles. The resulting categorification complex (4.1) is invariant up to homotopy under the second and third Reidemeister moves, but it acquires a degree shift under the first Reidemeister move:

$$\langle\langle \text{cap} \rangle\rangle = \langle\langle \text{cup} \rangle\rangle \left[-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right]. \quad (4.5)$$

It is easy to see that the whole categorification complex (4.1) has a homogeneous degree \deg_2 .

¹ Our degree shift is defined in such a way that if an object M has a homogeneous degree n , then the shifted object $M[1]$ has a homogeneous degree $n + 1$.

4.2. The universal categorification of the Temperley-Lieb algebra. D. Bar-Natan [BN05] described the universal category TL , whose Grothendieck K_0 -group is the Temperley-Lieb algebra TL considered as a $\mathbb{Z}[q, q^{-1}]$ -module. We will use this category with obvious adjustments required by the new grading conventions.

4.2.1. A homotopy category of complexes of a \mathbb{Z} -graded additive category. A split Grothendieck group $G(\mathcal{C})$ of an additive category \mathcal{C} is generated by the images $G(A)$ of its objects modulo the additive relation $G(A \oplus B) = G(A) + G(B)$.

A Grothendieck group $K_0(\mathcal{C})$ of an abelian category \mathcal{C} is generated by the images $K_0(A)$ of its objects modulo the exact sequence relation: $K_0(A) - K_0(B) + K_0(C) = 0$, if there is an exact sequence $A \rightarrow B \rightarrow C$.

A Grothendieck group $K_0(\mathcal{C})$ of a triangulated category \mathcal{C} is generated by the images $K_0(A)$ of its objects modulo the translation relation $K_0(A[1]) = -K_0(A)$ and the exact triangle relation: $K_0(A) - K_0(B) + K_0(C) = 0$, if A , B and C form an exact triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$.

If the category \mathcal{C} is \mathbb{Z} -graded, that is, there is a shift functor $[1]_q$ and modules $\text{Hom}(A, B)$ are \mathbb{Z} -graded, then the groups $G(\mathcal{C})$ and $K_0(\mathcal{C})$ are modules over $\mathbb{Z}[q, q^{-1}]$, the multiplication by q corresponding to the shift $[1]_q$.

For an additive category \mathcal{C} , let $K^b(\mathcal{C})$ denote the homotopy category of bounded complexes and $K^-(\mathcal{C})$ – the similar category of bounded from above complexes over \mathcal{C} . These categories are triangulated. Suppose that \mathcal{C} is \mathbb{Z} -graded and generated, as an additive category, by objects E_1, \dots, E_N and their \mathbb{Z} -grading shifts, that is, an object A of \mathcal{C} has a form

$$A = \bigoplus_{1 \leq a \leq N} \bigoplus_{j \in \mathbb{Z}} m_j^a E_a [j]_q, \tag{4.6}$$

where $m_j^a \in \mathbb{Z}_{\geq 0}$ are multiplicities of the shifted objects $E_a [j]_q$. Suppose further that $K_0(E_1), \dots, K_0(E_N)$ generate freely $G(\mathcal{C})$ as a module over $\mathbb{Z}[q, q^{-1}]$, so the presentation (4.6) is unique.

An object of $K^b(\mathcal{C})$ has the form

$$\mathbf{A} = (\dots \rightarrow A_i \rightarrow A_{i+1} \rightarrow \dots), \quad A_i = \bigoplus_{1 \leq a \leq N} \bigoplus_{j \in \mathbb{Z}} m_{i,j}^a E_a [j]_q. \tag{4.7}$$

We call the objects A_i *chain ‘modules’* and we refer to objects E_a with non-zero multiplicities as *constituent objects* of the complex \mathbf{A} . The functor $\mathcal{C} \hookrightarrow K(\mathcal{C})$, $A \mapsto (0 \rightarrow A \rightarrow 0)$ generates the isomorphism of modules $G(\mathcal{C}) = K_0(K(\mathcal{C}))$ and

$$K_0(\mathbf{A}) = \sum_{i \in \mathbb{Z}} \sum_{1 \leq a \leq N} \sum_{j \in \mathbb{Z}} (-1)^i m_{i,j}^a q^j K_0(E_a). \tag{4.8}$$

Define the q^+ -order of an object (4.6) as $|A|_q = \min \{j: m_j^a \neq 0\}$. A complex \mathbf{A} in $\mathbb{T}\mathbb{L}^-$ is q^+ -bounded if $\lim_{i \rightarrow \infty} |A_{-i}|_q = +\infty$. Let $\mathbb{K}^{-/+}(\mathcal{C}) \subset \mathbb{K}^-(\mathcal{C})$ be the full subcategory of q^+ -bounded complexes. The category $\mathbb{K}^{-/+}(\mathcal{C})$ is triangulated. Define $\mathbb{K}_0^+(\mathbb{K}^{-/+}(\mathcal{C}))$ as a module over formal Laurent series $\mathbb{Q}[[q, q^{-1}]]$ freely generated by the elements $\mathbb{K}_0^+(E_a)$, $1 \leq a \leq N$ and define the map $\mathbb{K}_0^+ : \text{Ob } \mathbb{K}^{-/+}(\mathcal{C}) \rightarrow \mathbb{K}_0^+(\mathbb{K}^{-/+}(\mathcal{C}))$ by the formula similar to eq.(4.8):

$$\mathbb{K}_0^+(\mathbf{A}) = \sum_{i \in \mathbb{Z}} \sum_{1 \leq a \leq N} \sum_{j \in \mathbb{Z}} (-1)^i m_{i,j}^a q^j \mathbb{K}_0^+(E_a). \quad (4.9)$$

Since the complex \mathbf{A} is q^+ -bounded, the sum over i in this equation is well-defined.

4.2.2. *The additive category $\check{\mathbb{T}}\mathbb{L}$.* For two $\mathbb{T}\mathbb{L}$ (m, n) -tangles λ_1 and λ_2 , let

$$\lambda_1 \# \lambda_2 = S_{(1)}^1 \sqcup \cdots \sqcup S_{(k)}^1 \quad (4.10)$$

denote a union of disjoint circles produced by gluing together the matching end-points of λ_1 and λ_2 . A *planar cobordism* Σ from λ_1 to λ_2 is a compact orientable surface with a specified diffeomorphism between its boundary and $\lambda_1 \# \lambda_2$.

Let $\text{Cob}(\lambda_1, \lambda_2)$ be a $\mathbb{Z} \oplus \mathbb{Z}_2$ -graded module with free generators $\hat{\Sigma}$ associated with planar cobordisms Σ and having degrees

$$\text{deg}_q \hat{\Sigma} = \text{deg}_2 \hat{\Sigma} = \frac{1}{2}(m + n) - \chi(\Sigma),$$

where $\chi(\Sigma) = 2 - \#\text{holes} - \#\text{handles}$ is the Euler characteristic of Σ .

Let p_1, p_2, p_3, p_4 be 4 distinct points inside Σ . An associated 4Tu relation is a formal relation

$$\hat{\Sigma}_{12} + \hat{\Sigma}_{34} = \hat{\Sigma}_{13} + \hat{\Sigma}_{24},$$

where Σ_{ij} is a planar cobordism constructed by cutting small neighborhoods of p_i and p_j out of Σ and then gluing together the boundaries of the cuts.

By definition, $\check{\mathbb{T}}\mathbb{L}_{m,n}$ is an additive $\mathbb{Z} \oplus \mathbb{Z}_2$ -graded category generated by objects $\langle\langle \lambda \rangle\rangle$ which are indexed by $\mathbb{T}\mathbb{L}$ (m, n) -tangles λ . A module of morphisms is

$$\text{Hom}_{\check{\mathbb{T}}\mathbb{L}_{m,n}}(\langle\langle \lambda_1 \rangle\rangle, \langle\langle \lambda_2 \rangle\rangle) = \text{Cob}(\lambda_1, \lambda_2) / (4\text{Tu relations}).$$

A planar cobordism Σ is called *reduced* if it is a disjoint union of connected surfaces $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$, such that each Σ_i is either a 2-disk \mathbb{B}^2 or a 2-torus with a hole; since each Σ_i has a single boundary component, there are specified diffeomorphisms between the boundaries $\partial \Sigma_i$ and circles $S_{(i)}^1$ of eq.(4.10).

Theorem 4.1 ([Kho02], [BN05]). *The module $\text{Hom}_{\check{\mathbb{T}}\mathbb{L}_{m,n}}(\langle\langle \lambda_1 \rangle\rangle, \langle\langle \lambda_2 \rangle\rangle)$ is generated freely by reduced planar cobordisms.*

The following corollary of this theorem is easily established by relating two types of reduced planar cobordisms (a 2-disk and a 2-torus with a hole) to the generators 1 and x of the module (4.2).

Corollary 4.2. *There is a canonical isomorphism*

$$\mathrm{Hom}_{\check{\mathrm{TL}}_{m,n}}(\langle\langle\lambda_1\rangle\rangle, \langle\langle\lambda_2\rangle\rangle) = \langle\langle\lambda_2 \circ \lambda_1^\nabla; \mathbb{S}^3\rangle\rangle \left[0, \frac{m+n}{2}, \frac{m+n}{2}\right], \quad \lambda_1, \lambda_2 \in \mathfrak{T}_{m,n}^{\mathrm{TL}}. \quad (4.11)$$

Moreover, since $(\lambda_2 \circ \lambda_1^\nabla; \mathbb{S}^3) = (\lambda_1 \circ \lambda_2^\nabla; \mathbb{S}^3) = (\lambda_2^\nabla \circ \lambda_1; \mathbb{S}^3)$, there is a canonical isomorphism $\langle\langle\lambda_2 \circ \lambda_1^\nabla; \mathbb{S}^3\rangle\rangle = \langle\langle\lambda_1 \circ \lambda_2^\nabla; \mathbb{S}^3\rangle\rangle = \langle\langle\lambda_1^\nabla \circ \lambda_2; \mathbb{S}^3\rangle\rangle$ and in view of eq. (4.11) there are canonical isomorphisms between the modules of morphisms

$$\mathrm{Hom}_{\check{\mathrm{TL}}_{m,n}}(\langle\langle\lambda_1\rangle\rangle, \langle\langle\lambda_2\rangle\rangle) = \mathrm{Hom}_{\check{\mathrm{TL}}_{m,n}}(\langle\langle\lambda_2\rangle\rangle, \langle\langle\lambda_1\rangle\rangle) = \mathrm{Hom}_{\check{\mathrm{TL}}_{m,n}}(\langle\langle\lambda_1^\nabla\rangle\rangle, \langle\langle\lambda_2^\nabla\rangle\rangle). \quad (4.12)$$

An *elementary planar cobordism* is a reduced cobordism which is either a saddle cobordism or an x -multiplication, which is a connected sum of an identity cobordism and a 2-dimensional torus.

Proposition 4.3. *Every reduced planar cobordism can be presented as a composition of elementary planar cobordisms.*

A composition of TL tangles generates a bifunctor $\circ : \check{\mathrm{TL}}_{m,n} \otimes \check{\mathrm{TL}}_{l,m} \rightarrow \check{\mathrm{TL}}_{l,n}$, if we apply the categorified version of the rule (3.1) in order to remove disjoint circles:

$$\langle\langle\bigcirc\rangle\rangle = \langle\langle\lambda_\emptyset\rangle\rangle [0, 1, 1] + \langle\langle\lambda_\emptyset\rangle\rangle [0, -1, 1], \quad (4.13)$$

Thus the category $\check{\mathrm{TL}} = \bigoplus_{m,n} \check{\mathrm{TL}}_{m,n}$ acquires the monoidal structure.

4.2.3. *The universal category TL.* The universal category $\mathrm{TL}_{m,n}$ is the homotopy category of bounded complexes over $\check{\mathrm{TL}}_{m,n}$: $\mathrm{TL}_{m,n} = \mathbf{K}^b(\check{\mathrm{TL}}_{m,n})$. In other words, in accordance with the general formula (4.7), an object of TL is a complex

$$\mathbf{A} = (\cdots \rightarrow A_i \rightarrow A_{i+1} \rightarrow \cdots), \quad A_i = \bigoplus_{\substack{j \in \mathbb{Z} \\ \mu \in \mathbb{Z}_2}} \bigoplus_{\lambda \in \mathfrak{T}_{m,n}^{\mathrm{TL}}} m_{i,j,\mu}^\lambda \langle\langle\lambda\rangle\rangle [0, j, \mu]. \quad (4.14)$$

The total category TL is a formal sum of categories: $\mathrm{TL} = \bigoplus_{m,n} \mathrm{TL}_{m,n}$. It inherits the monoidal structure of $\check{\mathrm{TL}}$ which comes from the composition of tangles.

A categorification map $\langle\langle - \rangle\rangle : \mathfrak{T} \rightarrow \mathrm{TL}$ turns a framed tangle diagram τ into a complex $\langle\langle\tau\rangle\rangle$ according to the rules (4.2) and (4.4), the morphism f in the complex (4.4) being the saddle cobordism.

We use a shortcut $\mathrm{TL}_n = \mathrm{TL}_{n,n}$. The category TL_0 is generated by a single object $\langle\langle\lambda_\emptyset\rangle\rangle$. Hence it is equivalent to the homotopy category of free $\mathbb{Z} \oplus \mathbb{Z}_2$ -graded modules and it has

a homology functor. The homology functor applied to a complex $\langle\langle L \rangle\rangle$ of a link $L \subset \mathbb{S}^3$ yields, by definition, the link homology: $H_\bullet(\langle\langle L \rangle\rangle) = H_\bullet^{\text{Kh}}(L)$. A circular closure $(\tau; \mathbb{S}^3)$ of (n, n) -tangles τ within \mathbb{S}^3 extends to a \mathbb{S}^3 -closure functor

$$\text{TL}_n \xrightarrow{(-; \mathbb{S}^3)} \text{TL}_0 = \text{K}^b(\mathbb{Z} - \text{gmod}). \quad (4.15)$$

4.2.4. *Grothendieck group map K_0 .* In accordance with the general rules of subsection 4.2.1, $K_0(\text{TL}) = G(\check{\text{TL}}) = \text{TL}$ and there is a commutative diagram

$$\begin{array}{ccc} & & \text{TL} \\ & \langle\langle - \rangle\rangle \nearrow & \downarrow K_0 \\ \mathfrak{T} & & \text{TL} \\ & \langle - \rangle \searrow & \\ & & \text{TL} \end{array} \quad (4.16)$$

where the map K_0 turns the complex (4.14) into the sum (3.3):

$$K_0(\mathbf{A}) = \sum_{\lambda \in \mathfrak{T}_n^{\text{TL}}} \sum_{j \in \mathbb{Z}} a_{\lambda, j} q^j \langle \lambda \rangle, \quad a_{\lambda, j} = \sum_{\substack{i \in \mathbb{Z} \\ \mu \in \mathbb{Z}_2}} (-1)^{i+\mu} m_{i, j, \mu}^\lambda. \quad (4.17)$$

In addition to TL we consider the categories TL^- and $\text{TL}^{-/+} \subset \text{TL}^-$ defined in accordance with subsection 4.2.1. Obviously, $K_0^+(\text{TL}^{-/+}) = \text{TL}^+$.

4.2.5. *Involutive functors \diamond , ∇ and $\bar{}$.* Let TL^{op} denote the category TL in which the composition of tangles is performed in reverse order. We define a covariant functor \diamond and contravariant functors ∇ , $\bar{}$

$$\diamond, \nabla: \text{TL} \longrightarrow \text{TL}^{\text{op}}, \quad \text{TL}_{m, n} \longrightarrow \text{TL}_{n, m}, \quad \bar{}: \text{TL} \longrightarrow \text{TL}, \quad \text{TL}_{m, n} \longrightarrow \text{TL}_{m, n} \quad (4.18)$$

by the requirement that their action on generating objects $\langle\langle \lambda \rangle\rangle$ matches their action on underlying TL tangles λ and that \diamond should preserve degree shifts while ∇ and $\bar{}$ should invert them. The action of the functors (4.18) on morphisms is established with the help of isomorphisms (4.12).

It is easy to see by checking the action of \diamond and ∇ on eqs.(4.4) and (4.13) that the maps of the diagram (4.16) intertwine the actions of \diamond and ∇ on \mathfrak{T} , TL and TL .

For two complexes \mathbf{A} and \mathbf{B} in $\text{TL}_{m, n}$ there is a canonical isomorphism extending that of eq.(4.11):

$$\text{Hom}_{\text{TL}_{m, n}}(\mathbf{A}, \mathbf{B}) = \langle\langle \mathbf{B} \circ \mathbf{A}^\nabla; \mathbb{S}^3 \rangle\rangle \left[0, \frac{m+n}{2}, \frac{m+n}{2} \right].$$

4.2.6. *A split subcategory.* An additive split category $\check{\mathbb{T}}\mathbb{L}_{2m,2n}^{\text{spl}} \subset \check{\mathbb{T}}\mathbb{L}_{2m,2n}$ is a full subcategory whose objects are split TL tangles $\langle\langle \beta \circ \alpha^\diamond \rangle\rangle$, where $\alpha \in \mathfrak{C}_m$ and $\beta \in \mathfrak{C}_n$. There is an obvious functor

$$\check{\mathbb{T}}\mathbb{L}_{2m,0} \otimes \check{\mathbb{T}}\mathbb{L}_{0,2n} \longrightarrow \check{\mathbb{T}}\mathbb{L}_{2m,2n}^{\text{spl}}, \quad (\langle\langle \beta \rangle\rangle, \langle\langle \alpha^\diamond \rangle\rangle) \mapsto \langle\langle \beta \circ \alpha^\diamond \rangle\rangle. \quad (4.19)$$

A split category $\mathbb{T}\mathbb{L}_{2m,2n}^{\text{spl}} \subset \mathbb{T}\mathbb{L}_{2m,2n}$ is a full subcategory whose objects are $\langle\langle \beta \circ \alpha^\diamond \rangle\rangle$, where $\alpha \in \mathfrak{C}_m$ and $\beta \in \mathfrak{C}_n$. There is an obvious functor

$$\mathbb{T}\mathbb{L}_{2m,0} \otimes \mathbb{T}\mathbb{L}_{0,2n} \longrightarrow \mathbb{T}\mathbb{L}_{2m,2n}^{\text{spl}}, \quad (\langle\langle \beta \rangle\rangle, \langle\langle \alpha^\diamond \rangle\rangle) \mapsto \langle\langle \beta \circ \alpha^\diamond \rangle\rangle. \quad (4.20)$$

Theorem 4.4. *The functor (4.19) is an equivalence of categories.*

Proof. Obviously, this functor acts bijectively on objects and injectively on morphisms. The surjectivity of its action on morphisms is established with the help of Theorem 4.1: a reduced planar cobordism between the tangles $\alpha_1 \circ \beta_1^\diamond$ and $\alpha_2 \circ \beta_2^\diamond$ is a disjoint union of a cobordism between α_1 and α_2 and a cobordism between β_1 and β_2 . \square

The additive category equivalence (4.19) implies the equivalence of homotopy categories

$$\mathbb{K}^{-/+}(\check{\mathbb{T}}\mathbb{L}_{2m,0} \otimes \check{\mathbb{T}}\mathbb{L}_{0,2n}) \longrightarrow \mathbb{T}\mathbb{L}_{2m,2n}^{\text{spl},-/+}, \quad (4.21)$$

where $\mathbb{T}\mathbb{L}_{2m,2n}^{\text{spl},-/+} = \mathbb{K}^{-/+}(\check{\mathbb{T}}\mathbb{L}_{2m,2n}^{\text{spl}})$ is the full subcategory of $\mathbb{T}\mathbb{L}_{2m,2n}^{-/+}$.

