

A complete algorithm for determining copositive matrices[☆]

Jia Xu^{a,*}, Yong Yao^b

^a*College of Computer Science and Technology, Southwest University for Nationalities,
Chengdu, Sichuan 610041, PR China*

^b*Chengdu Institute of Computer Applications, Chinese Academy of Sciences, Chengdu,
Sichuan 610041, PR China*

Abstract

In this paper, we present a complete algorithm called COPOMATRIX for determining the copositivity of an $n \times n$ matrix. The core of this algorithm is decomposition theorem, which is used to deal with simplicial subdivision of $\widehat{T}^- = \{y \in \Delta_m | \beta^T y \leq 0\}$ on standard simplex Δ_m , where each component of the vector β^T is either -1 or 0 or 1.

Keywords: Copositive matrices, Copositive quadratic forms, Simplicial subdivision of convex polytope, Complete algorithm

2000 MSC: 15A48, 15A57, 15A63, 65F30

1. Introduction

QUESTION 1 Let A be a given $n \times n$ real symmetric matrix, \mathbb{R}_+ be the set of nonnegative real numbers, and

$$Q(X) = X^T A X, \quad X \neq 0$$

be a quadratic form. What conditions should A satisfy for $[\forall X \in \mathbb{R}_+^n, Q(X) \geq 0 (> 0)]$?

[☆]This research was supported by the National Natural Science Foundation of China(11001228,10901116).

*Corresponding author.

Email addresses: j.jia.xu@gmail.com (Jia Xu), yaoyong@casit.ac.cn (Yong Yao)

If $[\forall X \in \mathbb{R}_+^n, Q(X) \geq 0(> 0)]$, the quadratic form $Q(X)$ is called a (strictly) copositive quadratic form with corresponding matrix A being called a (strictly) copositive matrix.

Copositive matrices have numerous applications in diverse fields of applied mathematics, especially in mathematical programming and graph theory(see [3, 5, 7, 11, 13, 15, 16, 25, 26, 29, 39]). Therefore copositivity has been studying thoroughly since 1950s(see[1, 6, 14, 17, 20, 22, 23, 27, 28, 33, 41, 42, 48, 49]).

Generally, it is an NP-complete problem to determine whether a given $n \times n$ symmetric matrix is not copositive[37, 38]. That means, every algorithm that solves the problem, in the worst case, will require at least an exponential number of operations, unless P=NP. For that reason, it is still valuable for so many incomplete algorithms discussing some special kinds of matrices (see[3, 4, 10, 12, 18, 21, 24, 30, 38]). For small values of $n(\leq 6)$, some necessary and sufficient conditions have been constructed(see[1, 14, 30, 49]). However, it is still a hard problem to construct a complete algorithm for an arbitrary n . From another viewpoint, **QUESTION 1** is a typical real quantifier elimination problem[2, 8, 9, 32, 35, 40, 44, 45], which can be solved by standard tools of real quantifier elimination(e.g., using CAD)[2, 8, 9, 46, 47]. Thus, there is a complete algorithm for determining copositive matrices in theory. Unfortunately, this algorithm is not efficient in practice for real quantifier elimination algorithm is doubly-exponentially complex (see[2, 8, 9]). In this paper, we will construct a complete algorithm with exponentially complex.

The standard simplex $\Delta_m(m \geq 2)$ is defined as the following set

$$\Delta_m = \{(y_1, \dots, y_m)^T \mid y_1 + \dots + y_m = 1, y_1 \geq 0, \dots, y_m \geq 0\}.$$

It is well known that the dimension of Δ_m is $m - 1$. Denote the vertices of Δ_m as e_1, \dots, e_m , namely, $e_1 = (1, 0, \dots, 0)^T, \dots, e_m = (0, 0, \dots, 1)^T$.

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and be partitioned as

$$A = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha^T \\ \alpha & A_2 \end{bmatrix}.$$

Define $B = \alpha_{11}A_2 - \alpha\alpha^T$. It is easy to see the following facts(cf.[1])

1. If $\alpha_{1i} \geq 0, i = 2, \dots, n$, then
 A is(strictly)copositive $\iff \alpha_{11} \geq 0(> 0)$ and A_2 is(strictly)copositive.

