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Self-Intersecting Periodic Curves in the Plane

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Abstract

Suppose a smooth planar curve γ is 2π-periodic in the x direction and the length of one period is ℓ. It is shown that if γ self-intersects, then it has a segment of length $\ell - 2\pi$ on which it self-intersects and somewhere its curvature is at least $2\pi/(\ell - 2\pi)$. The proof involves the projection Γ of γ onto a cylinder. (The complex relation between γ and Γ was recently observed analytically in [1], see also [5, Ch. 10]). When γ is in general position there is a bijection between self-intersection points of γ modulo the periodicity, and self-intersection points of Γ with winding number 0 around the cylinder. However, our proof depends on the observation that a loop in Γ with winding number 1 leads to a self-intersection point of γ.

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Let a smooth 2π-periodic curve γ in the (x, y)-plane be parametrized by arc-length as follows:

$$\begin{cases} \gamma = \{p(s) : s \in \mathbb{R}\}, \quad p(s) = (u(s), v(s)), \\ u(s+\ell) = 2\pi + u(s), \quad v(s+\ell) = v(s), \qquad\qquad s \in \mathbb{R}. \\ u'(s)^2 + v'(s)^2 = 1, \end{cases}$$

The length of one period of γ is ℓ and $q \in \gamma$ is called a *crossing* if $q = p(s_1) = p(s_2)$ and $s_1 \neq s_2$. Note that crossings exist if and only if p is not injective. A crossing q is called *simple* if there are exactly two real numbers $s_1 \neq s_2$ with $p(s_1) = p(s_2) = q$ and if $p'(s_1) \neq p'(s_2)$ when $p(s_1) = p(s_2)$ and $s_1 \neq s_2$. Note that the smooth curve γ can be approximated arbitrarily closely by smooth curves in *general position*, that is with all crossings simple. If γ is in general position, then it follows from the smoothness that the set of crossings is discrete, and hence finite by compactness. Let $p'(s) = (\cos\vartheta(s), \sin\vartheta(s)),\ s \in \mathbb{R}$, where ϑ is smooth [3, Prop. 2.2.1]. The goal is to establish the following which is intuitively obvious. (A *periodic segment* of γ is a segment of the form $\{p(t) : t \in [a, a+\ell]\}$.)

Proposition. *Suppose that all crossings of γ are simple.*

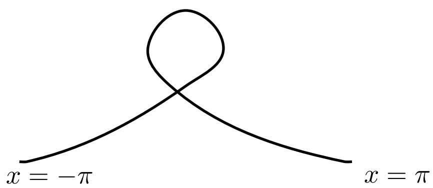
(a) If p is injective on every interval of length $\ell - 2\pi$, p is injective.

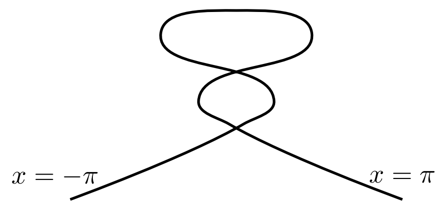
(b) If p is not injective its curvature is somewhere at least $2\pi/(\ell - 2\pi)$.

(c) If p is not injective and ϑ is periodic, then γ has a periodic segment which contains two crossings.

The global problem of bounding from below the maximum curvature of a self-intersecting periodic planar curve arose in a study of water waves beneath an elastic sheet. In the model [6], the sheet energy increases with the curvature and, roughly speaking, the conclusion needed was that sheets of certain energies could not self-intersect.

Remark. Periodicity of ϑ in the Proposition does not follow from that of p, as the first diagram below shows. Part (c) of the Proposition is illustrated in the second diagram, where ϑ is periodic.





□

For a proof, we project γ onto the cylinder $\mathbf{C} = S^1 \times \mathbb{R}$, where $S^1 = \{e^{i\phi} : \phi \in \mathbb{R}\}$. Let $P : \mathbb{R} \to \mathbf{C}$ be given by $P(s) = (e^{iu(s)}, v(s))$ and let $\Gamma = \{P(s) : s \in [0, \ell]\}$. Thus the projection of the periodic, non-compact curve γ in $\mathbb{R}^2$ onto $\mathbf{C}$ is the compact curve Γ. Now Γ has a crossing Q if $P(s_0) = P(t_0) = Q$ for some $0 \le t_0 < s_0 < \ell$ and we note that

