Dark Energy as Double N-flation—Observational Predictions

J. Richard Gott, III and Zachary Slepian

November 12, 2010

Department of Astrophysical Sciences, Princeton University, Princeton, NJ, 08544

E-mail: jrg@astro.princeton.edu, zslepian@princeton.edu

Abstract

We propose a simple model for dark energy useful for comparison with observations. This is based on the idea that dark energy and inflation should be caused by exactly the same physical process. Linde's simple chaotic inflation $V = \frac{1}{2}m^2\phi^2$ produces values of $n_s = 0.967$ and $r = 0.13$ which are consistent with the WMAP 1σ error bars. We therefore propose $V = \frac{1}{2}m_2^2\phi_2^2 + \frac{1}{2}m_1^2\phi_1^2$ with $m_1 \sim 10^{-5}$ and $m_2 \leq 10^{-60}$. The fine tuning problem is thus only half as bad as if one wanted dark energy to be produced by a constant $V_0 \sim 10^{-120}$. For comparison, neutrino masses are of order 10^{-29}. The field ϕ_1 drives inflation and has damped by now ($\phi_{1,0} = 0$), while ϕ_2 in slow roll produces dark energy with values today of $\delta w_0 \equiv w_0 + 1 \approx 4/(3\phi_{2,0}^2 + 2)$. Our numerical results are well fit by $\delta w(z) \approx \delta w_0 \left(H_0/H(z)\right)^2$. This should be true in any slow roll inflation. Our potential can be easily realized in N-flation models with many fields. This model is easily falsifiable by upcoming experiments—for example, if Linde's chaotic inflation is ruled out. But if r values consistent with Linde's chaotic inflation are detected then one should take this model seriously indeed.

1 Introduction

Measuring dark energy is one of the most exciting problems in cosmology today. There are a number of expensive programs underway to measure w, the ratio of the pressure to the energy density in dark energy. If dark energy is a pure cosmological constant, then $w = constant = -1$. Currently, measurements of w are compared to a toy model in which w changes linearly with expansion factor a: $w = w_0 + w_a(1 - a)$. Observational programs are judged by a figure of merit which includes their ability to measure the quantities w_0 and w_a in this toy model (Albrecht, et al 2006). If one parameterizes $w(a)$ in terms of the above toy model, the current 1σ limits from the 7-year WMAP data combined with BAO+Ho+SN are $w_0 = -.93 \pm .13$, and $w_a = -.41 \pm .72$ (Komatsu et al. 2010). These values are consistent at the 1σ level with $w_0 = -1$ and $w_a = 0$, which would be a pure cosmological constant. The Sloan III survey should measure the Baryon Acoustic Oscillation (BAO) scale to high accuracy using LRG galaxies, and should lead to a measurement of w_0 to an accuracy of 3%. Using this same dataset, we can use genus topology to measure w_0 independently to an accuracy of 5% (Zunckel, Gott, and Lunnan 2010). Supernova studies can achieve similar results (c.f. Albrecht et al 2006). The Euclid satellite mission may be able to achieve an accuracy of 1%. Such observations may continue to point to $w_0 = -1$ and $w_a = 0$ with higher and higher accuracy, which would be an important result, but would leave us still in the dark as to the exact nature of dark energy.

More exciting would be if a detectable difference δw_0 between w_0 and -1 is found (i.e. $\delta w_0 = w_0 + 1$). For this reason, it is desirable to consider simple models, consistent with current data and falsifiable in principle the near future, in which there is a chance that such a detectable difference may be found. Such models may be used as a guide for interpreting the observations. We therefore propose such a simple model, based on the idea that inflation and dark energy must come from the same physical mechanism. This hypothesis is not present in most of the standard theories popular today.

Today, the most popular theory for dark energy is that a string landscape exists with many metastable vacuua with different values of V_0. In this picture, we are currently sitting at the bottom of a potential well whose low point has a vacuum energy density of V_0. Thus, the accelerated expansion we see in the universe today is attributed to our current static location at a metastable well in the potential, while in contrast, we have evidence that the accelerated expansion we see in the early universe is due to slow-roll inflation, where the field is slowly rolling down a potential hill.

Furthermore, for dark energy today, there is supposed to be a complicated potential that is a function of many fields ϕ_i rather than merely the one field in slow-roll inflation. The kinetic energy in these ϕ_i fields has damped out and they are no longer changing. A problematic feature of this model is the value of V_0: 10^{-120} in Planck units, an extraordinarily small number. The common solution to this problem is to propose that there are of order 10^{500} vacuum states with values of $-1 < V_0 < 1$. Then one evokes anthropic effects to argue that it would be difficult for intelligent life to evolve unless $V_0 \sim 10^{-120}$. This is not as satisfying to some physicists as an actual prediction of the amount of dark energy we observe made directly from the physical model.

A final problem with this model is the appearance of Boltzmann brains (see discussion in Gott 2008 and references therein). Briefly, in the far future, the universe comes to a finite Gibbons and Hawking temperature $T = 1/(2\pi r_0)$, where $r_0 = (3/V_0)^{1/2}$, and Boltzmann brains appear. While this problem may be manageable depending on what measure one uses (c.f. deSimone et al. 2010), it is still a problem that must be addressed in the popular V_0 model.

In contrast to the V_0 model, our model is based on the idea that inflation and dark energy must be the result of exactly the same physical process. Why? When inflation was proposed by Guth (1981), one might have asked why we should believe such an extraordinary thing as that one had a period of accelerating expansion in the early universe. After all, Hawking (1967) had proven that the universe began with a singularity by assuming that gravity in the early universe was always attractive–so what could push outward against gravity?

However, when the current acceleration of the universe was discovered, we suddenly had a persuasive reason for believing inflation: we see repulsive exponential expansion starting today. Inflation in the early universe became more plausible because we are actually observing a low-grade inflation today. Of course, by now there is other, stronger evidence for inflation: WMAP has shown that the power spectrum of fluctuations in the microwave background is in rather exact agreement with the predictions of inflation. Furthermore, the fluctuations are accurately Gaussian random phase, which the theory of inflation predicts. And the universe is measured to be flat to within the observable limits, another implication of inflation. With all of this in mind, it seems the simplest model of dark energy would be one in which it and the inflation we encounter in the early universe are essentially identical.