4.3. Bimodule categorification.

4.3.1. *The rings H_n .* M. Khovanov [Kho02] defined the algebras H_n as the sums of rings of morphisms between the objects $\langle\langle \alpha \rangle\rangle$ or, equivalently, $\langle\langle \alpha^\nabla \rangle\rangle$, $\alpha \in \mathfrak{C}_n$:

$$H_n = \bigoplus_{\alpha, \beta \in \mathfrak{C}_n} \text{Hom}_{\mathbb{T}\mathbb{L}_{2n,0}}(\langle\langle \alpha \rangle\rangle, \langle\langle \beta \rangle\rangle) = \bigoplus_{\alpha, \beta \in \mathfrak{C}_n} \text{Hom}_{\mathbb{T}\mathbb{L}_{2n,0}}(\langle\langle \alpha^\nabla \rangle\rangle, \langle\langle \beta^\nabla \rangle\rangle), \quad (4.22)$$

the isomorphisms being particular cases of those of eq. (4.12). The rings H_n have $\mathbb{Z} \oplus \mathbb{Z}_2$ -grading associated with q -related \mathbb{Z} -grading and \mathbb{Z}_2 -grading of the category $\mathbb{T}\mathbb{L}$ (in fact, $\text{deg}_2 H_n = 0$). The first of isomorphisms (4.12) provides a canonical isomorphism

$$\diamond: H_n \rightarrow H_n^{\text{op}}. \quad (4.23)$$

Denote $H_{n,m} = H_n \otimes H_m^{\text{op}}$. We use abbreviated notations $H_n^e = H_n \otimes H_n^{\text{op}}$ and $\mathbb{K}^b(H_{n,m}) = \mathbb{K}^b(H_{n,m} - \mathbf{gmod})$. The isomorphism (4.23) generates the isomorphism $\diamond: H_{n,m} \rightarrow H_{m,n}$ and consequently an equivalence functor

$$\hat{\diamond}: \mathbb{K}^b(H_{n,m}) \rightarrow \mathbb{K}^b(H_{m,n}). \quad (4.24)$$

4.3.2. *Bimodules from tangles.* The categorification maps

$$\langle\langle - \rangle\rangle_{\mathbb{K}}: \mathfrak{T}_{2m,2n} \rightarrow \mathbb{K}^b(H_{n,m}) \quad (4.25)$$

are defined in [Kho02] by the formula

$$\langle\langle \tau \rangle\rangle_{\mathbb{K}} = \bigoplus_{\substack{\alpha \in \mathfrak{C}_m \\ \beta \in \mathfrak{C}_n}} \text{Hom}_{\text{TL}_{2m,2n}}(\langle\langle \beta \rangle\rangle, \langle\langle \tau \circ \alpha \rangle\rangle) = \bigoplus_{\substack{\alpha \in \mathfrak{C}_m \\ \beta \in \mathfrak{C}_n}} \langle\langle \beta^\nabla \circ \tau \circ \alpha \rangle\rangle [n]_{q,2}. \quad (4.26)$$

Comparing eqs.(4.22) and (4.26), it is easy to see that

$$\langle\langle \overline{\vdots} \ 2n \rangle\rangle_{\mathbb{K}} = H_n, \quad (4.27)$$

where H_n is considered as a module over H_n^e .

Theorem 4.5 ([Kho02]). *The map $\langle\langle - \rangle\rangle_{\mathbb{K}}$ translates the composition of tangles into the tensor product over the intermediate ring: if τ_1 is a $(2l, 2m)$ -tangle and τ_2 is a $(2m, 2n)$ -tangle, then there is a canonical isomorphism*

$$\langle\langle \tau_2 \circ \tau_1 \rangle\rangle_{\mathbb{K}} = \langle\langle \tau_2 \rangle\rangle_{\mathbb{K}} \otimes_{H_m} \langle\langle \tau_1 \rangle\rangle_{\mathbb{K}}. \quad (4.28)$$

The map (4.25) restricted to TL tangles extends to a functor $\mathcal{F}_{\mathbb{K}}: \text{TL}_{2m,2n} \rightarrow \mathbb{K}^b(H_{n,m})$, which maps an object $\langle\langle \lambda \rangle\rangle$ to $\langle\langle \lambda \rangle\rangle_{\mathbb{K}}$ and translates planar cobordisms between TL tangles λ and λ' into homomorphisms between the modules $\langle\langle \lambda \rangle\rangle_{\mathbb{K}}$ and $\langle\langle \lambda' \rangle\rangle_{\mathbb{K}}$. Moreover, the categorification maps (4.25) can be threaded through the universal category:

$$\begin{array}{ccc} & \mathfrak{T}_{2m,2n} & \\ \langle\langle - \rangle\rangle \swarrow & & \searrow \langle\langle - \rangle\rangle_{\mathbb{K}} \\ \text{TL}_{2m,2n} & \xrightarrow{\mathcal{F}_{\mathbb{K}}} & \mathbb{K}^b(H_{n,m}) \end{array} \quad (4.29)$$

The covariant involutive functor \diamond and contravariant involutive functors ∇ and $\bar{}$

$$\diamond, \nabla: \mathbb{K}^b(H_{n,m}) \rightarrow \mathbb{K}^b(H_{m,n}), \quad \bar{}: \mathbb{K}^b(H_{n,m}) \rightarrow \mathbb{K}^b(H_{n,m}) \quad (4.30)$$

are defined by the formulas

$$\mathbf{M}^\diamond = \mathbf{M}^{\hat{\diamond}} [m-n]_{q,2}, \quad \mathbf{M}^\nabla = \mathbf{M}^\vee [-m-n]_{q,2}, \quad \bar{\mathbf{M}} = (\mathbf{M}^{\hat{\diamond}})^\vee [-2m]_{q,2}, \quad (4.31)$$

where \mathbf{M} is a complex in $\mathbb{K}^b(H_{n,m})$, \mathbf{M}^\vee is the dual complex and $\hat{\diamond}$ is the equivalence functor (4.24). It is easy to see that the maps of the diagram (4.29) intertwine the actions (2.3), (4.18) and (4.30) of \diamond , ∇ and $\bar{}$. In particular,

$$\langle\langle \tau \rangle\rangle^\diamond = \langle\langle \tau^\diamond \rangle\rangle \quad (4.32)$$

for $\tau \in \mathfrak{T}_{2n,2n}$.

5. DERIVED CATEGORY OF $H_{n,m}$ -MODULES AND HOCHSCHILD HOMOLOGY

5.1. A quick review of derived categories of modules.

5.1.1. *Projective resolution formula.* For a (graded) ring R , $D^b(R)$ denotes the bounded derived category of (graded) R -modules, $R\text{-pr}$ denotes the category of (graded) projective modules of R , while $K_{\text{pr}}^b(R)$ and $K_{\text{pr}}^-(R)$ denote the bounded and bounded from above homotopy categories of (graded) projective R -modules. The following commutative diagram is a practical guide for working with $D^b(R)$:

$$\begin{array}{ccc}
 K_{\text{pr}}^b(R) & \hookrightarrow & K_{\text{pr}}^-(R) \\
 \downarrow & \nearrow \mathcal{P}_K & \uparrow \mathcal{P} \\
 K^b(R) & \xrightarrow{\mathcal{F}_{\text{KD}}} & D^b(R)
 \end{array} \tag{5.1}$$

Here the arrows \hookrightarrow denote full subcategory injective functors, the arrow \rightarrow denotes a functor with surjective action on objects, while \mathcal{P} and \mathcal{P}_K are functors of projective resolution.

If \mathbf{M} is a complex in $K^b(R)$, then usually, with a slight abuse of notations, $\mathcal{F}_{\text{KD}}(\mathbf{M})$ is denoted simply as \mathbf{M} , and we will follow this convention.

Since the functor \mathcal{P} is fully injective, while \mathcal{F}_{KD} is surjective, the structure of the derived category $D^b(R)$ is completely determined by the functor \mathcal{P}_K , which has a convenient presentation. Denote $R^e = R \otimes R^{\text{op}}$ and define a projective resolution of R as R^e -module to be a complex of R^e -modules

$$\mathcal{P}(R) = (\cdots \rightarrow \mathcal{P}(R)_{-2} \rightarrow \mathcal{P}(R)_{-1} \rightarrow \mathcal{P}(R)_0)$$

with a homomorphism $\mathcal{P}(R)_0 \rightarrow R$ such that the total complex $\cdots \rightarrow \mathcal{P}(R)_{-1} \rightarrow \mathcal{P}(R)_0 \rightarrow R$ is acyclic. Then \mathcal{P}_K acts on the complexes of $K^b(R)$ by tensoring them with $\mathcal{P}(R)$ over R :

$$\mathcal{P}_K = \mathcal{P}(R) \otimes_R -. \tag{5.2}$$

The isomorphism of objects \mathbf{M} and \mathbf{N} of $D^b(R)$ is called *quasi-isomorphism* and is denoted as $\mathbf{M} \simeq \mathbf{N}$.

5.1.2. *Semi-projective bimodules.* Suppose that R is a tensor product: $R = R_1 \otimes R_2^{\text{op}}$. A $R_1 \otimes R_2^{\text{op}}$ -module is called *semi-projective* if it is projective separately as a R -module and as

a R_2^{op} -module. Denote by $\mathbf{K}_{\text{sp}}^{\text{b}}(R_1 \otimes R_2^{\text{op}})$ the bounded homotopy category of semi-projective $R_1 \otimes R_2^{\text{op}}$ -modules and consider the following version of the commutative diagram (5.1):

$$\begin{array}{ccc}
 \mathbf{K}_{\text{sp}}^{\text{b}}(R_1 \otimes R_2^{\text{op}}) & \xrightarrow{\mathcal{P}_{\text{sp}}} & \mathbf{K}_{\text{pr}}^-(R_1 \otimes R_2^{\text{op}}) \\
 \downarrow & \nearrow \mathcal{P}_{\mathbf{K}} & \downarrow \mathcal{P} \\
 \mathbf{K}^{\text{b}}(R_1 \otimes R_2^{\text{op}}) & \xrightarrow{\mathcal{F}_{\text{KD}}} & \mathbf{D}^{\text{b}}(R_1 \otimes R_2^{\text{op}})
 \end{array} \tag{5.3}$$

The projective resolution functor $\mathcal{P}_{\mathbf{K}}$ can be expressed with the help of eq. (5.2): $\mathcal{P}_{\mathbf{K}} = \mathcal{P}(R_1) \otimes_{R_1} (-) \otimes_{R_2} \mathcal{P}(R_2)$. However, its restriction \mathcal{P}_{sp} to complexes of semi-projective modules has a simpler expression, because it is sufficient to tensor with a projective resolution of one of two rings:

$$\mathcal{P}_{\text{sp}} = \mathcal{P}(R_1) \otimes_{R_1} - = - \otimes_{R_2} \mathcal{P}(R_2). \tag{5.4}$$

5.1.3. *Hochschild homology.* For any ring R consider a ring $R^e = R \otimes R^{\text{op}}$. The Hochschild homology and cohomology functors are defined by the diagrams

$$\begin{array}{ccc}
 \mathbf{K}_{\text{pr}}^-(R^e) & & \mathbf{K}_{\text{pr}}^-(R^e) \\
 \uparrow \mathcal{P} & \searrow \mathbf{H}_{\bullet}(- \otimes_{R^e} R) & \uparrow \mathcal{P} \\
 \mathbf{D}^{\text{b}}(R^e) & \xrightarrow{\mathbf{HH}_{\bullet}(-)} & (\mathbb{Q} - \mathbf{gmod})^- & \mathbf{D}^{\text{b}}(R^e) & \xrightarrow{\mathbf{HH}^{\bullet}(-)} & (\mathbb{Q} - \mathbf{gmod})^+
 \end{array} \tag{5.5}$$

where $(\mathbb{Q} - \mathbf{gmod})^-$ and $(\mathbb{Q} - \mathbf{gmod})^+$ denote the categories of \mathbb{Z} -graded vector spaces over \mathbb{Q} with homological \mathbb{Z} -grading bound from above and from below. In other words, Hochschild homology and cohomology are functors $\text{Tor}_{R^e}(-, R)$ and $\text{Ext}_{R^e}(-, R)$.

If \mathbf{M} is a bounded complex of R^e -modules, then

$$\mathbf{HH}_{\bullet}(\mathbf{M}) = \mathbf{H}_{\bullet}(\mathbf{M} \otimes_{R^e} \mathcal{P}(R)). \tag{5.6}$$

5.2. Split TL tangles and projective $H_{n,m}$ -modules.

5.2.1. *Derived category of $H_{n,m}$ -modules.* Let $\mathbf{K}_{\text{sp}}^{\text{b}}(H_{n,m}) \subset \mathbf{K}^{\text{b}}(H_{n,m})$ denote the bounded homotopy category of semi-projective $H_{n,m}$ -modules. The following is easy to prove:

Theorem 5.1 ([Kho02]). *For any TL $(2n, 2m)$ -tangle λ , the $H_{n,m}$ -module $\langle\langle \lambda \rangle\rangle_{\mathbf{K}}$ is semi-projective with possible degree shift and, consequently, the image of the functor $\mathcal{F}_{\mathbf{K}}$ lies within $\mathbf{K}_{\text{sp}}^{\text{b}}(H_{n,m})$.*

The next theorem strengthens this result for split TL tangles:

Theorem 5.2 ([Kho02]). *The $H_{n,m}$ -modules $P_{\beta,\alpha} = \langle\langle \beta \circ \alpha^\nabla \rangle\rangle_{\mathbb{K}} [m]_{q,2}$, $\alpha \in \mathfrak{C}_m$, $\beta \in \mathfrak{C}_n$ form a complete list of indecomposable projective $H_{n,m}$ -modules.*

Since $H_0 = \mathbb{Z}$ and, consequently, $H_{n,0} = H_n$, Theorem 5.2 implies that $\langle\langle \alpha \rangle\rangle_{\mathbb{K}}$, $\alpha \in \mathfrak{C}_n$ form the full list of indecomposable projective modules of H_n . This fact has three corollaries. The first one is obvious:

Corollary 5.3. *The map $\langle\langle - \rangle\rangle_{\mathbb{K}}: \mathfrak{T}_{0,n} \rightarrow \mathbf{K}^b(H_n)$ threads through the homotopy category of projective modules:*

$$\begin{array}{ccc} & \xrightarrow{\langle\langle - \rangle\rangle_{\mathbb{K}}} & \\ \mathfrak{T}_{0,n} & \xrightarrow{\langle\langle - \rangle\rangle_{\mathbb{K}}} & \mathbf{K}_{\text{pr}}^b(H_n) \hookrightarrow \mathbf{K}^b(H_n) \end{array}$$

The second corollary is a consequence of an obvious relation

$$\langle\langle \beta \circ \alpha^\nabla \rangle\rangle_{\mathbb{K}} = \langle\langle \beta \rangle\rangle_{\mathbb{K}} \otimes \langle\langle \alpha^\nabla \rangle\rangle_{\mathbb{K}}. \quad (5.7)$$

Corollary 5.4. *The isomorphism (5.7) generates a canonical equivalence of categories*

$$H_{n,m}\text{-pr} = (H_n\text{-pr}) \otimes (H_m^{\text{op}}\text{-pr})$$

and consequently

$$\mathbf{K}_{\text{pr}}^{-/+}(H_{n,m}) = \mathbf{K}^{-/+}\left((H_n\text{-pr}) \otimes (H_m^{\text{op}}\text{-pr})\right). \quad (5.8)$$

The third corollary comes from the combination of eqs. (4.11), (4.28) and (4.31): for $\alpha, \beta \in \mathfrak{C}_n$

$$\begin{aligned} \text{Hom}_{\text{TL}_{2n,0}}(\langle\langle \alpha \rangle\rangle, \langle\langle \beta \rangle\rangle) &= \langle\langle \alpha^\nabla \circ \beta \rangle\rangle [n]_{q,2} = \langle\langle \alpha^\nabla \rangle\rangle_{\mathbb{K}} \otimes_{H_n} \langle\langle \beta \rangle\rangle_{\mathbb{K}} [n]_{q,2} \\ &= \langle\langle \alpha \rangle\rangle_{\mathbb{K}}^\vee \otimes_{H_n} \langle\langle \beta \rangle\rangle_{\mathbb{K}} = \text{Hom}_{\mathbf{K}_{\text{pr}}^b(H_n)}(\langle\langle \alpha \rangle\rangle_{\mathbb{K}}, \langle\langle \beta \rangle\rangle_{\mathbb{K}}). \end{aligned}$$

Corollary 5.5. *The functor*

$$\mathcal{F}_{\mathbb{K}}: \check{\text{TL}}_{0,2n} \xrightarrow{=} H_n\text{-pr}, \quad \text{TL}_{0,2n} \xrightarrow{=} \mathbf{K}_{\text{pr}}^b(H_n) \quad (5.9)$$

establishes an equivalence of categories.