$$P(s_0) = P(t_0) \text{ if and only if } p(s_0) = p(t_0) + k(2\pi, 0) = p(t_0 + k\ell), \quad k \in \mathbb{Z},$$

where $k = \#(\Gamma_Q)$, the winding number around $\mathbf{C}$ of

$$\Gamma_Q = \{P(s) : s \in [t_0, s_0]\}, \tag{1}$$

a loop at Q. Crossings of Γ with winding number k correspond to the existence of horizontal chords with length $2|k|\pi$ connecting points of γ. Significantly for the Proposition, there is a one-to-one correspondence between crossings of γ and crossings of Γ with winding number zero. Note that $\#(\Gamma) = 1$, since $P(\ell) = P(0)$ and $p(\ell) = p(0) + (2\pi, 0)$.

Lemma 1. *Suppose that $\#(\Gamma_Q) \in \{0, 1\}$ for a crossing Q of Γ. Then p is not injective on some interval of length ℓ.*

Proof. By hypothesis $\Gamma_Q := \{P(s) : s \in [t_0, s_0]\}$, $[t_0, s_0] \subset [0, \ell)$ and

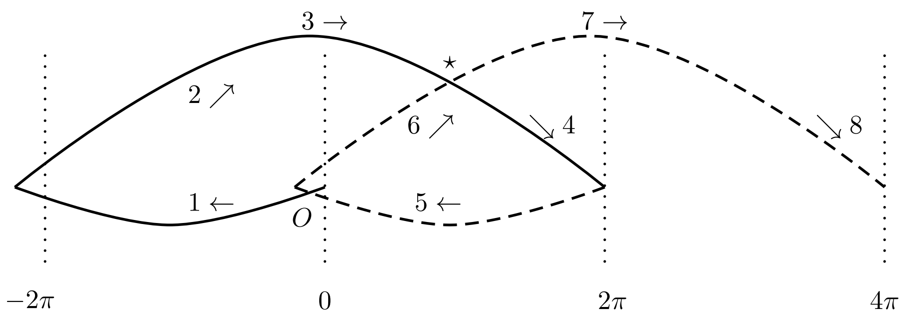
$$u(s_0) = u(t_0) + 2k\pi \text{ for } k \in \{0, 1\}, \quad v(s_0) = v(t_0).$$

If $k = 0$, $p(s_0) = p(t_0)$ and the conclusion holds. If $k = 1$,

$$p(s_0) = p(t_0 + \ell), \quad 0 < t_0 + \ell - s_0 < \ell,$$

and again the conclusion holds. □

Remark. Note that if $\#(\Gamma_Q) = -1$, the proof of Lemma 1 leads only to the conclusion that there is an interval of length 2ℓ on which p is not injective, as illustrated in the example below.



The segment $1 \to 2 \to 3 \to 4$, in which arrows denote increasing arc-length, represents one period of γ in $\mathbb{R}^2$. The dashed curve $5 \to 6 \to 7 \to 8$ represents the next period. The segment numbered 1 contains a sub-loop of Γ on $\mathbf{C}$ with winding number -1 and the construction just described leads to the crossing O on γ. However, the length of the corresponding closed sub-arc of $1 \to 2 \to 3 \to 4 \to 5$ in $\mathbb{R}^2$ lies between ℓ and 2ℓ which does not vindicate the Proposition. However, there is another crossing $\star$ on γ, and the closed loop $4 \to 5 \to 6$ satisfies the conclusion of the Proposition. □

The following is the key.

Lemma 2. *Suppose the crossings of* Γ *are all simple. For any loop at* $\widetilde{Q}$ *of the form* $\Gamma_{\widetilde{Q}} = \{P(s) : s \in [a, b]\}$, $P(a) = P(b) = \widetilde{Q}$, *with* $\#(\widetilde{\Gamma}) > 1$, *there exists a sub-loop at* $\widetilde{Q}_1$ *of the form* $\Gamma_{\widetilde{Q}} := \{P(s) : s \in [a_1, b_1]\}$, $P(a_1) = P(b_1) = \widetilde{Q}_1$, $a \le a_1 < b_1 < b$, *with* $\#(\Gamma_{\widetilde{Q}_1}) = 1$.

Proof. Since $\#(\Gamma_{\widetilde{Q}}) > 1$ it follows from the topology of the cylinder that $\Gamma_{\widetilde{Q}}$ has a crossing. The proof is by induction on the number of crossings.

