L^1 COHOMOLOGY OF BOUNDED SUBANALYTIC MANIFOLDS

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Abstract. We prove some de Rham theorems on bounded subanalytic submanifolds of $\mathbb{R}^n$ (not necessarily compact). We show that the L^1 cohomology of such a submanifold is isomorphic to its singular homology. In the case where the closure of the underlying manifold has only isolated singularities this implies that the L^1 cohomology is Poincaré dual to L^∞ cohomology (in dimension $j < m-1$). In general, Poincaré duality is related to the so-called L^1 Stokes' Property. For oriented manifolds, we show that the L^1 Stokes' property holds if and only if integration realizes a nondegenerate pairing between L^1 and L^∞ forms. This is the counterpart of a theorem proved by Cheeger on L^2 forms.

1. introduction

Given a Riemannian manifold M, the L^1 forms are the differential forms ω on M satisfying

$$\int_M |\omega|\, dvol_M < \infty, \tag{1.1}$$

where $|\omega|$ is the norm of the differential form ω derived from the Riemannian metric of M. The smooth L^1 forms having an L^1 exterior derivative constitute a cochain complex which gives rise to cohomology groups, **the L^1 cohomology groups of M**.

In this paper we first prove a de Rham theorem for the L^1 cohomology:

Theorem 1.1. *Let M be a bounded subanalytic submanifold of $\mathbb{R}^n$. The L^1 cohomology of M is isomorphic to its singular cohomology.*

Here, M is equipped with the Riemannian metric inherited from the ambient space. In particular, the L^1 cohomology groups are finitely generated and are topological invariants of M.

Forms with integrability conditions have been the focus of interest of many authors. Let us mention, among many others, [BGM, C1, C2, C3, CGM, D, We, HP, S1, S2, Y]. First, integration is necessary to construct a pairing, crucial to define a Poincaré duality morphism which we study below. Secondly, integrability conditions are of foremost importance in geometric analysis and differential equations on manifolds.

The L^1 condition if of metric nature. The metric geometry of singularities is much more challenging than the study of their topology. For instance it is well known that subanalytic sets may be triangulated and hence are locally homeomorphic to cones. This property is very important for it reduces the study of the topology of the singularity to the study of

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the topology link. The story is more complicated is one is interested in the description of the aspect of singularities from the metric point of view. A triangulation may not be achieved without affecting drastically the metric structure the singular set. The proof of this theorem thus requires new techniques for we do not restrict ourselves to metrically conical singularities.

In [V1, V2], the author introduced and constructed some triangulations enclosing enough information to determine al the metric properties of the singularities. The idea was to control the way the metric is affected by the triangulation. The proof of Theorem 1.1 requires an acurate description of the metric type of subanalytic singularities. Using the techniques developped in [V1] [V2] and [V3] we show that the conical structure of subanalytic singularities is not only topological but Lipschitz in a very explicit sense that we shall define in this paper. This is achieved in section 2 of this paper and it is the keystone of the proof of Theorem 1.1. This section is of its own interest, offering a nice new description of the Lipschitz geometry of subanalytic sets. We improve the results of [V3] where it was shown that every subanalytic germ may be retracted in a Lipschitz way (see also [SV]).

The history of L^p forms on singular varieties began when J. Cheeger started constructing a Hodge theory for singular compact varieties. He first computed in [C1, C2] the L^2 cohomology groups for varieties with metrically conical singularities. It turned out to be related to intersection cohomology making of it a good candidate to get a generalized Hodge theory on singular varieties [C3, C4, C5, CGM].

Since Cheeger's work on L^2 forms, many authors have investigated L^p forms on singular varieties [BGM, D, We, HP, S1, S2, Y] (among many others). Nevertheless all of them focus on particular classes of Riemmanian manifolds, with strong restrictions on the metric near the singularities, like in the case of the so-called f-horns or metrically conical singularities. In the present paper we only assume that the given set is subanalytic.

Recently, the author of the present paper computed the L^∞ cohomology groups for any subanalytic pseudomanifold. Let us recall the de Rham theorem achieved in [V3].

Theorem 1.2. [V3] *Let X be a compact subanalytic pseudomanifold. Then, for any j:*

$$H^j_\infty(X_{reg}) \simeq I^t H^j(X).$$

Furthermore, the isomorphism is induced by the natural map provided by integration on allowable simplices.

Here, $H^\bullet_\infty$ denotes the L^∞ cohomology and $I^t H^j(X)$ the intersection cohomology of X in the maximal perversity. The definitions of these cohomology theories are recalled in sections 5.1 and 5.2 below. We write X_{reg} for the nonsingular part of X, i. e. the set of points at which X is a smooth manifold.

Intersection homology was discovered by M. Goresky and R. MacPherson who computed these homology groups. What makes it very attractive is that they showed in their fundamental paper [GM1] that it satisfies Poincaré duality for a quite large class of sets (recalled in Theorem 5.2), enclosing all the complex analytic sets (see also [GM2]).

In view of the above paragraph, the two above de Rham theorems raise the very natural question of whether we can hope for Poincaré duality between L^1 and L^∞ cohomology. Actually, the two above theorems, via Goresky and MacPherson's generalized Poincaré duality, admit the following corollary.

Corollary 1.3. *Let X be an oriented subanalytic pseudomanifold with isolated singularities. Then, L^1 cohomology is Poincaré dual to L^∞ cohomology in dimension $j < m-1$, i. e. for any $j < m-1$:*

$$H^j_{(1)}(X_{reg}) \simeq H^{m-j}_\infty(X_{reg}).$$

More generally, if the singular locus is of dimension k then the L^1 cohomology is dual to the L^∞ cohomology in dimension $j < m-k-1$. This is due to the fact that in this case intersection homology coincides with the usual homology of X_{reg} (in dimension j). Intersection homology turns out to be very useful to assess the lack of duality between L^1 and L^∞ cohomology. We see that the obstruction for this duality to hold is of purely topological nature. Although the L^1 and L^∞ conditions are closely related to the metric structure of the singularities, the above theorems show that the knowledge of the topology of the singularities is enough to enure Poincaré duality. It is worthy of notice that the only data of the topology of X_{reg} is not enough.

In his study of L^2 cohomology, Cheeger also pointed out a problem that may arise on singular varieties, even with conical singularities: the L^2 Stokes' property may fail. Roughly speaking, this property says that the exterior differential operator is self-adjoint on L^2 forms (up to sign, considering $(m-j)$ -forms as the dual of j-forms, see (1.2)). This property is crucial in Hodge theory, which yields Poincaré duality as a byproduct. Cheeger investigated the case of conical singularities in [C2] and completely clarified the situation. He showed that the L^2 Stokes' property holds on conical singularities if and only if Poincaré duality holds. Thus, in this case, a nice Hodge theory may be performed and Cheeger was able to prove that every cohomology class has a unique harmonic representative. Cheeger's L^2 Stokes' property is also crucial because it allows to define a pairing on the L^2 cohomology groups by integrating wedge products of forms. The Poincaré duality isomorphism on L^2 cohomology then results from this pairing which provides a very natural isomorphism.

The L^p Stokes' property has been then studied by Y. Youssin on f-horns in [Y], who obtained an analogous result.

Therefore, in our framework, the latter duality for L^1 cohomology very naturally raises the question on whether the L^1 Stokes property holds and whether integration provides an isomorphism between L^1 and L^∞ cohomology. In order to be more specific, let us explicitly define the L^1 **Stokes' property** by saying that it holds (in dimension j) on a C^∞ manifold M of dimension m whenever for any C^∞ L^1 j-form α, with $d\alpha$ L^1 we have:

$$\int_M \alpha \wedge d\beta = (-1)^{j+1} \int_M d\alpha \wedge \beta, \tag{1.2}$$

for any L^∞ $(m-j)$-form β with $d\beta$ L^∞.

For smooth forms on compact manifolds without boundary this is always true by Stokes' formula. Somehow, the question is whether the singularities behave like a boundary or if the closure of M may behave like a manifold. This question occurs especially in the case where the singular locus of the closure of M is of low dimension.

We shall answer this question in a very precise way, giving a L^1 counterpart of Cheeger's theorem on the L^2 Stokes' property. Again, our theorems on L^1 cohomology hold for any subanalytic bounded manifold (metrically conical or not).

Given a submanifold $M \subset \mathbb{R}^n$ we shall write δM for the set $cl(M) \setminus M$, where $cl(M)$ stands for the topological closure of M. We shall prove:

Theorem 1.4. *Let $j < m$ and let M be a bounded subanalytic oriented manifold. The L^1 Stokes' property holds for j-forms iff $\dim \delta M < m - j - 1$.*

In particular, if $cl(M)$ has only isolated singularities then the L^1 Stokes' property holds in any dimension $j < m-1$. In this case, integration of forms induces the Poincaré duality isomorphism of Corollary 1.3.

Noteworthy, the obstruction for the L^1 Stokes property to hold is also purely topological. The only knowledge of the dimension of the singular locus is enough to ensure that this property holds, no matter how fast the volume is collapsing near the singularities.

Dirichlet L^1 cohomology. Let $M \subset \mathbb{R}^n$ be a subanalytic bounded submanifold. We just explained that in the case of non closed oriented manifolds, the L^1-Stokes' property may fail. This "boundary phenomenon" may appear near the singularities preventing the L^1 classes from being Poincaré dual to the L^∞ classes.

On compact manifolds with boundary, "ideal boundary conditions" are usually put in order overcome this kind of problems. They give rise so-called **Dirichlet cohomology**. The Dirichlet forms are those whose restriction to the boundary is identically zero. These are also the forms satisfying the L^2 Stokes' property.

In our setting, if ω is a form defined on M, it does not make sense to require that it vanishes on δM. **Dirichlet L^1 cohomology** is thus usually defined (see for instance [IM]) as the cohomology of the L^1 forms α (with $d\alpha$ L^1) satisfying (1.2). This is the biggest space of L^1 forms on which d is self-adjoint (up to sign, identifying L^∞ with the dual of L^1). This cohomology theory is discussed in section 7.

We will denote the Dirichlet L^1 cohomology of a submanifold $M \subset \mathbb{R}^n$ by $H^{m-j}_{(1)}(M; \delta M)$. Now, as in the case of manifolds with boundary, Lefschetz-Poincaré duality holds in general:

Theorem 1.5. *For any bounded subanalytic orientable submanifold $M \subset \mathbb{R}^n$:*

$$H^j_{(1)}(M; \delta M) \simeq H^{m-j}_\infty(M).$$

It is worthy of notice that this duality is a general fact on bounded subanalytic manifolds: we do not assume that the closure of M is a pseudomanifold. The version stated in Theorem 6.8 is actually even stronger.

In particular, by Goresky and MacPherson's generalized Poincaré duality, the Dirichlet L^1 cohomology is isomorphic to intersection homology in the zero perversity and Theorem 1.2 and Theorem 1.5 admit the following immediate interesting corollary.

Corollary 1.6. *(De Rham theorem for Dirichlet L^1 cohomology) Let X be a subanalytic bounded orientable pseudomanifold. We have:*

$$H^j_{(1)}(X_{reg}; X_{sing}) \simeq I^0 H^j(X_{reg}).$$

Here X_{sing} stands for the singular locus and X_{reg} denotes its complement in X.

Content of the paper. Section 2 introduces and yields the "Lipschitz conic structure of subanalytic sets" (definition 2.6). We prove in section 3 some basic results on L^1 cohomology and establish Theorem 1.1 in section 4. Poincaré duality for L^1 cohomology is then discussed in section 5. In section 6 we introduce the L^1 Dirichlet cohomology groups and establish the relative form of Lefschetz-Poincaré duality claimed in Theorem 1.5. We then study the L^1 Stokes' property, proving Theorem 1.4 in section 7. We end this paper with a concrete example, the suspension of the torus, on which we discuss all the results of this paper.

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Notations and conventions. In the sequel, all the considered sets and maps will be subanalytic (if not otherwise specified) except the differential forms.

By "**subanalytic**" we mean "globally subanalytic", i. e. which remains subanalytic after compactifying $\mathbb{R}^n$ (by $\mathbb{P}^n$).

Given a set $X \subset \mathbb{R}^n$, we denote by $C^j(X)$ the singular cohomology cochain complex and by $H^j(X)$ the cohomology groups. Simplices are defined as continuous (subanalytic) maps $\sigma : \Delta_j \to X$, where Δ_j is the standard simplex.

Given two nonnegative functions $\xi : X \to \mathbb{R}$ and $\eta : X \to \mathbb{R}$ we will write $\xi \sim \eta$ if there is a positive constant C such that $\xi \leq C\eta$ and $\eta \leq C\xi$. We write $[\xi;\eta]$ for the set $\{x \in X \times \mathbb{R} : \xi(x) \leq y \leq \eta(x)\}$ and define similarly the open interval $(\xi;\eta)$.

Given a (subanalytic) set X, we denote by X_{reg} the set of point near which X is a C^∞ manifold and by X_{sing}, its complement in X. The subsets δM and $cl(M)$ are also as explained above. By manifold we will mean C^∞ manifold.

We shall say that a function $\xi : X \to \mathbb{R}$ is **Lipschitz** if there is a constant C such that for any x and x' in X:

$$|\xi(x) - \xi(x')| \leq C|x - x'|.$$

A map $f : X \to \mathbb{R}^k$ is Lipschitz if all its components are Lipschitz and a homeomorphism h is **bi-Lipschitz** if both h and h^{-1} are Lipschitz.

We shall write $S^{n-1}(x_0;\varepsilon)$ for the sphere of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^n$ and $B^n(x_0;\varepsilon)$ for the corresponding ball. We will write $L(x_0;X)$ for the **link** of X at x_0. It is the subset $S^{n-1}(x_0;\varepsilon) \cap X$, where $\varepsilon > 0$ is small enough. By [V2] this subset is, up to a subanalytic bi-Lipschitz map, independent of $\varepsilon > 0$.

2. On the Lipschitz geometry of subanalytic sets.

The results of this section will be very important to compute the L^1 cohomology groups later on.

It is well known that subanalytic sets are locally homeomorphic to cones. It is not true that subanalytic germs of singularities are bi-Lipschitz homeomorphic to cones. We describe the metric types of subanalytic germs in a very precise way. This is very important since the L^1 condition heavily relies on the metric. Roughly speaking, we show that, given a subanalytic germ X, we can find a subanalytic homeomorphism from a cone (over the link) such that the eigenvalues of the pullback of the metric induced by $\mathbb{R}^n$ on X by this homeomorphism are increasing as we are wandering away from the origin. This improves significantly the results of [V1] [V3] where a Lipschitz strong deformation retraction onto the origin was constructed.

Given $n > 1$ and a positive constant R we set:

$$\mathcal{C}_n(R) := \{(x_1; x') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \le |x'| \le R x_1 \}.$$

For $n = 1$, we just define $\mathcal{C}_1$ as the positive x_1-axis.

2.1. **Regular lines.** We start by recalling a result of [V3].