2. If at least one of α_{1i} is negative, then we are going to focus on the set of points $T^- = \{y \in \Delta_{n-1} | \alpha^T y \leq 0\}$. It is well known that T^- is a convex polytope on Δ_{n-1} (see[1]). The polytope T^- can be subdivided into the simplices S_1, \dots, S_p , that is

$$T^- = \bigcup_{i=1}^p S_i, \text{ int}(S_i) \cap \text{int}(S_j) = \emptyset, \text{ for } i \neq j,$$

where $\text{int}(S_i)$ denotes the interior of simplex S_i . The coordinates of the vertices that span the simplex S_i constitute a matrix denoted as W_i . Andersson et al.([1],p.23) proved the following results.

Lemma 1.1. (a) A is copositive iff $\alpha_{11} \geq 0$ and $A_2, W_1^T B W_1, \dots, W_p^T B W_p$ are all copositive.

(b) A is strictly copositive iff $\alpha_{11} > 0$ and $A_2, W_1^T B W_1, \dots, W_p^T B W_p$ are all strictly copositive.

In order to formulate the algorithm of Lemma 1.1, we first consider that how to obtain the simplicial subdivision of the polytope $T^- = \{y \in \Delta_{n-1} | \alpha^T y \leq 0\}$. For small values of $n(\leq 6)$, Andersson et al.[1] and Yang and Li[49] give the simplicial subdivision of T^- . However, all of them are not able to propose the simplicial subdivision of T^- for all values of n , which will be solved in this paper. We propose a simplicial subdivision of T^- for all values of n , and consequently construct a complete algorithm for determining the copositivity of an $n \times n$ matrix.

We will adopt a flexible approach. Unlike subdividing T^- into simplices(of course our method is also valid for subdividing T^- into simplices), we first transform the matrix A into the following matrix called \hat{A} .

Let $\alpha = (\alpha_{12}, \dots, \alpha_{1n})^T$ and diagonal matrix $D = \text{diag}(d_2, \dots, d_n)$, where

$$d_i = \begin{cases} 1, & \text{if } \alpha_{1i} = 0; \\ 1/|\alpha_{1i}|, & \text{if } \alpha_{1i} \neq 0. \end{cases}$$

Then

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \hat{\alpha}^T \\ \hat{\alpha} & D A_1 D \end{bmatrix}. \quad (1)$$

where $\hat{\alpha} = (\text{sign}(\alpha_{12}), \dots, \text{sign}(\alpha_{1n}))^T$ with sign denoting symbolic function.

Obviously, A is (strictly)copositive $\iff \widehat{A}$ is (strictly) copositive. Use Lemma 1.1 for \widehat{A} . Let

$$\beta_1 = \text{sign}(\alpha_{12}), \dots, \beta_{n-1} = \text{sign}(\alpha_{1n}).$$

Thus we just need to subdivide \widehat{T}^- into simplices, here

$$\widehat{T}^- = \{y \in \Delta_{n-1} | (\beta_1, \dots, \beta_{n-1})y \leq 0, \beta_i \in \{-1, 0, 1\}\}.$$

Next we make further simplification: separate -1,0,1 from $\beta_1, \dots, \beta_{n-1}$, namely let

$$\begin{aligned} \beta_{a_1} = \dots = \beta_{a_s} = 1, \beta_{b_1} = \dots = \beta_{b_t} = -1, \beta_{c_1} = \dots = \beta_{c_r} = 0. \\ \{a_1, \dots, a_s, b_1, \dots, b_t, c_1, \dots, c_r\} = \{1, \dots, n-1\}, \\ r, s, t \geq 0, t \geq 1, r + s + t = n-1. \end{aligned}$$

In geometry it is easy to see that the convex polytope \widehat{T}^- is the convex hull of its surface S^- and its vertices e_{c_1}, \dots, e_{c_r} , that is,

$$\widehat{T}^- = \text{conv}\{e_{c_1}, \dots, e_{c_r}, S^-\}. \quad (2)$$

$$\begin{aligned} S^- = \{(y_1, \dots, y_{n-1})^T \in \Delta_{n-1} | y_{a_1} + \dots + y_{a_s} - y_{b_1} - \dots - y_{b_t} \leq 0, \\ (y_{a_1}, \dots, y_{a_s}, y_{b_1}, \dots, y_{b_t})^T \in \Delta_{s+t}\}. \end{aligned} \quad (3)$$

If the simplicial subdivision of S^- is known, the simplicial subdivision of \widehat{T}^- is directly obtained by (2). So we just need to study the simplicial subdivision of the polytope S^- .