If $\Gamma_{\widetilde{Q}}$ has only one crossing, $\Gamma_{\widetilde{Q}}$ is the union of two loops, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, based at a point of $\Gamma_{\widetilde{Q}}$. Since they have no crossings, each has winding number ± 1 or 0. Since the sum of their winding numbers is $\#(\Gamma_{\widetilde{Q}}) > 1$, each has winding number 1 and $\#(\Gamma_{\widetilde{Q}}) = 2$. If $\widetilde{Q} \in \widetilde{\Gamma}_2$, then the sub-path $\widetilde{\Gamma}_1$ satisfies the conclusion of the lemma, and vice versa.

Now we make the inductive hypothesis that the lemma holds for any loop $\Gamma_{\widetilde{Q}}$ of the form in the lemma with no more than $N - 1$ crossings, $N \ge 2$.

Suppose a loop $\Gamma_{\widehat{Q}} = \{P(s) : s \in [\widehat{a}, \widehat{b}]\}$, $P(\widehat{a}) = P(\widehat{b}) = \widehat{Q}$, has N crossings. Choose one of them, $P(s_1) = P(t_1) =: \widetilde{Q}$, say. This splits $\Gamma_{\widehat{Q}}$ into two loops, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, based at $\widetilde{Q}$. If they both have winding number 1, then the result follows, exactly as in the case $N = 1$ above. Otherwise one of them, $\widetilde{\Gamma}_1$ say, has winding number at least 2 and no more than $N - 1$ crossings.

Now, momentarily, let $\widetilde{Q}$ be the origin of arc length so that $\widetilde{\Gamma}_1 = \{P(s) : s \in [0, \widetilde{t}]\}$ where s is arc length measured from $\widetilde{Q}$ along $\widetilde{\Gamma}_1$. Then, by induction, there is a loop $\widetilde{\Gamma}_{11}$ in $\widetilde{\Gamma}_1$, satisfying the conclusion of the lemma with $[0, \widetilde{t}]$ instead of $[a, b]$, and winding number 1.

If $\widetilde{\Gamma}_{11}$ does not contain $\widehat{Q}$, then $\widetilde{\Gamma}_{11}$ with the original parametrization satisfies the conclusion of the lemma.

If $\widetilde{\Gamma}_{11}$ does contain $\widehat{Q}$, then its complement in $\widehat{\Gamma}$ is a sub-path $\widetilde{\Gamma}_{12} = \{P(s) : s \in [a', b'] \subset [a, b]\}$ of $\widehat{\Gamma}$, with winding number not smaller than 1 and no more than $N - 1$ crossings.

If the winding number of $\widetilde{\Gamma}_{12}$ is 1, then we are done. If it exceeds 1, then the required conclusion follows from the inductive hypothesis. □

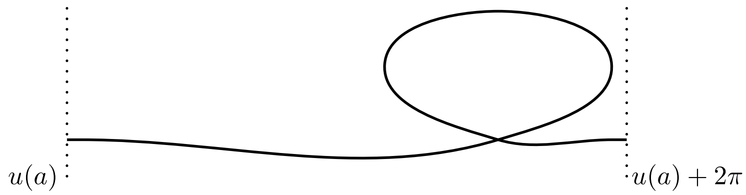
Lemma 3. *If $\#(\Gamma_Q) > 1$ for a crossing Q of Γ, then p is not injective on some closed interval of length ℓ.*

Proof. Assume first that all the crossings of the original curve Γ are simple. Putting $\widetilde{\Gamma} = \Gamma_Q$ in Lemma 2 gives the existence of a crossing of Γ with winding number 1. The required result follows by Lemma 1 when all the crossings of Γ are simple. If the crossings of Γ are not all simple, apply the conclusion of Lemma 2 to a uniform periodic approximation γ_1 of γ parametrized by a smooth periodic function p_1 with the property that each crossing of Γ_1 is simple and close to a crossing of Γ. The required result in the general case will follow by a simple limiting argument. □

Proof of the Proposition. (a) If p is not injective, Γ has a crossing, Q. Suppose $P(t_0) = P(s_0)$, $0 \le t_0 < s_0 < \ell$, Then, in the notation of (1), $\Gamma_Q = \{P(s) : s \in [t_0, s_0]\}$ and there is a minimal sub loop $\Gamma_{Q_1} = \{P(s) : s \in [t_1, s_1]\}$ of Γ_Q (a loop in Γ_Q which has no proper sub loop) $[t_1, s_1] \subset [t_0, s_0]$, $P(s_1) = P(t_1) =: Q_1$. Since Γ_{Q_1} has no crossings, $|\#(\Gamma_{Q_1})| \le 1$.