Definition 2.1. Let X be a subset of $\mathbb{R}^n$. An element λ of S^{n-1} is said to be **regular for** X if there is a positive number α such that:

$$dist(\lambda; T_x X_{reg}) \ge \alpha,$$

for any x in X_{reg}.

Regular lines do not always exist, as it is shown by the simple example of a circle. Nevertheless, given a subanalytic set of empty interior, up to a bi-Lipschitz homeomorphism, we can get a line which is regular. This is what is established by theorem 3.13 of [V1]. This theorem has then been improved in [V3] into a statement that we shall need in its full generality. It is recalled in Lemma 2.3. To state this lemma, we need the following definition.

Definition 2.2. Let $A, B \subset \mathbb{R}^n$. A map $h : A \to B$ is x_1**-preserving** if it preserves the first coordinate in the canonical basis of $\mathbb{R}^n$.

We denote by $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the canonical projection. In the Lemma below all the considered germs are germs at the origin.

Lemma 2.3. [V3] *Given germs $X_1, \dots, X_s \subset \mathcal{C}_n(R)$, there exist a germ of x_1-preserving bi-Lipschitz homeomorphism $h : \mathcal{C}_n(R) \to \mathcal{C}_n(R)$, with $R > 0$, and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that:*

1. *$\mathcal{D}$ is compatible with $h(X_1), \dots, h(X_s)$*
2. *e_n is regular for any cell of $\mathcal{D}$ in $\mathcal{C}_n(R)$ which is a graph over a cell of $\mathcal{C}_{n-1}(R)$ of $\mathcal{D}$*
3. *Given finitely many germs of nonegative functions $\xi_1, \dots, \xi_l$ on $\mathcal{C}_n(R)$, we may assume that on each cell D of $\mathcal{D}$, every germ $\xi_i \circ h$ is $\sim$ to a function of the form:*

$$|y - \theta(x)|^r a(x) \tag{2.3}$$

(for $(x; y) \in \mathbb{R}^{n-1} \times \mathbb{R}$) where $a, \theta : \pi_n(D) \to \mathbb{R}$ are functions with θ Lipschitz and $r \in \mathbb{Q}$.

Remark 2.4. Given a family of Lipschitz functions $f_1, \dots, f_k$ defined over $\mathbb{R}^n$ we can find some Lipschitz functions $\xi_1 \le \dots \le \xi_l$ and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n-1}$ such that over each cell $D \in \mathbb{R}^{n-1}$ delimited by the graphs of two consecutive functions $[\xi_{i|D}; \xi_{i+1|D}]$, with

$D \in \mathcal{D}$, the functions $|q_{n+1} - f_i(x)|$ (where $q = (x; q_{n+1})$) are comparable to each other (for relation $\leq$) and comparable to the functions $f_i \circ \pi_n$. Indeed, it suffices to choose a cell decomposition $\mathcal{D}$ compatible with the sets $f_i = f_j$ and to add the graphs of the functions f_i, $f_i + f_j$ and $\frac{f_i+f_j}{2}$. We may then use min and max to transform this family into an ordered family (for $\leq$).

2.2. **Lipschitz conic structure of subanalytic sets.** This section is crucial in the proof of our de Rham theorems. We introduce and establish what we call "the Lipschitz conic structure" of subanalytic sets.

Let $X \subset \mathbb{R}^n$ of dimension m and let $x_0 \in cl(X)$.

Definition 2.5. A **tame basis** on a manifold M is a basis $\lambda_1, \dots, \lambda_m$ ($m = \dim M$) of bounded subanalytic 1-forms on M such that:

$$| \wedge_{i=1}^m \lambda_i| \geq \varepsilon > 0, \tag{2.4}$$

on M.

Let us make a point that we do not assume the tame bases to be continuous, but, as they are assumed to be subanalytic, they are indeed implicitly required to be smooth almost everywhere. This will be enough for us, since, for integrability conditions, only the behavior almost everywhere is relevant. Alike, in the definition below, the φ_i's do not need to be continuous, but indeed only the generic values of these functions really matter since (3) of the definition below is required almost everywhere. We shall also pull-back the forms via subanalytic maps. The pullback will be well defined almost everywhere since, once again, subanalytic mappings are smooth generically.

Definition 2.6. We say that X is **Lipschitz conical** at $x_0 \in X$ if there exist a positive real number ε and a Lipschitz homeomorphism

$$h : (0; \varepsilon) \times L(x_0; X) \to X \cap B^n(x_0; \varepsilon) \setminus \{x_0\},$$

with $d(x_0; h(t; x)) = t$, such that we can find some positive functions $\varphi_1, \dots, \varphi_{m-1}$: $(0; \varepsilon) \times L(x_0; X) \to \mathbb{R}$, for which we have:

(1) The $\varphi_i(t; x)$'s are decreasing to zero as t is going to zero for any x,
(2) The $\varphi_i(t; x)$'s are bounded below on any closed set disjoint from $\{t = 0\}$.
(3) There is a tame basis $\lambda_1, \dots, \lambda_{m-1}$ of $L(x_0; X_{reg})$ such that if $\theta_i := h^{-1*}(\varphi_i \cdot \lambda_i)$ then $(h^{-1*}dt; \theta_1; \dots; \theta_{m-1})$ is a tame basis on a dense subset of X_{reg}.

Theorem 2.7. *Every (subanalytic) set is Lipschitz conical at any point.*

Proof. We shall consider sets $A \subset \mathbb{R}^n$ as families parameterized by x_1 and write A^ε for the "fiber" at $\varepsilon \in \mathbb{R}$, that is to say:

$$A^\varepsilon := \{x \in \mathbb{R}^{n-1} : (\varepsilon; x) \in A\}.$$

We will actually prove by induction on n the following statements.

($\mathbf{A}_n$) Let $X_1, \dots, X_s$ be finitely many subsets of $\mathcal{C}_n(R)$ and let $\xi_1, \dots, \xi_l$ be some bounded functions.

There exist positive real numbers R and ε, together with a Lipschitz x_1-preserving homeomorphism

$$h : (0;\varepsilon) \times B^{n-1}(0;R) \to \mathcal{C}_n(R) \setminus \{0\},$$

such that for every $j \in \{1,\dots,s\}$, we can find some positive functions $\varphi_{1,j},\dots,\varphi_{\mu_j-1,j}$ ($\mu_j := \dim X_j$) on $(0;\varepsilon) \times X^{\varepsilon}_{j,reg}$ with:

(1) $h((0;\varepsilon) \times X^{\varepsilon}_j) = X_j \cap \{0 < x_1 < \varepsilon\}$
(2) The $\varphi_{i,j}(t;x)$'s are decreasing to zero as t goes to zero, for any $x \in X^{\varepsilon}_{j,reg}$
(3) The $\varphi_i(t;x)$'s are bounded below on any closed set disjoint from $\{t = 0\}$
(4) There is a tame basis $\lambda_{1,j},\dots,\lambda_{m-1,j}$ of $X^{\varepsilon}_{j,reg}$ such that if $\theta_{i,j} := h^{-1*}(\varphi_{i,j} \cdot \lambda_{i,j})$ then $(h^{-1*}dt;\theta_{1,j};\dots;\theta_{m-1,j})$ is a tame basis of a dense subset of $X_{j,reg}$.
(5) There is a constant C such that for any $i \le l$ and any $0 < \tau \le u \le t$ we have:

$$C_\tau \xi_i(h(\tau;x)) \le \xi_i(h(u;x)) \le C\xi_i(h(t;x)). \tag{2.5}$$

for some positive constant C_τ.

Before proving these statements, let us make it clear that this implies the desired result. Let $X \subset \mathbb{R}^n$. We can assume that $0 \in X$ and work nearby the origin. The set

$$\hat{X} := \{(x_1;x) \in \mathbb{R} \times X : |x| = x_1\}$$

is a subset of $\mathcal{C}_{n+1}(R)$ (for $R > 1$) to which we can apply $(\mathbf{A}_{n+1})$. Observe that $\hat{X}$ is bi-Lipschitz equivalent to X. This means that it is enough to check the properties $(1-3)$ of definition 2.6 for $\hat{X}$. But they are implied by (2), (3) and (4) of $(\mathbf{A}_{n+1})$.

The assertion (5) is not necessary to prove that X is Lipschitz conical. It is assumed so as to perform the proof of (2) during the induction step.

As $(\mathbf{A}_n)$ obviously holds in the case where $n = 1$ (h being the identity map), we fix some $n > 1$. We fix some subsets $X_1,\dots,X_s$ of $\mathcal{C}_n(R)$, for $R > 0$, and some subanalytic bounded functions $\xi_1,\dots,\xi_l : \mathcal{C}_n(R) \to \mathbb{R}$.

Apply Lemma 2.3 to the family constituted by the X_i's and the union of the zero loci of the ξ_i's. We get a x_1-preserving bi-Lipschitz map $h : \mathcal{C}_n(R) \to \mathcal{C}_n(R)$ and a cell decomposition $\mathcal{D}$ such that (1), (2), and (3) of the latter lemma hold. As we may work up to a x_1-preserving bi-Lipschitz map we will identify h with the identity map. Hence, thanks to (3) of the latter Lemma, we may assume that the functions ξ_i's are $\sim$ to a function like in (2.3).

Let Θ be a cell of $\mathcal{D}$ in $\mathcal{C}_n(R)$ which is the graph of a function $\eta : \Theta' \to \mathbb{R}$, with $\Theta' \in \mathcal{D}$. By (2) of Lemma 2.3, η is then necessarily a Lipschitz function. Consequently, it may be extended to a Lipschitz function on the whole $\mathcal{C}_{n-1}(R)$ whose graph still lies in $\mathcal{C}_n(R)$. Repeating this for all the cells of $\mathcal{D}$ which are graphs over a cell of $\mathcal{D}$ in $\mathbb{R}^{n-1}$ we get a family of functions $\eta_1,\dots,\eta_v$. Using the operators min and max we may transform this family in an ordered one (for $\le$), so that, keeping the same notations for the new family, we will assume that it satisfies $\eta_1 \le \dots \le \eta_v$.

Fix an integer $1 \le j < v$ and a connected component B of $(\eta_j;\eta_{j+1})$. Let Θ be a cell of $\mathcal{D}$ and set for simplicity $D := \Theta \cap B$.

Up to constants, the functions ξ_k's are like in (2.3) on D, i. e. there exist $(n-1)$-variable functions on D, say θ_k and a_k, $k = 1, \dots, m$ with θ_k Lipschitz such that:

$$\xi_k(x; y) \sim (y - \theta_k(x))^{\alpha_k} a_k(x),$$

for $(x; y) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

We shall apply the induction hypothesis to all the a_k's (obtained for all such sets D). Unfortunately this is not enough if one wants to get that the ξ_k's satisfy (5), due to the term $(y - \theta_k)$ in the decomposition of the ξ_k's just above. Therefore, before applying the induction hypothesis, we need to complete the family to which we will apply (5) of the induction hypothesis by some extra bounded $(n-1)$ variable functions that we are going to introduce.

As the zero loci of the ξ_k's are included in the graphs of the η_i's, we have on D for every k, either $\theta_k \le \eta_j$ or $\theta_k \ge \eta_{j+1}$. We will assume for simplicity that $\theta_k \le \eta_j$.

This means that for $(x; y) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$:

$$\xi_k(x; y) \sim \min((y - \eta_j(x))^{\alpha_k} a_k(x); (\eta_j - \theta_k(x))^{\alpha_k} a_k(x)), \tag{2.6}$$

if α_k is negative and

$$\xi_k(x; y) \sim \max((y - \eta_j(x))^{\alpha_k} a_k(x); (\eta_j - \theta_k(x))^{\alpha_k} a_k(x)), \tag{2.7}$$

in the case where α_k is nonnegative.

First, consider the following functions:

$$\kappa_k(x) := (\theta_k(x) - \eta_j(x))^{\alpha_k} a_k(x), \qquad k = 1, \dots, l. \tag{2.8}$$

For every k, the function κ_k is bounded for it is equivalent to the function $\xi_k(x; \eta_j(x))$ which is bounded since ξ_k is. Complete the family κ by adding the functions $(\eta_{j+1} - \eta_j)$ as well as the functions $\min(a_k; 1)$. The union of all these families (the just obtained family κ depends on D), obtained for every such set D (intersection of a connected component of $(\eta_j; \eta_{j+1})$, for some j, with some cell D of $\mathcal{D}$) provides us a finite collection of functions $\sigma_1, \dots, \sigma_p$.

We now turn to the construction of the desired homeomorphism. The cell decomposition $\mathcal{D}$ induces a cell decomposition of $\mathbb{R}^{n-1}$. Refine it into a cell decomposition $\mathcal{E}$ compatible with the zero loci of the functions $(\eta_j - \eta_{j+1})$. Apply induction hypothesis to the family constituted by the cells of $\mathcal{E}$ which lie in $\mathcal{C}_{n-1}(R)$. This provides a homeomorphism

$$h : (0; \varepsilon) \times B^{n-2}(0; R) \to \mathcal{C}_{n-1}(R).$$

We first are going to lift h to a homeomorphism $\tilde{h} : (0; \varepsilon) \times B^{n-1}(0; R) \to \mathcal{C}_n(R)$.

Thanks to the induction hypothesis, we may assume that the functions $\sigma_1, \dots, \sigma_p$ satisfy (2.5).

We lift h as follows. For simplicity we define η'_j as the restriction of η_j to $\mathcal{C}_n(R) \cap \{x_1 = \varepsilon\}$. On $(\eta_j; \eta_{j+1})$ we set

$$\nu(q) := \frac{y - \eta_j(x)}{\eta_{j+1}(x) - \eta_j(x)},$$

where $q = (x; y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then, for $(t; q) \in (0; \varepsilon) \times (\eta'_j; \eta'_{j+1})$

$$\widetilde{h}(t; q) := (h(t; x); \nu(q)(\eta_{j+1}(h(t; x)) - \eta_j(h(t; x))) + \eta_j(h(t; x))).$$

In virtue of the induction hypothesis, the inequality (2.5) is fulfilled by the functions $(\eta_{i+1} - \eta_i)$. Therefore, as h is Lipschitz, we see that $\widetilde{h}$ is Lipschitz as well. As (1) holds by construction for every cell, it holds for all the X_j's.

We now turn to define the functions $\varphi_{i,j}$. Actually, as all the X_i's are unions of cells, it is enough to carry out the proof on every cell $E \in \mathcal{E}$, i. e. to define some functions $\varphi_{1,E}, \dots, \varphi_{\mu-1,E}$ (where $\mu = dim\, E$), decreasing to 0 with respect to t, and a tame basis $\lambda_{1,E}, \dots, \lambda_{\mu-1,E}$ such that the family $(\widetilde{h}^{-1*}dt; \theta_{1,E}; \dots; \theta_{\mu-1,E})$, where $\theta_{i,E} := \widetilde{h}^{-1*}(\varphi_{i,E} \cdot \lambda_{i,E})$, is a tame basis of E.

Indeed, the desired functions $\varphi_{i,j}$ can then be defined as the functions induced by all the functions $\varphi_{i,E}$ (defined on $\widetilde{h}^{-1}(E)$), for all the cells E of dimension μ_j included in X_j (as pointed out before definition 2.6 only the generic values of $\varphi_{i,j}$ actually matter).