2. Simplicial subdivision algorithm of convex polytope S^-

2.1. Fundamental notations

The notation S^- is simple, but it can not reveal the information of convex polytopes. In order to simplify the descriptions, we will introduce a new notation, which is fundamental to our study.

Definition 2.1. Suppose that two sequences of positive integers $[a_1, a_2, \dots, a_s]$, $[b_1, b_2, \dots, b_t]$ satisfying

$$\{a_1, \dots, a_s, b_1, \dots, b_t\} \subseteq \{1, 2, \dots, m\}, \quad s \geq 0, t \geq 1, m \geq s + t \geq 2,$$

where all of $s + t$ elements of $\{a_1, \dots, a_s, b_1, \dots, b_t\}$ are different from each other. Then the notation $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is defined as the polytope S^- (see(3)).

For example, let us compare the polytope $[[2, 3], [5]]_5$ and the polytope $[[2, 3], [5]]_6$. $[[2, 3], [5]]_5$ denotes the polytope

$$\{(y_1, \dots, y_5)^T \in \Delta_5 | y_2 + y_3 - y_5 \leq 0, (y_2, y_3, y_5)^T \in \Delta_3\}.$$

Here $(y_2, y_3, y_5)^T \in \Delta_3$ implies that $y_1 = 0, y_4 = 0$. $[[2, 3], [5]]_6$ indicates the polytope

$$\{(y_1, \dots, y_6)^T \in \Delta_6 | y_2 + y_3 - y_5 \leq 0, (y_2, y_3, y_5)^T \in \Delta_3\}.$$

Here $(y_2, y_3, y_5)^T \in \Delta_3$ implies that $y_1 = 0, y_4 = 0, y_6 = 0$. It is clear that $[[2, 3], [5]]_5$ and $[[2, 3], [5]]_6$ are congruent, although they are set in simplices of different dimensions.

For $0 \leq k \leq m - 1$, the polytope L_k^- is defined as

$$L_k^- = \{(y_1, \dots, y_m)^T \in \Delta_m | y_1 + \dots + y_k - y_{k+1} - \dots - y_m \leq 0\}.$$

L_k^- is written as $[[1, \dots, k], [k + 1, \dots, m]]_m$ by the notation of Definition 2.1. L_k^- is a special case of S^- , but its properties have generality.

In the following we will study the basic geometric properties of convex polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$.

2.2. Geometric properties of S^-

Let e_1, \dots, e_m be vertices of the standard simplex Δ_m , and $M_{i,j} = (e_i + e_j)/2$ be the midpoint of the line segment $e_i e_j$.

Lemma 2.1. [1] Given a convex polytope L_k^- , then all of its vertices are

$$V = \{e_{k+1}, \dots, e_m, M_{i,j}, i = 1, 2, \dots, k, j = k + 1, \dots, m\}.$$

And the number of the vertices is $|V| = (k + 1)(m - k)$.

PROOF. Note that the convex polytope L_k^- is obtained by cutting the standard simplex Δ_m with the hyperplane

$$L_{=0} : y_1 + \dots + y_k - y_{k+1} - \dots - y_m = 0.$$

Therefore the vertices of the polytope L_k^- come from two parts: one part is vertices of Δ_m , that is, $\{e_{k+1}, \dots, e_m\}$; while the other part is the intersection points of the hyperplane $L_{=0}$ and the edges of standard simplex Δ_m .

First consider the intersection point of $L_{=0}$ and the edge $ae_1 + be_{k+1}$ ($a, b \geq 0$, $a + b = 1$). Substitute $ae_1 + be_{k+1}$ into the following equations,

$$y_1 + \cdots + y_k - y_{k+1} - \cdots - y_m = 0, \quad y_1 + \cdots + y_m = 1$$

Therefore, the solutions are $a = 1/2, b = 1/2$, namely, the intersection point is $M_{1,k+1}$.