A combination of equivalences (5.8), (4.19) and (5.9) leads to the equivalence of categories

$$\mathcal{F}_{\mathbb{K}}: \text{TL}_{2m,2n}^{\text{spl},-/+} \xrightarrow{=} \mathbf{K}_{\text{pr}}^{-/+}(H_{n,m}). \quad (5.10)$$

Let $\langle\langle - \rangle\rangle_{\mathbb{D}}$ be a composition of the categorification map $\langle\langle - \rangle\rangle_{\mathbb{K}}$ and the functor $\mathcal{F}_{\mathbb{K}\mathbb{D}}$:

$$\begin{array}{ccc} & \xrightarrow{\langle\langle - \rangle\rangle_{\mathbb{D}}} & \\ \mathfrak{T}_{2m,2n} & \xrightarrow{\langle\langle - \rangle\rangle_{\mathbb{K}}} & \mathbf{K}_{\text{sp}}^b(H_{n,m}) \xrightarrow{\mathcal{F}_{\mathbb{K}\mathbb{D}}} \mathbf{D}^b(H_{n,m}) \end{array}$$

Theorem 5.6. *The elements $K_0^Q(\langle\langle\beta \circ \alpha^\nabla\rangle\rangle_D)$, $\alpha \in \mathfrak{C}_m$, $\beta \in \mathfrak{C}_n$ form a basis of the $\mathbb{Q}(q)$ vector space $K_0^Q(\mathcal{D}^b(H_{n,m}))$.*

Corollary 5.7. *There are canonical isomorphisms*

$$K_0^Q(\mathcal{D}^b(H_{n,m})) = \mathbb{Q}\mathrm{TL}_{2m,2n}^{\mathrm{spl}}, \quad K_0^+(\mathcal{D}^b(H_{n,m})) = \mathrm{TL}_{2m,2n}^{\mathrm{spl},+}$$

which identify the basis elements $K_0^Q(\langle\langle\beta \circ \alpha^\nabla\rangle\rangle_D)$ and $\langle\beta \circ \alpha^\nabla\rangle$.

From now on throughout the paper we will use $\mathbb{Q}\mathrm{TL}_{2m,2n}^{\mathrm{spl}}$ and $\mathrm{TL}_{2m,2n}^{\mathrm{spl},+}$ in place of Grothendieck groups of $\mathcal{D}^b(H_{n,m})$.

5.2.2. A universal projective resolution.

Theorem 5.8. *The projective resolution functor $\mathcal{P}: \mathcal{D}^b(H_{n,m}) \rightarrow K_{\mathrm{pr}}^-(H_{n,m})$ threads through q^+ -bounded complexes, so there is a commutative diagram*

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & & \curvearrowright & & \\ \mathcal{D}^b(H_{n,m}) & \xrightarrow{\mathcal{P}^+} & K_{\mathrm{pr}}^{-/+}(H_{n,m}) & \hookrightarrow & K_{\mathrm{pr}}^-(H_{n,m}) \\ & \searrow \mathcal{P}_{\mathrm{TL}} & \uparrow \mathcal{F}_K = & & \\ & & \mathrm{TL}_{2m,2n}^{\mathrm{spl},-/+} & & \end{array}$$

which defines the functor $\mathcal{P}_{\mathrm{TL}}$.

Let $\mathbf{P}_n = \mathcal{P}_{\mathrm{TL}}(H_n)$ be a ‘universal’ projective resolution of the H_n^e -module H_n . Since the functor \mathcal{F}_K transforms the tangle composition into the tensor product, projective resolution (5.4) of semi-projective $H_{n,m}$ -modules can be performed universally with the help of the following commutative diagram

$$\begin{array}{ccc} \mathrm{TL}_{2m,2n} & \xrightarrow{\hat{\mathbf{P}}_*} & \mathrm{TL}_{2m,2n}^{\mathrm{spl},-/+} \\ \mathcal{F}_K \downarrow & & \downarrow \mathcal{F}_K = \\ K_{\mathrm{sp}}^b(H_{n,m}) & \xrightarrow{\mathcal{P}_{\mathrm{sp}}} & K_{\mathrm{pr}}^{-/+}(H_{n,m}) \\ \mathcal{F}_{\mathrm{KD}} \searrow & & \nearrow \mathcal{P} \\ & \mathcal{D}^b(H_{n,m}) & \end{array} \tag{5.11}$$

where the universal projective resolution functor $\hat{\mathbf{P}}_*$ is defined similarly to eq.(5.4):

$$\hat{\mathbf{P}}_*(-) = \mathbf{P}_n \circ - = - \circ \mathbf{P}_m. \tag{5.12}$$

In fact, the commutativity of the square in the diagram (5.11) together with the second equality of eq.(5.4) proves that $\mathbf{P}_n \circ - = - \circ \mathbf{P}_m$.

5.2.3. *Hochschild homology.* Let $(\mathbb{Q} - \mathbf{gmod})^{-/+}$ denote the category of $\mathbb{Z} \oplus \mathbb{Z}$ -graded vector spaces $V = \bigoplus_{i,j \in \mathbb{Z}} V_{i,j}$ whose homological grading is bounded from above and which are also q^+ -bounded (see the definition in subsection 4.2.1). Obviously, $K_0^+(\mathbb{Q} - \mathbf{gmod})^{-/+} = \mathbb{Z}[[q, q^{-1}]]$ and K_0^+ acts on the objects of $(\mathbb{Q} - \mathbf{gmod})^{-/+}$ as the graded Euler characteristic χ_q . Let $(\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}} \subset (\mathbb{Q} - \mathbf{gmod})^{-/+}$ denote the full subcategory whose objects have the property that their Euler characteristic is a rational function of q , that is, χ_q threads through $\mathbb{Q}(q)$:

$$\begin{array}{ccc} & \xrightarrow{\chi_q} & \\ & \text{---} & \\ (\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}} & \xrightarrow{\chi_q} & \mathbb{Q}(q) \hookrightarrow \mathbb{Q}[[q, q^{-1}]] \end{array}$$

Theorem 5.9. *The Hochschild homology of $D^b(H_n^e)$ lies within $(\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}}$:*

$$\begin{array}{ccc} & \xrightarrow{\text{HH}_\bullet(-)} & \\ & \text{---} & \\ D^b(H_n^e) & \xrightarrow{\text{HH}_\bullet(-)} & (\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}} \hookrightarrow (\mathbb{Q} - \mathbf{gmod})^{-/+} \end{array}$$

6. RESULTS

6.1. **Categorification.** Admissible boundaries in stable restricted topological WRT theory are 2-spheres S_n^2 with $2n$ marked points. Associated algebras are $\mathcal{A}(S_n^2) = H_n$, canonical involution (1.8) being the homomorphism (4.23). Then by the axiom (1.10)

$$\mathcal{C}(S_n^2) = D^b(H_n), \quad \mathcal{C}(S_m^2 \sqcup S_n^2) = D^b(H_{n,m}),$$

where in the last equation we assume that the boundary component S_m^2 is ‘in’, while the boundary component S_n^2 is ‘out’. Gluing formulas force us to define the category $\mathcal{C}(\emptyset)$ associated with empty boundary as $(\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}}$, however we can not define the action of functors ∇ and $\bar{}$ on it.

The categorification map for 3-ball tangles is defined as the unique map $\langle\langle - \rangle\rangle: \mathfrak{T}_n(\mathbb{B}^3) \rightarrow D^b(H_n)$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathfrak{T}_{0,n} & \xrightarrow{=} & \mathfrak{T}_n(\mathbb{B}^3) \\ \downarrow \langle\langle - \rangle\rangle_{\mathcal{K}} & \searrow \langle\langle - \rangle\rangle_{\mathcal{D}} & \downarrow \langle\langle - \rangle\rangle \\ \mathcal{K}_{\text{pr}}^b(H_n) & \hookrightarrow & D^b(H_n) \end{array}$$

Since gluing two 3-balls together produces a 3-sphere, the gluing axiom determines the categorification map for links in S^3 . Derived tensor product in $D^b(H_n)$ coincides with the ordinary tensor product in $\mathcal{K}_{\text{pr}}^b(H_n)$, hence in view of Theorem 4.5, a link $L \subset S^3$ is mapped into its Khovanov homology: $\langle\langle L, \mathbb{B}^3 \rangle\rangle = \langle\langle L \rangle\rangle$.

The categorification map $\langle\langle - \rangle\rangle: \mathfrak{T}(\mathbb{S}^2 \times \mathbb{I}) \rightarrow \mathcal{D}^b(H_{n,m})$ for rough tangles in $\mathbb{S}^2 \times \mathbb{I}$ is defined with the help of the following theorem:

Theorem 6.1. *There exists a unique categorification map $\langle\langle - \rangle\rangle: \mathfrak{T}(\mathbb{S}^2 \times \mathbb{I}) \rightarrow \mathcal{D}^b(H_{n,m})$ such that the following diagram is commutative:*

$$\begin{array}{ccc}
 \mathfrak{T}_{2m,2n} & \xrightarrow{\mathfrak{s}} & \mathfrak{T}_{2m,2n}(\mathbb{S}^2 \times \mathbb{I}) \\
 \langle\langle - \rangle\rangle_{\mathcal{K}} \downarrow & \searrow \langle\langle - \rangle\rangle_{\mathcal{D}} & \downarrow \langle\langle - \rangle\rangle \\
 \mathcal{K}_{\text{sp}}^b(H_{n,m}) & \xrightarrow{\mathcal{F}_{\mathcal{K}\mathcal{D}}} & \mathcal{D}^b(H_{n,m})
 \end{array} \tag{6.1}$$

Theorem 6.2. *The diagram (1.13)*

$$\begin{array}{ccc}
 & \mathfrak{T}_{2m,2n}(\mathbb{S}^2 \times \mathbb{I}) & \\
 \langle - \rangle \swarrow & & \searrow \langle\langle - \rangle\rangle \\
 \mathcal{D}^b(H_{n,m}) & \xrightarrow{K_0} & \text{QTL}_{2m,2n}^{\text{spl}}
 \end{array}$$

is commutative.

Since $\mathbb{S}^2 \times \mathbb{S}^1$ can be constructed by gluing together the boundary components of $\mathbb{S}^2 \times \mathbb{I}$, the gluing axiom requires that the categorification map for links in $\mathbb{S}^2 \times \mathbb{S}^1$ should be determined by the following theorem:

Theorem 6.3. *There exists a unique homology map $H_{\bullet}^{\text{Kh}}(-): \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) \rightarrow (\mathbb{Q} - \text{gmod})^{\mathcal{Q}}$ such that the following diagram is commutative:*

$$\begin{array}{ccc}
 \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I}) & \xrightarrow{(-; \mathbb{S}^2 \times \mathbb{S}^1)} & \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) \\
 \langle\langle - \rangle\rangle \downarrow & & \downarrow H_{\bullet}^{\text{st}}(-) \\
 \mathcal{D}^b(H_{n,m}) & \xrightarrow{\text{HH}_{\bullet}(-)} & (\mathbb{Q} - \text{gmod})^{\mathcal{Q}}
 \end{array}$$

Theorem 6.4. *The diagram (1.13)*

$$\begin{array}{ccc}
 & \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) & \\
 H_{\bullet}^{\text{st}}(-) \swarrow & & \searrow \langle - \rangle \\
 (\mathbb{Q} - \text{gmod})^{\mathcal{Q}} & \xrightarrow{\chi_q} & \mathbb{Q}(q)
 \end{array} \tag{6.2}$$

is commutative.

Next theorem shows that the universal resolution \mathbf{P}_n allows us to compute the homology $\langle\langle L, \mathbb{S}^2 \times \mathbb{S}^1 \rangle\rangle$ as a Khovanov-type homology within \mathbb{S}^3 and without reference to the algebras $H_{n,m}$.

Theorem 6.5. *The following diagram is commutative:*

$$\begin{array}{ccccc}
 \mathfrak{T}_{2n,2n} & \xrightarrow{s} & \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I}) & \xrightarrow{(-; \mathbb{S}^2 \times \mathbb{S}^1)} & \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) & (6.3) \\
 \langle\langle - \rangle\rangle \downarrow & & & & \searrow^{H_\bullet^{\text{st}}(-)} & \\
 \text{TL}_{2n,2n} & \xrightarrow{\hat{\mathbf{P}}_*} & \text{TL}_{2n,2n}^{\text{spl}, -/+} & \xrightarrow{H_\bullet(-; \mathbb{S}^3)} & (\mathbb{Q} - \text{gmod})^{-/+} & \longleftarrow (\mathbb{Q} - \text{gmod})^{\mathcal{Q}}
 \end{array}$$

In other words, if a link in $\mathbb{S}^2 \times \mathbb{S}^1$ is presented as a $\mathbb{S}^2 \times \mathbb{S}^1$ closure of a $(2n, 2n)$ -tangle τ , then its stable homology is isomorphic to the homology of the composition $\langle\langle \tau \rangle\rangle \circ \mathbf{P}_n$ closed within \mathbb{S}^3 :

$$H_\bullet^{\text{st}}(\tau, \mathbb{S}^2 \times \mathbb{S}^1) = H_\bullet(\langle\langle \tau \rangle\rangle \circ \mathbf{P}_n; \mathbb{S}^3). \quad (6.4)$$

The universal projective resolution complex \mathbf{P}_n is approximated by categorification complexes of torus braids with high twist. As a result, eq.(6.4) provides an effective method of computing stable homology of links in $\mathbb{S}^2 \times \mathbb{S}^1$ by approximating it with Khovanov homology of their ‘torus braid closures’ within \mathbb{S}^3 .

6.2. Infinite torus braid as a projective resolution of H_n .

6.2.1. *Torus braids yield a projective resolution of H_n .* Let \mathcal{A}_\angle be a set of pairs of integer numbers confined within a certain angle on a square lattice:

$$\mathcal{A}_\angle = \{(i, j) \in \mathbb{Z}^2 \mid i \geq 0, \ i \leq j \leq 2i\} \quad (6.5)$$

and let $\mathcal{A}_\angle[k, l]$ denote the shifted set \mathcal{A}_\angle :

$$\mathcal{A}_\angle[k, l] = \{(i, j) \mid (i - k, j - l) \in \mathcal{A}_\angle\}. \quad (6.6)$$

We also introduce a special notation

$$\mathcal{A}_\angle(t, n; m) = \mathcal{A}_\angle \left[\frac{1}{2}mt^2, \frac{1}{2}mt^2 + mt - \frac{1}{2}t + \frac{1}{2}n \right]. \quad (6.7)$$

A complex $\mathbf{A} \in \text{Ob TL}_{2n}^-$ of eq.(4.14) is called *angle-shaped* if the multiplicities $m_{-i,j,\mu}^\lambda$ are non-zero only when $(i, j) \in \mathcal{A}_\angle$. Obviously, in this case the complex is an object of $\text{TL}_{2n}^{-/+}$. A complex \mathbf{A} is called *split* if the multiplicities $m_{-i,j,\mu}^\lambda$ are non-zero only when the tangles λ are split.

A *truncation* of a complex \mathbf{A} is defined as follows: $\mathbf{t}_{\leq m}^- \mathbf{A} = (A_{-m} \rightarrow A_{-m+1} \rightarrow \dots)$.

Let \mathbf{A} be an object of TL_{2n}^- . We use a notation \mathbf{A}^\sharp for a particular complex with special properties, which represents a homotopy equivalence class of \mathbf{A} .

In Section 8 we construct special categorification complexes of torus braids.

Theorem 6.6. *There exists a sequence of special complexes $\langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$, $m = 1, 2, \dots$, such that*

- (1) *a TL tangle λ with a through degree t_λ may appear in $\langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ only within a shifted angle region:*

$$m_{-i,j,\mu}^\lambda > 0 \quad \text{only if } (i, j) \in \mathcal{A}_\angle(t_\lambda, n; m); \quad (6.8)$$

- (2) *there is an isomorphism of truncated complexes*

$$\mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m+1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp \cong \mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp. \quad (6.9)$$

According to eq.(6.9), braid complexes $\mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ ‘stabilize’ and it turns out that their ‘stable limit’ is the universal resolution \mathbf{P}_n .

Theorem 6.7. *There exists a particular angle-shaped complex $\mathbf{P}_n^\sharp \in \mathrm{Ob} \mathrm{TL}_{2n}^{-/+}$ representing the homotopy equivalence class of \mathbf{P}_n , such that the following truncated complexes are isomorphic:*

$$\mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp \cong \mathbf{t}_{\leq 2m-1}^- \mathbf{P}_n^\sharp. \quad (6.10)$$

A combination of this theorem with formula (6.4) leads to a practical method of computing homology of a link in $\mathbb{S}^2 \times \mathbb{S}^1$ presented as a closure of a $(2n, 2n)$ -tangle τ :

Theorem 6.8. *For a $(2n, 2n)$ -tangle τ there is a canonical isomorphism of homologies*

$$\mathrm{H}_i^{\mathrm{st}}(\tau, \mathbb{S}^2 \times \mathbb{S}^1) = \mathrm{H}_i^{\mathrm{Kh}}\left(\tau \circ \overset{m}{\mathcal{X}} \mathcal{X}_{2n}; \mathbb{S}^3\right) \quad (6.11)$$

for $i \geq |\langle\langle \tau \rangle\rangle|_{\mathfrak{h}}^+ - 2m + 2$.

Note that since the complexes \mathbf{P}_n^\sharp and $\langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ are trivial in positive homological degrees, both sides of eq.(6.11) are trivial for $i > |\langle\langle \tau \rangle\rangle|_{\mathfrak{h}}^+$.