Fix a cell $E \subset \mathcal{C}_n(R)$, set $E' := \pi(E)$ and $\mu' := dim\, E'$, where $\pi : \mathcal{C}_n(R) \to \mathcal{C}_{n-1}(R)$ is the obvious orthogonal projection. Let now $\varphi_{1,E'}, \dots, \varphi_{\mu'-1,E'}$ be the functions given by the induction hypothesis. We distinguish two cases.

First case: $\mu' = \mu - 1$ (where $\mu = dim\, E$). Let us set:

$$\varphi_{i,E}(t;x) := \varphi_{i,E'}(t;\pi(x)).$$

As $\mu' = \mu - 1$, the cell E is included in $[\eta_{j|E'}; \eta_{j+1|E'}]$, for some $j \le \lambda$, and we also set:

$$\varphi_{\mu-1,E}(t;x) := \frac{\eta_{j+1}(h(t;x)) - \eta_j(h(t;x))}{\eta_{j+1}(h(\varepsilon;x)) - \eta_j(h(\varepsilon;x))}, \tag{2.9}$$

Let us show that these functions satisfy (2) and (3).

Recall that we applied (5) of the induction hypothesis to the function $(\eta_{j+1} - \eta_j)(x)$. If a function ξ satisfies (2.5) then

$$\xi(h(s;x)) \sim \inf_{s \le t < \varepsilon} \xi(h(t;x)),$$

and consequently $\xi \circ h$ is $\sim$ to an increasing function.

Therefore, changing $\varphi_{i,E}$ for an equivalent function if necessary, we may assume that it is increasing with respect to t. As the graphs of the η_i's are included in $\mathcal{C}_n(R)$, the η_i's must vanish at the origin. Consequently $\varphi_{\mu-1,E}$ tends to zero, as t goes to zero for any $x \in E$, which yields (2).

As $(\eta_{j+1} - \eta_j)$ satisfy (2.5), the φ_i's are bounded away from zero on $(\tau; \varepsilon) \times E^\varepsilon$ for every $0 < \tau < \varepsilon$, showing (3).

We now are going to define our tame basis of 1-forms $\lambda_{i,E}$ in order to prove (4).

Denote by $\pi_E : E \to E'$ the restriction of the orthogonal projection. Let us now set on $(0;\varepsilon) \times E^\varepsilon$

$$\lambda_{i,E} := \pi^*_{|E} \lambda_{i,E'}. \tag{2.10}$$

Then set for $x \in E'$ and $a \in [0;1]$:

$$\eta_{j,a}(x) = (\eta_{j+1}(x) - \eta_j(x))a + \eta_j(x).$$

Denote by E_a the graph of $\eta_{j,a}$.

Put now
$$\lambda_{\mu-1,E}(q)(u) = 0,$$
if u is tangent to $(E_{\nu(q)})^\varepsilon$, and finally set
$$\lambda_{\mu-1,E}(q)(e_n) = 1.$$

As the $\eta_{i,a}$ are Lipschitz with a Lipschitz constant bounded with respect to a, the angle between e_n and the tangent to the graph of $\eta_{i,\nu(x)}$ is bounded below away from zero, and therefore the norm of $\lambda_{\mu-1,E}$ is bounded. The Lipschitz character of the $\eta_{j,a}$'s also implies that the family $\lambda_{1,E}, \dots, \lambda_{\mu-1,E}$ is a tame basis of E^ε.

By definition of $\widetilde{h}$ and $\varphi_{\mu-1,E}$ we have $d_{(t;x)}h(e_n) \sim \varphi_{\mu-1,E}(t;x)$ so that:
$$|\widetilde{h}^{-1*}\lambda_{i,E}| \sim \frac{1}{\varphi_{\mu-1,E}}.$$

The forms $\theta_{i,E} := \widetilde{h}^{-1*}(\varphi_{i,E} \cdot \lambda_{i,E})$ are thus bounded. For the same reasons as for the λ_i's, the family $(\widetilde{h}^{-1*}dt; \theta_{1,E}; \dots; \theta_{\mu-1,E})$ is a tame basis of E.

Second case: $\mu = \mu'$. In this case we only have to define $(\mu' - 1)$ functions and $(\mu' - 1)$ 1-forms. This may be done like in the first case (like in (2.9) and (2.10)). This is indeed much easier to check that (2) (3) and (4) hold, since, as π_E is bi-Lipschitz, the required properties which are true downstairs for h thanks to the induction hypothesis obviously continue to hold upstairs for $\widetilde{h}$. This completes the proof of (2) (3) and (4).

Finally, we have to check that the ξ_k's fulfill (2.5) for $\widetilde{h}$. As the ξ_k's are bounded this is enough to check it for the functions $\min(\xi_k; 1)$. We check it on a given cell $E \in \mathcal{E}$. Fix an integer $1 \le k \le l$. By the induction hypothesis we know that the κ_i's (see (2.8)) satisfy (2.5). Remark that the function $\nu(\widetilde{h}(t;q))$ is constant with respect to t.

Observe that by (2.6) and (2.7) it is enough to show that the functions $\min((y - \eta_j(x))^{\alpha_k} a_k(x); 1)$ and the functions $\min(|\theta_k - \eta_j|(x)^{\alpha_k} a_k(x); 1)$ satisfy (2.5). As for the latter functions this follows from the induction hypothesis and choice of the κ_i's, we only need to focus on the former ones.

For simplicity we set
$$F(x;y) := (y - \eta_j(x))^{\alpha_k} a_k(x),$$
and
$$G(x) := (\eta_{j+1} - \eta_j)(x)^{\alpha_k} \cdot a_k(x).$$

We have to show the desired inequality for $\min(F; 1)$. We have:
$$F(q) = \nu(q)^{\alpha_k} \cdot G(x), \tag{2.11}$$
where again $q = (x; y)$.

As $\nu(\widetilde{h}(t;q))$ is constant with respect to t, this implies that:
$$F(\widetilde{h}(t;q)) = \nu(q)^{\alpha_k} \cdot G(h(t;x)) \tag{2.12}$$

We assume first that α_k is negative. Thanks to the induction hypothesis we know that for $0 < \tau \le u \le t$:
$$C_\tau \min(G(h(\tau;x)); 1) \le \min(G(h(u;x)); 1) \le C \min(G(h(t;x)); 1),$$
for some positive constants C_τ, C.

But this implies (multiplying by ν^{α_k} and applying (2.11) and (2.12)) that

$$C_\tau \min(F(\widetilde{h}(\tau;q));\nu^{\alpha_k}(q);1) \ \le \min(F(\widetilde{h}(u;q));\nu^{\alpha_k}(q);1) \le C \min(F(\widetilde{h}(t;q));\nu^{\alpha_k}(q);1).$$

But, as α_k is negative, $\min(F;\nu^{\alpha_k};1) = \min(F;1)$ and we are done.

We now assume that α_k is nonnegative. This implies that F is a bounded function (by (2.7)). Moreover, by (2.7), it is enough to show the desired inequality for F, and thanks to (2.11), it actually suffices to show it for G. As G is one of the κ_i's, the result follows from the induction hypothesis. This yields (2.5) for $\widetilde{h}$, establishing (5). □

Remark 2.8. (1) As in [V1], the φ_i's could be expressed as quotients of sums of products of powers of the monomial t and distances to some subsets of the link.

(2) Observe that in the proof of the above the induction hypothesis is stronger than the theorem since we have proved the Lipschitz conic structure of finitely many sets simultaneously and that the homeomorphism is defined on the ambient space as well.

(3) Denote by ρ_X the Riemannian norm induced by the ambient space on X_{reg}. Condition (3) of definition 2.6 clearly implies the following:

$$h^*\rho_X^2 \approx dt^2 + \sum_{i=1}^{m-1} \varphi_i^2(t;x)\cdot\lambda_i^2(x), \tag{2.13}$$

for $(t;x)$ in a dense subset of $(0;\varepsilon)\times L(x_0;X_{reg})$. As the φ_i's are bounded below and above far away from $\{t=0\}$ we see that the above mapping h is thus a quasi-isometry on any closed subset of X_{reg} disjoint from $\{t=0\}$.

Theorem 2.9. *Let $x_0 \in X \subset \mathbb{R}^n$ and set $\rho(x) := |x-x_0|$. There exists $\varepsilon>0$ such that ρ is bi-Lipschitz trivial above $[\nu;\varepsilon]$ for any $0<\nu<\varepsilon$, i. e. for every $\nu>0$ we can find a bi-Lipschitz homeomorphism*

$$h : \rho^{-1}([\nu;\varepsilon]) \to \rho^{-1}(\varepsilon)\times[\nu;\varepsilon],$$

with $\pi_2(h(x)) = \rho(x)$, where $\pi_2 : \rho^{-1}(\varepsilon)\times[\nu;\varepsilon] \to [\nu;\varepsilon]$ is the projection onto the second factor.

This theorem is a particular case of the bi-Lipschitz version of Hardt's Theorem proved in [V1]. This is also easy to derive from the proof of Theorem 2.7. The subanalycity of the isotopy will be useful in section 3.4 (recall that, except the differential forms, everything is implicitly assumed to be subanalytic).

3. L^1 COHOMOLOGY GROUPS

In this section $M \subset \mathbb{R}^n$ stands for a bounded (subanalytic) submanifold. Such a manifold has a natural structure of Riemannian manifold giving rise to a measure on M that we denote $dvol_M$. Below, the word L^1 will always mean L^1 with respect to this measure.

The L^1 cohomology. As we said in the introduction, the L^1 **forms on** M are the forms ω on M satisfying (1.1). We denote by $(\Omega^{\bullet}_{(1)}(M); d)$ the differential complex constituted by the C^∞ L^1 forms ω such that $d\omega$ is L^1.

The L^1 cohomology groups, denoted $H^j_{(1)}(M)$ are the cohomology groups of the differential complex $(\Omega^{\bullet}_{(1)}(M); d)$.

We endow this de Rham complex with the natural norm:

$$|\omega|_1 := \int_M |\omega|\, dvol_M + \int_M |d\omega|\, dvol_M.$$

In this section we prove some preliminary results about L^1 cohomology that we shall need to establish our de Rham theorem in the next section.

3.1. L^1 **cohomology with compact support.** We now define the L^1 forms with compact support. We prove some basic facts, relying on the bi-Lipschitz triviality result presented in Theorem 2.9. Let us point out that our notion of forms with compact support is slightly different that the usual one since we allow the forms to be nonzero near the singularities of $cl(M)$. The support is indeed a subset of $cl(M)$.

Let $M \subset \mathbb{R}^n$ be a submanifold and let $X := cl(M)$.

Definitions 3.1. Let U be an open subset of M and let $V \supset U$ be an open subset of X. Let ω be a differential form on U. The **support of** ω **in** V is the closure in V of the set constituted by the points of U at which ω is nonzero.

We denote by $\Omega^j_{(1),V}(U)$ the C^∞ j-forms ω on U with compact support in V such that ω and $d\omega$ are L^1, and by $H^j_{(1),V}(U)$ the resulting cohomology groups.

For instance $\Omega^j_{(1),X}(M)$ stands for the L^1 j-forms (with an L^1 derivative) having compact support in X. Such forms have to be zero in a neighborhood of infinity (in M). However, they need not to be zero near the points of δM.

3.2. **Weakly differentiable forms.** The homeomorphism that we constructed in Theorem 2.7 is not smooth. Thus, we will need to work with weakly differentiable forms, just differentiable as currents. Therefore, the first step is to prove that the bounded weakly differentiable forms give rise to the same cohomology theory. We will follow an argument similar to the one used by Youssin in [Y].

Given a smooth manifold M (possibly with boundary), we denote by $\Omega^j_{0,\infty}(M)$ the set of C^∞ j-forms on M with compact support (in M).

Definition 3.2. Let U be an open subset of $\mathbb{R}^n$. A differential j-form α on U is called **weakly differentiable** if there exists a $(j+1)$-form ω such that for any form $\varphi \in \Omega^{n-j-1}_{0,\infty}(U)$:

$$\int_U \alpha \wedge d\varphi = (-1)^{j+1} \int_U \omega \wedge \varphi.$$

The form ω is then called **the weak exterior derivative of** α and we write $\omega = \overline{d}\alpha$. A continuous differential j-form α on M is called **weakly differentiable** if it gives rise to weakly differentiable forms via the coordinate systems of M.

We denote by $\overline{\Omega}^j_{(1)}(M)$ the set of measurable weakly differentiable j-forms, locally bounded in M, which are L^1 and which have an L^1 weak exterior derivative. Together with $\overline{d}$, they constitute a cochain complex. We denote by $\overline{H}^j_{(1)}(M)$ the resulting cohomology groups.

We endow this de Rham complex with the corresponding norm:

$$|\omega|_1 := \int_M |\omega|\, dvol_M + \int_M |\overline{d}\omega|\, dvol_M.$$

Similarly, we may introduce the theory of **weakly differentiable L^1 forms with compact support in V** that we shall denote $\overline{\Omega}^j_{(1),V}(U)$ and $\overline{H}^j_{(1),V}(U)$ (see definition 3.1).

In the case of compact smooth manifolds it is easily checked that the two cohomology theories coincide:

Lemma 3.3. *If K is a smooth compact manifold (possibly with boundary) then:*

$$\overline{H}^j_{(1)}(K) \simeq H^j(K). \tag{3.14}$$

Proof. The proof follows the classical argument. As in the case of smooth forms (see for instance [BT]) it is enough to show Poincaré Lemma. Both of the above cohomology theories are invariant under smooth homotopies. Any point of K has a smoothly contractible neighborhood. As K is compact, locally L^1 implies L^1. □

We now are going to see that the isomorphism also holds in the noncompact case:

Proposition 3.4. *Let $M \subset \mathbb{R}^n$ be a C^∞ submanifold and let V open in $cl(M)$. The inclusions $\Omega^\bullet_{(1)}(M) \hookrightarrow \overline{\Omega}^\bullet_{(1)}(M)$ and $\Omega^\bullet_{(1),V}(V\cap M) \hookrightarrow \overline{\Omega}^\bullet_{(1),V}(V\cap M)$ induce isomorphisms between the cohomology groups.*

Proof. As the proof is the same for the two inclusions, we shall focus on the former one. It is enough to show that, for any form $\alpha \in \overline{\Omega}^j_{(1)}(M)$ with $\overline{d}\alpha \in \Omega^{j+1}_{(1)}(M)$ (i. e. α is weakly smooth and $\overline{d}\alpha$ is smooth), there exists $\theta \in \overline{\Omega}^{j-1}_{(1)}(M)$ such that $(\alpha + \overline{d}\theta)$ is C^∞. For this purpose, we prove by induction on i the following statements.

