ℓ^2-Linear Independence for the System of Integer Translates of a Square Integrable Function

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Abstract

We prove that if the system of integer translates of a square integrable function is ℓ^2-linear independent then its periodization function is strictly positive almost everywhere. Indeed we show that the above inference is true for any square integrable function if and only if the following statement on Fourier analysis is true: For any (Lebesgue) measurable subset A of $[0,1]$, with positive measure, there exists a non trivial square summable function, with support in A, whose partial sums of Fourier series are uniformly bounded.

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1. Introduction

Given a square integrable function $\psi \in L^2(\mathbb{R})$, many properties of the system of integer translates

$$\mathscr{B}_\psi = \{T_k\psi, k \in \mathbb{Z}\}, \quad T_k\psi(x) = \psi(x-k), \quad x \in \mathbb{R},\, k \in \mathbb{Z},$$

can be completely described in terms of properties of the 1-periodic function

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\ \hat{\psi}(\xi+k)\ |^2, \quad \xi \in \mathbb{R},$$

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called the periodization function of ψ (note that $p_\psi \in L^1(\mathbb{T})$). Systems of integer translates arise in the context of wavelet analysis and, more generally, in the theory of shift invariant spaces. We refer the reader to the work of Hernández, Šikić, Weiss, and Wilson, [1] and to references contained therein for a comprehensive summary of these properties.

In this paper we focus on ℓ^2-linear independence. First of all, let us pass to recall different concepts of independence in great generality.

Definition 1. Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space H. We say that

(i) $(e_n)_{n\in\mathbb{N}}$ is *linearly independent* if every finite subsequence of $(e_n)_{n\in\mathbb{N}}$ is linearly independent.

(ii) $(e_n)_{n\in\mathbb{N}}$ is ℓ^2- *linearly independent* if whenever the series $\sum_{n=0}^{+\infty} c_n e_n$ is convergent and equal to zero for some coefficients $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, then necessarily $c_n = 0$ for all $n \in \mathbb{N}$.

(iii) $(e_n)_{n\in\mathbb{N}}$ is *ω-independent* if whenever the series $\sum_{n=0}^{+\infty} c_n e_n$ is convergent and equal to zero for some scalar coefficients $(c_n)_{n\in\mathbb{N}}$, then necessarily $c_n = 0$ for all $n \in \mathbb{N}$.

(iv) $(e_n)_{n\in\mathbb{N}}$ is *minimal* if for all $k \in \mathbb{N}$, $e_k \notin \overline{\operatorname{span}\{e_n,\, n \neq k\}}$.

Since we will not always be dealing with unconditionally convergent series, we order $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ as it is usually done with Fourier series.

Hence by $\sum_{k\in\mathbb{Z}} c_k T_k \psi = 0$ we mean $\lim_{n\to+\infty} \sum_{|k|\leq n} c_k T_k \psi = 0$.

So $\mathscr{B}_\psi$ is ℓ^2- linearly independent if and only if whenever $\{c_k\} \in \ell^2$ and

$$\lim_{n\to+\infty} \left\| \sum_{|k|\leq n} c_k\, T_k \psi \right\|_2 = 0,$$

then necessarily $c_k = 0$ for all $k \in \mathbb{Z}$.

Relations between the various type of independence for $\mathscr{B}_\psi$ and properties of the periodization function are summarized in the following scheme:

$$\begin{array}{|ccc|}\hline \mathscr{B}_\psi \text{ is minimal} & \Longleftrightarrow & \dfrac{1}{p_\psi} \in L^1(\mathbb{T}) \\ \Downarrow & & \\ \mathscr{B}_\psi \text{ is } \omega\text{-independent} & & \\ \Downarrow & & \\ \mathscr{B}_\psi \text{ is } \ell^2\text{-linearly independent} & \Longleftarrow & p_\psi(\xi) > 0 \text{ a.e.} \\ \Downarrow & & \\ \mathscr{B}_\psi \text{ is linearly independent} & & \text{Always true} \\ \hline \end{array}$$

In particular, it is known that

$$p_\psi(\xi) > 0 \text{ a.e. } \Rightarrow \mathscr{B}_\psi \text{ is } \ell^2\text{−linear independent.}$$

A question raised in [1] is the following: Is the converse true?

Šikić, and Speegle [5] have given a positive answer if p_ψ is bounded; see also Paluszyński's paper [4], where it is proved that $p_\psi(\xi) > 0$ a.e. is equivalent to L^2-Cesàro linear independence of $\mathscr{B}_\psi$. The latter means that if the Cesàro averages

$$\frac{1}{n}\sum_{h=0}^{n-1} S_h, \quad S_h = \sum_{|k|\le h} c_k T_k \psi, \quad \{c_k\} \in \ell^2,$$

tend to zero in L^2 norm, then necessarily $c_k = 0$, for all $k \in \mathbb{Z}$.