6.3. A conjecture about the structure of the Hochschild homology and cohomology of the algebra H_n . Hochschild homology and cohomology of H_n is related to the homology of the circular closure of the trivial braid within $S^2 \times S^1$. Indeed, combining eq.(4.27) and the definition of the categorification map $H_\bullet^{\text{st}}(-): \mathfrak{L}(S^2 \times S^1) \rightarrow (\mathbb{Q} - \mathbf{gmod})^Q$ via Theorem 6.3 we get the isomorphism

$$\text{HH}_\bullet(H_n) = H_\bullet^{\text{st}}\left(\overline{\underset{\cdot}{\cdot}}_{2n}; S^2 \times S^1\right). \tag{6.12}$$

The algebra H_n has a Frobenius trace with q -degree equal to $-2n$, hence its Hochschild cohomology is dual to its Hochschild homology up to a degree shift:

$$\text{HH}^\bullet(H_n) = \text{HH}_\bullet(H_n)^\vee [2n]_q. \tag{6.13}$$

This relation can be proved also with the help of universal categorification:

$$\begin{aligned} \text{HH}^\bullet(H_n) &= H_\bullet(\text{Hom}_{H_n^e}(\mathcal{P}(H_n), H_n)) = H_\bullet(\mathcal{P}(H_n)^\vee \otimes_{H_n^e} H_n) = H_\bullet(\mathcal{P}(H_n)^\nabla \otimes_{H_n^e} H_n) [2n]_q \\ &= H_\bullet(\mathcal{F}_K(\mathbf{P}_n^\nabla) \otimes_{H_n^e} H_n) [2n]_q = H_\bullet(\mathbf{P}_n^\nabla; \mathbb{S}^3) [2n]_q = H_\bullet(\mathbf{P}_n; \mathbb{S}^3)^\vee [2n]_q \\ &= H_\bullet(\mathcal{P}(H_n) \otimes_{H_n^e} H_n)^\vee [2n]_q = \text{HH}_\bullet(H_n)^\vee [2n]_q. \end{aligned}$$

Here the first and last equalities are the definitions (5.5) of the Hochschild homology and cohomology, the third equality comes from the second of equations (4.31), while the seventh and fifth equalities come from commutative triangles

$$\begin{array}{ccc} \text{TL}_{2n}^{\text{spl}, -} & \xrightarrow{(-; \mathbb{S}^3)} & (\mathbb{Q} - \mathbf{gmod})^- \\ \mathcal{F}_K \downarrow & & \\ \mathcal{K}_{\text{pr}}^-(H_n^e) & \xrightarrow{H_\bullet(- \otimes_{H_n^e} H_n)} & (\mathbb{Q} - \mathbf{gmod})^- \end{array} \qquad \begin{array}{ccc} \text{TL}_{2n}^{\text{spl}, +} & \xrightarrow{(-; \mathbb{S}^3)} & (\mathbb{Q} - \mathbf{gmod})^+ \\ \mathcal{F}_K \downarrow & & \\ \mathcal{K}_{\text{pr}}^+(H_n^e) & \xrightarrow{H_\bullet(- \otimes_{H_n^e} H_n)} & (\mathbb{Q} - \mathbf{gmod})^+ \end{array}$$

The first of these triangles is a part of the commutative diagram (7.10) (the q^+ -boundedness condition plays no role there), while the second diagram is its analog for complexes bounded from below.

Let $T_{n, nm}$ denote the torus link which can be presented as n -cabling of the unknot with framing number m : $T_{n, nm} = \left(\bigcirc_{2n; \mathbb{S}^3}^m\right)$, and let $T_{n, -nm} = T_{n, nm}^\nabla$ denote its mirror image. Note that the links are framed and each of $2n$ components of the link $T_{n, nm}$ has a self-linking number m . Combining Theorem 6.8 with eqs. (6.12) and (6.13) we arrive at the following theorem:

Theorem 6.9. *There are isomorphisms*

$$\text{HH}_{-i}(H_n) = H_{-i}^{\text{Kh}}(T_{n, nm}), \qquad \text{HH}^i(H_n) = H_i^{\text{Kh}}(T_{n, -nm}) [2n]_q$$

for $i \leq 2m - 2$

These isomorphisms were first observed by Jozef Przytycki [Prz10] in case of $n = 1$.

The homologies of torus links can be evaluated for sufficiently small n and m with the help of computer programs. This experimental data led us to a conjecture regarding the structure of the Hochschild cohomology of H_n as a commutative algebra.

Consider the following lists of variables:

$$\begin{array}{lll} \mathbf{x} = x_1, \dots, x_{2n} & \deg_q x_i = 2 & \deg_h x_i = 0 \\ \mathbf{a} = a_1, \dots, a_n & \deg_q a_i = -2i - 2 & \deg_h a_i = 2i \\ \boldsymbol{\theta} = \theta_1, \dots, \theta_n & \deg_q \theta_i = -2i + 2 & \deg_h \theta_i = 2i - 1 \end{array}$$

The variables $\boldsymbol{\theta}$ have odd homological degree, hence they anti-commute:

$$\theta_i \theta_j = -\theta_j \theta_i, \quad \theta_i^2 = 0.$$

Conjecture 6.10. *The Hochschild cohomology of the algebra H_n has the following graded commutative algebra structure:*

$$\mathrm{HH}^\bullet(H_n) = \mathbb{Q}[\mathbf{x}, \mathbf{a}, \boldsymbol{\theta}] / I_{\mathrm{rel}},$$

where I_{rel} is the ideal generated by relations

$$\begin{aligned} x_1^2 = \dots = x_{2n}^2 = 0, \quad x_1 + \dots + x_{2n} = 0, \\ a_i p_i(\mathbf{x}) = \theta_i p_i(\mathbf{x}) = 0 \quad \text{for } \forall p_i \in \mathbb{Q}[\mathbf{x}] \text{ such that } \deg_{\mathbf{x}} p_i(\mathbf{x}) = i, \end{aligned}$$

where in the second relation $p_i \in \mathbb{Q}[\mathbf{x}]$ is any polynomial of homogeneous \mathbf{x} -degree i .

7. PROOFS OF TQFT PROPERTIES

7.1. A derived category of $H_{n,m}$ -modules.

7.1.1. *A special \mathbb{Z} -graded algebra.* Consider an algebra R which has some special properties shared by all algebras $H_{n,m}$.

A finite-dimensional non-negatively \mathbb{Z} -graded algebra $R = \bigoplus_{j \geq 0} R_{|j}$ with an involution $\diamond: R \rightarrow R^{\mathrm{op}}$ is called *convenient* if its zero-degree subalgebra $R_{|0}$ is generated by a finite number of mutually orthogonal idempotents e_1, \dots, e_N :

$$e_a e_b = \begin{cases} e_a, & \text{if } a = b, \\ 0, & \text{if } a \neq b, \end{cases} \quad e_a + \dots + e_N = \mathbb{1}_R,$$

where $\mathbb{1}_R$ is the unit of R . A convenient algebra R has N indecomposable projective modules $P_a = R e_a$ and N one-dimensional irreducible modules $S_a = (R/R_{|>0}) e_a$, where $R_{|>0} = \bigoplus_{j \geq 1} R_{|j}$.

Theorem 7.1 ([Kho02]). *The modules P_a and S_a , $a = 1, \dots, N$ form a complete list of indecomposable projective and, respectively, irreducible R -modules. The elements $K_0(S_a)$ generate freely $K_0(D^b(R))$.*

Theorem 7.2. *Each R -module M has a resolution $\mathcal{P}(M) = (\dots \rightarrow A_1 \rightarrow A_0)$ such that $|A_{-i}|_q \geq i$.*

Proof. Let $\text{Hom}_j(P_a, P_b)$ denote the q -degree j part of $\text{Hom}(P_a, P_b)$. Then it is easy to see that

$$\text{Hom}_{<0}(P_a, P_b) = 0, \quad \text{Hom}_0(P_a, P_b) = \begin{cases} \mathbb{Q}, & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases} \quad (7.1)$$

and the generator of $\text{Hom}_0(P_a, P_a)$ is the identity homomorphism.

Consider a resolution of the R module M . If a constituent projective module of the resolution complex has a non-trivial zero-degree homomorphism coming from it to another module P_a , then this pair can be contracted. After these contractions, if a constituent projective module does not have a homomorphism originating from it, then it will contribute to homology, because its zero-degree part can not be annihilated by incoming homomorphisms. Since the homology of a resolution must be concentrated in the zero homological degree, it follows that $|A_{-i-1}|_q \geq |A_{-i}|_q$. \square

Corollary 7.3. *The projective resolution functor $\mathcal{P}: D^b(R) \rightarrow K_{\text{pr}}^-(R)$ threads through the subcategory of q^+ -bounded complexes:*

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & \curvearrowright & & \curvearrowleft & \\ D^b(R) & \xrightarrow{\mathcal{P}^+} & K_{\text{pr}}^{-/+}(R) & \hookrightarrow & K_{\text{pr}}^-(R) \end{array}$$

This corollary allows us to use projective resolutions for the calculation of K_0 of objects of $D^b(R)$. Indeed, there is a commutative diagram

$$\begin{array}{ccc} K_{\text{pr}}^{-/+}(R) & \xrightarrow{K_0^+} & K_0^+(K_{\text{pr}}^{-/+}(R)) \\ \mathcal{P}^+ \uparrow \int & & \downarrow \\ D^b(R) & \xrightarrow{K_0} & K_0^+(D^b(R)) \end{array} \quad (7.2)$$

where by definition $K_0^+(D^b(R)) = K_0(D^b(R)) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}[[q, q^{-1}]]$. The right vertical map is surjective, because $K_0^+(D^b(R))$ is generated by K_0 -images of objects in $D^b(R)$, but all those objects can be mapped into $K_0^+(D^b(R))$ through the resolution \mathcal{P}^+ .

Theorem 7.4. *The elements $K_0(P_a)$ form a basis in $K_0^Q(D^b(R))$.*

Proof. According to theorem 7.1, the elements $K_0(S_a)$ generate freely $K_0(\mathcal{D}^b(R))$, hence they form a basis of $K_0^+(\mathcal{D}^b(R))$ and $\dim K_0^+(\mathcal{D}^b(R)) = \dim K_0^Q(\mathcal{D}^b(R)) = N$. The right vertical map in the diagram (7.2) is surjective, hence N elements $K_0(P_a)$ generate $K_0^+(\mathcal{D}^b(R))$. Therefore they are linearly independent there and in $K_0^Q(\mathcal{D}^b(R))$, so they form a basis of $K_0^Q(\mathcal{D}^b(R))$. \square

Corollary 7.3 implies that the space $K_0^+(\mathcal{D}^b(R))$ has a $\mathbb{Q}[[q, q^{-1}]]$ -valued symmetric bilinear pairing

$$(K_0^+(\mathbf{A}), K_0^+(\mathbf{B}))_{\text{Tor}} = \chi_q(\text{Tor}_R(\mathbf{A}^\diamond, \mathbf{B})), \quad (7.3)$$

where, by definition, Tor_R is the homology of the derived tensor product:

$$\text{Tor}_R(\mathbf{A}^\diamond, \mathbf{B}) = H_\bullet(\mathcal{P}^+(\mathbf{A})^\diamond \otimes_R \mathbf{B}) = H_\bullet(\mathbf{A}^\diamond \otimes_R \mathcal{P}^+(\mathbf{B})).$$

Proposition 7.5. *The pairing (7.3) restricted to $K_0^Q(\mathcal{D}^b(R))$ takes values in $\mathbb{Q}(q)$.*

Proof. According to [Kho02], the irreducible modules S_a and projective modules P_a have the property

$$(K_0^+(S_a), K_0^+(P_b))_{\text{Tor}} = \dim_q(S_a^\diamond \otimes_R P_b) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta. \end{cases} \quad (7.4)$$

Since the elements $K_0^+(S_a)$ and $K_0^+(P_a)$ belong to $K_0^Q(\mathcal{D}^b(R))$ and form (dual) bases there, equation (7.4) implies that the pairing (7.3) on $K_0^Q(\mathcal{D}^b(R))$ takes values in its base field $\mathbb{Q}(q)$. \square

The following is obvious:

Proposition 7.6. *If algebras R_1 and R_2 are convenient then $R_1 \otimes R_2$ is also convenient.*

In particular, if R is convenient then $R^e = R \otimes R^{\text{op}}$ is also convenient.

Theorem 7.7. *The Hochschild homology of $\mathcal{D}^b(R^e)$ lies within $(\mathbb{Q} - \text{gmod})^Q$:*

$$\begin{array}{ccc} & \text{HH}_\bullet(-) & \\ & \curvearrowright & \\ \mathcal{D}^b(R^e) & \xrightarrow{\text{HH}_\bullet(-)} & (\mathbb{Q} - \text{gmod})^Q \hookrightarrow (\mathbb{Q} - \text{gmod})^{-/+} \end{array}$$

Proof. If \mathbf{M} is a bounded complex of R^e -modules, then according to the definition of the Hochschild homology,

$$\chi_q(\text{HH}_\bullet(\mathbf{M})) = \chi_q(\text{Tor}_{R^e}(\mathbf{M}, R)) = (\mathbf{M}, R^{\text{op}})_{\text{Tor}},$$

hence by Proposition 7.6 $\chi_q(\text{HH}_\bullet(\mathbf{M})) \in \mathbb{Q}(q)$ and by definition $\text{HH}_\bullet(\mathcal{F}_{\text{KD}}(\mathbf{M})) \in (\mathbb{Q} - \text{gmod})^Q$. \square

7.1.2. *Algebras $H_{n,m}$ and the universal resolution.* The algebras $H_{n,m}$ are convenient. In view of Proposition 7.6, it is sufficient to check that H_n is convenient. Indeed, it is easy to see from the definition (4.22) that the zero-degree part of H_n consists of identity endomorphisms of objects $\langle\langle \alpha \rangle\rangle$, and those morphisms are mutually orthogonal idempotents.

Since the algebras $H_{n,m}$ are convenient, Theorems 5.2, 5.6, 5.9 and 5.8 are particular cases of Theorems 7.1, 7.4, 7.7 and Corollary 7.3.

Theorem 7.8. *The map $K_0^+ : \mathcal{TL}_{2n}^{\text{spl},-/+} \rightarrow \mathcal{TL}_{2n}^{\text{spl},+}$ maps the universal projective resolution \mathbf{P}_n to the Jones-Wenzl projector $P_{2n,0}$:*

$$K_0^+(\mathbf{P}_n) = P_{2n,0}. \tag{7.5}$$

Proof. For $\alpha \in \mathfrak{C}_n$, the H_n -module $\langle\langle \alpha \rangle\rangle$ is projective. Projective resolution is unique up to homotopy, hence there is a homotopy equivalence $\mathcal{F}_K(\mathbf{P}_n) \otimes_{H_n} \langle\langle \alpha \rangle\rangle_K \sim \langle\langle \alpha \rangle\rangle_K$. The functor \mathcal{F}_K translates tangle composition into tensor product and establishes a category equivalence (5.10), hence $\mathbf{P}_n \circ \langle\langle \alpha \rangle\rangle \sim \langle\langle \alpha \rangle\rangle$. Applying K_0^+ to both sides of this relation, we find that $K_0^+(\mathbf{P}_n) \circ \langle\alpha\rangle = \langle\alpha\rangle$. Since $K_0^+(\mathbf{P}_n) \in \mathcal{TL}_{2m,2n}^{\text{spl},+}$, Theorem 3.1 implies eq.(7.5) \square

Corollary 7.9. *The solid arrows of following diagram are commutative:*

$$\begin{array}{ccc}
 \mathcal{K}_{\text{sp}}^b(H_{n,m}) & \xrightarrow{\mathcal{F}_{\text{KD}}} & \mathcal{D}^b(H_{n,m}) \\
 \uparrow \mathcal{F}_K & \swarrow \langle\langle - \rangle\rangle_K & \swarrow \langle\langle - \rangle\rangle_D \\
 & \mathfrak{T}_{2m,2n} & \xrightarrow{s} \mathfrak{T}_{2m,2n}(S^2 \times \mathbb{I}) \\
 & \swarrow \langle\langle - \rangle\rangle & \downarrow \mathcal{P}_{\text{TL}} \\
 \mathcal{TL}_{2m,2n} & \xrightarrow{\hat{\mathbf{P}}_*} & \mathcal{TL}_{2m,2n}^{\text{spl},-/+} \\
 & \downarrow \mathcal{K}_0 & \downarrow \mathcal{K}_0 \\
 & \mathcal{TL}_{2m,2n} & \xrightarrow{\hat{\mathbf{P}}_{*,0}} \mathcal{QTL}_{2m,2n}^{\text{spl}} \\
 & & \downarrow \mathcal{J} \\
 & & \mathcal{TL}_{2m,2n}^{\text{spl},+}
 \end{array} \tag{7.6}$$

Proof. The commutativity of the ‘skewed’ bottom horizontal square follows from Theorem 7.8. The vertical square is a part of the diagram (5.11). The commutativity of the right vertical triangle follows from the commutativity of the diagram (7.2) in the case when $R = H_{n,m}$. The vertical upper-left triangle is the diagram (4.29) and the vertical lower-left triangle is the diagram (4.16). Finally, the upper horizontal triangle is the definition of the functor $\langle\langle - \rangle\rangle_D$. \square

Remark 7.10. *Since $H_n \otimes_{H_n} H_n = H_n$, it follows that $\mathcal{P}(H_n) \otimes_{H_n} \mathcal{P}(H_n) \sim \mathcal{P}(H_n)$ and, consequently,*

$$\mathbf{P}_n \circ \mathbf{P}_n \sim \mathbf{P}_n.$$

This relation together with eq.(7.5) suggests that \mathbf{P}_n is a categorification of the projector $P_{2n,0}$, but we think that a proper setting for this statement would be a simultaneous categorification of all projectors $P_{n,m}$ which would allow the verification of the categorified orthogonality and completeness conditions (3.6), so we leave it as a conjecture.