($\mathbf{H}_i$) Fix a form $\alpha \in \overline{\Omega}^j_{(1)}(M)$ with $\overline{d}\alpha \in \Omega^{j+1}_{(1)}(M)$. Consider an exhaustive sequence of compact smooth manifolds with boundary $K_i \subset M$ such that for each i, K_i is included in the interior of K_{i+1} and $\cup K_i = M$. Then, for any integer i, there exists a closed form $\theta_i \in \overline{\Omega}^{j-1}_{(1)}(M)$ such that *supp* $\theta_i \subset Int(K_i) \setminus K_{i-2}$ and $|\theta_i|_1 \leq \frac{1}{2^i}$ and such that $\alpha_i := \alpha + \sum_{k=1}^i \overline{d}\theta_k$ is smooth in a neighborhood of K_{i-1}.

Before proving these statements observe that $\theta = \sum_{i=1}^\infty \theta_i$ is the desired exact form (this sum is locally finite).

Let us assume that θ_{i-1} has been constructed, $i \geq 1$ (we may set $K_0 = K_{-1} = K_{-2} = \emptyset$). Observe that by (3.14), there exists a smooth form $\beta \in \Omega^{(j-1)}_{(1)}(K_i)$ such that $d\beta = \overline{d}\alpha$. This means that $(\alpha_{i-1} - \beta)$ is $\overline{d}$-closed, and by (3.14) there is a smooth form $\beta' \in \Omega^{(j-1)}_{(1)}(K_i)$ such that

$$\alpha_{i-1} - \beta = \beta' + \overline{d}\gamma,$$

with $\gamma \in \overline{\Omega}^{(j-2)}_{(1)}(K_i)$ (if $j = 1$ then $\alpha_{i-1} - \beta$ is constant and then smooth). Thanks to the induction hypothesis there exists an open neighborhood V of K_{i-2} on which α_{i-1} is smooth. This implies that $\overline{d}\gamma$ is smooth on V. Therefore, by induction, we know that we can add an exact form $d\sigma$ to γ to get a form smooth on V. Multiplying σ by a function with support in V which is 1 in a neighborhood W of K_{i-2}, we get a form σ' on M such that $(\overline{d}\sigma' + \gamma)$ is smooth on W. This means that we can assume that γ is smooth on an open neighborhood W of K_{i-2} possibly replacing γ by $(\overline{d}\sigma' + \gamma)$. We will assume this fact without changing notations.

By means of a convolution product with bump functions, for any $\varepsilon > 0$, we may construct a smooth form γ_ε such that $|\gamma_\varepsilon - \gamma|_1 \leq \varepsilon$.

Consider a smooth function ϕ which is 1 on a neighborhood of $(M \setminus W) \cap K_{i-1}$ and with support in $int(K_i) \setminus K_{i-2}$. Then set:

$$\theta_i(x) := \phi(x)(\gamma_\varepsilon - \gamma)(x).$$

If ε is chosen small enough $|\theta_i|_1 + |d\theta_i|_1 \leq \frac{1}{2i}$. On a neighborhood of $(M \setminus W) \cap K_{i-1}$, because $\phi \equiv 1$, we have $\alpha_{i-1} + \overline{d}\theta_i = \beta + \beta' + d\gamma_\varepsilon$ which is smooth. The form $(\alpha_{i-1} + \overline{d}\theta_i)$ is smooth on W as well since α_{i-1} and θ_i are both smooth. □

3.3. **Weakly smooth forms and bi-Lipschitz maps.** Given two open subsets of $\mathbb{R}^n$, it is well known that any subanalytic map $h : U \to V$ is smooth almost everywhere. Therefore, any form ω on V may be pulled-back to a form $h^*\omega$ on U, defined almost everywhere.

We are going to see that in the case where h is locally bi-Lipschitz then the pull-back of a smooth form is weakly smooth (Proposition 3.7).

Definition 3.5. Let Σ be a stratification of $U \subset \mathbb{R}^k$ and let $h : U \to \mathbb{R}^n$ be smooth on strata. The map h is **horizontally C^1 (with respect to Σ)** if, for any sequence $(x_l)_{l\in\mathbb{N}}$ in a stratum S of Σ tending to some point x in a stratum S' and for any sequence $u_l \in T_{x_l}S$ tending to a vector u in T_xS', we have

$$\lim d_{x_l}h_{|S}(u_l) = d_x h_{|S'}(u).$$

Horizontally C^1 maps have been introduced by David Trotman and Claudio Murolo in [MT]. They will be useful to show that the pull-back of a weakly differentiable L^1 form by a subanalytic bi-Lipschitz map (not everywhere smooth) is weakly differentiable.

The following lemma will be needed. Similar results were proved in [SV] where the theory of stratified forms is investigated and a de Rham type theorem for these forms is proved.

Lemma 3.6. *Let $h : U \to \mathbb{R}^m$ be a Lipschitz map. There exists a stratification of U such that h is horizontally C^1 with respect to this stratification.*

Proof. Consider a Whitney (a) stratification Σ_h of the graph of h (see for instance [BCR, DS] for the definition of the Whitney (a) condition and the construction of such a stratification). Let π_1 (resp. π_2) be the projection on the source (resp. target) axis of h. The image of Σ_h under π_1 gives rise to a stratification Σ of U. Let us prove that h is horizontally C^1 with respect to this stratification. Fix a stratum S of this stratification, a sequence $x_l \in S$ tending to x belonging to a stratum S', as well as a sequence $u_l \in T_{x_l}S$

of vectors tending to some $u \in T_x S'$. Let Z be the stratum which projects onto S via π_1. For every l, there is a unique vector $v_l \in T_{(x_l;h(x_l))}Z$ which projects onto u_l. As h is Lipschitz the norm of v_l is bounded above and we may assume that v_l is converging to a vector v. The vector v then necessarily projects onto u.

We claim that v is tangent to the stratum Z' of Σ_h containing $(x; h(x))$. Indeed, if otherwise, there would be a vector w in $\tau = \lim T_{(x_l;h(x_l))}Z$ such that $(w - v)$ lies in the kernel of π_1, in contradiction with the fact that h is Lipschitz (the graph of Lipschitz map may no have a vertical limit tangent vector). This shows the claim, and consequently:

$$\lim d_{x_l} h_{|S}(u_l) = \lim \pi_2(v_l) = \pi_2(v) = d_x h_{|S'}(u),$$

since v is tangent to Z'. □

We shall need the following fact on subanalytic homeomorphisms. It seems that it could be improved but this will be enough for our purpose.

Proposition 3.7. *Let U be an open subset of $\mathbb{R}^n$ and let ω be a bounded weakly differentiable form on U with $\overline{d}\omega$ bounded. If $h : U \to V$ is a locally bi-Lipschitz map, then $h^*\omega$ is weakly differentiable and $\overline{d}h^*\omega = h^*\overline{d}\omega$, almost everywhere.*

Proof. Take $\varphi \in \Omega^{m-j}_{0,\infty}(U)$.

First case: assume that ω is smooth. Let ρ be the function defined by the distance to the boundary of U and set $U^\varepsilon := \{\rho \geq \varepsilon\}$.

By Lemma 3.6, h is horizontally C^1 with respect to some stratification of U. Consequently, the forms $h^*\omega$ and $h^*d\omega$ are continuous at almost every point of $cl(U^\varepsilon)$ (it is a manifold with boundary a. e.). Hence, so are $h^*\omega \wedge \varphi$ and $h^*d\omega \wedge \varphi$. The form $h^*\omega$ is smooth almost everywhere. By Stokes' Formula for stratified forms [SV] (see also [L2]),

$$\int_{U^\varepsilon} d(h^*\omega \wedge \varphi) = \int_{\rho=\varepsilon} h^*\omega \wedge \varphi = 0,$$

for $\varepsilon > 0$ small enough, since φ has compact support in U.

Now, integrating by parts we have for $\varepsilon > 0$ small enough:

$$(-1)^{j+1} \int_U h^*\omega \wedge d\varphi = \int_U dh^*\omega \wedge \varphi - \int_U d(h^*\omega \wedge \varphi) = \int_U dh^*\omega \wedge \varphi.$$

This completes the proof of our first case.

In general, if ω is not smooth but just weakly smooth, as φ is smooth and h^{-1} bi-Lipschitz, $h^{-1*}d\varphi$ is weakly smooth (by the *First case* applied to φ and h^{-1}) and we may write:

$$\int_U h^*\omega \wedge d\varphi = \int_V \omega \wedge h^{-1*}d\varphi = \int_V \omega \wedge \overline{d}h^{-1*}\varphi,$$

and, again integrating by parts:

$$\int_V \omega \wedge \overline{d}h^{-1*}\varphi = (-1)^{j+1} \int_V \overline{d}\omega \wedge h^{-1*}\varphi = (-1)^{j+1} \int_U h^*\overline{d}\omega \wedge \varphi.$$

□

3.4. **Subanalytic bi-Lipschitz maps and L^1 cohomology.** In general, if $f : M \to N$ is a weakly smooth map between smooth manifolds and if ω is a L^1 form on M then $f^*\omega$ is not necessarily a L^1 form on N, even if f has bounded derivatives. Nevertheless, if f is a diffeomorphism and if $|d_x f^{-1}|$ is bounded above then the pullback of a L^1 form is L^1.

In particular, if f is a subanalytic bi-Lipschitz map, by Proposition 3.7, $h^*\omega$ is a weakly smooth L^1 form (it is well defined almost everywhere). This means that any subanalytic bi-Lipschitz map $h : M \to N$ induces some maps

$$h^{*\bullet} : \overline{\Omega}^\bullet_{(1)}(N) \to \overline{\Omega}^\bullet_{(1)}(M),$$

pulling-back the forms. These mappings induce mappings in cohomology which are obviously isomorphisms since h is invertible.

Fix a C^∞ submanifold $M \subset \mathbb{R}^n$. Let $x_0 \in cl(M)$, and set $M^\varepsilon := B^n(x_0;\varepsilon)$ as well as $N^\varepsilon := M \cap S^{n-1}(x_0;\varepsilon)$.

Proposition 3.8. *For any ε positive small enough, there exists a fundamental system of neighborhoods $(U_i)_{i\in\mathbb{N}}$ of N^ε such that:*

$$H^\bullet_{(1)}(U_i \cap M^\varepsilon) \simeq H^\bullet_{(1)}(L(x_0;M)).$$

Proof. By Proposition 3.4, it is enough to show the result for the L^1 cohomology of weakly smooth forms. Apply Theorem 2.9 to $cl(M)$. Then set

$$U_i := \rho^{-1}((\varepsilon - \frac{\varepsilon}{2i}; 2\varepsilon)),$$

for i positive integer (with the notations of the latter theorem). Now the bi-Lipschitz homeomorphism provided by Theorem 2.9 induces an isomorphism (as explained in the paragraph preceding the proposition) between $\overline{H}^j_{(1)}(U_i \cap M^\varepsilon)$ and $\overline{H}^j_{(1)}(N^\nu \times (\varepsilon - \frac{\varepsilon}{2i}; \varepsilon))$, for any $\nu \in (\varepsilon - \frac{\varepsilon}{2i}; \varepsilon)$. It is a routine to check that the latter is isomorphic to $\overline{H}^j_{(1)}(N^\nu)$. □

Remark 3.9. We recall that the link is defined as the intersection of the set with a little sphere, say that it is N^ν. In the above proposition, the isomorphism is induced by restriction. Of course, the restriction of a L^1 form on M^ε has no reason to give rise to a L^1 form on N^ν but every class has a representative which is L^1 in restriction to N^ν, since the isomorphism

$$\overline{H}^j_{(1)}(N^\nu) \simeq \overline{H}^j_{(1)}(N^\nu \times (\varepsilon - \frac{\varepsilon}{2i}; \varepsilon))$$

involved in the above proof is itself induced by the restriction.

3.5. **An exact sequence nearby singularities.** The letter $M \subset \mathbb{R}^n$ still stands for a C^∞ submanifold. We shall point out an exact sequence nearby a singular point of the closure of M. Fix $x_0 \in X$ and set $M^\varepsilon := B^n(x_0;\varepsilon) \cap M$, $N^\varepsilon := S^{n-1}(x_0;\varepsilon) \cap N$ as well as $X^\varepsilon := B^n(x_0;\varepsilon) \cap X$.

By Proposition 3.8, for any ε small enough, there is a basis of neighborhoods $(U_i)_{i\in\mathbb{N}}$ of N^ε for which the restriction map (see remark 3.9) induces an isomorphism for every i:

$$H^j_{(1)}(U_i) \simeq H^j_{(1)}(N^\varepsilon). \tag{3.15}$$

Denote by $\hat{\Omega}^j_{(1)}(N^\varepsilon)$ the direct limit of $\Omega^j_{(1)}(U \cap M)$ where U runs over all the neighoborhoods of N^ε. Denote by $\hat{H}^j_{(1)}(N^\varepsilon)$ the resulting cohomology (these groups are indeed isomorphic to $H^j_{(1)}(N^\varepsilon)$ thanks to Proposition 3.8). The short exact sequences

$$0 \to \Omega^\bullet_{(1),X^\varepsilon}(M^\varepsilon) \to \Omega^\bullet_{(1)}(M^\varepsilon) \to \hat{\Omega}^\bullet_{(1)}(N^\varepsilon) \to 0,$$

give rise to the following long exact sequence:

$$\dots \to \hat{H}^{j-1}_{(1)}(N^\varepsilon) \to H^j_{(1),X^\varepsilon}(M^\varepsilon) \to H^j_{(1)}(M^\varepsilon) \to \dots .$$

Similarly let $C^\bullet_{X^\varepsilon}(M^\varepsilon)$ be the singular cohomology with compact support in X^ε, i. e. the singular cochains of M^ε whose support does not meet any neighborhood of $S^{n-1}(x_0;\varepsilon)$. Consider now the mappings:

$$\psi^\bullet_{M^\varepsilon,X^\varepsilon} : \Omega^\bullet_{(1),X^\varepsilon}(M^\varepsilon) \to C^\bullet_{X^\varepsilon}(M^\varepsilon),$$

obtained in the same way as $\psi^\bullet_{X^\varepsilon}$, by integrating the L^1 differential forms on simplices.

The above exact sequence, together with the analogous exact sequence in singular cohomology, provide the following commutative diagram:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H^j_{(1),X^\varepsilon}(M^\varepsilon) & \longrightarrow & H^j_{(1)}(M^\varepsilon) & \longrightarrow & H^j_{(1)}(N^\varepsilon) & \longrightarrow & H^{j+1}_{(1),X^\varepsilon}(M^\varepsilon) & \longrightarrow \dots \\
& \downarrow \psi^j_{M^\varepsilon,X^\varepsilon} & & \downarrow \psi^j_{M^\varepsilon} & & \downarrow \psi^j_{N^\varepsilon} & & \downarrow \psi^{j+1}_{M^\varepsilon,X^\varepsilon} & \\
\dots \longrightarrow & H^j_{X^\varepsilon}(M^\varepsilon) & \longrightarrow & H^j(M^\varepsilon) & \longrightarrow & H^j(N^\varepsilon) & \longrightarrow & H^{j+1}_{X^\varepsilon}(M^\varepsilon) & \longrightarrow \dots
\end{array}$$

diag. 1.

