G-COMPLETE REDUCIBILITY AND SEMISIMPLE MODULES

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Abstract. Let G be a connected reductive algebraic group defined over an algebraically closed field of characteristic $p > 0$. Our first aim in this note is to give concise and uniform proofs for two fundamental and deep results in the context of Serre's notion of G-complete reducibility, at the cost of less favourable bounds. Here are some special cases of these results: Suppose that the index $(H : H^\circ)$ is prime to p and that $p > 2\dim V - 2$ for some faithful G-module V. Then the following hold: (i) V is a semisimple H-module if and only if H is G-completely reducible; (ii) H° is reductive if and only if H is G-completely reducible.

We also discuss two new related results: (i) if $p \geq \dim V$ for some G-module V and H is a G-completely reducible subgroup of G, then V is a semisimple H-module – this generalizes Jantzen's semisimplicity theorem (which is the case $H = G$); (ii) if H acts semisimply on $V \otimes V^*$ for some faithful G-module V, then H is G-completely reducible.

1. Introduction

Throughout, G is a connected reductive linear algebraic group defined over an algebraically closed field of characteristic $p > 0$ and H is a closed subgroup of G. Following Serre [13], we say that H is *G-completely reducible* (G-cr for short) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P; for an overview of this concept see for instance [12] and [13]. Note that in case $G = \mathrm{GL}(V)$ a subgroup H is G-cr exactly when V is a semisimple H-module. Recall that if H is G-cr, then the identity component H° of H is reductive, [12, Property 4].

Let V denote a rational G-module and let $\rho : G \to \mathrm{GL}(V)$ be the representation of G afforded by V. Following Serre [12], we call V *non-degenerate* provided $(\ker\rho)^\circ$ is a torus.

First we consider two important and deep theorems in this context, [13, Thm. 5.4] and [13, Thm. 4.4], which provide necessary and sufficient conditions for a subgroup H of G to be G-cr provided p is sufficiently large.

Theorem 1.1. [13, Thm. 5.4] *Suppose that $p > n(V)$ for some rational G-module V.*

(i) *If H is G-completely reducible, then V is a semisimple H-module.*
(ii) *Suppose that V is non-degenerate. If V is semisimple as an H-module, then H is G-completely reducible.*

Here the invariant $n(V)$ is defined as follows: let T be a maximal torus of G and let λ be a T-weight of V. Define $n(\lambda) = \sum_{\alpha>0}\langle\lambda, \alpha^\vee\rangle$, where the sum is taken over all positive roots of G with respect to T. Then define $n(V) = \sup\{n(\lambda)\}$, where the supremum is taken over all T-weights λ of V, [13, §5.2]. The proof of Theorem 1.1 is elaborate and complicated; it depends on the full force of the following result.

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Theorem 1.2. [13, Thm. 4.4] *Suppose that $p \geq a(G)$ and that $(H : H^\circ)$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

Here the invariant $a(G)$ is defined as follows: for G simple, set $a(G) = \mathrm{rk}(G) + 1$, where $\mathrm{rk}(G)$ is the rank of G. For G reductive, let $a(G) = \sup(1, a(G_1), \dots, a(G_r))$, where $G_1, \dots, G_r$ are the simple components of G, cf. [13, §5.2].

We emphasize that Theorem 1.2 is a consequence of a number of deep theorems due to Jantzen [5] (Theorem 2.1) and McNinch [7] in case G is classical and Liebeck and Seitz [6] in case G is of exceptional type, where the latter involves complicated and long case-by-case analyses. Given that the proofs of both these theorems are intricate, it is desirable to have uniform arguments for them even under additional restrictions on p. We present new concise and uniform proofs of these two results in Theorems 3.3 and 3.5 with different bounds on p. Here we are particularly interested in obtaining short proofs for sufficient conditions for G-complete reducibility. Unfortunately, though not unexpectedly, the brevity and uniformity do come at the expense of less favourable bounds on p.