2 Chaotic Inflation

So let us begin with Linde's (1983) chaotic inflation, arguably the simplest model of inflation ever proposed. Linde's potential was of the form:

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{1}$$

This is a simple massive scalar field. Its equations of motion are:

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) = -m^2\phi \tag{2}$$

where H is the Hubble constant and $3H\dot{\phi}$ is a frictional term due to the expansion of the universe. Pretty soon, the field ϕ reaches an approximately constant velocity so that $\ddot{\phi} << 3H\dot{\phi}$, where $H^2 >> k/a^2$, making the universe effectively flat. Further, $\dot{\phi}^2 << m^2\phi^2$, so the equations can be simplified (Linde 2002):

$$H \approx m\phi/\sqrt{6}, \dot{\phi} \approx -m\sqrt{2/3}. \tag{3}$$

The expansion is approximately exponential: $a(t) \approx \exp(Ht)$. This is slow-roll inflation.

$$H = \frac{1}{a}\frac{da}{dt} = \frac{d\ln a}{dt}. \tag{4}$$

$$d\ln a = m\phi/\sqrt{6}dt = -[m\phi/\sqrt{6}][d\phi/(\sqrt{2/3}m)] = -\frac{1}{2}\phi d\phi. \tag{5}$$

If N is the number of e-folds of inflation then

$$N = \ln a_{final} - \ln a_{initial} = -\frac{1}{2}\int \phi d\phi = \frac{1}{4}[\phi^2_{initial} - \phi^2_{final}]. \tag{6}$$

Inflation continues in slow-roll approximation until $\phi^2_{final} = 1$ when the kinetic energy in the field $\frac{1}{2}\dot{\phi}^2$ becomes comparable with the potential energy $\frac{1}{2}m^2\phi^2$. At this point, the exponential expansion ends and the kinetic energy in the field is dumped into the thermal energy of particles. This ushers in the hot big bang epoch–a radiation dominated, thermal-energy filled universe. The ϕ field is damped and settles at $\phi = 0$ during this process. Now $V(\phi) = 0$ and the vacuum energy density in the massive scalar field becomes zero. (Or the tiny value of $V_0 = 10^{-120}$ if one adds a tiny constant to the formula such that $V(\phi) = V_0 + \frac{1}{2}m^2\phi^2$ to account for dark energy. But we will not be adding this V_0 term.) Fluctuations re-entering the causal horizon now left the causal horizon approximately $N = 60$ e-folds prior to the end of inflation, when according to the above equation the value of $\phi^2 = 4N + 1 = 241$. In this case, the value of the power law primordial tilt evaluated at $k_0 = .002\, Mpc^{-1}$ should be

$$n_s \approx 1 + 2(V''/V) - 3(V'/V)^2 \approx 1 - \frac{8}{\phi^2} \approx .967\, (predicted). \tag{7}$$

(cf. Easther and McAllister 2006) and the value of r, the ratio of tensor to scalar modes, should be

$$r \approx 8(V'/V)^2 \approx 32/\phi^2 \approx 8/N \approx .13\, (predicted) \tag{8}$$

(cf. Kim and Liddle 2006). Fitting the amplitude of the observed fluctuations requires

$$m = 7.8 \times 10^{-6} \tag{9}$$

(cf. Kim and Liddle 2006). Remarkably, the predicted values of n_s and r are consistent with the observed values from WMAP+BAO+Ho (Komatsu et al. 2010):

$$n_s = .968 \pm .012\, (observed) \tag{10}$$

and the 95% confidence level constraint

$$r < .24\, (observed). \tag{11}$$

The agreement between the predicted and observed values of n_s is especially impressive considering that potentials of the form $V(\phi) = (\lambda/4)\phi^4$ have been ruled out by predicting unacceptable values of $n_s = .95$ and $r = .26$ (cf. Komatsu et al. 2010). Given this, the search for the tensor modes ($r > 0$) is on–for instance, the Planck satellite hopes to improve the measurement of r. Polarization studies in the future should if successful offer a smoking-gun proof that the tensor modes are there. Such modes are not predicted by the Ekpyrotic/Cyclic scenario and, if found, they would offer a convincing proof of inflation (cf. Linde 2002).

It is remarkable that a model as simple as Linde's massive-scalar-field chaotic inflation is currently consistent with the observational data. Linde's model predicts in particular a value of n_s noticeably less than 1– a hallmark of slow-roll inflation that is indeed observed.

While more complicated potentials can produce lower values of r (cf. Kallosh and Linde 2010), this comes at the expense of adding more free parameters. The Linde theory, in contrast, is simple enough that it offers the possibility of being confirmed in a dramatic way if the observed value of r is .13. If that occurs, we will no doubt conclude that the inflation seen in the early universe is due to a massive scalar field. In that case, we argue here that we should expect a similar origin for dark energy as well.

3 Double Inflation

We propose the following simple double-inflation potential for inflation and dark energy:

$$V = \frac{1}{2}m_2^2\phi_2^2 + \frac{1}{2}m_1^2\phi_1^2 \tag{12}$$

with $m_1 \sim 10^{-5}$ and $m_2 \leq 10^{-60}$. Double inflation (Silk and Turner 1987) was introduced (typically with $m_1 \approx 5m_2$) to explain inflation alone. We will be using it with widely different mass scales to explain inflation and dark energy. The equations of motion are:

$$\ddot{\phi}_1 + 3H\dot{\phi}_1 + m^2\phi_1 = 0, \tag{13}$$

$$\ddot{\phi}_2 + 3H\dot{\phi}_2 + m^2\phi_2 = 0. \tag{14}$$

In the inflationary epoch when the universe is dominated by the vacuum energy density provided by V:

$$3H^2 = V = \frac{1}{2}m_2^2\phi_2^2 + \frac{1}{2}m_1^2\phi_1^2 \tag{15}$$

The slow-roll approximation is valid for both fields and the evolution of the two fields is given by

$$\ln\frac{\phi_2(t)}{\phi_{2,initial}} = \left(\frac{m_2^2}{m_1^2}\right)\ln\frac{\phi_1(t)}{\phi_{1,initial}} \tag{16}$$

(cf. Easther & Mcallister 2002). Since $m_2 << m_{1,}$ $\phi_2(t) \approx \textit{constant} \approx \phi_{2,initial}$ even though $\phi_1(t)/\phi_{1,initial}$ evolves considerably during inflation. There have been 60 e-folds of inflation since the perturbations now re-entering the horizon left the causal horizon, but there could have been more e-folds of inflation before that, so we expect $N > 60$ and thus $\frac{1}{4}[\phi_{1,initial}^2 - \phi_{1,final}^2] = N > 60$. Since $\phi_{1,final}^2 = 1$ marks the end of inflation, $\phi_{1,initial}^2 > (4 \times 60 + 1) = 241$. As inflation ends, the kinetic energy in the ϕ_1 field is converted into thermal particles, the motion in ϕ_1 damps and ϕ_1 comes to rest at a value of $\phi_1 = 0$. The dark energy seen today thus derives from the ϕ_2 field, and $V = \frac{1}{2}m_2^2\phi_2^2$. Since we observe a dark energy acceleration today consistent with $w \approx -1$ we expect slow-roll inflation to apply. Given the equation of motion for ϕ_2, if the universe today is flat, dark energy is the dominant source of energy density, and the field has reached an approximately constant velocity, the equations can be simplified using the slow-roll approximation as before:

$$H \approx m_2\phi_2/\sqrt{6},\ \frac{d\phi_2}{dt} \approx -(\frac{2}{3})^{1/2}m_2. \tag{17}$$

Then the value of w is given by:

$$w = \frac{\frac{1}{2}\dot{\phi}_2^2 - V}{\frac{1}{2}\dot{\phi}_2^2 + V}. \tag{18}$$

We thus have

$$w = \frac{m_2^2/3 - \frac{1}{2}m_2^2\phi_2^2}{m_2^2/3 + \frac{1}{2}m_2^2\phi_2^2} = \frac{-(3\phi_2^2 - 2)}{3\phi_2^2 + 2}. \tag{19}$$

The difference between w and -1 is given by

$$\delta w = w + 1 = \frac{4}{3\phi_2^2 + 2}. \tag{20}$$

At present, $\delta w_0 = 4/(3\phi_{2,0}^2 + 2)$. Current limits from WMAP suggest $\delta w_0 = .07 \pm .13$. Importantly, since in the future we expect ϕ_2 to eventually roll down to zero, leaving a vacuum energy density of zero (with no V_0 term), the Boltzmann brain problem disappears.