4. Proof of the de Rham theorem for L^1 cohomology.

The first step of the proof of Theorem 1.1 is to compute the cohomology groups locally. This requires to construct some homotopy operators and describe their properties.

The letter $M \subset \mathbb{R}^n$ still stands for a bounded submanifold. Set $X := cl(M)$ and take $x_0 \in X$. Set again for simplicity $M^\varepsilon := M \cap B^n(x_0;\varepsilon)$ and $N^\varepsilon := M \cap S^{n-1}(x_0;\varepsilon)$ as well as $X^\varepsilon := B^n(x_0;\varepsilon) \cap X$ (we do not match x_0 since it is arbitrary).

4.1. **Some operators on weakly smooth forms.** For $\varepsilon > 0$ small enough and $j > 0$ fixed, we are going to construct operators for weakly smooth forms.

For this purpose, apply Theorem 2.7 to X at x_0. Let $h : (0;\varepsilon) \times N^\varepsilon \to M^\varepsilon$ be the homeomorphism described in definition 2.6 and fix $\omega \in \overline{\Omega}^j_{(1)}(M^\varepsilon)$ with $j \geq 0$ (where ε is also provided by definition 2.6). Set now $Z := (0;\varepsilon) \times N^\varepsilon$.

We may define two forms ω_1 and $\omega_2 \in \overline{\Omega}^j_{(1)}(Z)$ by:

$$h^*\omega(t;x) := \omega_1(t;x) + dt \wedge \omega_2(t;x),$$

where ω_1 and ω_2 do not involve the differential term dt. The forms ω_1 and ω_2 are indeed only defined for almost every $(t;x) \in Z$.

Next, we set for almost every $(t;x) \in Z$ and $0 < \nu \leq \varepsilon$:

$$\alpha(t;x) := \int_\nu^t \omega_2(s;x)ds, \tag{4.16}$$

and

$$\mathcal{K}_\nu \omega := h^{-1*}\alpha. \tag{4.17}$$

We first show that $\mathcal{K}_\nu$ preserves the weakly smooth forms.

The mapping π_ν. Given $\omega \in \overline{\Omega}^j_{(1)}(M^\varepsilon)$ and $\nu \leq \varepsilon$, let $\pi_\nu := h \circ P_\nu \circ h^{-1}$, where $P_\nu(t;x) := (\nu;x)$. Given a differential form ω on M^ε we will denote by $\pi_\nu^*\omega$ the form given by the pull-back of ω by means of $\pi_\nu : M^\varepsilon \to M^\varepsilon$.

Lemma 4.1. *For M as above, $\mathcal{K}_\nu$ preserves the weakly smooth forms and satisfies on $\overline{\Omega}^j_{(1)}(M^\varepsilon)$:*

$$\overline{d}\mathcal{K}_\nu - \mathcal{K}_\nu\overline{d} = Id - \pi_\nu^*, \tag{4.18}$$

Proof. Take ω in $\overline{\Omega}^j_{(1)}(M^\varepsilon)$ and let us fix a form $\varphi \in \Omega^{m-j}_{0,\infty}(M)$. Let h be as above.

The mapping h is locally bi-Lipschitz in $h^{-1}(M^\varepsilon)$ (see Remark 2.8 (3)). By Proposition 3.7, the form $h^*\omega$ is weakly differentiable and $\overline{d}h^*\omega = h^*\overline{d}\omega$ and the same is true for φ. Let α be the form defined in (4.16) and set $\psi := h^*\varphi$. It is enough to show:

$$(-1)^j\int_Z \alpha\wedge\overline{d}\psi = \int_Z h^*\mathcal{K}_\nu d\omega\wedge\psi + \int_Z h^*\omega\wedge\psi - \int_Z h^*\pi_\nu^*\omega\wedge\psi.$$

For this purpose, note that we have (for relevant orientations):

$$\begin{aligned}(-1)^j\int_Z \alpha\wedge\overline{d}\psi &= (-1)^j\int_0^\varepsilon\Big(\int_{[\nu;t]\times N^\varepsilon} h^*\omega\wedge\overline{d}\psi\Big)\,dt\\ &= \int_0^\varepsilon\int_{[\nu;t]\times N^\varepsilon}\overline{d}h^*\omega\wedge\psi - \int_0^\varepsilon\int_{[\nu;t]\times N^\varepsilon}\overline{d}(h^*\omega(s;x)\wedge\psi(t;x)),\end{aligned}$$

(integrating by parts) and therefore if $\Delta_\nu := \{(s;t) : \nu \leq s \leq t < \varepsilon \text{ or } 0 < t \leq s \leq \nu\}$ we have:

$$(-1)^j\int_Z \alpha\wedge\overline{d}\psi = \int_Z h^*\mathcal{K}_\nu d\omega\wedge\psi - \int_{N^\varepsilon}\int_{\Delta_\nu}\overline{d}(h^*\omega(s;x)\wedge\psi(t;x)).$$

But, since ψ has compact support in M^ε, by Stokes' formula we have:

$$\int_{N^\varepsilon}\int_{\Delta_\nu}\overline{d}(h^*\omega(s;x)\wedge\psi(t;x)) = \int_Z h^*\pi_\nu^*\omega\wedge\psi - \int_Z h^*\omega\wedge\psi.$$

Together with the preceding equality this implies that

$$(-1)^j\int_Z \alpha\wedge\overline{d}\psi = \int_Z h^*\omega\wedge\psi + \int_Z h^*\mathcal{K}_\nu d\omega\wedge\psi - \int_Z h^*\pi_\nu^*\omega\wedge\psi,$$

as required. □

The homotopy operator $\mathcal{K}$. We derive from $\mathcal{K}_\nu$ a local homotopy operator $\mathcal{K}$.

Let $\varepsilon > 0$ be as above and let $j > 0$. We just saw that $\mathcal{K}_\nu$ preserves the weakly smooth forms. Observe that if ω has compact support in X^ε then $h^*\omega(\nu; x)$ is zero for $\nu < \varepsilon$ sufficiently close to ε. Therefore $\mathcal{K}_\nu$ induces an operator:

$$\mathcal{K} : \overline{\Omega}^j_{(1),X^\varepsilon}(M^\varepsilon) \to \overline{\Omega}^{j-1}_{(1),X^\varepsilon}(M^\varepsilon),$$

defined by the stationary limit $\mathcal{K}\omega := \lim_{\nu\to\varepsilon} \mathcal{K}_\nu$. Below we describe the properties of $\mathcal{K}$.

Proposition 4.2. *For M as above, $\mathcal{K}$ is a homotopy operator, in the sense that:*

$$\overline{d}\mathcal{K} - \mathcal{K}\overline{d} = Id, \tag{4.19}$$

bounded for the L^1 norm and satisfying for $j < m$:

$$\lim_{t\to 0} \int_{N^t} |\mathcal{K}\omega| = 0, \tag{4.20}$$

for any $\omega \in \overline{\Omega}^j_{(1),X^\varepsilon}(M^\varepsilon)$.

Proof. As observed in the paragraph preceding the proposition, if ω has compact support in X^ε then $h^*\omega(\nu; x)$ vanishes near $\nu = \varepsilon$, and thus $\pi_\nu^*\omega$ is zero if ν is sufficiently close to ε. As a matter of fact, equality (4.19) follows from (4.18). We have to check that $\mathcal{K}$ is bounded for the L^1 norm and show (4.20).

Some notations. We shall write $\mathcal{I}_k^m$ for the set all the multi-indices $I = (i_1, \dots, i_k)$ with $0 < i_1 < \dots < i_k < m$. Given $I \in \mathcal{I}_k$ we shall write $\hat{I}$ for the multi-index of $\mathcal{I}_{m-1-k}$ such that $I \cup \hat{I} = \{1, \dots, m-1\}$. Let $\lambda_1, \dots, \lambda_{m-1}$ be the tame basis of 1-forms provided by definition 2.6 (on a dense subset N' of N^ε) and set for any multi-index $\lambda_I := \lambda_{i_1} \wedge \dots \wedge \lambda_{i_k}$.

We now are going to show that the operator $\mathcal{K}$ is bounded for the L^1 norm.

As $(\lambda_1; \dots; \lambda_{m-1})$ is a tame basis of N^ε we may decompose $\alpha := \sum_{I\in\mathcal{I}_{j-1}} \alpha_I \lambda_I$ (where α is the form defined in (4.16)) and observe that by (3) of definition 2.6

$$|\mathcal{K}\omega| = |\sum_{I\in\mathcal{I}_{j-1}} h^{-1*}\alpha_I| \sim \sum_{I\in\mathcal{I}_{j-1}} \frac{|\alpha_I \circ h^{-1}|}{\varphi_I \circ h^{-1}}, \tag{4.21}$$

where $\varphi_I = \varphi_{i_1} \cdots \varphi_{i_k}$, and consequently it is enough to show that all the summands of the right hand side are L^1 on M^ε. Changing variables by means of h, this amounts to show that for any $I \in \mathcal{I}_k$:

$$\int_Z |\alpha_I| \cdot \frac{J_h}{\varphi_I} < \infty,$$

where J_h stands for the absolute value of the Jacobian determinant of h.

Alike, decompose

$$\omega_2 = \sum_{I\in\mathcal{I}_{j-1}} \omega_{2,I}\lambda_I,$$

(recalled that we decomposed $h^*\omega := \omega_1 + dt \wedge \omega_2$). As $(\lambda_1; \dots; \lambda_{m-1})$ is a tame basis of N^ε we have$|\omega_2| \sim \sum_{I\in\mathcal{I}_{j-1}} |\omega_{2,I}|$. For the same reasons as in (4.21):

$$|\omega_{2,I}(s; x)| \le C|\omega(h(s; x))| \cdot \varphi_I(s; x). \tag{4.22}$$

By (3) of definition 2.6 we have on Z:

$$\varphi_I \cdot \varphi_{\hat{I}} \sim J_h \tag{4.23}$$

Put $Y^t := \{t\} \times N^\varepsilon$. There is a constant C such that for almost every t and any $I \in \mathcal{I}_k$:

$$\begin{aligned}
\int_{Y^t} \frac{|\alpha_I|}{\varphi_I} \cdot J_h &\leq C \int_{x \in N^\varepsilon} |\alpha_I(t;x)| \cdot \frac{J_h(t;x)}{\varphi_I(t;x)} \\
&\leq C \int_{N^\varepsilon} |\alpha_I(t;x)| \cdot \varphi_{\hat{I}}(t;x) \quad \text{(by (4.23))} \\
&\leq C \int_{N^\varepsilon} \int_t^\varepsilon |\omega_{2,I}(s;x)| \cdot \varphi_{\hat{I}}(t;x) ds \quad \text{(by (4.16))} \\
&\leq C \int_{N^\varepsilon} \int_t^\varepsilon |\omega(s;x)| \cdot \varphi_I(s;x) \cdot \varphi_{\hat{I}}(t;x) ds \quad \text{(by (4.22))} \\
&\leq C \int_{N^\varepsilon} \int_t^\varepsilon |\omega(s;x)| \cdot \varphi_I(s;x) \cdot \varphi_{\hat{I}}(s;x) ds
\end{aligned} \tag{4.24}$$

since, by (1) of definition 2.6, $\varphi_{\hat{I}}(t;x)$ is nondecreasing with respect to t. We finally get:

$$\int_{Y^t} \frac{|\alpha_I|}{\varphi_I} \cdot J_h \leq C \int_{(t;\varepsilon) \times N^\varepsilon} |\omega(s;x)| \cdot J_h(s;x) = C \int_{h((t;\varepsilon) \times N^\varepsilon)} |\omega|. \tag{4.25}$$

which is bounded above uniformly in t since ω is a L^1 form, proving that $\frac{\alpha_I}{\varphi_I}$ is integrable. It remains to establish (4.20). For simplicity set

$$f_t(x) = \int_0^\varepsilon |\omega(s;x)| \cdot \varphi_I(s;x) \cdot \varphi_{\hat{I}}(t;x) ds.$$

As $\varphi_{\hat{I}}(t;x)$ is nondecreasing with respect to t, this family of functions is obviously bounded by the L^1 function $\int_0^\varepsilon |\omega(x;s)| \cdot \varphi_I(s;x) \cdot \varphi_{\hat{I}}(s;x) ds$.

Moreover, as $\varphi_{\hat{I}}$ goes to zero as t tends to zero (since $j < m$), we see that the function f_t tends to zero (pointwise) as t goes to zero (by the Lebesgue dominated convergence theorem). Hence, (applying a second time the Lebesgue dominated convergence theorem) we conclude:

$$\lim_{t \to 0} \int_{N^\varepsilon} f_t = 0.$$

By (4.24):

$$\lim_{t \to 0} \int_{Y^t} \frac{|\alpha_I|}{\varphi_I} \cdot J_h \leq C \lim_{t \to 0} \int_{N^\varepsilon} f_t = 0.$$

But, by (4.21) this establishes (4.20). □

Remark 4.3. Notice that equation (4.25) yields that thee is a constant C such that:

$$\int_{N^t} |\mathcal{K}\omega| \leq C|\omega|_1,$$

for any $t \leq \varepsilon$ and any form ω in $\Omega^j_{(1)}(M^\varepsilon)$.

4.2. **Local computations of the L^1 cohomology.** The following proposition may be considered as a "Poincaré Lemma for L^1 cohomology". This is an important step in the proof of Theorem 1.1.

Proposition 4.4. *For $\varepsilon > 0$ small enough we have for every j:*

$$H^j_{(1),X^\varepsilon}(M^\varepsilon) \simeq 0.$$

Proof. For $j = 0$, a closed form with compact support is the zero form and the result is clear. Fix a closed form $\omega \in \Omega^j_{(1),X^\varepsilon}(M^\varepsilon)$ with $j > 0$. Let $\mathcal{K}$ be the homotopy operator constructed in the previous section (see Proposition 4.2). As ω is closed with compact support $\overline{d}\mathcal{K}\omega = \omega$, showing that ω is $\overline{d}$-exact and thus exact by Proposition 3.4. □

4.3. **The mappings ψ^j_M.** As in the case of the classical de Rham theorem (for compact smooth manifolds), the isomorphism is given by integration on simplices. Let us define this natural map. Recall that singular simplices $\sigma : \Delta_j \to M$ are assumed to be subanalytic mappings. Therefore, see [V3] for details, we may define the following maps:

$$\begin{array}{rcl} \psi^j_M : \Omega^j_{(1)}(M) & \to & C^j(M) \\ \omega & \mapsto & [\psi^j_M(\omega) : \sigma \mapsto \int_\sigma \omega]. \end{array}$$

By Stokes' formula for singular simplices [P, SV, V3], this is a cochain map.

4.4. **De Rham theorem for L^1 cohomology.** We are now ready to prove the following theorem, which clearly implies Theorem 1.1.

Theorem 4.5. *The above mappings ψ^j_M induce isomorphisms between the respective cohomology groups for any bounded (subanalytic) manifold M.*

Proof. We prove the theorem by induction on m ($= \dim M$). For $m = 0$ the statement is vacuous.