In the same way, we get all intersection points of $L_{=0}$ and the edges of Δ_m . They are $\{M_{i,j}, i = 1, 2, \dots, k, j = k + 1, \dots, m.\}$

Hence the number of vertices of L_k^- is $|V| = m - k + k(m - k) = (k + 1)(m - k)$.

Likewise, we can prove that

Lemma 2.2. Given a convex polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$, then all of its vertices are

$$V = \{e_{b_1}, \dots, e_{b_t}, M_{a_i, b_j}, i = 1, 2, \dots, s, j = 1, \dots, t\}.$$

And the number of the vertices is $|V| = (s + 1)t$.

We see that the polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ and the polytope L_k^- are similar in many respects, which will be further discussed.

Lemma 2.3. The convex polytope L_k^- is simplicial iff $k = 0$, or $k = m - 1$.

PROOF. When $k = 0$, $L_0^- = \Delta_m$ is simplicial.

When $k = m - 1$, consider convex polytope,

$$L_{m-1}^- := \{(y_1, \dots, y_m)^T \in \Delta_m \mid y_1 + \cdots + y_{m-1} - y_m \leq 0\}.$$

By Lemma 2.1, we know that all vertices of L_{m-1}^- are $\{e_m, M_{i,m}, i = 1, 2, \dots, m - 1\}$. Obviously all the vectors of $\{M_{i,m} - e_m, i = 1, 2, \dots, m - 1\}$ are linearly independent, so L_{m-1}^- is simplicial.

Conversely, we know that the dimension of the polytope L_k^- is $m - 1$. If $k \neq 0, m - 1$, then by Lemma 2.1, the number of vertices of L_k^- is $(k + 1)(m - k) \neq m$, so L_k^- is not simplicial.

Lemma 2.4. The convex polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ (here the vertices are obtained by Lemma 2.2) is simplicial iff $s = 0$, or $t = 1$.

Lemma 2.5. The dimension of the polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is $(s + t - 1)$.

If the polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is not a simplex, we will subdivide it into simplices.

Lemma 2.6. If the polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is not a simplex, then there are only two $(s + t - 2)$ -dimensional surfaces that do not include the vertex M_{a_1, b_1} . They are

$$[[a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m, [[a_1, \dots, a_s], [b_2, \dots, b_t]]_m.$$

(obtained by deleting a_1, b_1 from array $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ respectively)

PROOF. All the $(s + t - 2)$ -dimensional surfaces of the convex polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ are obviously

$$\begin{aligned} & [[\widehat{a}_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m, \\ & [[a_1, \widehat{a}_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m, \\ & \dots, \\ & [[a_1, a_3, \dots, a_s], [b_1, b_2, \dots, \widehat{b}_t]]_m \end{aligned}$$

(where the notation $[[\widehat{a}_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is the polytope taken off a_1) and

$$\{(y_1, \dots, y_m)^T \in \Delta_m | y_{a_1} + \dots + y_{a_s} - y_{b_1} - \dots - y_{b_t} = 0, (y_{a_1}, \dots, y_{a_s}, y_{b_1}, \dots, y_{b_t})^T \in \Delta_{k+t}\}.$$

That makes $s + t + 1$ $(s + t - 2)$ -dimensional surfaces in all. We can verify that only

$$[[a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m, [[a_1, \dots, a_s], [b_2, \dots, b_t]]_m$$

do not include the vertex M_{a_1, b_1} .

Lemma 2.6 leads to the following decomposition theorem.

2.3. The decomposition process for polytope S^-

Theorem 2.1 (decomposition theorem). If the polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is not simplicial, then it can be decomposed into the union of two convex polytopes(not always simplicial). The expression is

$$\begin{aligned} & [[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m \\ &= \text{conv}\{M_{a_1, b_1}, [[a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m\} \cup \\ & \text{conv}\{M_{a_1, b_1}, [[a_1, a_2, \dots, a_s], [b_2, \dots, b_t]]_m\}. \end{aligned}$$

Here $\text{conv}\{S\}$ denotes the convex hull of the set of points S .