In [12] and [13], Serre gave an alternative proof of his tensor product theorem [10, Thm. 1] via the concept of G-cr subgroups. Theorems 1.1 and 1.2 are both part of this approach. In Section 3 we essentially argue the other way round: based on a special case of the aforementioned tensor product theorem (Theorem 2.2), we first derive a short proof of Theorem 1.1 (in Theorem 3.3) and in turn use part of that result to obtain a concise and uniform proof of Theorem 1.2 (in Theorem 3.5), with a worse bound on p.

Our next result generalizes Jantzen's semisimplicity Theorem 2.1 to G-cr subgroups of G.

Theorem 1.3. *If $p \geq \dim V$ and H is G-completely reducible, then V is a semisimple H-module.*

Theorem 1.3 is also of interest, as the bound $p > n(V)$ in Theorem 1.1(i) does not apply in case V admits a non-restricted composition factor, cf. Remark 4.4. The proof of Theorem 1.3 requires the force of Theorem 1.1(i) and [10, Thm. 1].

Serre's notion of saturation in $\mathrm{GL}(V)$ (see Definition 2.5) is an important tool in the theory of complete reducibility, see [10] and [12]. As an application of Theorem 1.3 we show in Corollary 4.2 that if $p \geq \dim V$ and H is a G-cr subgroup of $G \leq \mathrm{GL}(V)$, then the saturation of H in $\mathrm{GL}(V)$ is completely reducible in the saturation of G in $\mathrm{GL}(V)$.

Our final result is similar in spirit to Theorem 1.1(ii) giving a sufficient semisimplicity condition for H to be G-cr, but strikingly it does not require any restriction on p.

Theorem 1.4. *If H acts semisimply on $V \otimes V^*$ for some non-degenerate G-module V, then H is G-completely reducible and $\rho(H)$ is separable in $\rho(G)$.*

Recall that a subgroup H of G is said to be *separable in G* if its scheme-theoretic centralizer is smooth, i.e., if its global and infinitesimal centralizers have the same dimension, cf. [1, Def. 3.27]. In [2], we study the interaction between this notion of separability and the concept of G-complete reducibility. Several general theorems concerning G-complete reducibility require some separability hypothesis, e.g., see [1, Thm. 3.35], [1, Thm. 3.46]. In [2, Thm. 1.2], we show that any subgroup of G is separable in G provided p is very good for G. Note that the special case of Theorem 1.4 when $G = \mathrm{GL}(V)$ follows from [1, Thm. 3.46]. The proof of Theorem 1.4 is based on a variant of Richardson's tangent space argument (cf. the separability statement of [14, Thm. 1]), see Lemma 5.2.

2. Preliminaries

We maintain the notation from the Introduction. In particular, G is a connected reductive linear algebraic group defined over an algebraically closed field of characteristic $p > 0$ and H is a closed subgroup of G. Moreover, V is a rational G-module and $\rho : G \to \mathrm{GL}(V)$ is the representation of G afforded by V.

First we recall Jantzen's fundamental semisimplicity result, [5, Prop. 3.2].

Theorem 2.1. *If $\dim V \leq p$, then V is semisimple.*

We continue with the special case for connected reductive groups of Serre's seminal tensor product theorem, [10, Prop. 8].

Theorem 2.2. *Suppose $p > 2\dim V - 2$. If V is semisimple, then so is $V \otimes V^*$.*

Both Theorems 2.1 and 2.2 have conceptional and uniform proofs and both bounds are sharp (cf. [5, Rem.(2), p. 260] and [10, §1.3]).

Let H be a closed subgroup of G such that H° is reductive. We say that (G, H) is a *reductive pair* if the Lie algebra $\mathrm{Lie}\, H$ of H is an H-module direct summand of the Lie algebra $\mathrm{Lie}\, G$ of G, cf. [8]. Our next result is [1, Cor. 3.36].

Proposition 2.3. *Suppose that $(\mathrm{GL}(V), G)$ is a reductive pair. If V is a semisimple H-module, then H is G-completely reducible.*

Next we recall [2, Cor. 2.13].