Define a complex of presheaves on X by $\Omega^j_{(1)}(U) := \Omega^j_{(1)}(U \cap M)$, if U is an open subset of X and denote by $\mathcal{L}^j$ the resulting differential sheaf. This is the sheaf on X of *locally* L^1 forms of M (locally in X). Denote by $\mathcal{H}^\bullet(\mathcal{L}^\bullet)$ the derived complex of sheaves, i. e. the complex of sheaves obtained from the presheaves $H^j(\Omega^\bullet_{(1)}(U))$.

On the other hand, consider the complex of presheaves on X defined by $S^j(U) := C^j(M \cap U)$, for U open set of X, and denote by $\mathcal{S}^\bullet$ the associated complex of sheaves.

As the $\mathcal{L}^j$'s are soft sheaves, they are acyclic and it follows from the theory of spectral sequences (see for instance [B] IV Theorem 2.2) that, if the sheaf-mappings $\psi^j : \mathcal{H}^j(\mathcal{L}^\bullet) \to \mathcal{H}^j(\mathcal{S}^\bullet)$, induced by the morphisms of complexes of presheaves $\psi^j_U : \Omega^j_{(1)}(U) \to C^j(U)$, are all isomorphisms, then the mappings ψ^j_M must induce an isomorphism between the cohomology groups of the respective global sections of $\mathcal{S}^\bullet$ and $\mathcal{L}^\bullet$. Global sections of $\mathcal{L}^\bullet$ are L^1, since, as M is bounded, X is compact and then locally L^1 amounts to L^1.

To see that the mappings $\psi^j : \mathcal{H}^j(\mathcal{L}^\bullet) \to \mathcal{H}^j(C^\bullet)$ are all local isomorphisms, we have to show that for every point x_0 in X, the mapping ψ_{X^ε} is an isomorphism for any ε small enough. Indeed, by section 3.5, for any ε small enough, we have the following commutative diagram for any j:

$$\begin{array}{ccc} H^j_{(1)}(M^\varepsilon) & \longrightarrow & H^j_{(1)}(N^\varepsilon) \\ \psi^j_{M^\varepsilon}\downarrow & & \downarrow\psi^j_{N^\varepsilon} \\ H^j(M^\varepsilon) & \longrightarrow & H^j(N^\varepsilon) \end{array}$$

By Proposition 4.4 (see diag 1.), the horizontal arrows are isomorphisms for any ε small enough.

Observe also that N^ε is of dimension less than m. By induction on m, $\psi^j_{N^\varepsilon}$ induces an isomorphism on the cohomology groups and thus, the above commutative diagram clearly shows that the mapping $\psi^j_{M^\varepsilon}$ induces an isomorphism as well for any j. □

5. Poincaré duality for L^1 cohomology

We draw some consequences of Theorem 1.1, stating some duality results between L^1 and L^∞ cohomology. We start by recalling some results and providing basic definitions. We recall that, except the differential forms, all the sets and mappings are assumed to be (globally) subanalytic.

5.1. **Intersection homology.** We recall the definition of intersection homology as it was introduced by Goresky and Macpherson [GM1, GM2].

Definitions 5.1. A subset $X \subset \mathbb{R}^n$ is an m**-dimensional pseudomanifold** if X_{reg} is an m-dimensional manifold which is dense in X and $\dim X_{sing} < m-1$.

A **stratified pseudomanifold** is the data of a pseudomanifold together with a filtration:

$$X_0 \subset \cdots \subset X_{m-2} = X_{m-1} \subset X_m = X,$$

such that $X_i \setminus X_{i-1}$ is either empty or a smooth manifold of dimension i.

Throughout this section, the letter X will denote a stratified pseudomanifold.

A **perversity** is a sequence of integers $p = (p_2, p_3, \dots, p_m)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$. A subspace $Y \subset X$ is called $(p;i)$**-allowable** if $\dim Y \cap X_{m-k} \leq p_k + i - k$. Define $I^pC_i(x)$ as the subgroup of $C_i(X)$ consisting of those chains σ such that $|\sigma|$ is (p,i)-allowable and $|\partial\sigma|$ is $(p, i-1)$-allowable.

The i^{th} **intersection homology group of perversity** p, denoted $I^pH_j(X)$, is the i^{th} homology group of the chain complex $I^pC_\bullet(X)$. The i^{th} **intersection cohomology group of perversity** p, denoted $I^pH^j(X)$, is defined as $Hom(I^pH_j(X);\mathbb{R})$.

In [GM1, GM2] Goresky and MacPherson have proved that, if the stratification is sufficiently nice (i. e. if topological triviality holds along strata) then these homology groups are finitely generated and independent of the stratification. Since such stratifications exist for subanalytic sets [DS] we will admit this fact and shall work without specifying the stratification.

Furthermore, Goresky and MacPherson also proved that their theory satisfy a generalized version of Poincaré duality. We denote by t **the maximal perversity**, i. e. $t = (0; 1; \dots; m-2)$.

Theorem 5.2. *(Generalized Poincaré duality [GM1, GM2]) Let X be a compact oriented pseudomanifold and let p and q be perversities with $p + q = t$. Then:*

$$I^p H^j(X) = I^q H^{m-j}(X).$$

Example 5.3. We will be interested in the cases of the zero perversity $0 = (0; \dots ; 0)$ and the maximal perversity, which are complement perversities. By the above theorem, we have for any pseudomanifold X of dimension m:

$$I^0 H^j(X) = I^t H^{m-j}(X).$$

5.2. **L^∞-cohomology.** We recall the definition of the L^∞ cohomology groups that have been introduced by the author of the present paper in [V3]. Let $M \subset \mathbb{R}^n$ be a smooth oriented submanifold.

Definition 5.4. We say that a form ω on M is L^∞ if there exists a constant C such that for any $x \in M$:

$$|\omega(x)| \le C.$$

We denote by $\Omega^j_\infty(M)$ the cochain complex constituted by all the C^∞ j-forms ω such that ω and $d\omega$ are both L^∞.

The cohomology groups of this cochain complex are called the **L^∞-cohomology groups of** M and will be denoted by $H^\bullet_\infty(M)$. We may endow this cochain complex with the norm:

$$|\omega|_\infty := \sup_M |\omega| + \sup_M |d\omega|.$$

We also introduce the **locally L^∞ forms** as follows. Given an open subset U of $cl(M)$, let $\Omega^j_{\infty,loc}(U \cap M)$ be the de Rham complex constituted by the smooth forms on $U \cap M$ locally bounded in U which have a locally bounded (in U) exterior derivative. This gives rise to a cohomology theory that we shall denote $H^j_{\infty,loc}(U \cap M)$.

Similarly we define the de Rham complex $\overline{\Omega}^\bullet_\infty(M)$ as the L^∞ forms weakly smooth and almost everywhere continuous.

By Theorem 1.2, we know that the L^∞ cohomology of a pseudomanifold coincides with its intersection cohomology in the maximal perversity. We shall need the following theorem of [V3]. Set again $M^\varepsilon = M \cap B^n(x_0; \varepsilon)$ for some $x_0 \in cl(M)$ fixed.

Theorem 5.5. [V3]*(Poincaré Lemma for L^∞ cohomology) For ε positive small enough and any positive integer j:*

$$H^j_\infty(M^\varepsilon) \simeq 0.$$

5.3. **Poincaré duality for L^1 cohomology.** We give some corollaries of Theorem 1.1. Thanks to Goresky and MacPherson generalized Poincaré duality, we get an explicit topological criterion on the singularity to determine whether L^1 cohomology is Poincaré dual to L^∞ cohomology.

Corollary 5.6. *Let X be a compact oriented pseudomanifold. If $H^j(X_{reg}) \simeq I^0H^j(X)$ then L^∞ cohomology is Poincaré dual to L^1 cohomology in dimension j, i. e.*

$$H^j_\infty(X_{reg}) \simeq H^{m-j}_{(1)}(X_{reg}).$$

Proof. This is a consequence of Theorems 1.1, 1.2 and Goresky and MacPherson's generalized Poincaré duality. □

Corollary 5.7. *Let $M \subset \mathbb{R}^n$ be an oriented bounded C^∞ submanifold. If $\dim \delta M = k$ then L^1 cohomology is Poincaré dual to L^∞ cohomology in dimension $j < m - k - 1$, i. e. for any positive integer $j < m - k - 1$:*

$$H^j_{(1)}(M) \simeq H^{m-j}_\infty(M).$$

Proof. We may assume $k < m - 1$ since otherwise the result is trivial. Set $X = cl(M)$ and observe that X is a pseudomanifold. Fix a Whitney (b) stratification of X (see [BCR, DS] for the construction of such stratifications) such that X is a stratified pseudomanifold. By definition of 0-allowable chains (see section 5.1), the support of a singular chain $\sigma \in I^0C_j(X)$ may not intersect the strata of the singular locus of dimension less than $m - j$. If $j < m - k$ (and hence $k < m - j$) then there is no stratum of dimension bigger or equal to $(m - j)$ and thus $|\sigma|$ must lie entirely in X_{reg} and therefore

$$I^0C_j(X) = C_j(X_{reg}).$$

Hence if $j < m - k - 1$, the same applies to $(j + 1)$ and therefore

$$I^0H_j(X) = H_j(X_{reg}).$$

The result follows from the preceding corollary. □

This corollary clearly implies Corollary 1.3.

6. Lefschetz duality for L^1 cohomology.

We are going to investigate Lefschetz duality. It means that we are going to consider L^1 forms satisfying boundary conditions. Our duality result will relate the cohomology of these forms to the cohomology of L^∞ forms (Theorem 6.8).

We first define and study the de Rham complex of Dirichlet L^1 forms. In section 6.2, we establish Lefschetz duality for L^1 cohomology.

6.1. **Dirichlet L^1-cohomology groups.** In this section, M is an orientable submanifold of $\mathbb{R}^n$ (not necessarily bounded) and X will stand for its topological closure. We are going to consider L^1 forms with compact support. We recall that the support in X of a L^1 form on M is defined as the closure *in* X of the set of points at which this form is nonzero. Let $V \subset X$ be open.

Definition 6.1. We shall say that $\omega \in \overline{\Omega}^j_{(1)}(M)$ has the L^1 **Stokes' property in** V if for any $\alpha \in \overline{\Omega}^{m-j-1}_{\infty,V}(M)$ we have:

$$\int_M \omega \wedge \overline{d}\alpha = (-1)^{j+1} \int_M \overline{d}\omega \wedge \alpha. \tag{6.26}$$

The de Rham complex of weakly smooth L^1 forms of M satisfying this property (and whose weak exterior derivative satisfy this property as well) is called the complex of (weakly smooth) **Dirichlet L^1 forms on** M and is denoted $\overline{\Omega}^j_{(1)}(M; V \cap \delta M)$. The subcomplex of the C^∞ such forms is denoted $\Omega^j_{(1)}(M; V \cap \delta M)$.

As before, we denote by $\overline{\Omega}^j_{(1),X}(M; V \cap \delta M)$ and $\Omega^j_{(1),X}(M; V \cap \delta M)$ the subcomplexes of the forms having compact support in X.

Remark 6.2. If ω has compact support in V and satisfies the L^1 Stokes' property in V then clearly (6.26) holds for any $\alpha \in \overline{\Omega}^{m-j-1}_{\infty}(M)$.

If K denotes a compact manifold with boundary ∂K, the relative de Rham complex of differential forms $\Omega^j(K; \partial K)$ is usually defined as the set of j-forms ω on K such that $\omega_{|\partial K} \equiv 0$. However, the smooth forms of the pair $(K; \partial K)$ may also be characterized as the smooth forms satisfying (6.26) for any smooth L^∞ form α on M. The Dirichlet L^1 cohomology defined above is therefore completely analogous to the one of compact smooth manifolds.

In the case of non-compact manifolds, it is not possible to require that the forms vanish at the singularities since the forms are not defined on δM. If one wants a similar characterization as in the case of compact manifolds with boundaries, we have to require a condition near δM and pass to the limit.

For this purpose, choose an exhaustion function $\rho : X \to \mathbb{R}^+$, that is to say, a positive C^2 function on M tending to zero as we approach δM. Then $\{\rho \geq \varepsilon\}$ is a manifold with boundary $\{\rho = \varepsilon\}$. Given $\omega \in \overline{\Omega}^j_{(1)}(M)$, we may define an operator on $\overline{\Omega}^{m-j-1}_{\infty,X}(M)$ by:

$$l_\omega(\alpha) := \lim_{\varepsilon \to 0} \int_{\rho = \varepsilon} \omega \wedge \alpha, \tag{6.27}$$

for $\alpha \in \overline{\Omega}^{m-j-1}_{\infty,X}(M)$. It is easy to see (by Stokes' formula) that if $\alpha \in \overline{\Omega}^{m-j-1}_{\infty,X}(M)$ and $\omega \in \overline{\Omega}^j_{(1)}(M)$ then the latter limit exists and that:

$$\int_M \omega \wedge \overline{d}\alpha = (-1)^{j+1} \int_M \overline{d}\omega \wedge \alpha + l_\omega(\alpha).$$

In particular the limit in (6.27) is independent of the exhaustion function ρ. Observe also that l_ω is a bounded operator on $(\overline{\Omega}^j_{\infty,X}(M); |.|_\infty)$.

Definition 6.3. Set:

$$|\omega|_{1,\delta} := |l_\omega|,$$

where $|l_\omega|$ denotes the operator norm of l_ω.

Now, it follows from the definitions that if $|\omega|_{1,\delta} = 0$ if and only if the L^1 Stokes' property holds for ω. Hence, we get the following characterization of Dirichlet L^1 forms:

$$\overline{\Omega}^{\bullet}_{(1)}(M;\delta M) = \{\omega \in \overline{\Omega}^{\bullet}_{(1)}(M) : |\omega|_{1,\delta} = |\overline{d}\omega|_{1,\delta} = 0\}. \tag{6.28}$$

This characterization will be very useful to check that the L^1 Stokes property holds later on.

Proposition 6.4. *The inclusions* $\Omega^j_{(1),X}(M;\delta M) \hookrightarrow \overline{\Omega}^j_{(1),X}(M;\delta M)$ *and* $\Omega^j_{(1)}(M;\delta M) \hookrightarrow \overline{\Omega}^j_{(1)}(M;\delta M)$ *induce isomorphisms in cohomology.*

Proof. The argument used in the proof of Proposition 3.4 also applies for Dirichlet cohomology. □

Let $\delta M^\varepsilon := \delta M \cap X^\varepsilon$. We can also make use of Proposition 3.8 in the same way as in section 3.5 to get the following exact sequence:

$$\dots \to H^{j-1}_{(1)}(N^\varepsilon;\delta N^\varepsilon) \to H^j_{(1),X^\varepsilon}(M^\varepsilon;\delta M^\varepsilon) \to H^j_{(1)}(M^\varepsilon;\delta M^\varepsilon) \to \dots \tag{6.29}$$

6.2. **Lefschetz duality for Dirichlet L^1 cohomology and the de Rham theorem.** Let $M \subset \mathbb{R}^n$ be an orientable submanifold of dimension m, set $X = cl(M)$ and take $x_0 \in X$. Set again $M^\varepsilon := M \cap B^n(x_0;\varepsilon)$ and $N^\varepsilon := M \cap S^{n-1}(x_0;\varepsilon)$.