4 N-flation

A potential of the form in equation (12) governed by equations of motion (13) and (14) can be produced easily by models of N-flation. A possible criticism of the original Linde chaotic inflation is that it requires $\varphi > 1$ in Planck units. Linde argued that this was okay as long as $V < 1$. But it was felt that it would be difficult to produce values of $\phi > 1$ in string theory models. Thus, N-flation (Dimopoulos et al. 2005) has been proposed (cf. also Easther and McAllister 2006). Supersymmetric string theories allow of order 10^5 axion fields.

For such axion fields $V(\psi_i) = \mu^4[1 - \cos(\psi_i/f_i)]$ and for $\psi_i << 1$ (i.e. ψ_i significantly below the Planck mass–which we would like) we note that the potential is of the Linde quadratic form with effective mass $m = \mu^2/f_i$. As we have already pointed out, string theory allows the number of such fields to be large (Dimopoulos et al. 2005). Hence we will adopt $N = 10^4$. In this model there are N fields ψ_i (where $i = 1, \ldots, N$) with approximately equal masses $m_i \approx m$. Then the potential is given by $V = \sum \frac{1}{2}m_i^2\psi_i^2 \approx V(\phi) = \frac{1}{2}m^2\phi^2$where by definition $\phi^2 = \sum \psi_i^2$. This is so-called assisted inflation.

Since each $m_i \approx m$, all the ψ_i's evolve together via

$$\ln[\psi_i(t)/\psi_{i,initial}] = m_i^2/m_j^2 \ln[\psi_j(t)/\psi_{j,initial}] \approx \ln[\psi_j(t)/\psi_{j,initial}] \tag{21}$$

if $m_i \approx m_j$ for all i and j. All this requires is for the mass spectrum of the fields to be strongly peaked and densely packed:

$$m_i^2 = m^2 \exp[(i-1)/\sigma), \tag{22}$$

where $\sigma > 280$ for $N > 600$ (Kim and Liddle 2006). If the fields are strictly non-interacting the masses could in fact be exactly equal. Thus, in what follows we will assume that all the fields are essentially equal in mass.

We want double N-flation with a potential

$$V = \sum \frac{1}{2} m_2^2 \psi_{2,i}^2 + \sum \frac{1}{2} m_1^2 \psi_{1,i}^2 = \frac{1}{2} m_2^2 \phi_2^2 + \frac{1}{2} m_1^2 \phi_1^2 \tag{23}$$

where there are $N = 10^4$ ψ_2 fields each of mass $m_2 \approx 10^{-60}$ and $N = 10^4$ ψ_1 fields each of mass $m_1 \approx 10^{-5}$ (in keeping with our hypothesis that inflation and dark energy should arise from the same process) and by definition $\phi_2^2 = \sum \psi_{2,i}^2$ and analogously for ϕ_1^2. Initially we need $\phi_{1,initial}^2 > 241$ to produce enough inflation (> 60 e-folds) to explain our universe. With 10^4 fields, that just means that each $\psi_{1,i}^2 > .0241$ and so all the $\psi_{1,i}$'s can be sub-Planckian (< 1). This is good.

Are such low values of $m_2 \approx 10^{-60}$ plausible from the point of view of string theory? Interestingly, Kaloper and Sorbo (2005) have independently proposed just such an N field quiNtessence model for dark energy using ultralight pNGB [pseudo-Nambu Goldstone bosons] (axions) from string theory. They argue for potentials of the form $V(\psi_i) = \mu^4[1 - \cos(\psi_i/f_i)]$. Svrcek (2006) has also argued for multiple ultra-light axion fields with potentials of this form to explain dark energy.

We note in each case that for sub-Planckian $\psi_{i,initial}$'s, the potential is of the desired Linde quadratic form with effective mass $m^2 = \mu^2/f_i$. Svrcek notes that pseudoscalar axion fields have a shift symmetry and if this symmetry were exact it would set the potential to zero and the axions would be massless. In string theory the shift symmetry is broken only by nonperturbative effects. In string theory axions thus receive potential only from nonperturbative instanton effects which are exponentially suppressed by the instanton action. Hence, if the instantons have large actions they can give rise to a potential many orders of magnitude below the Planck scale.

Svrcek argues that $\mu^4 = M^4 \exp[-S_{inst}]$, where $M \sim 1$ and $S_{inst} \sim 280$, can create a vacuum energy density today comparable with what we observe for dark energy. Svrcek adds a V_0 term as well, which we eliminate as unnecessary. We would argue that if the axion fields are able to explain the amount of dark energy we observe today the V_0 term can be eliminated. Both Kaloper and Sorbo (2005) and Svrcek (2006) are explicitly creating quintessence models for dark energy. Both also note that single field models with sub-Planckian field values are unacceptable for quintessence and favor models with $N = 10^{4-5}$ fields.

We are proposing to combine these quintessence models that use N fields with the N-flation models to explain dark energy *and* inflation. Independently Svrcek (2006) also speculates "Hence, it could be that some of the string theory axions have driven inflation while others are currently responsible for [a] cosmological constant." We take the point of view here that there are $N = 10^4$ equal-mass ultra-light fields $\psi_{2,i}$ that create a slow-roll dark energy and $N = 10^4$ equal-mass heavy fields $\psi_{1,i}$ that create slow-roll inflation in the early universe.

If there are in addition singleton fields with intermediate masses, with sub-Planckian ψ_i's also, these would have not inflated but rolled down, as Svrcek notes, and some of these axion fields could have rolled down and created dark matter. They do not cause inflation because $\psi_i < 1$, so when the other thermal particles redshift so that the ψ_i vacuum field energy becomes dominant, $3H^2 = \frac{1}{2}m_i^2\psi_i^2$. Thus H is not large enough to cause the low velocity ($\dot{\psi_i}$) required for slow-roll inflation.

However, if there are many fields of essentially the same mass, $3H^2 = \sum \frac{1}{2}m_i^2\psi_i^2 \approx \frac{1}{2}m^2\phi^2$, where $\phi^2 > 1$. In this case, H is much higher (by a factor of $\sqrt{N}$) causing $\dot{\psi_i}$ to be lower and slow-roll inflation to occur. Thus, inflation only occurs when many fields congregate at the same mass scale.

In our model, then, we see two epochs of inflation: WMAP allows us to probe the last 60 e-folds of inflation in the early universe, while dark energy provides the beginning of a new epoch of inflation. Furthermore, it may allow us to learn something about the initial conditions on ϕ_1 and ϕ_2 by observing δw_0.