Proposition 2.4. *If $(\mathrm{GL}(V), G)$ is a reductive pair, then every subgroup of G is separable.*

Suppose that $p \geq \dim V$, so that every non-trivial unipotent element in $\mathrm{GL}(V)$ has order p. We recall Serre's notion of *saturation* in this instance, cf. [12]. Let $u \in \mathrm{GL}(V)$ be unipotent. Then there is a nilpotent element $\epsilon \in \mathrm{End}(V)$ with $\epsilon^p = 0$ such that $u = 1 + \epsilon$. For $t \in \mathbb{G}_a$ we can define u^t by $u^t = (1+\epsilon)^t = 1 + t\epsilon + \binom{t}{2}\epsilon^2 + \cdots + \binom{t}{p-1}\epsilon^{p-1}$. Then $\{u^t \mid t \in \mathbb{G}_a\}$ is a closed connected subgroup of $\mathrm{GL}(V)$ isomorphic to $\mathbb{G}_a$.

Definition 2.5. Suppose $p \geq \dim V$. Let H be a closed subgroup of $\mathrm{GL}(V)$. We say that H is *saturated* if for each unipotent $u \in H$ we have $u^t \in H$ for all $t \in \mathbb{G}_a$. The *saturated closure* H^{sat} of H in $\mathrm{GL}(V)$ is the smallest saturated subgroup of $\mathrm{GL}(V)$ containing H.

There is a notion of saturation for any connected reductive group G, but this is considerably more subtle, see [12] for details.

The next result is the special case when $G = \mathrm{GL}(V)$ in [13, Thm. 5.3]. It follows since parabolic and Levi subgroups of $\mathrm{GL}(V)$ are saturated.

Lemma 2.6. *Suppose that $p \geq \dim V$. Let H be a closed subgroup of $\mathrm{GL}(V)$. Then V is semisimple as an H-module if and only if it is semisimple as an H^{sat}-module.*

The following is one of the key properties of saturated subgroups, [12, Property 3].

Lemma 2.7. *Suppose H is a saturated subgroup of $\mathrm{GL}(V)$. Then $(H : H^\circ)$ is prime to p.*

3. Variations on Theorems 1.1 and 1.2

We begin by showing that Theorems 2.1 and 2.2 and the bound on p in the latter guarantee that $(\mathrm{GL}(V), \rho(G))$ is a reductive pair, which is crucial for some of our subsequent arguments.

Theorem 3.1. *Suppose that $p > 2\dim V - 2$. Then $(\mathrm{GL}(V), \rho(G))$ is a reductive pair.*

Proof. Since $p > 2\dim V - 2$, we also have $p \geq \dim V$. Thus V is semisimple, by Theorem 2.1. Thanks to Theorem 2.2, $V \otimes V^* \cong \mathrm{Lie}(\mathrm{GL}(V))$ is also semisimple. Consequently, $\mathrm{Lie}\,\rho(G)$ is a direct $\rho(G)$-module summand of $\mathrm{Lie}(\mathrm{GL}(V))$. □

Remark 3.2. The bound in Theorem 3.1 is sharp. For, let $p = 2$, $G = \mathrm{SL}_2$, and V the natural module for G. Since G is not separable in itself, Proposition 2.4 implies that $(\mathrm{GL}(V), G)$ is not a reductive pair. In fact, although Theorem 3.1 asserts that generically every representation V of G gives rise to a reductive pair $(\mathrm{GL}(V), \rho(G))$, this is *never* the case if p is bad for G and V is non-degenerate, cf. Remark 3.6.

The following is our variant of Theorem 1.1:

Theorem 3.3. *Let H be a closed subgroup of G and let V be a G-module.*

(i) *Suppose that $p \geq \dim V$ and that $(H : H^\circ)$ is prime to p. If H is G-completely reducible, then V is a semisimple H-module.*
(ii) *Suppose that V is non-degenerate and $p > 2\dim V - 2$. If V is semisimple as an H-module, then H is G-completely reducible.*

Proof. First suppose as in (i). Since H is G-cr, H° is reductive, by [12, Property 4]. Since $p \geq \dim V$ and $(H : H^\circ)$ is prime to p, it follows from Theorem 2.1 (applied to H°) and [5, Lem. 3.1] that V is a semisimple H-module.