PROOF. The proof is immediately completed by Lemma 2.6.

Based on Theorem 2.1, the polytope S^- can be easily subdivided into simplices.

Example 1. Please show the simplicial subdivision of the following convex polytope

$$L_2^- := \{(y_1, \dots, y_5) \mid y_1 + y_2 - y_3 - y_4 - y_5 \leq 0, (y_1, \dots, y_5)^T \in \Delta_5\}.$$

SOLUTION. Denote L_2^- as $[[1, 2], [3, 4, 5]]_5$. We know that $[[1, 2], [3, 4, 5]]_5$ is not simplicial by Lemma 2.4. Using Theorem 2.1 we have

$$[[1, 2], [3, 4, 5]]_5 = \text{conv}\{M_{1,3}, [[2], [3, 4, 5]]_5\} \cup \text{conv}\{M_{1,3}, [[1, 2], [4, 5]]_5\}.$$

By Lemma 2.4 we know that both $[[2], [3, 4, 5]]_5$ and $[[1, 2], [4, 5]]_5$ are not simplicial. Therefore we repeatedly apply Theorem 2.1 to them and have

$$\begin{aligned} & [[2], [3, 4, 5]]_5 \\ &= \text{conv}\{M_{2,3}, [[], [3, 4, 5]]_5\} \cup \text{conv}\{M_{2,3}, [[2], [4, 5]]_5\}. \\ &= \text{conv}\{M_{2,3}, [[], [3, 4, 5]]_5\} \cup \text{conv}\{M_{2,3}, M_{2,4}, [[], [4, 5]]_5\} \cup \\ & \text{conv}\{M_{2,3}, M_{2,4}, [[2], [5]]_5\} \\ &= \text{conv}\{M_{2,3}, e_3, e_4, e_5\} \cup \text{conv}\{M_{2,3}, M_{2,4}, e_4, e_5\} \cup \\ & \text{conv}\{M_{2,3}, M_{2,4}, M_{2,5}, e_5\}. \end{aligned}$$

$$\begin{aligned} & [[1, 2], [4, 5]]_5 \\ &= \text{conv}\{M_{1,4}, [[2], [4, 5]]_5\} \cup \text{conv}\{M_{1,4}, [[1, 2], [5]]_5\}. \\ &= \text{conv}\{M_{1,4}, M_{2,4}, [[], [4, 5]]_5\} \cup \text{conv}\{M_{1,4}, M_{2,4}, [[2], [5]]_5\} \cup \\ & \text{conv}\{M_{1,4}, [[1, 2], [5]]_5\} \\ &= \text{conv}\{M_{1,4}, M_{2,4}, e_4, e_5\} \cup \text{conv}\{M_{1,4}, M_{2,4}, M_{2,5}, e_5\} \cup \\ & \text{conv}\{M_{1,4}, M_{1,5}, M_{2,5}, e_5\}. \end{aligned}$$

Finally we get the expression of simplicial subdivision of $[[1, 2], [3, 4, 5]]_5$,

$$\begin{aligned} & [[1, 2], [3, 4, 5]]_5 \\ &= \text{conv}\{M_{1,3}, M_{2,3}, e_3, e_4, e_5\} \cup \text{conv}\{M_{1,3}, M_{2,3}, M_{2,4}, e_4, e_5\} \cup \\ & \quad \text{conv}\{M_{1,3}, M_{2,3}, M_{2,4}, M_{2,5}, e_5\} \cup \text{conv}\{M_{1,3}, M_{1,4}, M_{2,4}, e_4, e_5\} \cup \\ & \quad \text{conv}\{M_{1,3}, M_{1,4}, M_{2,4}, M_{2,5}, e_5\} \cup \text{conv}\{M_{1,3}, M_{1,4}, M_{1,5}, M_{2,5}, e_5\}. \end{aligned}$$

So $[[1, 2], [3, 4, 5]]_5$ is a union of six 4-dimensional simplices.

We summarize the decomposition process of Example 1 into the following algorithm.

Algorithm 1 (Vmatrix)

Input: The expression of polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$.