5 Why N-flation is superior

We expect all of the $\psi_{2,i}$'s and $\psi_{1,i}$'s to be sub-Planckian. If $N = 10^4$, this means that $\phi_{i,initial}^2 < 10^4$, which means (using equation 6) that there can be at most 2500 e-folds of inflation in our universe. Hence our universe today is less than exp[2440] times larger than the visible horizon. In Linde's original formulation of chaotic inflation, random quantum fluctuations allowed universes to give birth to universes with various values of $\phi_{1,initial}$. The ones with larger values of $\phi_{1,initial}$ grew faster until most of the volume of the multiverse was in the fastest expanding states, with $V = 1 = \frac{1}{2}m_1^2\phi_1^2$. That would mean $\phi_{1,initial}^2 \approx 10^{10}$ and the universe today would be $\exp[10^{10}]$ times larger than the part we can see.

For our model of dark energy, the simple Linde chaotic double-inflation picture would lead to eventually most of the volume of the multiverse being in the fastest expanding states, given by the ellipse

$$V = 1 = \frac{1}{2}m_2^2\phi_2^2 + \frac{1}{2}m_1^2\phi_1^2 = \frac{\phi_2^2}{2 \times (10^{60})^2} + \frac{\phi_1^2}{2 \times (10^5)^2}. \qquad (24)$$

This ellipse is very elongated in the ϕ_2 direction, and at random points on it the ϕ_2 field contributes just as much to the potential and to the inflation as the ϕ_1 field. Starting values $\phi_{2,initial} \approx (m_1/m_2)\phi_{1,initial}$ would be expected, and since ϕ_2 is slower to roll down than ϕ_1, we would not get a sub-dominant dark energy like we want.

If we use N-flation to realize the potential in equation (12) via equation (23) this problem does not occur. Since all of the fields are sub-Planckian, the fastest inflating regions are characterized by starting values of $\phi_2^2 \approx \phi_1^2$ that are bounded above by N, and since $m_2 << m_1$, the contribution of the ϕ_2 field to the potential is sub-dominant ($< \frac{1}{2}m_2^2 N$ for the ϕ_2 field versus $< \frac{1}{2}m_1^2 N$ for the ϕ_1 field).

In this picture we might expect the initial values for ϕ_2^2 and ϕ_1^2 to be comparable. What is the smallest $\phi_{1,initial}^2$ could be? Well, 241, to explain the at least 60 e-folds of inflation we see within the visible universe. By the above argument, we might expect $\phi_{2,initial}^2$ to be similar. Because of equation (16), ϕ_2 does not evolve much during the period of inflation. Its main chance to roll down is in the current epoch when H is low. But there have not been many e-folds of inflation during the current epoch, so we might expect $\phi_{2,0}^2$ to be only a little less than its initial value of about 241. That would give a value today of $\delta w_0 \approx 4/(3\phi_{2,0}^2 + 2) = .55\%$.

Since we expect the $\psi_{2,i}$'s and the $\psi_{1,i}$'s to be uncorrelated, we expect by the central limit theorem that $\phi_{2,initial}$ and $\phi_{1,initial}$ are Gaussian. If their magnitudes are comparable, in keeping with our hypothesis that the physical processes for inflation and dark energy are identical, we might a priori expect on average to find $< \phi_{2,initial}^2 > = < \phi_{1,initial}^2 >$.

Consider the probability function $P(\phi_{2,initial}, \phi_{1,initial}) d\phi_{2,initial} d\phi_{1,initial}$. If both variables are Gaussianly distributed, that distribution has circular probability contours in the ($\phi_{2,initial}$, $\phi_{1,initial}$) plane, and the probability of finding $|\phi_{2,initial}| < X|\phi_{1,initial}|$ for $X < 1$ is simply $P = (2/\pi) \arctan X$. If $\phi_{1,initial}^2 > 241$, the probability of observing a value of $\delta w_0 > Y$ is

$$P < \frac{2}{\pi} \arctan \left[\frac{1}{\sqrt{241}} \left(\frac{4}{3Y - 2/3} \right)^{1/2} \right] \tag{25}$$

via our approximate formula for δw_0. Thus, the probability of observing $\delta w_0 > 1\%$ is $P < 40\%$.

So we should not be surprised to find a small value of δw_0. On the assumption that the same physical processes led to both inflation and dark energy, if there were at least 60 e-folds of inflation in the early universe, there should be at least 60 e-folds of inflation due to dark energy ahead in the future, as indicated in the calculation above. However, the number of galaxies (and observers like ourselves) produced in our universe is proportional to

$$\exp[3N] = \exp \left[\frac{3}{4} \left(\phi_{1,initial}^2 - 1 \right) \right]. \tag{26}$$

So it might not be surprising for us to observe $< \phi_{2,initial}^2 >$ less than $< \phi_{1,initial}^2 >$, but the details depend on questions of measure which are unsettled.

Suffice it to say, one must find a measure that makes what we observe, namely $\phi_{2,initial}^2 > 6$ (since $\delta w_0 = 7\% \pm 13\%$ from WMAP) and$\phi_{1,initial}^2 > 241$

(to produce at least 60 e-folds of inflation), not particularly unlikely. For the time being, we can take an empirical approach and ask what future observations can tell us about $\phi_{2,initial.}$

6 Numerical Method

The above back-of-the-envelope calculations make the simplifying assumption that dark energy dominates the universe today (i.e. $\Omega_{DE} = 1$ today). In fact $\Omega_{DE} = .73$ and $\Omega_M = .27$. We can integrate the equation of motion for ϕ_2 (eq. 14) numerically by solving the Friedmann equation for $H(t)$ given these values. We can then calculate w directly from equations 12 and 18 assuming $\phi_1 = 0$. Note that, since the universe also has dark matter, $H(t)$ is larger, particularly in the past, than if we had dark energy alone. This implies even less movement in ϕ_2 from its initial value than in the back-of-the-envelope calculation above.

For starting conditions at $t = 0$ we will assume $d\phi_2/dt$=0 since $H(t)$ tends to infinity as t tends to zero. We discuss the starting condition on ϕ_2 itself later. Since in this section we will consider only the ϕ_2 field with mass m_2, for greater legibility we suppress the subscripts on ϕ_2 and m_2. We begin with the equation

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0.$$

where $H = \frac{1}{a}\frac{da}{dt}$ is the Hubble constant. Define a new variable for time $\tau = t/t_{H0} = tH_0$. Transforming $\ddot{\phi}$ and $\dot{\phi}$, we have $\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{d\tau^2}\frac{d\tau^2}{dt^2} = \phi'' H_0^2$ and $\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \phi' H_0$, where prime denotes a derivative with respect to the new time variable τ. We thus have upon substitution

$$H_0^2\phi'' + 3H(t)H_0\phi' + m^2\phi = 0.$$

Dividing through by H_0^2, we obtain

$$\phi'' + 3\frac{H(t)}{H_0}\phi' + \frac{m^2}{H_0^2}\phi = 0.$$

Evidently, H is still in terms of t; to change this, we now convert the Friedmann equation from time variable t to time variable τ because so doing will yield $H(\tau)$. We begin with the Friedmann equation in terms of t, where to good approximation $w = -1$ and the universe is flat:

$$H^2(t) \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 \approx H_0^2\left(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda\right).$$