The approach we used in addressing this problem was global in nature: rather than examining the assumptions to be put on a single ψ, we preferred to analyze the issue as a whole, for all $\psi \in L^2(\mathbb{R})$.

The result is that the converse is true for any ψ if and only if the following statement on Fourier analysis is true: For any (Lebesgue) measurable subset A of $[0, 1]$, with positive measure $|A|$, we can find a non trivial square summable function, with support in A, whose partial sums of Fourier series are uniformly bounded.

By support of $f \in L^1(\mathbb{T})$ (denoted supp$f$) we mean the smallest closed set S such that $f(\xi) = 0$ almost everywhere in the complement of S.

After the proof of the main result in Section 2, in Section 3 we discuss the existence of such a good function for any measurable set $A \subset [0, 1]$. As far as we know existence is obtained as a corollary of general results by Kislyakov and Vinogradov, although we realize that there may be other direct proofs that we are not aware of.

We end with some notations. If $A \subset [0,1]$ we set $A^c = [0,1] \setminus A$. For a measurable set E, χ_E is the characteristic function of E: $\chi_E(\xi) = 1$ if $\xi \in E$, zero otherwise. For $f \in L^2(\mathbb{T})$ the symmetric partial sums of the Fourier series are

$$S_n(f)(\xi) = \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k \xi}.$$

For $f \in L^1(\mathbb{R})$ the Fourier transform is

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx.$$

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2. Main result

In this section we prove the main result. The first step requires uniformly boundedness only in the complement of the set A.

Theorem 2. *The following are equivalent:*

a) For any $0 \neq \psi \in L^2(\mathbb{R})$ the following is true

$$\mathscr{B}_\psi \textit{ is } \ell^2\textit{-linear independent } \Rightarrow p_\psi(\xi) > 0 \text{ a.e.};$$

b) For every measurable $A \subset [0,1]$, $|A| > 0$, $|A^c| > 0$, there exists $0 \neq f \in L^2(\mathbb{T})$, such that

1. *$supp f \subset A$;*
2. *$\underset{t \notin A}{ess\ sup} |S_n(f)(t)| = \|S_n(f)\|_{L^\infty(A^c)}$ are uniformly bounded, i.e. there exists $M > 0$ such that $\|S_n(f)\|_{L^\infty(A^c)} \leq M$, for all $n \in \mathbb{N}$.*

Proof. **b)** $\Rightarrow$ **a)**
Let $0 \neq \psi \in L^2(\mathbb{R})$. We shall show that if the set where $p_\psi(\xi) = 0$ has positive measure, then $\mathscr{B}_\psi$ is not ℓ^2-linear independent.

Assume $|A| = | \{p_\psi(\xi) = 0\} | > 0$. Since $\psi \neq 0$ then $|A^c| > 0$.

By b) there exists $0 \neq f \in L^2(\mathbb{T})$, such that both 1. and 2. hold. Now a simple calculation shows

$$\| \sum_{|k| \leq n} \hat{f}(-k)\, T_k \psi \|_2^2 = \int_0^1 | \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k \xi} |^2 \;\chi_{A^c}(\xi)\, p_\psi(\xi)\, d\xi. \qquad (1)$$

By a.e. convergence of the partial sums to f, and $\operatorname{supp} f \subset A$, we get a.e.

$$\lim_n | \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k \xi} |^2 \;\chi_{A^c}(\xi)\, p_\psi(\xi) = 0.$$

By uniform boundedness of $S_n(f)$ in A^c, $p_\psi \in L^1(\mathbb{T})$, and by Lebesgue dominated convergence theorem we get

$$\lim_n \int_0^1 | \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k \xi} |^2 \;\chi_{A^c}(\xi)\, p_\psi(\xi) = 0.$$

So equality (1) gives us a non zero sequence $c_k = \hat{f}(-k)$ in $\ell^2(\mathbb{Z})$ such that

$$\sum_{k \in \mathbb{Z}} c_k\, T_k \psi = 0,$$

and so $\mathscr{B}_\psi$ is not ℓ^2-linear independent.
a) $\Rightarrow$ **b**)
Let $A \subset [0,1]$, $|A| > 0$, and $|A^c| > 0$. Without loss of generality, we can suppose $A^c \subset [0,1)$. Consider $\psi \in L^2(\mathbb{R})$ defined by

$$\hat{\psi}(\xi) = \begin{cases} 1, & \text{if } \xi \in A^c, \\ 0, & \text{if } \xi \in \mathbb{R} \setminus A^c. \end{cases}$$