7.1.3. Basic properties of Hochschild homology. For two algebras R_1 and R_2 , let \mathbf{M} and \mathbf{N} be complexes of $R_2 \otimes R_1^{\text{op}}$ -modules and, respectively, $R_1 \otimes R_2^{\text{op}}$ -modules. Then the Hochschild homologies of their derived tensor products are canonically isomorphic: $\text{HH}_\bullet(\mathbf{M} \overset{\text{L}}{\otimes}_{R_1} \mathbf{N}) = \text{HH}_\bullet(\mathbf{N} \overset{\text{L}}{\otimes}_{R_2} \mathbf{M})$. If \mathbf{M} and \mathbf{N} are semi-projective, then their derived tensor products coincide with the ordinary ones, so there is a simpler canonical isomorphism

$$\text{HH}_\bullet(\mathbf{M} \otimes_{R_1} \mathbf{N}) = \text{HH}_\bullet(\mathbf{N} \otimes_{R_2} \mathbf{M}). \quad (7.7)$$

Suppose that an algebra R has an involution $\diamond: R \rightarrow R^{\text{op}}$. It determines an involutive functor $\diamond: \mathbf{K}^b(R) \rightarrow \mathbf{K}^b(R^{\text{op}})$, which turns a R -module into a R^{op} -module with the help of the isomorphism \diamond . The algebra $R^e = R \otimes R^{\text{op}}$ is canonically isomorphic to its opposite, hence \diamond generates an automorphism $\diamond_e: R^e \rightarrow R^e$ and a corresponding autoequivalence functor $\diamond_e: \mathbf{K}^b(R^e) \rightarrow \mathbf{K}^b(R^e)$.

Theorem 7.11. *For a complex of R^e -modules \mathbf{M} there is a canonical isomorphism*

$$\text{HH}_\bullet(\mathbf{M}) = \text{HH}_\bullet(\mathbf{M}^{\diamond_e}). \quad (7.8)$$

Proof. It is easy to see that the isomorphism $\diamond: R \rightarrow R^{\text{op}}$ also establishes an isomorphism of R^e -modules $\diamond: R \rightarrow R^{\diamond_e}$. Now the isomorphism (7.8) is established by a chain of canonical isomorphisms

$$\text{HH}_\bullet(\mathbf{M}) = \text{Tor}_{R^e}(\mathbf{M}, R) = \text{Tor}_{R^e}(\mathbf{M}^{\diamond_e}, R^{\diamond_e}) = \text{Tor}_{R^e}(\mathbf{M}^{\diamond_e}, R) = \text{HH}_\bullet(\mathbf{M}^{\diamond_e}).$$

□

7.1.4. Hochschild homology and a closure within \mathbb{S}^3 .

Theorem 7.12. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{D}^b(H_n^e) & \xrightarrow{\text{HH}_\bullet(-)} & (\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}} \\ \mathcal{P}_{\text{TL}} \downarrow & & \downarrow \\ \text{TL}_{2n}^{\text{spl}, -/+} & \xrightarrow{(-; \mathbb{S}^3)} & (\mathbb{Q} - \mathbf{gmod})^{-/+} \end{array} \quad (7.9)$$

Proof. Consider the following diagram:

$$\begin{array}{ccc}
 \mathcal{K}_{\text{pr}}^{-/+}(H_n^e) & \xrightarrow{=} & \mathcal{K}^{-/+}\left((H_n^{\text{op-pr}} \otimes (H_n\text{-pr}))\right) & (7.10) \\
 \uparrow \mathcal{P}^+ & \searrow \mathbf{H}\bullet(-\otimes_{H_n^e} H_n) & \downarrow \mathbf{H}\bullet(-\otimes_{H_n} -) & \\
 \mathcal{F}_K = \mathcal{D}^b(H_n^e) & \xrightarrow{\text{HH}\bullet(-)} & (\mathbb{Q}\text{-gmod})^Q \hookrightarrow (\mathbb{Q}\text{-gmod})^{-/+} & = \mathcal{F}_K \otimes \mathcal{F}_K \\
 \downarrow \mathcal{P}_{\text{TL}} & \nearrow (-; S^3) & \uparrow -\circ- & \\
 \text{TL}_{2n}^{\text{spl}, -/+} & \xrightarrow{=} & \mathcal{K}^{-/+}(\check{\text{TL}}_{2n,0} \otimes \check{\text{TL}}_{0,2n}) &
 \end{array}$$

The lower left elementary triangle in it coincides with the diagram (7.9). All other elementary triangles, as well as the outer frame, are commutative, hence the lower left elementary triangle is also commutative. \square

7.2. Proof of categorification properties. The proof of Theorem 6.1 is based on a theorem which we will prove later:

Theorem 7.13. *The following complexes are homotopy equivalent within $\text{TL}_{2m,2n}^{\text{spl}, -/+}$:*

$$\hat{\mathbf{P}}_*\left(\left\langle\left\langle \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right\rangle\right\rangle_{2n}\right) \sim \hat{\mathbf{P}}_*\left(\left\langle\left\langle \begin{array}{c} 1 \\ \text{---} \\ \text{---} \end{array} \right\rangle\right\rangle_{2n}\right) \sim \hat{\mathbf{P}}_*\left(\left\langle\left\langle \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right\rangle\right\rangle_{2n}\right). \quad (7.11)$$

Proof of Theorem 6.1. The commutativity of solid arrows in the diagram (7.6) together with the injectivity of the functor \mathcal{P}_{TL} mean that homotopy equivalences (7.11) imply quasi-isomorphisms

$$\left\langle\left\langle \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right\rangle\right\rangle_{2n} \simeq \left\langle\left\langle \begin{array}{c} 1 \\ \text{---} \\ \text{---} \end{array} \right\rangle\right\rangle_{2n} \simeq \left\langle\left\langle \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right\rangle\right\rangle_{2n}. \quad (7.12)$$

According to Theorem 2.3, $\ker \mathfrak{s}$ is generated by the braids $\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array}; n$ and $\begin{array}{c} 1 \\ \text{---} \\ \text{---} \end{array}; n$, hence eq.(7.12) implies the existence of the map $\langle\langle - \rangle\rangle$. Its uniqueness follows from the surjectivity of the map \mathfrak{s} . \square

Proof of Theorem 6.2. It is easy to see that the commutativity of solid arrows in the diagram (7.6) together with the commutativity of the upper horizontal triangle with a dashed side (Theorem 6.1) and surjectivity of the map \mathfrak{s} implies the full commutativity of the diagram and, in particular, the commutativity of the right vertical triangle claimed by Theorem 6.2. \square

Proof of Theorem 6.3. Consider a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{T}_{2n,2n} & \xrightarrow{\mathfrak{s}} & \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I}) & \xrightarrow{(-; \mathbb{S}^2 \times \mathbb{S}^1)} & \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) \\
& & \downarrow \langle\langle - \rangle\rangle & & \downarrow \text{H}^{\text{st}}(-) \\
& \searrow \langle\langle - \rangle\rangle_{\mathbb{D}} & \text{D}^b(H_n^e) & \xrightarrow{\text{HH}_\bullet(-)} & (\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}}
\end{array}$$

The surjectivity of the maps \mathfrak{s} and $(-; \mathbb{S}^2 \times \mathbb{S}^1)$ together with Theorem 2.4 implies that the existence of the homology map $\text{H}_\bullet^{\text{Kh}}(-)$ would follow from the isomorphisms

$$\text{HH}_\bullet(\langle\langle \tau_2 \circ \tau_1 \rangle\rangle_{\mathbb{D}}) = \text{HH}_\bullet(\langle\langle \tau_1 \circ \tau_2 \rangle\rangle_{\mathbb{D}}), \quad \text{HH}_\bullet(\langle\langle \tau^\diamond \rangle\rangle_{\mathbb{D}}) = \text{HH}_\bullet(\langle\langle \tau \rangle\rangle_{\mathbb{D}}),$$

which should hold for any $\tau_1 \in \mathfrak{T}_{2n,2m}$, $\tau_2 \in \mathfrak{T}_{2m,2n}$ and $\tau \in \mathfrak{T}_{2n,2n}$. The first isomorphism follows from the isomorphisms (4.28) and (7.7), while the second isomorphism follows from the isomorphisms (4.32) and (7.8). \square

Proof of Theorem 6.4. The diagram (6.2) coincides with the right face of the solid cube in the following diagram:

$$\begin{array}{ccccc}
\text{D}^b(H_n^e) & \xrightarrow{\text{HH}_\bullet(-)} & (\mathbb{Q} - \mathbf{gmod})^{\mathcal{Q}} & & (7.13) \\
\downarrow \mathcal{P}^+ & \swarrow \langle\langle - \rangle\rangle & \uparrow \text{H}^{\text{st}}(-) & & \\
\mathfrak{T}_{2n,2n} & \xrightarrow{\mathfrak{s}} & \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I}) & \xrightarrow{(-; \mathbb{S}^2 \times \mathbb{S}^1)} & \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1) \\
\downarrow \langle\langle - \rangle\rangle & \downarrow \mathcal{P}^+ & \downarrow \langle\langle - \rangle\rangle & \downarrow \langle\langle - \rangle\rangle & \downarrow \langle\langle - \rangle\rangle \\
\text{TL}_{2n} & \xrightarrow{\hat{\mathcal{P}}_{*,0}} & \text{TL}_{2n}^{\text{spl}, -/+} & \xrightarrow{(-; \mathbb{S}^3)} & (\mathbb{Q} - \mathbf{gmod})^- & \xrightarrow{\chi_q} & \mathbb{Q}(q) \\
& & \downarrow \mathcal{K}_0^+ & \downarrow \langle\langle - \rangle\rangle & \downarrow \chi_q & \downarrow \chi_q & \downarrow \chi_q \\
& & \text{TL}_{2n}^{\text{spl}, +} & \xrightarrow{(-; \mathbb{S}^3)} & \mathbb{Q}[[q, q^{-1}]]
\end{array}$$

The commutativity of all other solid cube faces has been established (in particular, the vertical back face is the diagram (7.9)). Hence the commutativity of the right face follows from the surjectivity of the map $(-; \mathbb{S}^2 \times \mathbb{S}^1): \mathfrak{T}_{2n,2n}(\mathbb{S}^2 \times \mathbb{I}) \rightarrow \mathfrak{L}(\mathbb{S}^2 \times \mathbb{S}^1)$. \square

Proof of Theorem 6.5. The addition of dashed arrows to the solid cube in the diagram (7.13) preserves the commutativity, because the dashed arrows together with the left face of the solid cube form a part of the diagram (7.6). The diagram (6.3) is a part of the whole diagram (7.13). \square

7.3. Categorification complexes of quasi-trivial tangles. An elementary cobordism ϵ between two tangle diagrams τ and τ' is a cobordism of one of the following types: creation or annihilation of a disjoint circle, a saddle cobordism, a Reidemeister move. To an elementary cobordism of each type one associates a special morphism between the corresponding categorification complexes $\langle\langle\tau\rangle\rangle \xrightarrow{\hat{\epsilon}} \langle\langle\tau'\rangle\rangle$. In particular, to a saddle cobordism one associates the corresponding planar cobordism acting on constituent objects $\langle\langle\lambda\rangle\rangle$ in the complex $\langle\langle\tau\rangle\rangle$.

A *cobordism movie* is a sequence of elementary cobordisms: $\epsilon = \epsilon_1, \dots, \epsilon_k$. A morphism $\hat{\epsilon}$ associated to a movie is a composition of elementary morphisms: $\hat{\epsilon} = \hat{\epsilon}_k \cdots \hat{\epsilon}_1$.

Theorem 7.14. *If two cobordism movies ϵ and ϵ' yield isotopic cobordisms, then the corresponding morphisms are homotopy equivalent up to a sign: $\hat{\epsilon} \sim \pm \hat{\epsilon}'$.*

A Reidemeister movie is a cobordism movie which is a sequence of Reidemeister moves: $\rho = \rho_1, \dots, \rho_k$.

Definition 7.15. *A $(2n, 2n)$ -tangle τ is quasi-trivial if it satisfies two conditions:*

- (1) *For any $\alpha \in \mathfrak{T}_{2n,0}$ (that is, for α being a flipped crossingless matching) there is a Reidemeister movie ρ_α transforming $\alpha \circ \tau$ into α such that for any planar cobordism φ between two tangles α and α' the following diagram is commutative:*

$$\begin{array}{ccc}
 \alpha_1 \circ \tau & \xrightarrow{\varphi \circ \mathbb{1}_\tau} & \alpha_2 \circ \tau \\
 \downarrow \rho_{\alpha_1} & & \downarrow \rho_{\alpha_2} \\
 \alpha_1 & \xrightarrow{\varphi} & \alpha_2
 \end{array} \tag{7.14}$$

that is, the cobordism movies $\rho_{\alpha_1}(\varphi \circ \mathbb{1}_\tau)$ and $\varphi \rho_{\alpha_1}$ are isotopic ($\mathbb{1}_\tau$ denotes the identity cobordism between τ and τ).

- (2) *There exists a crossingless matching $\beta \in \mathfrak{C}_n$ and a Reidemeister movie ρ'_β transforming $\tau \circ \beta$ into β , such that for any flipped crossingless matching α the following Reidemeister movies are isotopic:*

$$\begin{array}{ccc}
 & \xrightarrow{\rho_\alpha \circ \mathbb{1}_\beta} & \\
 \alpha \circ \tau \circ \beta & \xrightarrow{\quad} & \alpha \circ \beta \\
 & \xleftarrow{\mathbb{1}_\alpha \circ \rho'_\beta} &
 \end{array} \tag{7.15}$$

Remark 7.16. *If ϵ is an x -multiplication, then the first condition of this definition is satisfied automatically, so if τ is quasi-trivial, then the second condition is satisfied for any elementary planar cobordism.*

To a $(2n, 2n)$ -tangle τ we associate a functor $\hat{\tau}: \check{\mathbb{T}}\mathbb{L}_{2n,0} \rightarrow \mathbb{T}\mathbb{L}_{2n,0}$ which acts by composing with $\langle\langle\tau\rangle\rangle$: $\hat{\tau} = - \circ \langle\langle\tau\rangle\rangle$.

Theorem 7.17. *If a $(2n, 2n)$ -tangle τ is quasi-trivial, then the tangle composition functor $\hat{\tau}$ is isomorphic to the injection functor $\check{\mathbb{T}}\mathbb{L}_{2n,0} \hookrightarrow \mathbb{T}\mathbb{L}_{2n,0}$.*

Proof. As an additive category, $\check{\mathbb{T}}\mathbb{L}_{2n,0}$ is generated freely by objects $\langle\langle\alpha\rangle\rangle$, where α are flipped crossingless matchings: $\alpha \in \mathfrak{T}_{2n,0}$.

According to Definition 7.15, for any α the tangles α and $\alpha \circ \tau$ are isotopic, hence there is a homotopy equivalence $\langle\langle\alpha\rangle\rangle \circ \langle\langle\tau\rangle\rangle \sim \langle\langle\alpha\rangle\rangle$. Hence, by the definition of functor isomorphism, it remains to prove that for every α there exists a particular homotopy equivalence $f_\alpha: \langle\langle\alpha \circ \tau\rangle\rangle \rightarrow \langle\langle\alpha\rangle\rangle$ such that for any pair of flipped crossingless matchings α_1, α_2 and for any planar cobordism φ between α_1 and α_2 the following diagram is commutative:

$$\begin{array}{ccc} \langle\langle\alpha_1 \circ \tau\rangle\rangle & \xrightarrow{\hat{\varphi} \circ \mathbb{1}_\tau} & \langle\langle\alpha_2 \circ \tau\rangle\rangle \\ \downarrow f_{\alpha_1} & & \downarrow f_{\alpha_2} \\ \langle\langle\alpha_1\rangle\rangle & \xrightarrow{\hat{\varphi}} & \langle\langle\alpha_2\rangle\rangle \end{array} \quad (7.16)$$

In view of Proposition 4.3, it is sufficient to prove this commutativity for φ being an elementary planar cobordism, that is, either an x -multiplication of a saddle cobordism.

Theorem 7.14 says that since the cobordism diagram (7.14) is commutative, the following diagram is commutative up to a sign:

$$\begin{array}{ccc} \langle\langle\alpha_1 \circ \tau\rangle\rangle & \xrightarrow{\hat{\varphi} \circ \mathbb{1}_\tau} & \langle\langle\alpha_2 \circ \tau\rangle\rangle \\ \downarrow \hat{\rho}_{\alpha_1} & \pm & \downarrow \hat{\rho}_{\alpha_2} \\ \langle\langle\alpha_1\rangle\rangle & \xrightarrow{\hat{\varphi}} & \langle\langle\alpha_2\rangle\rangle \end{array} \quad (7.17)$$

Since the Reidemeister movies (7.15) are isotopic, according to Theorem 7.14 there exists a sign factor μ_α such that

$$\mu_\alpha \hat{\rho}_\alpha \circ \mathbb{1}_\beta \sim \mathbb{1}_\alpha \circ \hat{\rho}'_\beta. \quad (7.18)$$

We choose homotopy equivalences as $f_\alpha = \mu_\alpha \hat{\rho}_\alpha$. The ‘up to a sign commutativity’ of the diagram (7.17) implies that there exists a sign factor $\mu_\varphi = \pm 1$ such that the diagram

$$\begin{array}{ccc} \langle\langle\alpha_1 \circ \tau\rangle\rangle & \xrightarrow{\hat{\varphi} \circ \mathbb{1}_\tau} & \langle\langle\alpha_2 \circ \tau\rangle\rangle \\ \downarrow \mu_{\alpha_1} \hat{\rho}_{\alpha_1} & & \downarrow \mu_{\alpha_2} \hat{\rho}_{\alpha_2} \\ \langle\langle\alpha_1\rangle\rangle & \xrightarrow{\mu_\varphi \hat{\varphi}} & \langle\langle\alpha_2\rangle\rangle \end{array} \quad (7.19)$$

is commutative. It remains to prove that $\mu_\varphi = 1$.