Substituting $\frac{da}{dt} = \frac{da}{d\tau}\frac{d\tau}{dt} = \frac{da}{d\tau}H_0$, we obtain

$$H^2(\tau) \equiv \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 \approx \left(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda\right) = \frac{H^2(t)}{H_0^2}.$$

We thus find the equation for ϕ

$$\phi'' + 3H(\tau)\phi' + m_*^2\phi = 0,$$

where we have defined $m_*^2 = m^2/H_0^2$. In principle, we can now solve the Friedmann equation numerically for $H(\tau)$ and using it obtain ϕ numerically. Note that we can compute τ_0, the value of our new time variable today, from WMAP-7 values:

$$\tau_0 = H_0 t_0 = 71.0\,km/s/Mpc \times 13.75\,Gyr = .998.$$

However, there is a constraint on $\phi_{2,0}$ that we must satisfy as we solve numerically. The universe is flat with $\Omega_{DE} + \Omega_M \approx 1$. Since dark matter is present, the value of H_0^2 is larger than it would be if only dark energy were present by a factor of Ω_{DE}^{-1}. WMAP-7 gives $\Omega_{DE} = .73$ (Komatsu et al. 2010). Thus,

$$3H_0^2 = \frac{1}{2\Omega_{DE}} m^2 \phi_0^2 X,$$

where $X = (\dot{\phi_0}^2 + m^2\phi_0^2)/m^2\phi_0^2$.
Using the definitions of w and δw, we find that

$$m_*^2 = \frac{3\Omega_{DE}(2 - \delta w_0)}{\phi_{2,0}^2}.$$

Hence we are not free to choose both m_* and $\phi_{2,initial}$ independently because $\phi_{2,initial}$ will determine $\phi_{2,0}$, which must be consistent with m_*. We therefore require a method of solving the equation of motion where we can set $\phi_{2,initial}$ (and hence $\phi_{2,0}$) after we already have a solution. This motivates us to observe that the equation can be dynamically rescaled by writing $\phi'/\phi = \zeta(\tau)$. It is evident that ϕ is always non-zero. Writing $\phi' = \zeta\phi$, we find that $\phi'' = \zeta\phi' + \zeta'\phi = \left(\zeta^2 + \zeta'\right)\phi$. Substituting into the rescaled equation for ϕ, we obtain

$$\left(\zeta^2 + \zeta'\right)\phi + 3H(\tau)\zeta\phi + m_*^2\phi = 0,$$

which leads to

$$\zeta^2 + \zeta' + 3H(\tau)\zeta + m_*^2 = 0.$$

We choose m_* and numerically solve this equation beginning at $\tau = 0$, where we have $\zeta = 0$ because of our earlier comment on $\dot{\phi}_{2,initial}$. This gives us δw_0, and using it, we can determine $\phi_{2,0}$ by inverting the formula for m_*^2.

7 Numerical Results

Figure 1: Plot comparing the slow-roll, dark energy-only approximate formula $\delta w_0 = 4/(3\phi_{2,0}^2 + 2)$ with the numerical results from solving the full Friedmann equation, using it to solve the equation of motion for ζ, and finally computing $\phi_{2,0}$ as outlined in Section 6. As we expect, δw_0 is smaller for the numerical than the approximate results because $\Omega_{DE} < 1$ makes H higher, although the agreement is fairly good nonetheless. It should particularly be noted that the approximate formula was derived on the assumption $\delta w << 1$, and the plot shows that the better this assumption is satisfied, the better the agreement between approximate and numerical results.

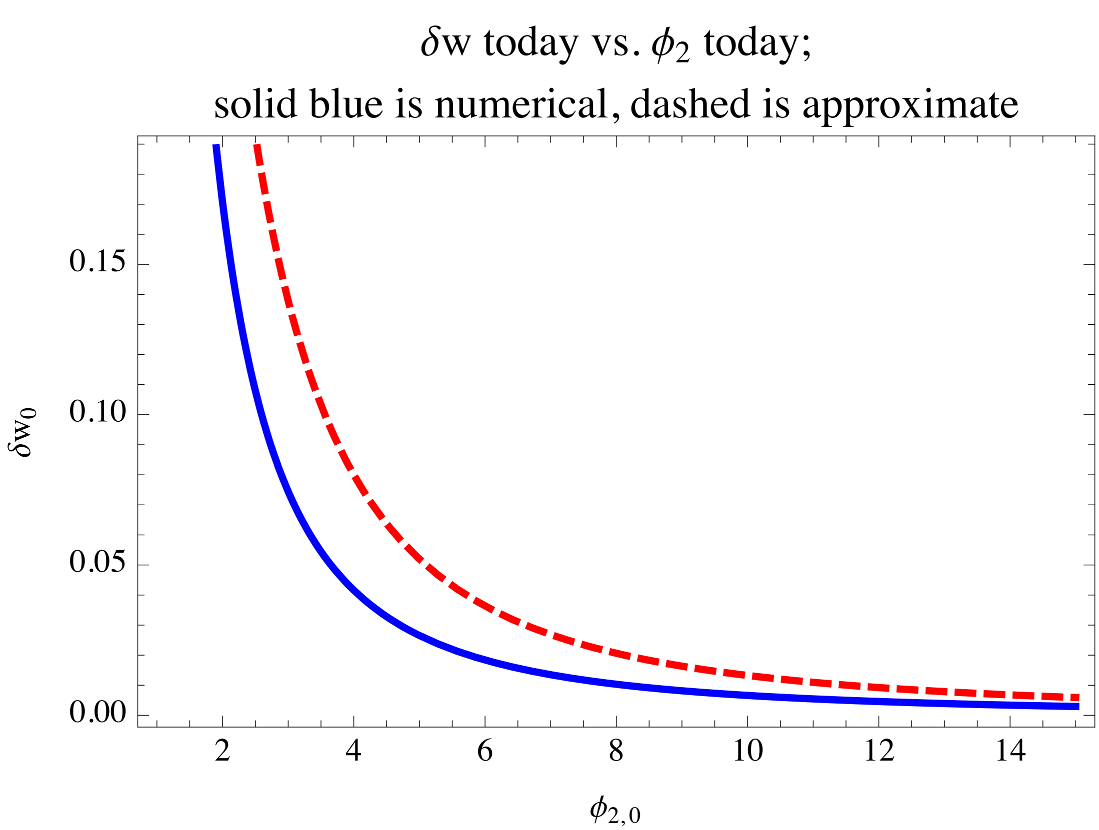


Figure 2: Plot of δw as a function of redshift for different values of $\phi_{2,0}$. The values of $\phi_{2,0}$ are listed in the order that the curves go bottom to top ($\phi_{2,0} = 15$ is the blue curve, $\phi_{2,0} = 11.6$ is the red curve, etc.) These values respectively correspond to $\phi_{2,initial} = 15.05,\ 11.64,\ 6.62,$ and 5.12. Notice that, in accord with our back-of-the-envelope calculation, larger values of $\phi_{2,0}$ lead to smaller values of δw. Note also that as the curves are rather steep functions of redshift, if our model is correct it will be more challenging to observe deviations from $w = -1$ than if we assumed that δw was constant. We chose $\phi_{2,0} = 15$ on the assumption that $\phi_{2,0} \simeq \phi_{2,initial} \simeq \phi_{1,initial}$ and $\phi_{1,initial} \simeq 15$ because that is the minimum value that can provide approximately 60 e-folds of inflation. The other values of $\phi_{2,0}$ were chosen because the approximate formula (see Section 3) implies that they should lead to $\delta w_0 \sim 1\%,\ 3\%,$ and 5%, respectively. As is evident from the $z = 0$ intercepts here, as well as from the previous plot (Figure 1), these estimates are somewhat over-generous.