The operator $\mathcal{K}_0$. We are going to construct a homotopy operator:

$$\mathcal{K}_0 : \overline{\Omega}^m_{(1)}(M^\varepsilon;\delta M^\varepsilon) \to \overline{\Omega}^{m-1}_{(1)}(M^\varepsilon;\delta M^\varepsilon),$$

($m = \dim M$) based on the operator $\mathcal{K}_\nu$ introduced in section 4.1.

Proposition 6.5. *On* $\overline{\Omega}^m_{(1)}(M^\varepsilon)$*:*

$$\lim_{\nu,t\to 0} |\mathcal{K}_\nu\omega - \mathcal{K}_t\omega|_1 = 0,$$

and consequently $\lim_{\nu\to 0}\mathcal{K}_\nu$ *defines a homotopy operator* $\mathcal{K}_0 : \overline{\Omega}^m_{(1)}(M^\varepsilon) \to \overline{\Omega}^{m-1}_{(1)}(M^\varepsilon)$.

Proof. Let $\omega \in \overline{\Omega}^m_{(1)}(M^\varepsilon)$. Let h be the homeomorphism used to define $\mathcal{K}_\nu$ (see section 4.1). As ω is an m-form, $h^*\omega$ is L^1. Clearly we have:

$$\lim_{t,\nu\to 0} \int_{M^\varepsilon} |\mathcal{K}_t\omega - \mathcal{K}_\nu\omega| = \lim_{t,\nu\to 0, t\le\nu} \int_t^\nu \int_{N^\varepsilon} |\omega_2| = 0,$$

since, as observed, $h^*\omega$ is L^1 on $h^{-1}(M^\varepsilon)$.

As ω is an m-form, it is identically zero in restriction to N^ν since this is an $(m-1)$-dimensional manifold. Consequently $\pi_\nu^*\omega$ is zero and, as $\overline{d}\omega = 0$, by (4.18) we have:

$$\overline{d}\mathcal{K}_\nu = Id_{\Omega^m_{(1)}(M^\varepsilon)}.$$

Passing to the limit we get that $\mathcal{K}_0\omega$ is weakly differentiable and that:

$$\overline{d}\mathcal{K}_0\omega = \omega,$$

as required. □

Proposition 6.6. *Let $\omega \in \overline{\Omega}^j_{(1)}(M^\varepsilon)$ satisfying the L^1 Stokes' property in X^ε.*

(i) *If $0 < j < m$ and ω has compact support in X^ε then $\mathcal{K}\omega$ satisfies the L^1 Stokes' property in X^ε.*
(ii) *If $j = m$, then $\mathcal{K}_0\omega$ satisfies the L^1 Stokes' property in X^ε.*

Proof. Let $\omega \in \overline{\Omega}^j_{(1),X^\varepsilon}(M^\varepsilon)$ be a form satisfying the L^1 Stokes' property. We have to check that $|\mathcal{K}\omega|_{1,\delta} = 0$ (see (6.28)).

Consider a C^2 nonnegative function $\rho_1(x) : N^\varepsilon \to \mathbb{R}$ zero on δN^ε and positive on N^ε. Set $\rho_2 = \rho_1 \circ h$ and denote by ρ the Euclidian distance to x_0. For μ and ν positive real numbers, let

$$M_{\mu,\nu} := \{x \in M^\varepsilon : \rho_2(x) \geq \mu,\ \rho(x) \geq \nu\}.$$

Then $M_{\mu,\nu}$ is a manifold with corners whose boundary is the union of $\{x \in N^\nu : \rho_2(x) \geq \mu\}$ with

$$W_{\mu,\nu} = \{x \in M^\varepsilon : \rho_2(x) = \mu,\ \rho(x) \geq \nu\}.$$

Define $Z_{\mu,\nu} := \partial W_{\mu\nu}$. Denote by $M'_{\mu,\nu}$, $W'_{\mu,\nu}$ and $Z'_{\mu,\nu}$ the respective images by h^{-1} of $M_{\mu,\nu}$, $W_{\mu,\nu}$ and $Z_{\mu,\nu}$. For the convenience of the reader, we gather all these notations on a picture:

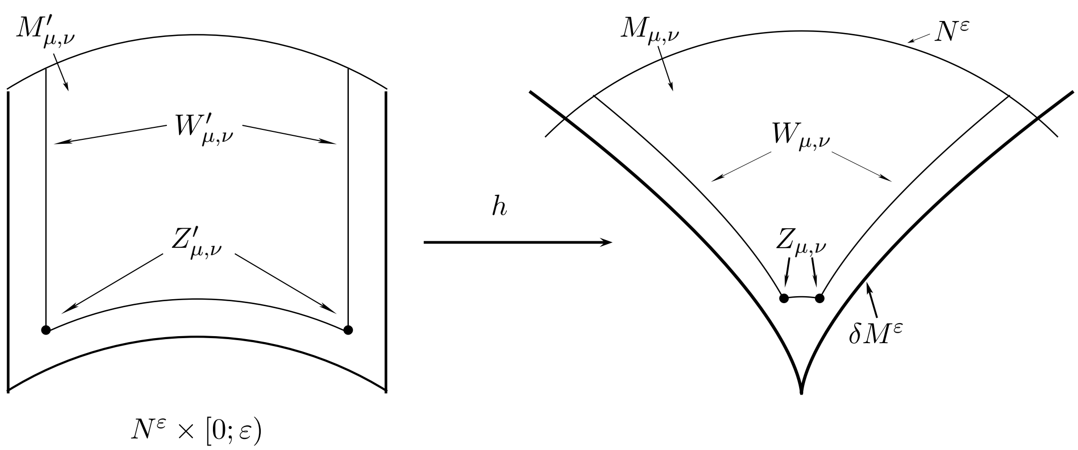

FIGURE 1. The Lipschitz conic structure of M^ε. Here $Z_{\mu,\nu}$ and $Z'_{\mu,\nu}$ are reduced to two points.

Observe that by construction (recall that $\rho(h(t;x)) = t$) we have $W'_{\mu,\nu} = Z'_{\mu,\nu} \times [\mu;\varepsilon)$.

By Proposition 4.2, we already know that:

$$\lim_{t\to 0} \int_{N^t} |\mathcal{K}\omega| = 0.$$

Therefore it is enough to check that for every positive real number ν:

$$\lim_{\mu\to 0_+}\int_{W_{\mu,\nu}} \mathcal{K}\omega\wedge\alpha = 0, \tag{6.30}$$

for any $\alpha \in \overline{\Omega}^{m-j-1}_{\infty,X^\varepsilon}(M^\varepsilon)$.

Fix such a form α. Write $\beta = h^*\alpha$ for simplicity, and decompose $\beta = \beta_1 + dt\wedge\beta_2$ as well as $h^*\omega = \omega_1 + dt\wedge\omega_2$. Observe that:

$$\beta_1\wedge\omega_2 = 0 \qquad \text{on } W'_{\mu,\nu}, \tag{6.31}$$

since this differential $(m-1)$-form does not involve dt.

$$\begin{aligned}
\int_{W_{\mu,\nu}} \mathcal{K}\omega\wedge\alpha &= \int_{(t;x)\in W'_{\mu,\varepsilon}} (\int_{s=t}^{\varepsilon} \omega_2(s;x)ds)\wedge\beta(t;x)\\
&= \int_{x\in Z'_{\mu,\varepsilon}}\int_{t=\nu}^{\varepsilon}\int_{s=t}^{\varepsilon} \omega_2(x;s)\wedge\beta_2(t;x)\,ds\,dt \quad (\text{by } (6.31))\\
&= \int_{Z'_{\mu,\varepsilon}}\int_{s=\nu}^{\varepsilon}\int_{t=\nu}^{s} \omega_2(s;x)\wedge\beta_2(t;x)\,dt\,ds \quad (\text{by Fubini})\\
&= \int_{s=\nu}^{\varepsilon}\int_{Z'_{\mu,\nu}} h^*\omega(x;s)\wedge\int_{t=\nu}^{s}\beta_2(t;x)\,dt. \qquad (6.32)
\end{aligned}$$

Define a form $\mathcal{K}'_\nu\alpha$ on $(0;\varepsilon)\times N^\varepsilon$ by

$$\mathcal{K}'_\nu\alpha(s;x) := \int_{t=\nu}^{s}\beta_2(t;x)\,dt$$

if $s\geq\nu$, and set $\mathcal{K}'_\nu\alpha(s;x)$ to be zero if $s\leq\nu$. By (2) of definition 2.6, h induces a quasi-isometry on $[\nu;\varepsilon)\times N^\varepsilon$ (see Remark 2.8 (3)) and therefore $h^{-1*}\mathcal{K}'_\nu\alpha$ is an L^∞ form. Moreover, in view of (6.32), we clearly have:

$$\int_{W_{\mu,\nu}} \mathcal{K}\omega\wedge\alpha = \int_{W'_{\mu,\nu}} h^*\omega\wedge\mathcal{K}'_\nu\alpha. \tag{6.33}$$

Now, as by definition $\mathcal{K}'_\nu\alpha$ is zero on $\partial M'_{\mu,\nu}\setminus W'_{\mu,\nu}$, this amounts to:

$$\int_{W_{\mu,\nu}} \mathcal{K}\omega\wedge\alpha = \int_{\partial M_{\mu,\nu}} \omega\wedge h^{-1*}\mathcal{K}'_\nu\alpha'$$

which tends to zero as μ goes to zero for ω satisfies the L^1 Stokes property and $\mathcal{K}'_\nu\alpha$ is an L^∞ form (see Remark 6.2), yielding (6.30) and establishing (i).

For a proof of (ii), observe that for any L^∞ $(m-j-1)$-form α with compact support in X^ε:

$$\lim_{t\to 0}\int_{N^t}|\mathcal{K}_0\omega\wedge\alpha| \leq C\lim_{t\to 0}\int_{(0;t)\times N^\varepsilon}|h^*\omega| = 0$$

(with $C = \sup|\alpha|$).

Therefore, like in the proof of (i), it is enough to show (6.30) for $\mathcal{K}_0$. By definition, $\mathcal{K}\omega$ is an $(m-1)$-form with no differential term involving dt. Thus $\mathcal{K}_0\omega$ must be identically zero on $W_{\mu,\nu}$ and consequently (6.30) is trivial in this case. □

Proposition 6.7. *(Poincaré Lemma for Dirichlet L^1 cohomology) For $j < m$ and $\varepsilon > 0$ small enough*

$$H^j_{(1),X^\varepsilon}(M^\varepsilon;\delta M^\varepsilon) \simeq 0 \simeq H^m_{(1)}(M^\varepsilon;\delta M^\varepsilon).$$

Proof. The case $j = 0$ is clear. Let $0 < j < m$ and let $\omega \in \Omega^j_{(1),X^\varepsilon}(M^\varepsilon;\delta M^\varepsilon)$ be a closed form. Then, by the preceding proposition $\mathcal{K}\omega$ satisfies the L^1 Stokes' property. Furthermore, $\overline{d}\mathcal{K}\omega = \omega$ and, by Proposition 6.4, $\mathcal{K}\omega$ satisfies the L^1 Stokes' property. The first isomorphism ensues.

To compute $H^m_{(1)}(M^\varepsilon;\delta M^\varepsilon)$, just use $\mathcal{K}_0$ and (ii) of the preceding proposition exactly in the same way. □

Lefschetz duality for L^1 cohomology. The setting is still the same as in section 6.2.

Theorem 6.8. *The pairing*

$$H^{m-j}_{(1),X}(M;\delta M) \otimes H^j_{\infty,loc}(M) \to \mathbb{R}$$

$$(\alpha;\beta) \mapsto \int_M \alpha \wedge \beta$$

is nondegenerate.

By "nondegenerate" we mean that for any L^1 differential form with compact support β there is a locally L^∞ differential form α such that $\int_M \alpha \wedge \beta = 1$ and for any closed locally L^∞ form α there is a form $\beta \in \Omega^{m-j}_{(1),X}(M;\delta M)$ for which the latter integral is nonzero as well.

Proof. We shall apply an argument which is similar to the one used in the proof of Theorem 4.5. As we may argue by induction on m, we shall assume that the theorem holds for manifolds of dimension $(m-1)$, $m \geq 1$.

Consider the complex of presheaves on X defined by $\Omega^j_{(1),U}(U\cap M;U\cap\delta M)^*$ (where $*$ denotes the algebraic dual vector space), if U is an open subset of X, and denote by $\mathcal{L}^j_{(1)}$ the resulting differential sheaf. Let $\mathcal{H}^\bullet(\mathcal{L}^\bullet_{(1)})$ be the derived sheaves. Similarly, denote by $\mathcal{L}^j_\infty$ the differential sheaf resulting from the presheaf $\Omega^j_{\infty,loc}(U\cap M)$.

For every subset $U \subset X$ and $j \leq m$, consider the mappings

$$\varphi^j_U : \Omega^j_{\infty,loc}(U\cap M) \to \Omega^{m-j}_{(1),U}(M\cap U;\delta M\cap U)^*,$$

defined by $\varphi^j_U(\alpha) : \beta \mapsto \int_{U\cap M} \alpha\wedge\beta$. It follows from the theory of spectral sequences (see for instance [B] IV Theorem 2.2) that, if the mapping of complex of differential sheaves induced by φ^j_U is a local isomorphism, then φ^j_M induces an isomorphism between the cohomology groups of the respective global sections of $\mathcal{L}^{m-j}_{(1)}$ and $\mathcal{L}^j_\infty$, as required.

Thus, we simply have to make sure that the mappings φ^j_U's induce local isomorphisms at any $x_0 \in cl(M)$. Notice that by Theorem 5.5 and Proposition 6.7, this is clear for $j > 0$.

It remains to deal with the case where $j = 0$.

As we can work separately on the connected components of M^ε we will assume that M^ε is connected. By Theorem 6.8 we have:

$$H^m_{(1)}(M^\varepsilon;\delta M^\varepsilon) \simeq 0.$$

By induction on the dimension, we know that Lefschetz duality holds for N^ε. Since N^ε is connected, by Theorem 5.2 we get:

$$H^{m-1}_{(1)}(N^\varepsilon;\delta N^\varepsilon) \simeq H^{m-1}_\infty(N^\varepsilon) \simeq I^t H^{m-1}(N^\varepsilon) \simeq \mathbb{R},$$

(see [GM1, GM2] for the local computations of the intersection homology groups).

Thanks to the long exact sequence (6.29), we deduce that:

$$H^m_{(1),X^\varepsilon}(M^\varepsilon;\delta M^\varepsilon) \simeq H^{m-1}_{(1)}(N^\varepsilon;\delta N^\varepsilon) \simeq \mathbb{R}.$$

Hence, it is enough to show that $\varphi^m_{M^\varepsilon}$ is onto. As the L^∞ closed 0-forms are reduced to the constant forms, it suffices to prove that for $x_0 \in cl(M)$ and $\varepsilon > 0$ small enough, we can find $\omega \in \Omega^m_{(1),X^\varepsilon}(M^\varepsilon;\delta M^\varepsilon)$ such that $\int_{M^\varepsilon}\omega \neq 0$.