Consider the following diagram:

$$\begin{array}{ccc}
 \langle\langle \alpha_1 \circ \tau \circ \beta \rangle\rangle & \xrightarrow{\varphi \circ \mathbb{1}_\tau \circ \mathbb{1}_\beta} & \langle\langle \alpha_2 \circ \tau \circ \beta \rangle\rangle \\
 \begin{array}{c} \curvearrowright \\ \mu_{\alpha_1} \hat{\rho}_{\alpha_1} \circ \mathbb{1}_\beta \\ \curvearrowleft \end{array} & & \begin{array}{c} \curvearrowright \\ \mu_{\alpha_2} \hat{\rho}_{\alpha_2} \circ \mathbb{1}_\beta \\ \curvearrowleft \end{array} \\
 \langle\langle \alpha_1 \circ \beta \rangle\rangle & \xrightarrow{\varphi \circ \mathbb{1}_\beta} & \langle\langle \alpha_2 \circ \beta \rangle\rangle \\
 \begin{array}{c} \curvearrowleft \\ \mathbb{1}_{\alpha_1} \circ \hat{\rho}'_\beta \\ \curvearrowright \end{array} & & \begin{array}{c} \curvearrowleft \\ \mathbb{1}_{\alpha_2} \circ \hat{\rho}'_\beta \\ \curvearrowright \end{array}
 \end{array}$$

The inner squeezed square is commutative, because cobordisms φ and $\hat{\rho}'_\beta$ act on different parts of the composite tangles $\alpha_1 \circ \tau \circ \beta$ and $\alpha_2 \circ \tau \circ \beta$. The commutativity of the left and right faces is equivalent to eq.(7.18) for α_1 and α_2 . Hence the whole diagram is commutative. Compare the commutativity of its outer bloated square with the tangle composition of the diagram (7.19) with β :

$$\begin{array}{ccc}
 \langle\langle \alpha_1 \circ \tau \circ \beta \rangle\rangle & \xrightarrow{\hat{\varphi} \circ \mathbb{1}_\tau \circ \mathbb{1}_\beta} & \langle\langle \alpha_2 \circ \tau \circ \beta \rangle\rangle \\
 \begin{array}{c} \downarrow \\ \mu_{\alpha_1} \hat{\rho}_{\alpha_1} \circ \mathbb{1}_\beta \\ \downarrow \end{array} & & \begin{array}{c} \downarrow \\ \mu_{\alpha_2} \hat{\rho}_{\alpha_2} \circ \mathbb{1}_\beta \\ \downarrow \end{array} \\
 \langle\langle \alpha_1 \circ \beta \rangle\rangle & \xrightarrow{\mu_\varphi \hat{\varphi} \circ \mathbb{1}_\beta} & \langle\langle \alpha_2 \circ \beta \rangle\rangle
 \end{array}$$

Since φ is an elementary planar cobordism, the linear map $\hat{\varphi} \circ \mathbb{1}_\beta: \langle\langle \alpha_1 \circ \beta \rangle\rangle \rightarrow \langle\langle \alpha_2 \circ \beta \rangle\rangle$ is non-zero, hence $\mu_\varphi = 1$. \square

Corollary 7.18. *If a $(2n, 2n)$ -tangle τ is quasi-trivial, then the tangle composition functor*

$$\hat{\tau}: \mathbf{TL}_{2n}^{\text{spl}, -/+} \longrightarrow \mathbf{TL}_{2n}^{\text{spl}, -/+}, \quad \hat{\tau} = - \circ \langle\langle \tau \rangle\rangle$$

is isomorphic to the identity functor and, in particular, for any complex \mathbf{A} in $\mathbf{TL}_{2n, 2m}^{\text{spl}, -}$ there is a homotopy equivalence

$$\mathbf{A} \circ \tau \sim \mathbf{A}. \tag{7.20}$$

Proof. Use the category equivalence (4.21) to replace $\mathbf{TL}_{2n}^{\text{spl}, -/+}$ with $\mathbf{K}^{-/+}(\check{\mathbf{T}}\mathbf{L}_{2n,0} \otimes \check{\mathbf{T}}\mathbf{L}_{0,2n})$. The functor $\hat{\tau}$ acts as identity on the $\check{\mathbf{T}}\mathbf{L}_{2n,0}$ factor and its action on the $\check{\mathbf{T}}\mathbf{L}_{0,2n}$ factor is equivalent to identity by Theorem 7.17 \square

Proof of Theorem 7.13. According to the definition (5.12) of the functor $\hat{\mathbf{P}}_*$, the homotopy equivalences (7.11) are explicitly

$$\mathbf{P}_n \circ \left\langle\left\langle \frac{\hat{\tau}}{\partial^2} \right\rangle\right\rangle_{2n} \sim \mathbf{P}_n \circ \left\langle\left\langle \overset{1}{\text{X}} \right\rangle\right\rangle_{2n} \sim \mathbf{P}_n. \tag{7.21}$$

It is easy to see that the tangles $\frac{\overline{\text{f}}}{\text{e}} \uparrow_2^{2n}$ and $\overline{\text{X}} \uparrow_2^{2n}$ are quasi-trivial (in fact, for these tangles any $\beta \in \mathfrak{C}_n$ satisfies the second condition of Definition 7.15). Since \mathbf{P}_n is an object of $\text{TL}_{2n}^{\text{spl}, -/+}$, the homotopy equivalences (7.21) follows from that of eq.(7.20). \square

8. PROPERTIES OF THE UNIVERSAL CATEGORIFICATION COMPLEX OF A TORUS BRAID

8.1. A multi-cone structure of a chain complex. In many instances, in order to simplify a complex through homotopy equivalence, we will use its presentation as a multi-cone, that is, as a multiple application of cone construction. A general theory of this procedure is related to Postnikov structures and it is described, *e.g.*, in [GM96]. We need only a tiny bit of this theory, as it applies to homotopy categories.

Let $\mathbf{Ch}^-(\mathbf{A})$ be a category of bounded from above chain complexes associated with an additive category \mathbf{A} . An object of $\mathbf{Ch}^-(\mathbf{A})$ is a chain complex

$$(\mathbf{A}, \mathbf{d}) = (\cdots \rightarrow A_i \xrightarrow{d_i} A_{i+1} \rightarrow \cdots \rightarrow A_k),$$

and a morphism between two chain complexes is a sequence of morphisms $\mathbf{f} = (\dots, f_i, \dots)$:

$$\begin{array}{ccccccc} \mathbf{A} & & \cdots & \xrightarrow{d_{i-1}} & A_i & \xrightarrow{d_i} & A_{i+1} & \xrightarrow{d_{i+1}} & \cdots \\ & & & & \downarrow f_i & & \downarrow f_{i+1} & & \\ \mathbf{B} & & \cdots & \xrightarrow{d'_{i-1}} & B_i & \xrightarrow{d'_i} & B_{i+1} & \xrightarrow{d'_{i+1}} & \cdots \end{array} \quad (8.1)$$

An associated homotopy category $\mathbf{K}^-(\mathbf{A})$ has the same objects as $\mathbf{Ch}^-(\mathbf{A})$, while the morphisms are chain morphisms (that is, morphisms which commute with differentials) up to homotopy.

A multi-cone in the category $\mathbf{Ch}^-(\mathbf{A})$ is a family of complexes $(\mathbf{A}_\lambda)_{\lambda \in \Lambda}$, where Λ is an index set with a grading function $h: \Lambda \rightarrow \mathbb{Z}$ such that $h(\Lambda)$ is bounded from above and $h^{-1}(n)$ is finite for any $n \in \mathbb{Z}$. The complexes of the multi-cone are connected by morphisms $\mathbf{A}_\lambda[-1] \xrightarrow{\mathbf{f}_{\lambda, \lambda'}} \mathbf{A}_{\lambda'}$ if $h(\lambda) < h(\lambda')$, and the morphisms satisfy the condition

$$\mathbf{f}_{\lambda, \lambda''} \mathbf{d}_\lambda + \mathbf{d}_{\lambda''} \mathbf{f}_{\lambda, \lambda''} + \sum_{\substack{\lambda' \in \Lambda \\ h(\lambda) < h(\lambda') < h(\lambda'')}} \mathbf{f}_{\lambda', \lambda''} \mathbf{f}_{\lambda, \lambda'} = 0.$$

It guarantees that the multi-cone determines a ‘total complex’ (\mathbf{A}, \mathbf{d}) in $\mathbf{Ch}(\mathbf{A})$, whose chain ‘modules’ are direct sums of chain ‘modules’ of \mathbf{A}_λ and differentials are the sums of morphisms $\mathbf{f}_{\lambda, \lambda'}: \mathbf{A} = \bigoplus_{\lambda \in \Lambda} \mathbf{A}_\lambda$, $\mathbf{d} = \sum_{\lambda, \lambda' \in \Lambda} \mathbf{f}_{\lambda, \lambda'}$. If a complex (\mathbf{A}, \mathbf{d}) of $\mathbf{Ch}(\mathbf{A})$ is presented as a total complex of a multi-cone, then we say that (\mathbf{A}, \mathbf{d}) has a multi-cone structure and we refer to \mathbf{A}_λ as constituent complexes.

The following easy proposition explains why the multi-cone structure helps to simplify a complex within its homotopy equivalence class.

Proposition 8.1. *For a homotopy equivalent family of complexes $\mathbf{A}'_\lambda \sim \mathbf{A}_\lambda$ with the same index set Λ there exist morphisms $\mathbf{f}'_{\lambda,\lambda'}$ such that the resulting multi-cone is homotopy equivalent to the original one: $(\bigoplus_{\lambda \in \Lambda} \mathbf{A}_\lambda, \sum_{\lambda,\lambda' \in \Lambda} \mathbf{f}_{\lambda,\lambda'}) \sim (\bigoplus_{\lambda \in \Lambda} \mathbf{A}'_\lambda, \sum_{\lambda,\lambda' \in \Lambda} \mathbf{f}'_{\lambda,\lambda'})$.*

In this paper multi-cone structures emerge when categorification complexes of tangle compositions are considered. For example, consider a composition $\langle\langle \tau \rangle\rangle \circ \mathbf{A}$ of two complexes in \mathbf{TL}_n , the first object being a categorification complex of a (n, n) -tangle τ . Constituent complexes of $\langle\langle \tau \rangle\rangle \circ \mathbf{A}$ are formed by the compositions $\langle\langle \tau \rangle\rangle \circ \langle\langle \lambda \rangle\rangle = \langle\langle \tau \circ \lambda \rangle\rangle$, where λ are constituent tangles of \mathbf{A} . Now we can use Reidemeister moves in order to simplify the tangles $\tau \circ \lambda$, knowing that the morphisms of the multi-cone can be adjusted accordingly, so that the modified total complex will be homotopy equivalent to the original complex $\langle\langle \tau \rangle\rangle \circ \mathbf{A}$.

8.2. A cup sliding trick. In this section we will prove Theorem 6.6 by induction over m . In proving it for $m = 1$ and in deducing the case of $m + 1$ from the case of m we will use a special trick.

Let $\gamma_i, i = 1, 2, \dots$ be a sequence of (i, i') -tangles (in our applications $i' = i$ or $i' = i + 2$) such that any cup-tangle can slide through them:

$$\gamma_i \circ i \overset{\mathbf{I}}{\curvearrowright} \approx i' \overset{\tilde{\mathbf{I}}}{\curvearrowright} \circ \gamma_{i-2d}, \tag{8.2}$$

where d is the number of cup arcs in $i \overset{\mathbf{I}}{\curvearrowright}$. Suppose that all tangles γ_i have special categorification complexes $\langle\langle \gamma_i \rangle\rangle^\sharp$ and we have to construct a special categorification complex for the composition $\tau' = \gamma_n \circ \tau$, where τ is a (m, n) -tangle with a special categorification complex $\langle\langle \tau \rangle\rangle^\sharp$.

First, we represent $\langle\langle \tau' \rangle\rangle$ by the tangle-composition of complexes $\langle\langle \gamma_n \rangle\rangle \circ \langle\langle \tau \rangle\rangle^\sharp$ (the choice of $\langle\langle \gamma_n \rangle\rangle$ does not matter). We view this composition as a multi-cone constructed by replacing every constituent object $\langle\langle \lambda \rangle\rangle$ of $\langle\langle \tau \rangle\rangle^\sharp$ with the composition $\langle\langle \gamma_n \rangle\rangle \circ \langle\langle \lambda \rangle\rangle$. In order to transform the multi-cone $\langle\langle \gamma_n \rangle\rangle \circ \langle\langle \tau \rangle\rangle^\sharp$ homotopically into the special complex $\langle\langle \tau' \rangle\rangle^\sharp$, we use the presentation (2.4) for λ and isotopy (8.2) in order to replace this composition with a homotopy equivalent one

$$\langle\langle n' \overset{\tilde{\mathbf{I}}}{\curvearrowright} \rangle\rangle \circ \langle\langle \gamma_{t_\lambda} \rangle\rangle^\sharp \circ \langle\langle \mathbb{C}^{\mathbf{J}} m \rangle\rangle, \tag{8.3}$$

where t_λ is a through degree of λ . The homotopy equivalence transformation of complexes $\langle\langle \gamma_n \rangle\rangle \circ \langle\langle \lambda \rangle\rangle$ into the complexes (8.3) may change the morphisms of the multi-cone in a rather non-trivial way, but this does not matter, since in this section we are interested mostly in the

position of a constituent object within the complex $\langle\langle\tau'\rangle\rangle^\sharp$. This position is determined by the position of the original object $\langle\langle\lambda\rangle\rangle$ within $\langle\langle\tau\rangle\rangle^\sharp$ and the structure of the special complex $\langle\langle\gamma_{t_\lambda}\rangle\rangle^\sharp$:

Proposition 8.2. *If an object $\langle\langle\lambda'\rangle\rangle$ appears in a position $\langle\langle\lambda'\rangle\rangle [i, j]$ in the complex $\langle\langle\gamma_{t_\lambda}\rangle\rangle^\sharp$, then the corresponding object in the composition (8.3) appears in the same position $\langle\langle n' \overline{\mathbb{C}} \rangle\rangle \circ \lambda' \circ \overline{\mathbb{C}}^m [i, j]$.*

Proof. The homological degree \deg_h is additive with respect to the tangle composition. The q -degree \deg_q is not additive because of eq. (4.13) required to remove disjoint circles, but the tangle composition $n' \overline{\mathbb{C}} \circ \lambda' \circ \overline{\mathbb{C}}^m$ does not contain disjoint circles, hence \deg_q is also additive in (8.3). Now the claim of the proposition follows from the fact that the degrees \deg_h and \deg_q of cap and cup tangles in (8.3) are zero. \square

8.3. A special categorification complex of a single twist torus braid. Recall the definitions (6.5)–(6.7) of the set \mathcal{A}_\angle and its shifts. We also use an abbreviated notation $\mathcal{A}_\angle(t, n) = \mathcal{A}_\angle(t, n; 1) = \mathcal{A}_\angle[\frac{1}{2}t^2, \frac{1}{2}(t^2 + t + n)]$.

Theorem 8.3. *For a torus braid $\overline{\mathbb{C}}^n$ there exists a special categorification complex $\langle\langle \overline{\mathbb{C}}^n \rangle\rangle^\sharp$ with the following property:*

$$m_{-i,j,\mu}^\lambda > 0, \quad \text{only if } (i, j) \in \mathcal{A}_\angle(t_\lambda, n).$$

The proof of this theorem is based on two lemmas.

Define the following tangle notations:

$$\overline{\mathbb{C}}^n = \begin{array}{c} | \cdots | \\ \text{---} \\ | \cdots | \\ 1 \quad 2 \quad n-1 \quad n \end{array}, \quad n \overline{\mathbb{C}} = \begin{array}{c} 1 \quad 2 \quad n-1 \quad n \\ \text{---} \\ | \cdots | \\ \text{---} \\ | \cdots | \end{array}, \quad (8.4)$$

$$\overline{\mathbb{C}} \overline{\mathbb{C}}^n = n \overline{\mathbb{C}} \circ \overline{\mathbb{C}}^n = \begin{array}{c} \text{---} \\ | \cdots | \\ \text{---} \\ | \cdots | \\ \text{---} \\ 1 \quad 2 \quad n-1 \quad n \end{array}. \quad (8.5)$$

and a ray shape: $\mathcal{A}_\nearrow = \{(i, i) \mid i \in \mathbb{Z}, i \geq 0\}$.

Lemma 8.4. *The tangle $\overline{\mathbb{C}} \overline{\mathbb{C}}^n$ has a special categorification complex $\langle\langle \overline{\mathbb{C}} \overline{\mathbb{C}}^n \rangle\rangle^\sharp$ with two properties:*

- (1) $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$ has a ray shape $\mathcal{A}_{\nearrow} [2 - n]_{h,q}$;
- (2) the object $\langle\langle \overline{\underline{\cdot}} n \rangle\rangle$ does not appear in $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$.

Proof. Let $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$ denote the standard categorification complex of $\overline{\underline{\cdot}} \overline{\underline{\cdot}} n$. Since Kauffman splicing of crossings in the diagram of this tangle does not produce disjoint circles, it follows from eq.(4.4) that $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$ has the shape $\mathcal{A}_{\nearrow} [\frac{2-n}{2}]_{h,q}$. By the same argument, the standard categorification complex $\langle\langle n \overline{\underline{\cdot}} \overline{\underline{\cdot}} \rangle\rangle^\sharp$ has the same shape.

Now we use the trick of subsection 8.2, where $\gamma_i = i \overline{\underline{\cdot}} \overline{\underline{\cdot}}$, $\tau = \overline{\underline{\cdot}} \overline{\underline{\cdot}} n$ and $\tau' = \overline{\underline{\cdot}} \overline{\underline{\cdot}} n$. After the sliding of cups, a constituent object of $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$ corresponding to a TL tangle $\lambda = n-2 \overline{\underline{\cdot}} \overline{\underline{\cdot}} \circ \overline{\underline{\cdot}} \overline{\underline{\cdot}} n$ turns into the complex

$$\langle\langle n \overline{\underline{\cdot}} \overline{\underline{\cdot}} \rangle\rangle \circ \langle\langle t_\lambda + 2 \overline{\underline{\cdot}} \overline{\underline{\cdot}} \rangle\rangle^\sharp \circ \langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle, \tag{8.6}$$

where t_λ is a through degree of λ . The first claim of the lemma follows from the fact the middle complex of the composition (8.6) has the shape $\mathcal{A}_{\nearrow} [-\frac{t_\lambda}{2}]_{h,q}$. The second claim of the lemma follows from the fact that the middle tangle in the composition (8.6) is of type $(t_\lambda, t_\lambda + 2)$: since $t_\lambda \leq n - 2$, only the tangle with through degree up to $n - 2$ can form in the composition.