Next suppose as in (ii). Since $(\mathrm{GL}(V), \rho(G))$ is a reductive pair, by Theorem 3.1, and V is a semisimple H-module, it follows from Proposition 2.3 that $\rho(H)$ is $\rho(G)$-cr. Since V is non-degenerate, [1, Lem. 2.12(ii)(b)] implies that H is G-cr. □

Remark 3.4. Note that Theorem 1.1(ii) (and thus Theorem 3.3(ii)) holds in particular cases for considerably weaker bounds. For instance, in [2, Thm. 1.7], we showed that if p is very good for G and H acts semisimply on the adjoint module $\mathrm{Lie}\,G$, then H is G-cr. The reverse implication fails under this bound, cf. [1, Rem. 3.43(iii)]. Here $n(\mathrm{Lie}\,G) = 2h - 2$, where h is the Coxeter number of G, cf. [13, Cor. 5.5].

Here is our variation of Theorem 1.2:

Theorem 3.5. *Suppose that $p > 2\dim V - 2$ for a non-degenerate G-module V and that $(H : H^\circ)$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

Proof. First suppose that H° is reductive. Since $p > 2\dim V - 2$, we also have $p \geq \dim V$. Thus V is a semisimple H°-module, by Theorem 2.1 (applied to H°). Moreover, since $(H : H^\circ)$ is prime to p, it follows from [5, Lem. 3.1] that V is a semisimple H-module. The result now follows from Theorem 3.3(ii).

The reverse implication is immediate, by [12, Property 4]. □

Clearly, our bound in Theorem 3.5 is much worse than the bound $a(G)$ in Theorem 1.2.

Remark 3.6. It follows from Theorem 3.1 and Proposition 2.4 that only separable subgroups of G are captured in Theorems 3.3(ii) and 3.5. In [4, Prop. 4.3], the second author showed that if p is bad for G, then there always exists a non-separable subgroup of G (likewise if G is simple and p is not very good for G). Consequently, in this case $(\mathrm{GL}(V), \rho(G))$ can't be a reductive pair for *any* non-degenerate rational G-module V, by Proposition 2.4, cf. [4, Rem. 4.4]. Each of the non-separable subgroups constructed in [4, §4] is a regular reductive subgroup of G which is G-cr, thanks to [1, Prop. 3.20].

4. Proof of Theorem 1.3

We now come to our generalization of Jantzen's semisimplicity Theorem 2.1 to G-cr subgroups of G; Theorem 2.1 is the special case of Theorem 1.3 when $H = G$.

Proof of Theorem 1.3. Suppose that $p \geq \dim V$ and H is G-cr. By Theorem 2.1, V is a semisimple G-module. So to show that V is also semisimple as an H-module, we may assume that $V = L(\lambda)$ is simple of highest weight $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_r$, with restricted weights λ_i. Set $L_i := L(p^i\lambda_i) \cong L(\lambda_i)^{[i]}$, the ith *Frobenius twist* of $L(\lambda_i)$. Then $V = L_0 \otimes L_1 \otimes \cdots \otimes L_r$, by Steinberg's tensor product theorem. Since $p \geq \dim V$, we also have $p \geq \dim L_i = \dim L(\lambda_i)$ for each i, so that $p > n(\lambda_i) = n(L(\lambda_i))$ for each i, according to [5, Lem. 1.2]. By Theorem 1.1(i), each $L(\lambda_i)$ is a semisimple H-module and hence so is each Frobenius twist L_i. Moreover, $p \geq \dim V$ also implies that $p > \sum_i(\dim L_i - 1)$. We therefore may apply Serre's tensor product theorem [10, Thm. 1] to deduce that V is a semisimple H-module. □

Note that Theorem 1.3 shows that Theorem 3.3(i) is valid even without the restriction on the index of H° in H.

Remark 4.1. Proposition 2.3 asserts that under the assumption that $(\mathrm{GL}(V), G)$ is a reductive pair, H is G-cr provided V is a semisimple H-module. In Theorem 1.3 we prove the reverse implication under the assumption that $p \geq \dim V$.