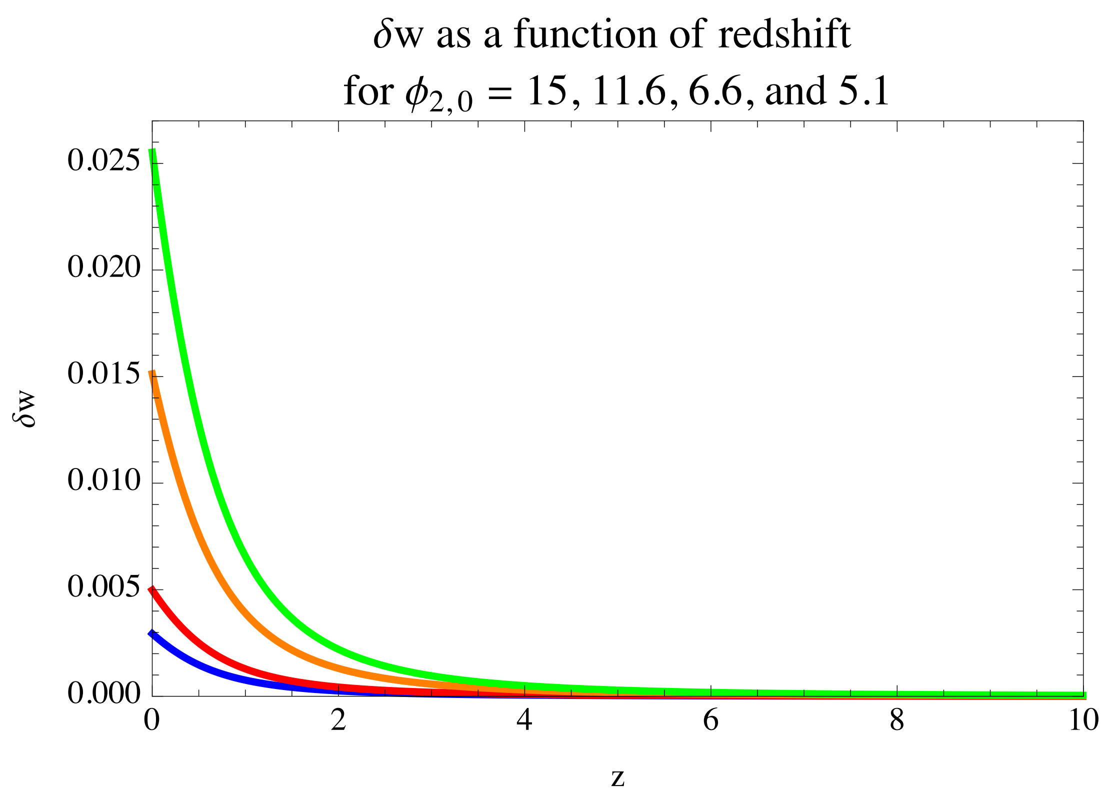


Figure 3: Plot of dw/dz as a function of redshift. The values of $\phi_{2,0}$ are listed in the order that the curves go top to bottom ($\phi_{2,0} = 15$ is the blue curve, $\phi_{2,0} = 11.6$ is the red curve, etc), and chosen to be the same as those in the previous plot (Figure 2). Since in our model $w = -1 + \delta w$, this is equivalent to a plot of $d\delta w/dz$, and therefore upon comparison to the previous plot of δw vs. z (Figure 2), it is hardly surprising that dw/dz is only significantly non-zero close to the present day. The most important result this plot shows is that $dw/dz \sim -\delta w$.