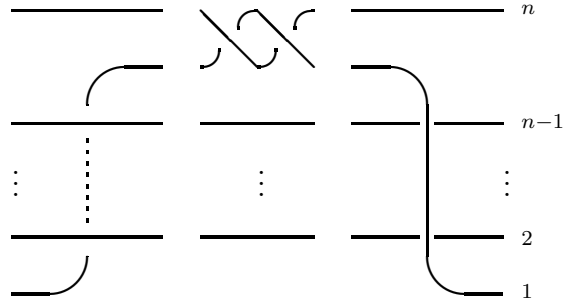
□

Lemma 8.5. *The tangle $\overline{\underline{\cdot}} \overline{\underline{\cdot}} n$ has a special categorification complex $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$ with the following properties:*

- $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} n \rangle\rangle^\sharp$ has an angle shape $\mathcal{A}_\angle [-n + 1, -n]$ if $n \geq 2$;
- the object $\langle\langle \overline{\underline{\cdot}} n \rangle\rangle$ appears in $\langle\langle \overline{\underline{\cdot}} \overline{\underline{\cdot}} 2n \rangle\rangle^\sharp$ only in the position $\langle\langle \overline{\underline{\cdot}} n \rangle\rangle [-1, 1]^{n-1}$ and its multiplicity there is 1.

Proof. We prove the lemma by induction over n . When $n = 1$ the claim is obvious.

Consider now a general value of n . The tangle $\underline{\underline{\mathbb{Z}}}[n]$ can be presented as a composition of three tangles



(8.7)

We use the categorification complex for the middle tangle which is based on the following categorification complex of the double-crossing $(2, 2)$ -tangle:

$$\langle\langle \text{two crossings} \rangle\rangle = \left(\langle\langle \text{two crossings} \rangle\rangle [-1, 1] \rightarrow \langle\langle \text{two crossings} \rangle\rangle \rightarrow \langle\langle \text{two crossings} \rangle\rangle [1, -2] \right).$$

The special complex $\langle\langle \underline{\underline{\mathbb{Z}}}[n] \rangle\rangle^\#$ is constructed by tangle-composing the objects of this complex with the first and third tangles of the decomposition (8.7) and using their special categorification complexes. As a result, $\langle\langle \underline{\underline{\mathbb{Z}}}[n] \rangle\rangle^\#$ is a ‘double-cone’ composed out of the following three complexes

$$\overline{\langle\langle \underline{\underline{\mathbb{Z}}}[n-1] \rangle\rangle^\#} [-1, 1], \quad \langle\langle \underline{\underline{\mathbb{Z}}}[n] \rangle\rangle^\#, \quad \langle\langle \underline{\underline{\mathbb{Z}}}[n] \rangle\rangle^\# [1, -2],$$

where the first tangle $\overline{\underline{\underline{\mathbb{Z}}}[n-1]}$ is the tangle $\underline{\underline{\mathbb{Z}}}[n-1]$ together with an extra straight strand on top. Now the claims of the lemma follow from the induction assumption applied to the first complex and Lemma 8.4 applied to the second and third complexes. \square

Proof of Theorem 8.3. We prove the theorem by induction over n . When $n = 1$, its claim is obvious.

In order to construct a special complex for the torus braid $\underline{\underline{\mathbb{X}}}[n]$, we present it as a composition of two braids:

$$\underline{\underline{\mathbb{X}}}[n] \approx \underline{\underline{\mathbb{Z}}}[n] \circ \underline{\underline{\mathbb{X}}}[n-1], \tag{8.8}$$

where the braid $\underline{\underline{\mathbb{X}}}[n-1]$ is constructed by adding an extra straight strand to the braid $\underline{\underline{\mathbb{X}}}[n-1]$. For a (k, l) -tangle τ , let $\underline{\underline{\tau}}$ denote a $(k + 1, l + 1)$ -tangle constructed by adding an extra straight line at the bottom of τ . We construct the special complex $\langle\langle \underline{\underline{\mathbb{X}}}[n-1] \rangle\rangle^\#$ by

replacing every constituent object $\langle\langle \lambda \rangle\rangle$ of the special complex $\langle\langle \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^{n-1} \rangle\rangle^\sharp$, which exists by the assumption of induction, with the object $\langle\langle \underline{\lambda} \rangle\rangle$. Then we use the trick of subsection 8.2 with $\tau = \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^{n-1}$, $\gamma_i = \underbrace{\bigcirc}_{\cdot}^i$ and $\tau' = \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^n$ with a slight modification: the cup-tangles that slide through γ_i must be of the form $\underbrace{\bigcirc}_{\cdot}^{n-1} \bigcirc_{\cdot}^{\mathbf{I}}$, but these are exactly the cup-tangles that appear in the decomposition of constituent tangles $\underline{\lambda}$ of the complex $\langle\langle \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^{n-1} \rangle\rangle$. In other words, if $\lambda = \underbrace{\bigcirc}_{\cdot}^{n-1} \bigcirc_{\cdot}^{\mathbf{I}} \circ \underbrace{\bigcirc}_{\cdot}^{\mathbf{J}} \bigcirc_{\cdot}^{n-1}$ is a constituent tangle of the special complex $\langle\langle \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^{n-1} \rangle\rangle^\sharp$, then the sliding isotopy is

$$\underbrace{\bigcirc}_{\cdot}^i \bigcirc_{\cdot}^n \circ \underbrace{\bigcirc}_{\cdot}^{n-1} \bigcirc_{\cdot}^{\mathbf{I}} \circ \underbrace{\bigcirc}_{\cdot}^{\mathbf{J}} \bigcirc_{\cdot}^{n-1} \approx \underbrace{\bigcirc}_{\cdot}^{n-1} \bigcirc_{\cdot}^{\mathbf{I}} \circ \underbrace{\bigcirc}_{\cdot}^i \bigcirc_{\cdot}^{t_\lambda+1} \circ \underbrace{\bigcirc}_{\cdot}^{\mathbf{J}} \bigcirc_{\cdot}^{n-1}$$

and we choose the special categorification complex

$$\langle\langle \underbrace{\bigcirc}_{\cdot}^i \bigcirc_{\cdot}^n \circ \underline{\lambda} \rangle\rangle^\sharp = \langle\langle \underbrace{\bigcirc}_{\cdot}^{n-1} \bigcirc_{\cdot}^{\mathbf{I}} \rangle\rangle \circ \langle\langle \underbrace{\bigcirc}_{\cdot}^i \bigcirc_{\cdot}^{t_\lambda+1} \rangle\rangle^\sharp \circ \langle\langle \underbrace{\bigcirc}_{\cdot}^{\mathbf{J}} \bigcirc_{\cdot}^{n-1} \rangle\rangle. \tag{8.9}$$

By the assumption of induction, the object $\langle\langle \lambda \rangle\rangle$ of the complex $\langle\langle \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^{n-1} \rangle\rangle^\sharp$ lies within the shape $\mathcal{A}_\perp(t_\lambda, n-1)$. According to Lemma 8.5 and Proposition 8.2, the complex (8.9) that λ yields, has the shape $\mathcal{A}_\perp[\frac{1}{2} - t_\lambda]_{h,q}$. Combining these two shapes together, we find that the ultimate contribution of λ to $\langle\langle \underbrace{\bigcirc}_{\cdot}^1 \bigcirc_{\cdot}^n \rangle\rangle^\sharp$ lies within the shape $\mathcal{A}_\perp(t_\lambda - 1, n)$, which corresponds to (n, n) -tangle of through degree $t_\lambda - 1$. The complex (8.9) contains only the tangles of through degree $t \leq t_\lambda + 1$ and parity opposite to that of t_λ . Since $\mathcal{A}_\perp(t, n) \subset \mathcal{A}_\perp(t', n)$ for $t \geq t'$, the tangles with through degree $t \leq t_\lambda - 1$ satisfy the claim of Theorem 8.3.

The tangles with through degree $t = t_\lambda + 1$ originate from the object $\langle\langle \underbrace{\bigcirc}_{\cdot}^{t_\lambda+1} \rangle\rangle$ within the complex $\langle\langle \underbrace{\bigcirc}_{\cdot}^i \bigcirc_{\cdot}^{t_\lambda+1} \rangle\rangle^\sharp$. According to the second claim of Lemma 8.5, it has a special degree shift relative to the vertex of $\mathcal{A}_\perp[\frac{1}{2} - t_\lambda]_{h,q}$. Because of this shift, the tangles with through degree $t = t_\lambda + 1$ fall within the shape $\mathcal{A}_\perp(t_\lambda + 1, n)$, hence they also satisfy the claim of Theorem 8.3. \square

8.4. A categorification complex of a torus braid as an approximation to the universal resolution of H_n .

Proof of Theorem 6.6. We define the complexes $\langle\langle \underbrace{\bigcirc}_{\cdot}^m \bigcirc_{\cdot}^{2n} \rangle\rangle^\sharp$ recursively and prove the theorem by induction over m . Theorem 8.3 establishes the first property for $m = 1$. Suppose

that the theorem holds for m . We construct the special complex $\langle\langle \overset{m+1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ by presenting the braid $\overset{m+1}{\mathcal{X}} \mathcal{X}_{2n}$ as a composition

$$\overset{m+1}{\mathcal{X}} \mathcal{X}_{2n} = \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \circ \overset{m}{\mathcal{X}} \mathcal{X}_{2n}.$$

Next, we split the special complex $\langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ into two pieces by homological degree:

$$\langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp = \text{Cone}\left(\mathbf{t}_{\geq 2m}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp[1] \longrightarrow \mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp\right), \quad (8.10)$$

and tangle-compose these pieces individually with $\langle\langle \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle$. By the assumption of induction, the complex $\mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ is split, so according to eq.(7.20), there is a homotopy equivalence

$$\langle\langle \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle \circ \mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp \sim \mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp.$$

Thus it remains to construct a special categorification complex

$$\left(\langle\langle \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle \circ \mathbf{t}_{\geq 2m}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp\right)^\sharp \quad (8.11)$$

and substitute it for the first term in the cone (8.10). We build this complex through the trick of subsection 8.2: we slide the cups of a constituent tangle $\lambda = 2n \overset{\mathbf{I}}{\mathcal{C}} \circ \overset{\mathbf{J}}{\mathcal{C}} 2n$ of the truncated special complex $\mathbf{t}_{\geq 2m}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ through $\overset{1}{\mathcal{X}} \mathcal{X}_{2n}$:

$$\overset{1}{\mathcal{X}} \mathcal{X}_{2n} \circ \overset{\mathbf{I}}{\mathcal{C}} 2n \circ \overset{\mathbf{J}}{\mathcal{C}} 2n \approx \overset{\mathbf{J}}{\mathcal{C}} 2n \circ \overset{\mathbf{I}}{\mathcal{C}} 2n \circ \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \circ \overset{\mathbf{J}}{\mathcal{C}} 2n,$$

(note the cancelation of framing twists), thus constructing a special complex for the composition $\overset{1}{\mathcal{X}} \mathcal{X}_{2n} \circ \lambda$:

$$\langle\langle \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \circ \lambda \rangle\rangle^\sharp = \langle\langle \overset{\mathbf{I}}{2n} \rangle\rangle \circ \langle\langle \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp \circ \langle\langle \overset{\mathbf{J}}{\mathcal{C}} 2n \rangle\rangle. \quad (8.12)$$

The special complex (8.11) consisting of complexes (8.12) satisfies the first property of Theorem 6.6. Indeed, a constituent object $\langle\langle \lambda' \rangle\rangle$ of the complex (8.12) satisfies the property $t_{\lambda'} \leq t_\lambda$ and by Theorem 8.3 it lies within the shape $\mathcal{A}_\mathcal{L}(t_{\lambda'}, 2n)$. At the same time, by the assumption of induction, the object $\langle\langle \lambda \rangle\rangle$ of $\mathbf{t}_{\geq 2m}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp$ lies within the shape $\mathcal{A}_\mathcal{L}(t_\lambda, 2n; m)$. Now it is easy to check that the place of the object $\langle\langle \lambda' \rangle\rangle$ in the complex (8.11) will be within the shape $\mathcal{A}_\mathcal{L}(t_{\lambda'}, 2n; m)$.

Finally, it is obvious that the cone

$$\langle\langle \overset{m+1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp = \text{Cone}\left(\left(\langle\langle \overset{1}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle \circ \mathbf{t}_{\geq 2m}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp\right)^\sharp[1] \longrightarrow \mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \mathcal{X}_{2n} \rangle\rangle^\sharp\right)$$

satisfies the second property of Theorem 6.6. \square

Proof of Theorem 6.7. Isomorphisms (6.9) imply the existence of a special complex $\mathbf{C}_n^\sharp \in \mathbf{TL}_{2n}^{\text{spl}, -}$ such that

$$\mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle^\sharp \cong \mathbf{t}_{\leq 2m-1}^- \mathbf{C}_n^\sharp. \quad (8.13)$$

The property (6.8) implies that truncated complexes $\mathbf{t}_{\leq 2m-1}^- \langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle^\sharp$ are angle-shaped and split, hence \mathbf{C}_n^\sharp is angle-shaped and split.

It remains to show that \mathbf{C}_n^\sharp satisfies the defining property of the complex \mathbf{P}_n : $\mathcal{F}_\mathbb{K}(\mathbf{C}_n^\sharp) \simeq H_n$. The homotopy equivalence (7.11) implies the quasi-isomorphism $\langle\langle \overset{1}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle_{\mathbb{D}} \simeq H_n$. Since the map $\langle\langle - \rangle\rangle_{\mathbb{D}} : \mathfrak{T}_{2n, 2n} \rightarrow \mathbf{D}^b(H_n^e)$ converts the tangle composition into the tensor product, this relation implies $\langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle_{\mathbb{D}} \simeq H_n$, which means that the homology of the complex $\langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle_{\mathbb{K}}$ is zero in all degrees except at degree zero where it is isomorphic to H_n :

$$H_i(\langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle_{\mathbb{K}}) = \begin{cases} 0 & \text{if } i < 0, \\ H_n & \text{if } i = 0. \end{cases}$$

In view of the isomorphisms (8.13), $\mathcal{F}_H(\mathbf{C}_n^\sharp)$ has the same property:

$$H_i(\mathcal{F}_\mathbb{K}(\mathbf{C}_n^\sharp)) = \begin{cases} 0 & \text{if } i < 0, \\ H_n & \text{if } i = 0. \end{cases} \quad (8.14)$$

Since \mathbf{C}_n^\sharp is split, its image $\mathcal{F}_\mathbb{K}(\mathbf{C}_n^\sharp)$ is projective, so eq. (8.14) means that $\mathcal{F}_\mathbb{K}(\mathbf{C}_n^\sharp)$ is a projective resolution of H_n and we set $\mathbf{P}_n^\sharp = \mathbf{C}_n^\sharp$. \square

Proof of Theorem 6.8. The isomorphism 6.4 and the use of special complexes for \mathbf{P}_n and $\langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle$ allows us to rewrite eq. (6.11) as the isomorphism between homologies

$$H_i(\langle\langle \tau \rangle\rangle \circ \mathbf{P}_n^\sharp; \mathbb{S}^3) = H_i(\langle\langle \tau \rangle\rangle \circ \langle\langle \overset{m}{\mathcal{X}} \circ \mathcal{X}_{2n} \rangle\rangle^\sharp; \mathbb{S}^3).$$

According to eq. (6.10), the chain ‘modules’ in both complexes are canonically isomorphic in homological degrees up to $|\langle\langle \tau \rangle\rangle|_{\mathbb{h}}^+ - 2m + 1$, hence their homologies are isomorphic in degrees up to $|\langle\langle \tau \rangle\rangle|_{\mathbb{h}}^+ - 2m + 2$. \square

APPENDIX A. JONES-WENZL PROJECTORS AND WRT POLYNOMIAL

A.1. Jones-Wenzl projectors. The most famous Jones-Wenzl projector P_n is the idempotent element of the Temperley-Lieb algebra QTL_n generated by $\text{TL}(n, n)$ -tangles over the field $\mathbb{Q}(q)$ of rational functions of q and defined by the property that for any $\text{TL}(n, n)$ -tangle λ ,

$$P_n \circ \langle \lambda \rangle = \langle \lambda \rangle \circ P_n = \begin{cases} P_n, & \text{if } \lambda = \overline{\begin{array}{c} \vdots \\ n \end{array}}, \\ 0, & \text{if } \lambda \neq \overline{\begin{array}{c} \vdots \\ n \end{array}}. \end{cases} \quad (\text{A.1})$$

To define the other Jones-Wenzl projectors of QTL_n , recall that as a \mathbb{Z} -module, QTL_n is freely generated by TL tangles, and each TL tangle has a presentation (2.4). Since this presentation plays a central role in our calculations, we will use a special notation

$$\lambda_{\mathbf{I}\mathbf{J}|m} = \overline{\begin{array}{c} \mathbf{I} \\ \vdots \\ n \end{array}} \circ \overline{\begin{array}{c} \mathbf{J} \\ \vdots \\ n \end{array}}, \quad \mathbf{I}, \mathbf{J} \in \mathcal{I}_{n,m}, \quad 0 \leq m \leq n, \quad n - m \in 2\mathbb{Z}. \quad (\text{A.2})$$

Now we define a ‘projected tangle’ $\langle \lambda \rangle_{\text{pr}}$ by inserting the Jones-Wenzl projector P_{t_λ} between the cup and cap parts of $\lambda_{\mathbf{I}\mathbf{J}|m}$:

$$\langle \lambda_{\mathbf{I}\mathbf{J}|m} \rangle_{\text{pr}} = \langle \overline{\begin{array}{c} \mathbf{I} \\ \vdots \\ n \end{array}} \rangle \circ P_m \circ \langle \overline{\begin{array}{c} \mathbf{J} \\ \vdots \\ n \end{array}} \rangle. \quad (\text{A.3})$$

Since $P_n - \langle \overline{\begin{array}{c} \vdots \\ n \end{array}} \rangle$ can be presented as a linear combination of TL tangles with through degree less than n , the relation between the QTL_n generators $\langle \lambda \rangle$ and the projected tangles $\langle \lambda \rangle_{\text{pr}}$ is upper-triangular: $\langle \lambda \rangle_{\text{pr}} - \langle \lambda \rangle$ is a linear combination of TL tangles with through degree less than t_λ . Hence projected tangles $\langle \lambda \rangle_{\text{pr}}$ also form a set of free generators of QTL_n . For m such that $m < n$ and $n - m$ is even, let $\text{QTL}_{n|m} \subset \text{QTL}_n$ be a submodule generated by all $\langle \lambda \rangle_{\text{pr}}$ such that $t_\lambda = m$. Since $\langle \lambda \rangle_{\text{pr}}$ generate QTL_n , the latter is a direct sum

$$\text{QTL}_n = \bigoplus_m \text{QTL}_{n|m}.$$

We will see shortly that $\text{QTL}_{n|m} \subset \text{QTL}_n$ are, in fact, two-sided ideals.