Output: Simplex D_1, D_2, \dots, D_p (denoted by matrices) such that

$$[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m = \bigcup_{i=1}^p D_i, \quad \text{int}(D_i) \cap \text{int}(D_j) = \emptyset, \text{ for } i \neq j.$$

V1: Let $F := \{[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m\}$, $\text{temp} := \{ \}$.

V2: When $F \neq \emptyset$, repeat the following procedures

V21: If the polytope $A \in F$ is simplicial, then $\text{temp} := \text{temp} \cup \{A\}$.

V22: If the polytope $A \in F$ is not simplicial, then by Theorem 2.1 decompose it into two convex polytope B_1, B_2 , and add them into the set F together with deleting A from F . Go to step V2.

V3: Return temp .

We have written a function in Maple[31] to implement the above algorithm.

Lastly, we will present a formula for computing the number of simplices given by polytope L_k^- subdivision.

Lemma 2.7. According to algorithm Vmatrix, the convex polytope $[[1, \dots, k], [k+1, \dots, m]]_m$ ($0 \leq k \leq m-1, m \geq 2$) can be subdivided just into $f(k, m)$ simplices, here

$$f(k, m) = \binom{m-1}{k} = \frac{(m-1)!}{k!(m-1-k)!}.$$

We know that $f(k, m)$ have the same recurrence formula as binomial coefficients by Theorem 2.1. Thus the proof of Lemma 2.7 is easy via mathematical induction. This formula will be used to estimate the cost of Algorithm 2 in next section.

3. Determining algorithm for copositive matrices

In this section, we will present the complete determining algorithm of a copositive matrix.

Given an $n \times n$ symmetric matrix

$$A = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha^T \\ \alpha & A_2 \end{bmatrix},$$

compute \widehat{A} (see (1))

$$\widehat{A} = \begin{bmatrix} \alpha_{11} & \widehat{\alpha}^T \\ \widehat{\alpha} & DA_2D \end{bmatrix}.$$

Let $B = \alpha_{11}DA_2D - \widehat{\alpha}\widehat{\alpha}^T$, and let

$$\widehat{\alpha} = (\text{sign}(\alpha_{12}), \dots, \text{sign}(\alpha_{1n}))^T = (\beta_1, \dots, \beta_{n-1})^T.$$

Define the projection operator **Proj** of the matrix A as follows,

- If $\beta_i \geq 0$, $i = 1, \dots, n-1$, then

$$\text{Proj}(A) = \{DA_2D\}.$$

- If there is at least one -1 in β_i , then

$$\text{Proj}(A) = \{DA_2D, W_1^T B W_1, \dots, W_p^T B W_p\}.$$

Here the matrices W_1, \dots, W_p is fixed by the simplicial subdivision of the convex polytope \widehat{T}^- (see (2)).

Algorithm 2 (COPOMATRIX)

Input: Symmetric matrix $A \in \mathbf{R}^{n \times n} (n \geq 2)$.

Output: A is copositive, or A is not copositive.

C1: Let $F := \{A\}$.

C2: Repeat the following steps for the set F .

C21: If the set F is empty, then return “ A is copositive”.

C22: Check the $(1,1)^{th}$ entry of every matrix K in set F . If at least one of them is negative, then return “ A is not copositive”.

C23: Compute the projective set $P := \bigcup_{K \in F} \text{Proj}(K)$ of set F . Delete the nonnegative matrices of P , and let the rest matrices consist F . Go to step C21.

Note that the above algorithm is also valid for 2×2 matrices. Furthermore, for strictly copositive matrices we can also formulate similar algorithm.

The correctness of the algorithm COPOMATRIX is guaranteed by Lemma 1.1, and the algorithm obviously terminate. The cost of the algorithm mainly depends on the number of simplicial subdivision of the polytope. According to Lemma 2.7, we can estimate the worst case is at most:

$$\begin{aligned} & \left(\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} + 1 \right) \left(\binom{n-3}{\lfloor \frac{n-3}{2} \rfloor} + 1 \right) \cdots \left(\binom{2}{1} + 1 \right) \\ & \leq (2^{n-3})(2^{n-4}) \cdots (2)(2) \\ & = 2^{(n-2)(n-3)/2+1}. \end{aligned}$$

The bound $O(2^{(n-2)(n-3)/2+1})$ has already much lower than doubly-exponential cost of CAD [2,9]. We have written a function in Maple to implement the algorithm COPOMATRIX. For non-commercial request, we will offer for free. Please sent e-mail to the address

yaoyong@casit.ac.cn, or,
j.jia.xu@gmail.com.