Even under the seemingly stronger condition that $(\mathrm{GL}(V), G)$ is a reductive pair, the statement of Theorem 1.3 is false without the restriction on p. Such an example is already known thanks to a construction from unpublished work of Serre, cf. [2, Ex. 4.7]. We now give a different example: Let $p = 3$, $q = 9$ and let $G = \mathrm{SL}_2$. Set $H = G(q)$. Clearly, H is G-cr. The simple G-module $V = L(1+q+q^2)$ is isomorphic to $L(1) \otimes L(1)^{[1]} \otimes L(1)^{[2]}$, by Steinberg's tensor product theorem, where the superscripts denote q-twists. Then $\dim V = 8 > p$. One readily checks that $V \otimes V^*$ is a semisimple G-module, so that $(\mathrm{GL}(V), G)$ is a reductive pair. However, as a $G(q)$-module, V is isomorphic to the G-module $L(1) \otimes L(1) \otimes L(1)$ which admits the non-simple indecomposable Weyl module of highest weight 3 as a constituent, and the latter is not semisimple for $G(q)$, e.g., see [16, (2D)]. Consequently, V is not semisimple as a $G(q)$-module.

It follows from [1, Lem. 2.12(ii)(a)] that if H is G-cr, then $\rho(H)$ is $\rho(G)$-cr. The following result shows that the same holds for the saturation $\rho(G)^{\mathrm{sat}}$ of the image of G in $\mathrm{GL}(V)$.

Corollary 4.2. *Suppose that $p \geq \dim V$. Then $\rho(G)^{\mathrm{sat}}$ is connected and reductive. If H is G-completely reducible, then $\rho(H)^{\mathrm{sat}}$ is $\rho(G)^{\mathrm{sat}}$-completely reducible.*

Proof. By Theorem 2.1, V is a semisimple G-module. Lemma 2.6 then shows that $\rho(G)^{\mathrm{sat}}$ is $\mathrm{GL}(V)$-cr and thus $(\rho(G)^{\mathrm{sat}})^\circ$ is reductive, by [12, Property 4]. Consider the subgroup M

of $\mathrm{GL}(V)$ generated by $\rho(G)$ and the closed connected subgroups $\{\rho(u)^t \mid t \in \mathbb{G}_a\} \cong \mathbb{G}_a$ of $\mathrm{GL}(V)$ for each unipotent element $u \in G$. By definition, $M \leq \rho(G)^{\mathrm{sat}}$. If $M \neq \rho(G)^{\mathrm{sat}}$, then by repeating this process with M (possibly several times), we eventually generate all of $\rho(G)^{\mathrm{sat}}$ by $\rho(G)$ and closed connected subgroups of $\mathrm{GL}(V)$ isomorphic to $\mathbb{G}_a$. It thus follows from [15, Cor. 2.2.7] that $\rho(G)^{\mathrm{sat}}$ is connected.

Now suppose that H is G-cr. Theorem 1.3 then implies that V is a semisimple H-module and thus, by Lemma 2.6, V is also semisimple as a $\rho(H)^{\mathrm{sat}}$-module. Thanks to [12, Cor. 1], we have $n_{\rho(G)^{\mathrm{sat}}}(V) \leq n_{\mathrm{GL}(V)}(V) = \dim V - 1 < p$. It thus follows from Theorem 1.1(ii) that $\rho(H)^{\mathrm{sat}}$ is $\rho(G)^{\mathrm{sat}}$-cr, as desired. □

We note that a variation of Corollary 4.2 is valid for the general notion of saturation replacing $\mathrm{GL}(V)$ with an arbitrary connected reductive group G. We will return to this in a future publication.

Remark 4.3. If $p > 2\dim V - 2$ and V is non-degenerate, then also the converse of the final assertion of Corollary 4.2 holds. For, if $p > 2\dim V - 2$, then $p \geq \dim V$. Thus $\rho(G)^{\mathrm{sat}}$ is connected and reductive, by the first part of Corollary 4.2. Now suppose that $\rho(H)^{\mathrm{sat}}$ is $\rho(G)^{\mathrm{sat}}$-cr. It then follows from Lemma 2.7 and Theorem 3.3(i) that V is a semisimple $\rho(H)^{\mathrm{sat}}$-module, and thus V is a semisimple H-module, by Lemma 2.6. The result now follows from Theorem 3.3(ii).