For $\mathbf{I}, \mathbf{J} \in \mathcal{I}_{n,m}$ the composition $\overline{\begin{array}{c} \mathbf{J} \\ \vdots \\ n \end{array}} \circ \overline{\begin{array}{c} \mathbf{I} \\ \vdots \\ n \end{array}}$ is a (m, m) -tangle. Let k denote the number of disjoint circles in it, then

$$\langle \overline{\begin{array}{c} \mathbf{J} \\ \vdots \\ n \end{array}} \circ \overline{\begin{array}{c} \mathbf{I} \\ \vdots \\ n \end{array}} \rangle = (-q - q^{-1})^k \langle \lambda \rangle, \quad (\text{A.4})$$

where λ is the TL (m, m) -tangle, which corresponds to the ‘connected’ part of the composition. Define the $\mathbb{Z}[q, q^{-1}]$ -valued coefficients $(B_{\mathbf{IJ}})_{\mathbf{I}, \mathbf{J} \in \mathcal{I}_{n, m}}$ of a symmetric matrix B by the formula

$$B_{\mathbf{IJ}} = \begin{cases} (-q - q^{-1})^k, & \text{if } \lambda = \overline{\overline{\vdots}} m, \\ 0 & \text{otherwise.} \end{cases}$$

where \dots stands for a linear combination of TL (m, m) -tangles whose through degree is less than m .

Proposition A.1. *Projected tangles satisfy a simple composition formula:*

$$\langle \lambda_{\mathbf{IJ}|m} \rangle_{\text{pr}} \circ \langle \lambda_{\mathbf{I}'\mathbf{J}'|m'} \rangle_{\text{pr}} = \delta_{mm'} \langle \lambda_{\mathbf{IJ}'|m} \rangle_{\text{pr}}, \tag{A.5}$$

where $\delta_{mm'}$ is the Kronecker symbol.

Proof. We use the formula eq.(A.4) for the composition of cup and cap tangles:

$$\langle \overbrace{\left(\begin{array}{c} \mathbf{J} \\ \text{C} \\ n \circ n \\ \mathbf{I}' \end{array} \right)} \rangle = (-q - q^{-1})^k \langle \lambda'' \rangle$$

Then the composition in the *r.h.s.* of (A.5) takes the form

$$\langle \lambda_{\mathbf{IJ}|m} \rangle_{\text{pr}} \circ \langle \lambda_{\mathbf{I}'\mathbf{J}'|m'} \rangle_{\text{pr}} = (-q - q^{-1})^k \langle \overbrace{\left(\begin{array}{c} \mathbf{I} \\ \text{C} \\ n \\ \mathbf{J}' \end{array} \right)} \rangle \circ P_m \circ \langle \lambda'' \rangle \circ P_{m'} \circ \langle \overbrace{\left(\begin{array}{c} \mathbf{J} \\ \text{C} \\ n \end{array} \right)} \rangle.$$

The property (A.1) of Jones-Wenzl projectors indicates that this expression is equal to zero, unless $m = m'$ and $\lambda'' = \overline{\overline{\vdots}} m$, in which case $(-q - q^{-1})^k = B_{\mathbf{I}'\mathbf{J}}$ and we obtain the *r.h.s.* of eq.(A.5). □

Corollary A.2. *Submodules $QTL_{n|m} \subset QTL_n$ are two-sided ideals.*

Define the special elements of $QTL_{n|m}$

$$P_{n, m} = \sum_{\mathbf{I}, \mathbf{J} \in \mathcal{I}_{n, m}} B_{\mathbf{IJ}}^{-1} \langle \lambda_{\mathbf{IJ}|m} \rangle_{\text{pr}} = \sum_{\mathbf{I}, \mathbf{J} \in \mathcal{I}_{n, m}} B_{\mathbf{IJ}}^{-1} \langle \overbrace{\left(\begin{array}{c} \mathbf{I} \\ \text{C} \\ n \\ \mathbf{J} \end{array} \right)} \rangle \circ P_m \circ \langle \overbrace{\left(\begin{array}{c} \mathbf{J} \\ \text{C} \\ n \end{array} \right)} \rangle, \tag{A.6}$$

where B^{-1} is the inverse matrix of B .

Proposition A.3. *An element $P_{n, m}$ satisfies the following property:*

$$P_{n, m} \circ \langle \lambda \rangle_{\text{pr}} = \langle \lambda \rangle_{\text{pr}} \circ P_{n, m} = \begin{cases} \langle \lambda \rangle_{\text{pr}}, & \text{if } t_\lambda = m, \\ 0, & \text{if } t_\lambda \neq m \end{cases}$$

Proof. Present $\langle \lambda \rangle_{\text{pr}}$ in the form (A.3) and use the formula (A.5). □

Corollary A.4. *The element $P_{n,m}$ is an idempotent projecting QTL_n onto $\text{QTL}_{n|m}$ and*

$$\sum_{\substack{0 \leq m \leq n \\ n-m \in 2\mathbb{Z}}} P_{n,m} = \langle\langle \overline{\vdots} \ n \rangle\rangle. \quad (\text{A.7})$$

Hence $P_{n,m}$ are Jones-Wenzl projectors mentioned in subsection 3.2.

Consider a left TL_n -module $\text{TL}_{m,n}^{\text{pr}} = \text{TL}_{m,n} \circ P_m$. An element $x \in \text{QTL}_n$ determines a multiplication endomorphism $\text{TL}_{m,n}^{\text{pr}} \xrightarrow{m_x} \text{TL}_{m,n}^{\text{pr}}$, $m_x(y) = x \circ y$.

Proposition A.5. *The \mathbb{S}^3 closure of the composition $x \circ P_{n,m}$ is proportional to the trace of m_x over $\text{TL}_{m,n}^{\text{pr}}$:*

$$(P_{n,m} \circ x; \mathbb{S}^3) = (P_m; \mathbb{S}^3) \text{Tr}_{\text{TL}_{m,n}^{\text{pr}}} m_x. \quad (\text{A.8})$$

Proof. If $x \in \text{QTL}_{n|m'}$ and $m' \neq m$, then both sides of this equation are equal to zero, hence we may assume that $x \in \text{QTL}_{n|m}$. Moreover, since $\text{QTL}_{n|m}$ is generated by the elements (A.3), we may assume that $x = \langle \lambda_{\mathbf{IJ}|m} \rangle_{\text{pr}}$. The the *l.h.s.* of eq. (A.8) can be calculated:

$$\begin{aligned} (P_{n,m} \circ \lambda_{\mathbf{IJ}|m}; \mathbb{S}^3) &= (\lambda_{\mathbf{IJ}|m}; \mathbb{S}^3) \\ &= \left(\langle \overset{\mathbf{I}}{\mathcal{C}} \ n \rangle \right) \circ P_m \circ \left(\langle \overset{\mathbf{J}}{\mathcal{C}} \ n \rangle \right) = \left(P_m \circ \left(\langle \overset{\mathbf{J}}{\mathcal{C}} \ n \rangle \circ \langle \overset{\mathbf{I}}{\mathcal{C}} \ n \rangle \right) \right) = B_{\mathbf{IJ}}(P_m; \mathbb{S}^3). \end{aligned} \quad (\text{A.9})$$

The module $\text{TL}_{m,n}^{\text{pr}}$ is generated freely by the elements

$$\langle \overset{\mathbf{I}'}{\mathcal{C}} \ n \rangle \circ P_m, \quad \mathbf{I}' \in \mathcal{I}_{n,m}. \quad (\text{A.10})$$

The matrix elements $(m_x)_{\mathbf{I}'\mathbf{J}'}$ of the endomorphism m_x with respect to this basis are easy to evaluate:

$$\begin{aligned} m_x(\langle \overset{\mathbf{I}'}{\mathcal{C}} \ n \rangle \circ P_m) &= \langle \lambda_{\mathbf{IJ}|m} \rangle \circ \langle \overset{\mathbf{I}'}{\mathcal{C}} \ n \rangle \circ P_m \\ &= \langle \overset{\mathbf{I}}{\mathcal{C}} \ n \rangle \circ P_m \circ \left(\langle \overset{\mathbf{J}}{\mathcal{C}} \ n \rangle \circ \langle \overset{\mathbf{I}'}{\mathcal{C}} \ n \rangle \right) \circ P_m = B_{\mathbf{IJ}} \langle \overset{\mathbf{I}}{\mathcal{C}} \ n \rangle \circ P_m, \end{aligned}$$

hence $(m_x)_{\mathbf{I}'\mathbf{J}'} = \delta_{\mathbf{IJ}'} B_{\mathbf{IJ}}$ and

$$\text{Tr}_{\text{TL}_{m,n}^{\text{pr}}} m_x = \sum_{\mathbf{I}' \in \mathcal{I}_{n,m}} (m_x)_{\mathbf{I}'\mathbf{I}'} = B_{\mathbf{IJ}}. \quad (\text{A.11})$$

This equation together with eq. (A.9) prove the proposition. \square

Corollary A.6. *If $x \in \text{TL}_n$, then $(P_{n,m} \circ x; \mathbb{S}^3) \in \mathbb{Z}[q, q^{-1}]$.*

In other words, although the expression for the Jones-Wenzl projector $P_{n,m}$ involves rational functions of q , the closure $(P_{n,m} \circ x; \mathbb{S}^3)$ is purely polynomial.

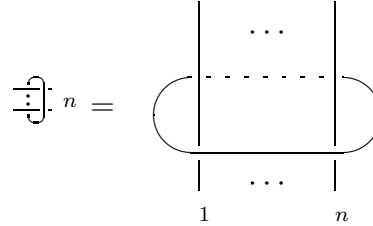
Proof of Corollary A.6. We are going to use eq.(A.8). First of all, note that

$$(P_m; \mathbb{S}^3) = (-1)^m \frac{q^{m+1} - q^{-m-1}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}],$$

hence we just have to prove $\text{Tr}_{\text{TL}_{m,n}^{\text{pr}}} m_x \in \mathbb{Z}[q, q^{-1}]$. We will show that the matrix elements $(m_x)_{\mathbf{I}, \mathbf{J}}$ with respect to the basis (A.10) belong to $\mathbb{Z}[q, q^{-1}]$. It is sufficient to verify this claim for $x = \lambda_{\mathbf{I}\mathbf{J}|m}$, and we leave the details to the reader. \square

Proof of Theorem 3.5. This theorem is a particular case ($m = 0$) of Corollary A.6. \square

A.2. The WRT invariant of links in $S^2 \times S^1$. Let us introduce another (n, n) -tangle notation:



We refer to the circle in this picture as *meridian*. Let $\langle \frac{\cdot}{\cdot} \cdot n \rangle$ denote the element of QTL_n constructed by replacing the meridian of $\frac{\cdot}{\cdot} \cdot n$ with a k -cable on which the Jones-Wenzl projector P_k is placed.

The dependence of the Temperley-Lieb algebra element $\langle \frac{\cdot}{\cdot} \cdot n \rangle_k$ on the value of k can be made more explicit, if we multiply it by the *l.h.s.* of eq.(A.7), slide the cups of eq.(A.6) through the meridian and use the well-known formula

$$P_m \circ \langle \frac{\cdot}{\cdot} \cdot m \rangle_k = (-1)^{k(m+1)} \frac{q^{(k+1)(m+1)} - q^{-(k+1)(m+1)}}{q^{m+1} - q^{-(m+1)}} P_m.$$

Then we obtain the following formula for $\langle \frac{\cdot}{\cdot} \cdot n \rangle_k$:

$$\begin{aligned} \langle \frac{\cdot}{\cdot} \cdot n \rangle_k &= \left(\sum_{\substack{0 \leq m \leq n \\ n-m \in 2\mathbb{Z}}} P_{n,m} \right) \circ \langle \frac{\cdot}{\cdot} \cdot n \rangle_k = \sum_m \sum_{\mathbf{I}, \mathbf{J} \in \mathcal{L}_{n,m}} B_{\mathbf{I}\mathbf{J}}^{-1} \langle n \mathbf{I} \mathbf{J} \rangle \circ P_m \circ \langle \frac{\cdot}{\cdot} \cdot m \rangle_k \circ \langle \mathbb{C} \cdot n \rangle \\ &= \sum_{\substack{0 \leq m \leq n \\ n-m \in 2\mathbb{Z}}} (-1)^{k(m+1)} \frac{q^{(k+1)(m+1)} - q^{-(k+1)(m+1)}}{q^{m+1} - q^{-(m+1)}} P_{n,m}. \end{aligned} \tag{A.12}$$

Proof of Theorem 3.6. The closure $(\tau; S^2 \times S^1)$ of a (n, n) -tangle τ in $S^2 \times S^1$ can be constructed by a surgery on a meridian of the link $(\tau \circ \frac{\cdot}{\cdot} \cdot n; S^3)$. Hence, the WRT invariant

of $(\tau; \mathbb{S}^2 \times \mathbb{S}^1)$ is expressed by the Reshetikhin-Turaev surgery formula:

$$Z_r(\tau, \mathbb{S}^2 \times \mathbb{S}^1) = -\frac{q - q^{-1}}{2r} \sum_{k=0}^{r-2} (-1)^k (q^{k+1} - q^{-k-1}) \left(\langle \tau \rangle \circ \left\langle \frac{\tau}{k}; n \right\rangle; \mathbb{S}^3 \right) \Big|_{q=\exp(\pi i/r)}.$$

If we replace $\left\langle \frac{\tau}{k}; n \right\rangle$ by the formula (A.12) and use the formula

$$\sum_{k=0}^{r-2} (-1)^{km} (q^{(k+1)(m+1)} - q^{-(k+1)(m+1)}) (q^{k+1} - q^{-k-1}) \Big|_{q=\exp(\pi i/r)} = \begin{cases} -2r, & \text{if } m = 2rl, l \in \mathbb{Z}, \\ 2r, & \text{if } m = 2rl - 2, l \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

then we find that $Z_r(\tau, \mathbb{S}^2 \times \mathbb{S}^1) = 0$ when n is odd and obtain the formula

$$Z_r(\tau, \mathbb{S}^2 \times \mathbb{S}^1) = \left(\sum_{0 \leq l \leq \frac{n}{r}} (P_{2n, 2lr} \circ \langle \tau \rangle; \mathbb{S}^3) + \sum_{1 \leq l \leq \frac{n+1}{r}} (P_{2n, 2lr-1} \circ \langle \tau \rangle; \mathbb{S}^3) \right) \Big|_{q=\exp(\pi i/r)}$$

when n is even. If $r \geq n + 2$ then only one term survives in the sums of the *r.h.s.* of this equation, and we get eq.(3.24). \square

A.3. Torus braids and the Jones-Wenzl projector $P_{2n,0}$. Define the q^+ order of an element

$$x = \sum_{\lambda \in \mathfrak{T}^{\text{TL}}} \sum_{j \in \mathbb{Z}} a_{\lambda, j}(\tau) q^j \langle \lambda \rangle \in \text{TL}^+ \quad (\text{A.13})$$

as $|x|_q = \inf \{j : \exists \lambda : a_{\lambda, j}(\tau) \neq 0\}$.

Definition A.7. A sequence of elements $x_1, x_2, \dots \in \text{TL}^+$ has a limit $\lim_{i \rightarrow \infty} x_i = x$ if $\lim_{i \rightarrow \infty} |x - x_i|_q = +\infty$.

In other words, the limit $\lim_{i \rightarrow \infty} x_i = x$ means that the coefficients at lower powers of q in the presentations (A.13) of the elements x_i stabilize progressively as i grows, and x is the Laurent series in q formed by the stable coefficients.

The following is obvious:

Proposition A.8. If the limit $\lim_{i \rightarrow \infty} x_i$ exists, then it is unique.

Theorem A.9. The Temperley-Lieb algebra elements corresponding to torus braids with high twist converge to the Jones-Wenzl projector $P_{2n,0}$:

$$\lim_{m \rightarrow \infty} \left\langle \bigcirc^m \bigcirc \right\rangle = P_{2n,0}. \quad (\text{A.14})$$

Proof. We multiply $\langle \bigcirc^m_{\cdot} \bigcirc_{\cdot} 2n \rangle$ by the identity element (A.7) and use the well-known formula

$$\langle \bigcirc^1_{\cdot} \bigcirc_{\cdot} n \rangle \circ P_n = (-1)^n q^{\frac{1}{2}n(n+2)} P_n$$

in order to express $\langle \bigcirc^m_{\cdot} \bigcirc_{\cdot} n \rangle$ as a sum of projectors with growing powers of q :

$$\langle \bigcirc^m_{\cdot} \bigcirc_{\cdot} 2n \rangle = \langle \bigcirc^m_{\cdot} \bigcirc_{\cdot} 2n \rangle \circ \sum_{m=0}^{2n} P_{2n,2m} = \sum_{m=0}^{2n} q^{2mn(n+1)} P_{2n,2m}.$$

Obviously, all terms in the sum in the *r.h.s.* tend to zero except the term with $m = 0$, which carries the zeroth power of q . □

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