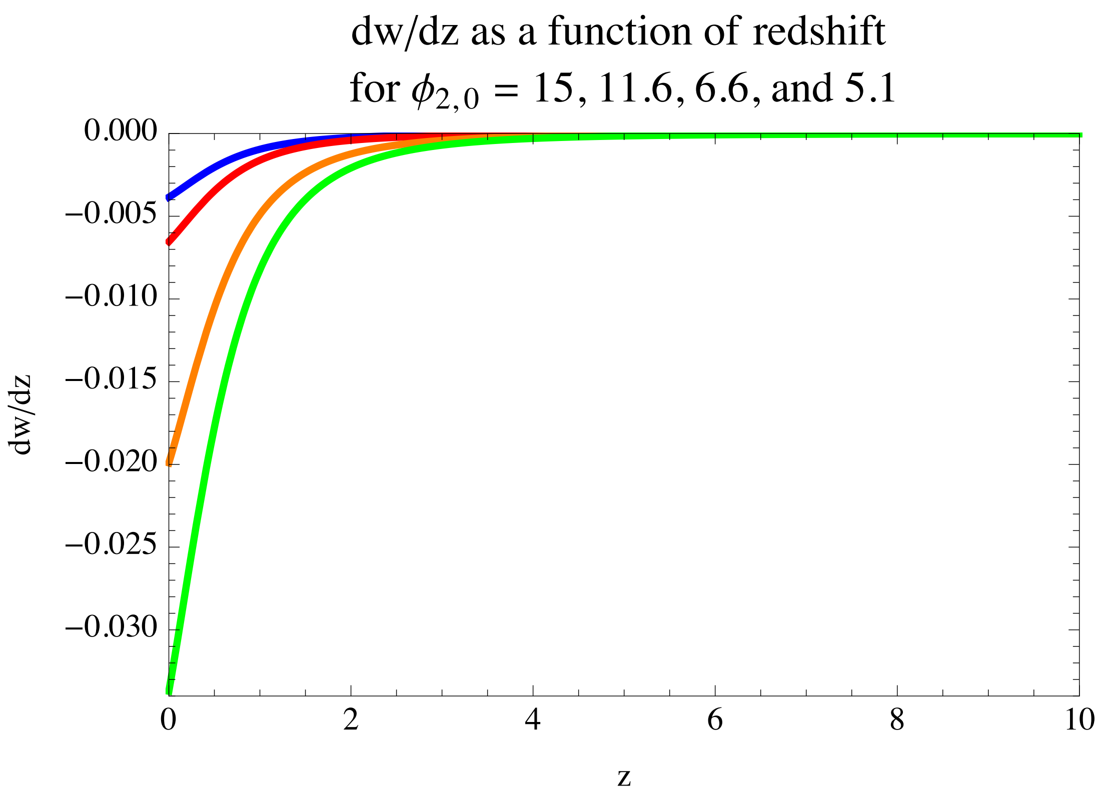


Figure 4: Plot of a lower bound on the probability that δw_0 is less than a given value. We obtained this by assuming that $\phi_{2,initial}$ and $\phi_{1,initial}$ are Gaussian random variables with zero means and standard deviations $< \phi_{1,initial}^2 >^{1/2} = < \phi_{2,initial}^2 >^{1/2}$ and that $\phi_{2,0} \simeq \phi_{2,initial}$. It can then be shown that the distribution of their ratio is Cauchy, and from this we compute the probability that ϕ_2 is larger than a certain multiple of ϕ_1, which corresponds to δw less than a certain value $\delta w'$. This is a lower bound on the cumulative distribution function because in fact $\phi_{1,initial}$ could be much greater than 15 (chosen to provide approximately 60 e-folds of inflation), leading to larger values of ϕ_2. However, the plot is moderately encouraging: there is still a nearly 10% chance that δw_0 could be as large as 2%, which would likely be detectable.

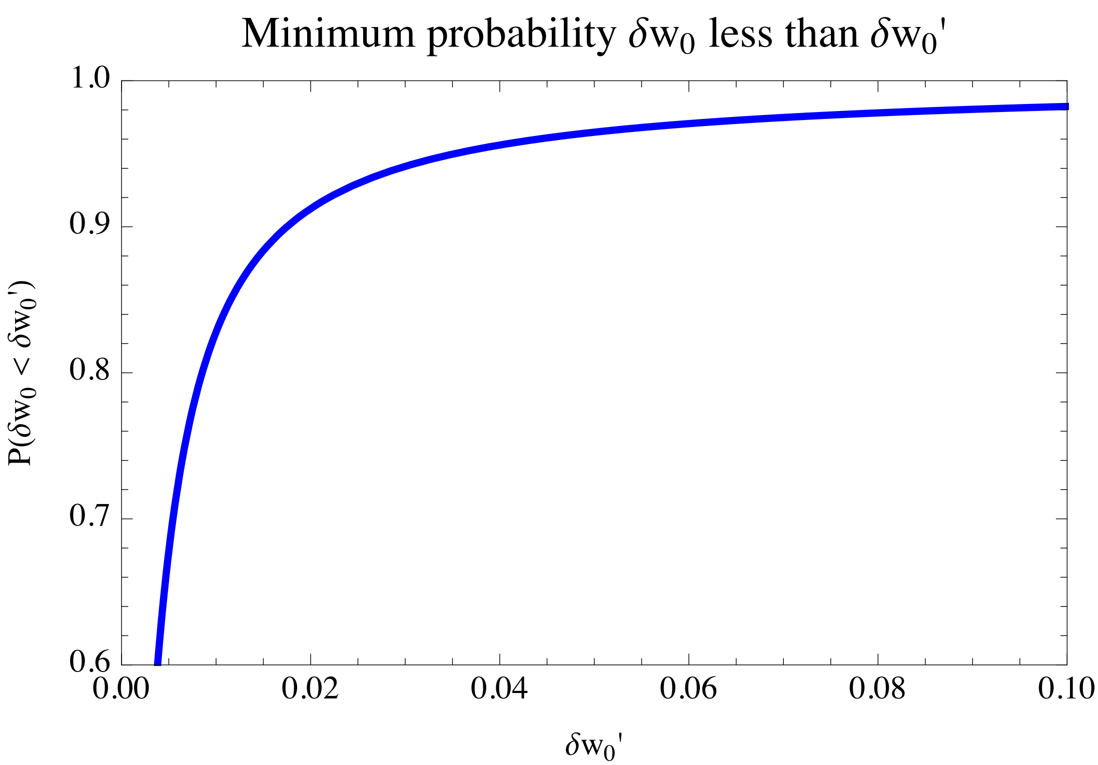


Figure 5: Here, we compare our model's predictions for w as a function of redshift to those of the popular toy model $w = w_0 + w_a(1 - a)$, where w_0 and w_a are constants and a is the scale factor appearing in the Friedmann equation. The toy model is the shallower, upper curve. We choose the parameters w_0 and w_a so that the toy model agrees with ours at $a = 1$ and at $a = 0$ (today and at the Big Bang). This leads to $w_0 = -1 + \delta w_0$ and $w_a = -\delta w_0$. The comparison shows that it will be harder to observe deviations from $w = -1$ in the past if our model is correct than if the toy model is correct. The value of $\phi_{2,0}$ we have chosen corresponds to a predicted δw_0 of 3% (approximate formula) and 1.5% (numerical).