Remark 4.4. We compare the bounds $p > n(V)$ from Theorem 1.1(i) and $p \geq \dim V$ from Theorem 1.3. Let $L(\mu_1), \dots, L(\mu_m)$ be the non-isomorphic simple factors of a composition series of V.

First, suppose that all μ_j are restricted. Then $p > n(L(\mu_j))$ for each j, by [5, Lem. 1.2]. In particular, $p > \sup\{n(L(\mu_j))\} = n(V)$, by [13, §5.2], so that Serre's bound applies. In general, $\dim V$ is considerably larger than $n(V)$ in this situation. For instance, let G be simple of type E_6 and let $V = L(\omega_1)$ be the simple G-module of highest weight ω_1, the first fundamental dominant weight. Here we have $n(V) = 16$, while $\dim V = 27$.

Next assume that one of the μ_j is not restricted, i.e., say $\mu_j = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_r$, with restricted weights λ_i and at least one $\lambda_i \neq 0$, $(i > 0)$. According to [13, §5.2], we find that $n(V) \geq n(\mu_j) = n(\lambda_0) + pn(\lambda_1) + \cdots + p^r n(\lambda_r) \geq p$, so the bound $p > n(V)$ does not apply.

Now suppose in addition that $p \geq \dim V$ and H is G-cr. Then Theorem 1.3 shows that V is a semisimple H-module. We can also argue as in the proof of Corollary 4.2: Since $p \geq \dim V$, we can saturate the image of G in $\mathrm{GL}(V)$. Then, because H is G-cr, it follows from Corollary 4.2 that $\rho(H)^{\mathrm{sat}}$ is $\rho(G)^{\mathrm{sat}}$-cr. Thanks to Lemma 2.7 and Theorem 3.3(i), we see that V is a semisimple $\rho(H)^{\mathrm{sat}}$-module, and thus V is a semisimple H-module, by Lemma 2.6. In place of Lemma 2.7 and Theorem 3.3(i), we can use Theorem 1.1(i) directly: [12, Cor. 1] implies that $n_{\rho(G)^{\mathrm{sat}}}(V) \leq n_{\mathrm{GL}(V)}(V) = \dim V - 1 < p$, so that Serre's condition is satisfied for $\rho(G)^{\mathrm{sat}}$ even though it is not satisfied for G itself.

5. Proof of Theorem 1.4

Let $H \leq K$ be closed subgroups of G. The normalizer of K in G is denoted by $N_G(K)$. By $C_G(H) = \{g \in G \mid gxg^{-1} = x \ \forall x \in H\}$ and $C_K(H) = C_G(H) \cap K$ we denote the centralizer of H in G and the centralizer of H in K, respectively. Analogously, we denote the centralizer of H in $\mathfrak{g} = \operatorname{Lie} G$ by $\mathfrak{c}_{\mathfrak{g}}(H) = \{y \in \mathfrak{g} \mid \mathrm{Ad}(x)y = y \ \forall x \in H\}$ and the centralizer of H in $\mathfrak{k} = \operatorname{Lie} K$ by $\mathfrak{c}_{\mathfrak{k}}(H) = \mathfrak{c}_{\mathfrak{g}}(H) \cap \mathfrak{k}$, respectively.

Given $n \in \mathbb{N}$, we let G act diagonally on G^n by simultaneous conjugation:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).$$

We require the notion of a generic tuple, [3, Def. 5.4]. Let $G \hookrightarrow \mathrm{GL}_m$ be an embedding of algebraic groups. Then $\mathbf{h} = (h_1, \ldots, h_n) \in H^n$ is called a *generic tuple of H for the embedding* $G \hookrightarrow \mathrm{GL}_m$ if the h_i generate the associative subalgebra of $\mathrm{Mat}_m = \mathfrak{gl}_m = \mathrm{Lie}\,\mathrm{GL}_m$ spanned by H. We call $\mathbf{h} \in H^n$ a *generic tuple of H* if it is a generic tuple of H for some embedding $G \hookrightarrow \mathrm{GL}_m$. Generic tuples exist for any embedding $G \hookrightarrow \mathrm{GL}_m$ if n is sufficiently large. The relevance to G-cr subgroups of this notion is as follows: For $\mathbf{h} \in H^n$ a generic tuple of H, [3, Thm. 5.8] asserts that the orbit $G \cdot \mathbf{h}$ is closed in G^n if and only if H is G-cr.