Now we observe that if p is not injective, then it is not injective on some interval of length ℓ. If $\#(\Gamma_{Q_1}) \in \{0, 1\}$, the observation holds by Lemma 1. If $\#(\Gamma_{Q_1}) = -1$, since $\#(\Gamma) = 1$, the complement of Γ_{Q_1} in Γ has winding number 2 and the observation holds, by Lemma 3.

Now consider an interval $[a, a + \ell]$ on which p is not injective. Since $p(a + \ell) = p(a) + (2\pi, 0)$, it follows easily (from the diagram below!) that the length of any loop in this periodic segment of γ does not exceed $\ell - 2\pi$. Hence there is an interval of length $\ell - 2\pi$ on which p is not injective.

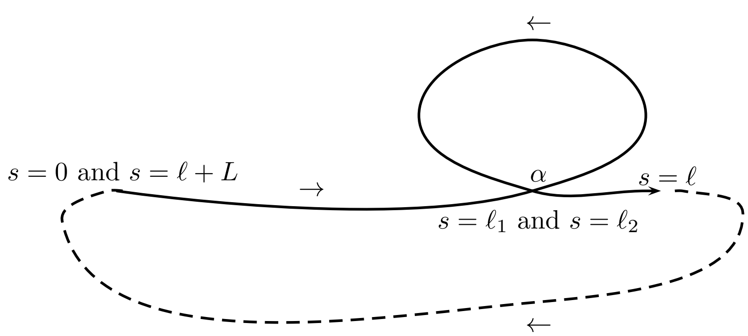


(b) A classical result [4] in the case of plane curves is the following [2, Remark on p. 38].

Axel Schur (1921). *Suppose that* $\Upsilon_i = \{\upsilon_i(s) : s \in [0, S]\}$*,* $i = 1, 2$*, are two plane curves parametrized by arc length, with the same length* S *and with curvatures* $\kappa_i(s)$ *at* $\upsilon_i(s)$*. Suppose that* Υ_1 *has no self-intersections and, along with the chord from* $\upsilon_1(0)$ *to* $\upsilon_1(S)$*, bounds a convex region. Furthermore, suppose that* $|\kappa_2| \leq \kappa_1$ *on* $[0, S]$*. Then* $|\upsilon_2(s) - \upsilon_2(0)| \geq |\upsilon_1(s) - \upsilon_1(0)|$*,* $s \in [0, S]$*.*

Let Υ_2 be a closed loop in γ with length S no greater than $\ell - 2\pi$ and suppose that at every point its curvature $|\kappa_2| \leq 2\pi(1-\epsilon)/(\ell - 2\pi)$ for some $\epsilon > 0$. Let Υ_1 be the segment of length S of a circle of radius $(\ell - 2\pi)/(2\pi(1-\epsilon))$. Now $|\kappa_2| \leq \kappa_1$, Υ_1 is not closed but Υ_2 is closed, which contradicts Schur's result. Hence no such ϵ exists, which proves (b).

(c) Consider a periodic segment of γ with only one crossing at an angle α, as illustrated by the solid line in the diagram. Now extend this segment as a smooth closed curve of length $\ell + L$ with no further crossings (the extension is the dashed curve $\widetilde{\gamma}$).



By the hypothesis of part (c),

$$\int_0^\ell \vartheta'(s)\, ds = 0, \text{ and by construction, } \int_\ell^{\ell+L} \vartheta'(s) ds = -2\pi.$$

So, from the hypothesis, the integral of ϑ' around the oriented loop $\gamma \cup \widetilde{\gamma}$ is -2π. On the other hand, by the Hopf's Umlaufsatz for curvilinear polygons [3,

§13.2],

$$\left|\int_{\ell_1}^{\ell_2} \vartheta'(s)ds\right| = \pi + \alpha = \left|\int_{\ell_2}^{\ell+L} \vartheta'(s)ds + \int_0^{\ell_1} \vartheta'(s)ds\right|.$$

This is impossible since $\alpha \notin \{0, \pi\}$, because all crossings are simple. This contradiction completes the proof.

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