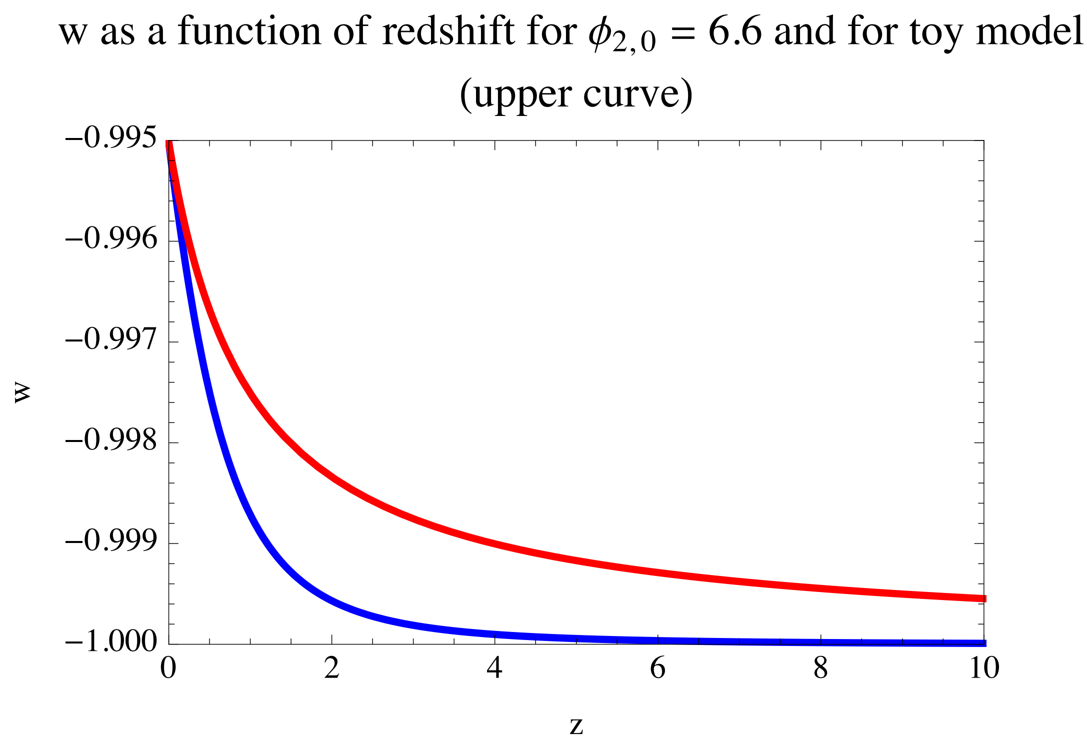
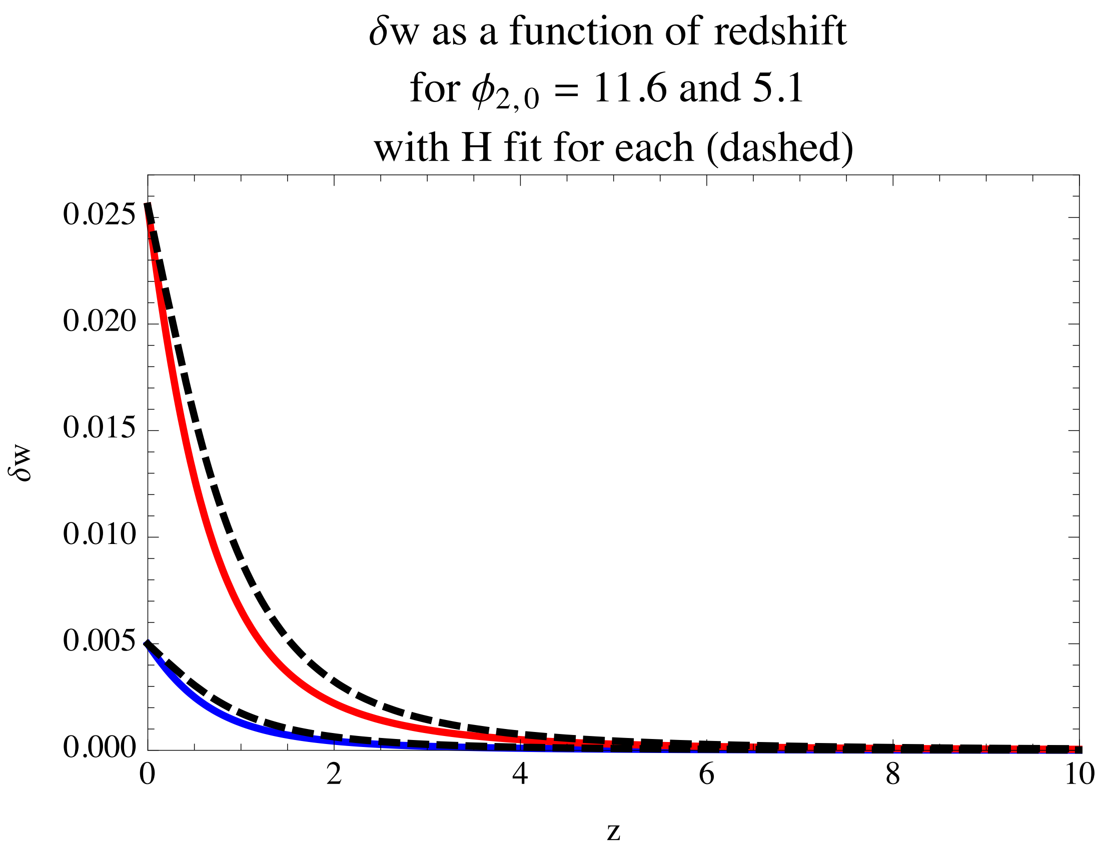


Figure 6: Here, we show our numerical results for δw as a function of redshift and also the slow-roll formula (dashed) $\delta w = \delta w_0 H^{-2}(\tau(z))$. Notice that it is a better approximation for smaller δw, which makes sense because this is where slow roll is most valid. $\phi_{2,0} = 11.6$ corresponds to the lower solid curve, $\phi_{2,0} = 5.1$ to the upper solid curve.



As Figure 6 notes, our numerical results are fit well by the approximate formula $\delta w(z) \approx \delta w_0 \left(\frac{H_0}{H(z)}\right)^2$. This should be true in any slow roll inflation model, because if $\ddot{\phi}$ is small then $3H\dot{\phi} \simeq -m^2\phi$. Thus, if $m^2\phi$ doesn't vary much, $\dot{\phi} \propto H^{-1}$. For small δw, $\delta w \propto \dot{\phi}^2 \propto H^{-2}$. Slow roll inflation is the best chance to observe a detectable value of δw_0. Knowing the functional form of $w(z)$ may be useful to observational programs.

8 Acknowledgements

The authors thank Bharat Ratra for helpful conversations.

9 References

Albrecht, A., Bernstein, G., Cahn, R., Freedman, W., Hewitt, J., Hu, W., Huth, J., Kamionkowski, Kolb, , Knox, L., Mather, J., Staggs, S., Suntzeff, N. B., "Report of the Dark Energy Task Force," arXiv:astro-ph/0609591 (2006)

de Simone, A., Guth, A. H., Linde, A., Noorbala, M., Salem, M. P., Vilenkin, A., "Boltzmann brains and the scale-factor cutoff measure of the multiverse," Physical Rev. D, Vol. 82, Issue 6, id 063520.

Dimopoulos, S., Kachru, S., McGreevy, J., & Wacker, J.G., " N-flation," arXiv:hep-th/0507205 (2005).

Easther, R. & McAllister, L., "Random Matrices and the Spectrum of N-flation," arXiv:hep-th/0512102 (2006).

Gott, J. R., "Boltzmann Brains—I'd rather see than be one", arXiv: 0802.0233 (2008).

Guth, A. H., "Inflationary universe: A possible solution to the horizon and flatness problems," Physical Review D, Vol. 23, Issue 2, 347-356 (1981).

Hawking, S. W. "The Occurrence of Singularities in Cosmology," Proc. Royal Soc. London, Series A, Mathematical and Physical Sciences, Volume 294, Issue 1439, 511-521 (1966).

Kallosh, R, and Linde, A., "New Models of Chaotic Inflation in Supergravity," arXiv: 1008.3375 [hep-ph] (2010).

Kaloper, N., & Sorbo, L., "Of pNGB QuiNtessence," arXiv: astro-ph/0511543 (2005).

Kim, S. A. & Liddle, R., "Nflation: multifield dynamics perturbations." arXiv: astro-ph/060560 (2006).

Komatsu, E., et al. "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation" Submitted to Astrophysical Journal Supplement Series (2010).

Linde, A. D., "Chaotic Inflation," Phys. Lett. B. 129, 177 (1983).

Linde, A. D., "Inflationary Theory versus Ekpyrotic/Cyclic Scenario," arXiv: hep-th/0205259 (2002).

Silk, J., Turner, M.S., "Double Inflation," Physical Rev. D, Vol. 35, Issue 2, 419-428 (1987).

Svrcek, P., "Cosmological Constant and Axions in String Theory," arXiv:hep-th/0607086 (2006).

Zunckel, C., Gott, J.R., Lunnan, R., " Using the Topology of Large Scale Structure to constrain Dark Energy," accepted $MNRAS$, arXiv:1005.3631[astro-ph](2010).