Our first result in this section generalizes [1, Rem. 3.31].

Lemma 5.1. *Let $H \leq K \leq G$ be closed subgroups of G. Let $\mathbf{h}$ be a generic tuple for H. Then the orbit map $K \to K \cdot \mathbf{h}$ is separable if and only if H is separable in K.*

Proof. According to [3, Lem. 5.5(i)], a generic tuple $\mathbf{h}$ for H satisfies the identity $C_K(H) = C_K(\mathbf{h})$. By the same argument, we obtain $\mathfrak{c}_{\mathfrak{k}}(H) = \mathfrak{c}_{\mathfrak{k}}(\mathbf{h})$.

Let $\pi : K \to K \cdot \mathbf{h}$ be the orbit map. Then π is separable if and only if $d_e\pi : \mathfrak{k} \to T_e(K \cdot \mathbf{h})$ is surjective. Using $\dim T_e(K \cdot \mathbf{h}) = \dim K \cdot \mathbf{h} = \dim K - \dim C_K(\mathbf{h})$ and $\dim \mathrm{im}(d_e\pi) = \dim \mathfrak{k} - \dim \mathfrak{c}_{\mathfrak{k}}(\mathbf{h}) = \dim K - \dim \mathfrak{c}_{\mathfrak{k}}(\mathbf{h})$, we find that the surjectivity of $d_e\pi$ is equivalent to the equality $\dim C_K(\mathbf{h}) = \dim \mathfrak{c}_{\mathfrak{k}}(\mathbf{h})$. By the first paragraph of the proof, this is equivalent to the separability of H in K. □

The following generalizes part of [2, Thm. 1.3] (which is the special case of Lemma 5.2 when (G, K) is a reductive pair and the h_i lie in K).

Lemma 5.2. *Let $K \leq G$ be a closed subgroup. Let $\mathbf{h} = (h_1, \ldots, h_n) \in N_G(K)^n$. Suppose that there is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ that is $\mathrm{Ad}_G(h_i)$-stable for $i = 1, \ldots, n$, and that the orbit map $\pi' : G \to G \cdot \mathbf{h} \subseteq G^n$ is separable. Then the orbit map $\pi : K \to K \cdot \mathbf{h}$ is separable.*

Proof. Since $\pi' : G \to G \cdot \mathbf{h}$ is separable, the differential $d_e\pi' : \mathfrak{g} \to T_{\mathbf{h}}(G \cdot \mathbf{h})$ is surjective. Since $T_{\mathbf{h}}(K \cdot \mathbf{h}) \subseteq T_{\mathbf{h}}(G \cdot \mathbf{h})$, any $y \in T_{\mathbf{h}}(K \cdot \mathbf{h})$ has a preimage $z \in \mathfrak{g}$ such that $d_e\pi'(z) = y$. Let $z = z_1 + z_2$ be a decomposition of z in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\mu : G^n \to G^n$ be the automorphism of varieties that sends a tuple $(g_1, \ldots, g_n)$ to $(g_1h_1^{-1}, \ldots, g_nh_n^{-1})$. Applying the differential of μ to y, we get $d_{\mathbf{h}}\mu(y) = d_{\mathbf{h}}\mu \circ d_e\pi'(z) = d_e(\mu \circ \pi')(z_1) + d_e(\mu \circ \pi')(z_2) \in \mathfrak{g}^n$. Due to our assumption on the h_i, the map μ sends $K \cdot \mathbf{h}$ to K^n, so that $d_{\mathbf{h}}\mu(y) \in \mathfrak{k}^n$. Likewise, $d_e(\mu \circ \pi')(z_1) \in \mathfrak{k}^n$. However, $d_e(\mu \circ \pi')(z_2) = ((1 - \mathrm{Ad}\, h_1)(z_2), \ldots, (1 - \mathrm{Ad}\, h_n)(z_2)) \in \mathfrak{m}^n$, according to the stability of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We deduce that $d_e(\mu \circ \pi')(z_2) = 0$ and hence $d_e(\mu \circ \pi')(z_1) = d_{\mathbf{h}}\mu(y)$. Since μ is an automorphism, this implies that $y = d_e\pi'(z_1) = d_e\pi(z_1)$. We have thus shown that $d_e\pi$ is surjective, i.e., that π is separable. □

Our next result generalizes [2, Thm. 1.4] (which is the special case of Corollary 5.3 when (G, K) is a reductive pair).