References

- [1] L.E.Andersson, G.Chang, T.Elfving, Criteria for copositive matrices using simplices and barycentric coordinates, Linear Algebra Appl. 220 (1995) 9-30.
- [2] S. Basu, R.Pollack, M.F.Roy, Algorithms in Real Algebraic Geometry (2nd), Springer-Verlag, New York (Berlin, Heidelberg), 2006, 159-194, 533-562.
- [3] I.M.Bomze, Block pivoting and shortcut strategies for detecting copositivity, Linear Algebra Appl. 248 (1996) 161-184.

- [4] I.M.Bomze, Linear-time copositivity detection for tridiagonal matrices and extension to block-tridiagonality, *SIAMJ. Matrix Anal. Appl.* 21 (2000) 840-848.
- [5] I.M.Bomze, M.Dür, E.de Klerk, C.Roos, A.J.Quist, T. Terlaky, On copositive programming and standard quadratic optimization problems, *J. Global Optim.* 18 (2000) 301-320.
- [6] S.Bundfuss, M.Dür, Algorithmic copositivity detection by simplicial partition, *Linear Algebra Appl.* 428 (2008) 1511-1523.
- [7] S.Burer, On the copositive representation of binary and continuous non-convex quadratic programs. *Math. Programming.*120(2009) 479-495.
- [8] G.E.Collins, Quantifier elimination for real closed fields by cylindric algebraic decomposition, In: *Second GI Conference on Automata Theory and Formal Languages, Lecture Notes in Computer Science*, vol. 33, 134-183, Springer- Verlag, Berlin,1975.
- [9] G.E.Collins, H.Hong, Partial cylindrical algebraic decomposition for quantifier elimination, *J.Symb.Comput.* 12(1991) 299-328.
- [10] R.W.Cottle, G.J.Habetler, C.E.Lemke, On classes of copositive matrices, *Linear Algebra Appl.* 3 (1970) 295-310.
- [11] G.Danninger, Role of copositivity in optimality criteria for nonconvex optimization problems, *J. Optim. Theory Appl.* 75 (1992) 535-558.
- [12] G.Danninger, A recursive algorithm to detect (strict) copositivity of a matrix, in: U. Rieder, A. Peyerimhoff, F.J.Rademacher (Eds.), *Methods of Operations Research*, vol. 62, 1990, pp. 45-52.
- [13] G.Eichfelder, J. Jahn, Set-semidefinite optimization. *J. Convex Anal.* 15 (2008) 767-801.
- [14] K.P.Hadeler, On copositive matrices, *Linear Algebra Appl.* 49 (1983) 79-89.
- [15] P.Hadjicostas, Copositive matrices and simpson's paradox, *Linear Algebra Appl.* 264 (1997) 475-488.

- [16] M.Hall Jr., M.Newman, Copositive and completely positive quadratic forms, Proc. Cambridge Philos. Soc. 59 (1963) 329-339.
- [17] E.Haynsworth, A.J.Hoffman, Two remarks on copositive matrices. Linear Algebra Appl. 2 (1969) 387-392.
- [18] A. J. Hoffman, F. Pereira, On copositive matrices with - 1, 0, 1 entries, J. Combin. Theory Ser. A 14 (1973) 302-309.
- [19] L.Hogben, C.R.Johnson, R.Reams, The copositive completion problem, Linear Algebra Appl. 408 (2005) 207-211.
- [20] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1992
- [21] K.D.Ikramov, Linear-time algorithm for verifying the copositivity of an acyclic matrix, Comput. Math. Math. Phys. 42 (2002) 1701-1703.
- [22] K.D.Ikramov, N.V.Savel'eva, Conditionally definite matrices, J. Math. Sci. 98 (2000) 1-50.
- [23] C.R.Johnson, R.Reams, Constructing copositive matrices from interior matrices, Electronic Journal of Linear Algebra, 17 (2008) 9-20.
- [24] C.R.Johnson, R.Reams, Spectral theory of copositive matrices, Linear Algebra Appl. 395 (2005) 275-281.
- [25] B.Jüttler, Arbitrarily weak linear convexity conditions for multivariate polynomials, Stud. Sci. Math. Hungar. 36(2000) 165-183.
- [26] E.de Klerk, D.V. Pasechnik, Approximation of the stability number of a graph via copositive programming, SIAMJ. Optim. 12 (2002) 875-892.
- [27] W.Kaplan, A test for copositive matrices, Linear Algebra Appl. 313 (2000) 203-206.
- [28] W.Kaplan, A copositivity probe, Linear Algebra Appl. 337 (2001) 237-251.
- [29] C.E.Lemke, Bimatrix equilibrium points and mathematical programming, Manage. Sci. 11 (1965) 681-689.