As M^ε is orientable we can find a volume form on M^ε. We may multiply this form by a bump function to get a form with compact support in X^ε. The integral on M^ε of this form is then necessarily nonzero. This shows that $\varphi^m_{M^\varepsilon}$ is onto. □

Of course, when M is bounded, $H^j_{\infty,loc}(M)$ (resp. $H^j_{(1),X}(M;\delta M)$) and $H^j_\infty(M)$ (resp. $H^j_{(1)}(M;\delta M)$) coincide so that the latter paring induces in the case of bounded manifold an isomorphism between $H^j_\infty(M)$ and the dual vector space of $H^j_{(1)}(M;\delta M)$, establishing Theorem 1.5.

Remark 6.9. As explained in the introduction, Theorem 1.5 and Generalized Poincaré duality imply the de Rham theorem for Dirichlet L^1 cohomology (Corollary 1.6). In this section we assumed that M is orientable. This is necessary to prove Lefschetz duality for L^1 cohomology (Theorem 6.8). Nevertheless, the de Rham theorem for L^1 cohomology could be proved directly (independently of Lefschetz duality) and then orientability is unnecessary.

7. On the L^1 Stokes' property

Let $M \subset \mathbb{R}^n$ be a bounded orientable submanifold. The latter theorem raises a natural question: when do we have the L^1 Stokes' property on a subanalytic manifold ? This amounts to wonder when the Dirichlet L^1 forms and the L^1 forms coincide not only as cohomology groups, but also as cochains complexes. The following theorem answers very explicitly. The L^1 Stokes' property holds for j-forms iff δM is of dimension less than $(m-j-1)$.

In particular, if a subanalytic compact set $X \subset \mathbb{R}^n$ has only isolated singularities, then the L^1 Stokes' property holds for any L^1 j-form on X_{reg}, $j < m-1$. Below we adopt the convention that $\dim \emptyset = -1$.

Theorem 7.1. *Let $j < m$. The L^1 Stokes' property holds for j-forms iff $\dim \delta M < m-j-1$. In this case, L^1 cohomology is naturally dual to L^∞ cohomology in dimension j, i. e. the pairing:*

$$H^j_{(1)}(M) \otimes H^{m-j}_\infty(M) \to \mathbb{R}$$

$$(\alpha;\beta) \mapsto \int_M \alpha \wedge \beta$$

is (well defined and) nondegenerate.

Proof. We first focus on the if part. Write $X := cl(M)$.

As pointed out in section 6.1 (see (6.28)), it is enough to show that for any $\omega \in \overline{\Omega}^j_{(1)}(M)$ we have $|\omega|_{1,\delta} = 0$. We shall prove by induction the following statements.

($\mathbf{A}_k$) Let $a < b$ be real numbers and let k and l be integers. Let M be a bounded manifold with $\dim \delta M = k$. Set $\mathbb{D} := [a;b]^l$. Write $\overline{\Omega}^j_{(1),X\times\mathbb{D}}(M\times\mathbb{D})$ for the weakly smooth forms ω on $M\times\mathbb{D}$, with compact support in $X\times\mathbb{D}$, such that ω and $\overline{d}\omega$ are continuous near almost every point of $M\times\partial\mathbb{D}$ and L^1 on $M\times\mathbb{D}$ and on $M\times\partial\mathbb{D}$.

Let $\theta : X \to \mathbb{R}$ be a C^2 nonegative function with $\theta^{-1}(0) = \delta M$. For $\omega \in \overline{\Omega}^j_{(1),X\times\mathbb{D}}(M\times\mathbb{D})$ and $\alpha \in \overline{\Omega}^{m-j+l-1}_\infty(M\times\mathbb{D})$ we have:

$$\lim_{\nu\to 0} \int_{\{\theta=\nu\}\times\mathbb{D}} \omega\wedge\alpha = 0.$$

The 'if part' of the theorem follows from the case where l is zero. The product by $\mathbb{D}$ will be useful to perform the induction step. Note that the case where $\dim \delta M = -1$ is obvious since in this case $\{\theta = \nu\}$ is empty for ν small enough.

Fix ω and α like in ($\mathbf{A}_k$), $k \geq 0$. It suffices to prove ($\mathbf{A}_k$) for the forms $\varphi_i\omega$, if φ_i is a partition of unity. This means that we can work locally and assume that the support of ω in X is included in a little ball $B^n(x_0;\varepsilon)\times\mathbb{D}$ with $\varepsilon > 0$ and $x_0 \in X$.

We adopt the same notations as in the proof of Proposition 6.6 that we recall (see fig. 1). Consider a C^2 nonnegative function $\rho_1(x) : N^\varepsilon \to \mathbb{R}$ zero on δN^ε and positive on N^ε. Set $\rho_2 = \rho_1 \circ h^{-1}$ (recall that h is the local mapping provided by Theorem 2.7) and denote by ρ the Euclidian distance to x_0. For μ and ν positive real numbers, let

$$M_{\mu,\nu} := \{x \in M^\varepsilon : \rho_2(x) \geq \mu,\ \rho(x) \geq \nu\}.$$

Then $M_{\mu,\nu}$ is a manifold with corners (for μ and ν generic) whose boundary is the union of the set $\{x \in N^\nu : \rho_2(x) \geq \mu\}$ with the set

$$W_{\mu,\nu} = \{x \in M^\varepsilon : \rho_2(x) = \mu,\ \rho(x) \geq \nu\}.$$

Denote by Z_μ the set $\{x \in N^\varepsilon : \rho_2(x) = \mu\}$.

We shall show that

$$\lim_{\nu\to 0}\lim_{\mu\to 0} \int_{\partial M_{\mu,\nu}\times\mathbb{D}} \omega\wedge\alpha = 0. \tag{7.34}$$

Extend trivially the mapping h to a mapping $h' : N^\varepsilon \times [0;\varepsilon]\times\mathbb{D} \to M^\varepsilon\times\mathbb{D}$ and let $\omega' := h'^*(\omega)$ and $\alpha' := h'^*(\alpha)$. Note that as $h^{-1}(W_{\mu,\nu}) = Z_\mu \times [\nu;\varepsilon]$:

$$\lim_{\mu\to 0}\int_{W_{\mu,\nu}\times\mathbb{D}} \omega\wedge\alpha = \lim_{\mu\to 0}\int_{Z_\mu\times[\nu;\varepsilon]\times\mathbb{D}} \omega'\wedge\alpha',$$

which tends to zero thanks to the induction hypothesis (since $\dim \delta N^\varepsilon < k$). It thus remains to show that:

$$\lim_{\nu\to 0}\int_{N^\nu\times\mathbb{D}} \omega\wedge\alpha = 0. \tag{7.35}$$

We shall again make use of the homotopy operator $\mathcal{K}$. We extend $\mathcal{K}$ to a operator on $\overline{\Omega}^{j+l}_{(1),X\times\mathbb{D}}(M^\varepsilon\times\mathbb{D})$, considering the extra variables in $\mathbb{D}$ as parameters (if a form $\omega(x;t)$ on $M^\varepsilon\times\mathbb{D}$ is L^1 then the form $\omega_t(x) := \omega(x;t)$ is L^1 on M^ε for almost every $t\in\mathbb{D}$). For almost every t, $\mathcal{K}\omega_t$ is a L^1 form of M^ε. Moreover, by remark 4.3, the forms $\beta(x;t) := \mathcal{K}\omega_t(x)$ and $\beta'(x;t) := \mathcal{K}d\omega_t(x)$ are L^1 forms on $M^\varepsilon\times\mathbb{D}$. Then (4.19) continue to hold for L^1 forms with compact support in $X^\varepsilon\times\mathbb{D}$.

This identity entails that (7.35) splits into:

$$\lim_{\nu\to 0}\int_{N^\nu\times\mathbb{D}} \mathcal{K}\overline{d}\omega_t\wedge\alpha = 0. \tag{7.36}$$

and

$$\lim_{\nu\to 0}\int_{N^\nu\times\mathbb{D}} \overline{d}\mathcal{K}\omega_t\wedge\alpha = 0. \tag{7.37}$$

In virtue of $(\mathbf{A}_{k-1})$ the L^1 Stokes' property holds on $N^\nu\times\mathbb{D}$ and, integrating by parts, the latter equation may be rewritten as:

$$\lim_{\nu\to 0}\left[\int_{N^\nu\times\mathbb{D}} \mathcal{K}\omega\wedge\overline{d}\alpha + \int_{N^\nu\times\partial\mathbb{D}} \mathcal{K}\omega\wedge\alpha\right] = 0. \tag{7.38}$$

Observe that (4.20) holds for ω_t and $\overline{d}\omega_t$ for almost every t, i. e. that we have for almost every t in $\mathbb{D}$:

$$\lim_{\nu\to 0}\int_{N^\nu} |\mathcal{K}\omega_t| = \lim_{\nu\to 0}\int_{N^\nu} |\mathcal{K}\overline{d}\omega_t| = 0.$$

Therefore, as α and $\overline{d}\alpha$ are L^∞, (7.36) and (7.37) (via (7.38)) both come down from the Lebesgue dominated convergence theorem.

For the statement on Poincaré duality, observe now that the condition $\dim\delta M < m - j - 1$ ensures that $(j-1)$ and j forms satisfy the L^1 Stokes' property. Hence,

$$H^j_{(1)}(M) \simeq H^j_{(1)}(M;\delta M)$$

and the statement follows from Theorem 6.8.

It remains to prove that if the L^1 Stokes' property holds for all j-forms then $\dim\delta M < m - j - 1$. Fix $j < m$. We shall indeed establish the contraposition.

Let $k := \dim\delta M$. Assume $k \geq m - j - 1$ and take a regular point x_0 of δM.

Up to a local diffeomorphism we may identify a neighborhood W of x_0 in δM with an open subset of $\mathbb{R}^k$ (that we will still denote W). Also, thanks to subanalytic bi-Lipschitz triviality [V1], there is a subanalytic by-Lipschitz map H sending a contractible neighborhood U of x_0 in X onto a product $W\times X'$, with X' having only an isolated singularity. We can also assume that $H(M\cap U)$ is a product $W\times M'$.

By Proposition 3.7, subanalytic bi-Lipschitz maps induce a one-to-one correpondence between weakly smooth forms and consequently, M satifies the L^1 Stokes' property if and only if so does $W \times M'$. Therefore it is enough to show the result on $W \times M'$.

Observe that

$$H^{m-k}_{(1),X'}(M') \simeq 0,$$

while $H^{m-k}_{(1),X'}(M';\delta M \cap X')$ is nonzero (by Corollary 1.6). Consequently there must be a form $\omega \in \Omega^{(m-1-k)}_{(1),X'}(M')$ which does not satisify the L^1 Stokes' property. Define an L^1 j-form on M by:

$$\alpha := \omega \wedge dx_1 \wedge \cdots \wedge dx_{j-m+k+1},$$

where $dx_1, \dots, dx_k$ is the canonical basis of 1-forms on W ($(j-m+k+1)$ is nonnegative by assumption). We claim that α does not satisfy the L^1 Stokes' property in $W \times X'$. We will exhibit a form $\beta \in \Omega^{m-j-1}_{\infty,W\times X'}(W \times M')$ such that $l_\alpha(\beta) \neq 0$.

For this purpose, recall that since the L^1 Stokes' property fails for ω on M', there exists a form $\theta \in \overline{\Omega}^0_{\infty,X'}(M')$ for which $l_\omega(\theta) \neq 0$. Define a form on $W \times M'$ by:

$$\theta' := \theta \, dx_{j-m+k+2} \wedge \cdots \wedge dx_m.$$

As θ' does not have compact support in $W \times X'$, we shall multiply it by a bump function. Let $\psi : W \to [0;1]$ be a smooth nonegative compactly supported function which takes value 1 at x_0 and set $\beta := \psi\theta'$. By Fubini

$$l_\omega(\beta) = l_\omega(\theta) \int_W \psi(y)dy \neq 0,$$

as required. □

Remark 7.2. The argument used in the above proof was essentially local. Therefore, if we replace L^∞ by L^∞_{loc} and L^1 by L^1 with compact support in X the theorem goes over unbounded manifolds as well.

8. An example.

We end this paper by an example on which we discuss all the results of this paper. Let X be the suspension of the torus.

This is the set constituted by two cones over a torus that are attached along this torus. It is the most basic example on which Poincaré duality fails for singular homology but holds for intersection homology [GM1]. Let x_0 and x_1 be the two isolated singular points.

This set is a pseudomanifold. It has very simple singularities (metrically conical). However, the results of this paper show that if they were not conical (say cuspidal), this would not affect the cohomology groups which only depend on the topology of the underlying singular space. This simple example is already enough to illustrate how the singularities affect Poincaré duality for L^1 cohomology.

The different cohomology groups considered in this paper are gathered in the table below.

Cohomology groups $j =$	0	1	2	3
$I^tH^j(X)$ and $H^j_\infty(X_{reg})$	$\mathbb{R}$	0	$\mathbb{R}^2$	$\mathbb{R}$
$I^0H^j(X)$ and $H^j_{(1)}(X_{reg}; X_{sing})$	$\mathbb{R}$	$\mathbb{R}^2$	0	$\mathbb{R}$
$H^j(X_{reg})$ and $H^j_{(1)}(X_{reg})$	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}$	0

All these results may be obtained from the isomorphisms given in the introduction and a triangulation. Below, we interpret them geometrically.

Let $T \subset X$ be the original torus and let σ and τ be the suspension of the (support of the) two generators of $H_1(T)$. Write $\sigma^\varepsilon := \{x \in |\sigma| : d(x; \{x_0, x_1\}) \le \varepsilon\}$.

If ω is an L^∞ 2-form zero near the singular points and satisfying

$$\int_\sigma \omega = 1, \tag{8.39}$$

and if $\omega = d\alpha$ then $\int_{\sigma^\varepsilon} \alpha \equiv 1$ (by Stokes' formula). As the volume of σ^ε tends to zero, α cannot be bounded. Consequently if ω is a L^∞ closed 2-form zero near the singularities satisfying (8.39), it must represent a nontrivial class. In fact, every nontrivial class may be represented by a shadow form [BGM].

However, the form α may be L^1. The only nontrivial L^1 class of 2 forms is actually provided by those forms whose integral on T is nonzero, but these forms obviously do not satisfy the L^1 Stokes property (see (6.28)). We see that the singularities induce a gap between L^1 and Dirichlet L^1 cohomology, making the L^1 Stokes' property fail.

We also see that L^∞ cohomology is dual to L^1 cohomology in dimension 2 and 3 (as it is established by Theorem 1.3). However, $H^1_\infty(X_{reg})$ is not isomorphic to $H^2_{(1)}(X_{reg})$.

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