Corollary 5.3. *Let $H \leq K \leq G$ be closed subgroups. Suppose that there is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as an H-module and that H is separable in G. Then H is separable in K.*

Proof. Let $\mathbf{h} \in G^n$ be a generic tuple of H. Since H is separable in G, the orbit map $G \to G \cdot \mathbf{h}$ is separable, thanks to Lemma 5.1. It then follows from Lemma 5.2 that $K \to K \cdot \mathbf{h}$ is separable and thus that H is separable in K, again by Lemma 5.1. □

Next we give an immediate consequence of Corollary 5.3 and [2, Thm. 1.2].

Corollary 5.4. *Suppose that p is very good for G. Let $H \leq K \leq G$ be closed subgroups. Suppose that there is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as an H-module. Then H is separable in K.*

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Thanks to [1, Lem. 2.12(ii)(b)], we may assume that V is a faithful G-module so that $G \leq \mathrm{GL}(V)$. By assumption, H acts semisimply on $\mathrm{Lie}(\mathrm{GL}(V)) \cong V \otimes V^*$, and it is automatically separable in $\mathrm{GL}(V)$ (cf. [1, Ex. 3.28]). Since the H-submodule $\mathfrak{g}$ must have a complement in $\mathrm{Lie}(\mathrm{GL}(V))$, we can use Corollary 5.3 (with "$G = \mathrm{GL}(V)$" and "$K = G$") to deduce that H is separable in G. Moreover, as an H-submodule of $\mathrm{Lie}(\mathrm{GL}(V))$, $\mathfrak{g}$ is also semisimple. Finally, [1, Thm. 3.46] implies that H is G-cr. □

We discuss some consequences of Theorem 1.4.

Remark 5.5. It follows readily from Theorem 1.4 that a linearly reductive subgroup of G is G-cr and separable in G, [1, Lem. 2.6] and [9, Lem. 4.1]. In particular, Theorem 1.4 gives an alternative proof for the first fact without any cohomology considerations, cf. [9, §6].

Remark 5.6. Assume as in Theorem 1.4 that H is a closed subgroup of G acting semisimply on $V \otimes V^*$ for some faithful G-module V. Since H is separable in $\mathrm{GL}(V)$ (cf. [1, Ex. 3.28]), and semisimple on $\mathrm{Lie}(\mathrm{GL}(V)) \cong V \otimes V^*$, it follows from [2, Thm. 1.6] that H is $\mathrm{GL}(V)$-cr, i.e., V is a semisimple H-module (cf. [11, (2.2.2), Prop. 3.2]).

Moreover, by Theorem 1.4, H is G-cr and thus H° is reductive and the proof of Theorem 1.4 shows that both $(\mathrm{GL}(V), H)$ and (G, H) are reductive pairs. Note that in general $(\mathrm{GL}(V), G)$ need not be a reductive pair.

Example 5.7. Let $p = 2$, $G = \mathrm{SL}_2$ and let T be a maximal torus of G. Let $H = N_G(T)$. If $\phi\colon G \to G'$ is a non-degenerate epimorphism (i.e., $(\ker \phi)^\circ$ is a torus), then $G' = \mathrm{SL}_2$ or $G' = \mathrm{PGL}_2$ and it is easily checked that $\phi(H) = N_{G'}(T')$, where $T' := \phi(T)$ is a maximal torus of G'. Hence $\phi(H)$ is not separable in G'. It follows from Theorem 1.4 that H does not act semisimply on $V \otimes V^*$ for *any* non-degenerate G-module V.

Remark 5.8. The converse of Theorem 1.4 is false. For instance, let $p = 2$, let $H = G = \mathrm{GL}_2$ and let V be the natural module for G. Then clearly H is G-cr and separable in G. But $V \otimes V^*$ is not H-semisimple.

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