- [30] P.Li, Y.Y.Feng, Criteria for copositive matrices of order four, *Linear Algebra Appl.* 194(1993) 109-124.
- [31] Maple 14 user manual, Maplesoft. http://www.maplesoft.com/documentation/_center/.
- [32] M.Marshall, *Positive Polynomials and Sums of Squares*, AMS Press, NewYork 2008.
- [33] D.H. Martin, Finite criteria for conditional definiteness of quadratic forms, *Linear Algebra Appl.* 39 (1981) 9-21.
- [34] D.H. Martin, D.H. Jacobson, Copositive matrices and definiteness of quadratic forms subject to homogeneous linear inequality constraints, *Linear Algebra Appl.* 35 (1981) 227-258.
- [35] B.Mishra, *Algorithmic Algebra*, Springer-Verlag, New York, 1993, 333-354.
- [36] T.S.Motzkin, Copositive quadratic forms, National Bureau of Standards Report 1818, (1952) 11-22.
- [37] K.G.Murty, S.N. Kabadi, Some NP-complete problems in quadratic and nonlinear programming, *Math. Program.*39 (1987) 117-129.
- [38] P.A. Parrilo, Semidefinite programming based test for matrix copositivity, *Proceedings of the 39th IEEE Conference on Decision and Control* 5 (2000) 4624-4629.
- [39] A.J.Quist, E.de Klerk, C.Roos, T.Terlaky, Copositive relaxation for general quadratic programming, *Optim.Methods Softw.* 9 (1998) 185-208.
- [40] A.Tarski, *A Decision method for elementary algebra and geometry*, University of California Press , California, 1951.
- [41] H.Väliäho, Criteria for copositive matrices, *Linear Algebra Appl.* 81 (1986) 19-34.
- [42] H.Väliäho, Testing the definiteness of matrices on polyhedral cones, *Linear Algebra Appl.* 101 (1988) 135-165.

- [43] H.Väliäho, Quadratic programming criteria for copositive matrices, *Linear Algebra Appl.* 119 (1989) 163-182.
- [44] L.Yang, B.Xia, Computational Real Algebraic Geometry, In: D.M. Wang (eds), *Selected lecture in symbolic computation*, Tsinghua Univ. Press, Beijing, 2003. (in Chinese)
- [45] L.Yang, B.Xia, *Automated Proving and Discovering on Inequalities*, Science Press, Beijing, 2008. (in Chinese)
- [46] L.Yang, J.Zhang, A practical program of automated proving for a class of geometric inequalities, *Automated Deduction in Geometry, Lecture Notes in Artificial Intelligence 2061*, Springer-Verlag , 2001, 41-57.
- [47] L.Yang, Recent advances in automated theorem proving on inequalities, *J. Comput Sci & Technol.* 14(1999), 434-446.
- [48] S.Yang, X.Li, Some simple criteria for copositive matrices, in: *Proceedings of the Seventh International Conference on Matrix Theory and Applications, Advances in Matrix Theory and Applications*, World Academic Union, 2006.
- [49] S.Yang, X.Li, Algorithms for determining the copositivity of a given symmetric matrix, *Linear Algebra Appl.* 430 (2009) 609-618.