Note that,

$$p_\psi(\xi) = \sum_{k \in \mathbb{Z}} | \hat{\psi}(\xi + k) |^2 = | \hat{\psi}(\xi) |^2, \quad \text{(periodic)},$$

so that $A = \{\xi \in [0,1] \,:\, p_\psi(\xi) = 0\}$. Hypothesis a) applied to ψ implies that $\mathscr{B}_\psi$ is not ℓ^2-linear independent, so there exists a non zero sequence $\{c_k\} \in \ell^2(\mathbb{Z})$ such that

$$\lim_n \| \sum_{|k| \leq n} c_k\, T_k \psi \|_2 = 0. \qquad (2)$$

There is a unique $f \in L^2(\mathbb{T})$ such that $\hat{f}(k) = c_{-k}$. We shall show that the partial sums $S_n(f)$ are uniformly bounded in A^c. Once we have proved this, it is easy to show that $f(\xi) = 0$ a.e. in A^c. Indeed

$$\begin{aligned} \| \sum_{|k|\leq n} c_k \, T_k \psi \|_2^2 &= \int_{\mathbb{R}} | \sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k \xi} |^2 \; \hat{\psi}(\xi) \, d\xi \\ &= \int_{A^c} | \sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k \xi} |^2 \; d\xi. \end{aligned}$$

So, for $n \to +\infty$, the left hand side tends to zero by (2), while, by uniform boundedness and the a.e. convergence of the partial sums, the right hand side tends to $\int_{A^c} | f(\xi) |^2 \; d\xi$.

In order to prove the uniform boundedness of the partial sums, we shall apply the uniform boundedness principle to the following bounded linear operators

$$T_n : L^1(\mathbb{T}) \to \mathbb{C},$$

$$T_n(g) = \int_0^1 \sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k \xi} \, \chi_{A^c}(\xi) \, g(\xi) \, d\xi.$$

First, it is known that

$$\|T_n\| = \|S_n(f)\|_{L^\infty(A^c)}.$$

Furthermore, for g in the dense subspace $L^1(\mathbb{T}) \cap L^2(\mathbb{T})$ of $L^1(\mathbb{T})$, we have by (2)

$$| \, T_n(g) \, | \leq \left(\int_{\mathbb{R}} | \sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k \xi} \, \hat{\psi}(\xi) \, |^2 \; d\xi \right)^{1/2} \|g\|_2 \underset{n \to +\infty}{\to} 0.$$

So, see Hutson and Pym [2], there exists a unique

$$T : L^1(\mathbb{T}) \to \mathbb{C}$$

such that $T_n(g) \underset{n}{\to} T(g)$, for all $g \in L^1(\mathbb{T})$. In particular, for any fixed $g \in L^1(\mathbb{T})$, we have definitively

$$| \, T_n(g) \, | \leq | \, T_n(g) - T(g) \, | + | \, T(g) \, | \leq 1 + | \, T(g) \, |,$$

which implies

$$\sup\{|\, T_n(g)\, | :\ n \in \mathbb{N}\} < +\infty.$$

By uniform boundedness principle we get

$$\sup \|S_n(f)\|_{L^\infty(A^c)} = \sup \|T_n\| < +\infty,$$

and everything is proved. □

Example. 1. If $A \subset [0,1]$ contains an interval I, the function $f = \chi_J$, for any interval J interior to I, verifies conditions 1. and 2. of previous theorem. Indeed, by the localization principle, $S_n(f)$ converges uniformly (to zero) in every closed interval contained in J^c. But J^c consists of, at most, two intervals strictly containing A^c, hence the uniform boundedness.

2. Take as A the set of irrationals in $[0, 1/2]$. Then A is totally disconnected but the function $f = \chi_{A\cap[0,1/4]}$ has the same Fourier series of $\chi_{[0,1/4]}$ and works well as in the previous case.
3. Let A be a Cantor-like set, in $[0,1]$, of positive measure. Let f be any function with support in A. Then $S_n(f)$ converges uniformly in any closed subinterval of the intervals (a_n, b_n) contiguous to A, but a priori nothing can be said for $\bigcup_n (a_n, b_n) = A^c$.

Remark 3. At the light of Example 1, it is easy to prove that the statement *b*) in Theorem 2 is equivalent to the apparent stronger requirement

c) For every measurable $A \subset [0,1]$, $|A| > 0$, $|A^c| > 0$, there exists $0 \neq f \in L^2(\mathbb{T})$, such that
   1. $supp f \subset A$;
   2. $\|S_n(f)\|_{L^\infty(\mathbb{T})}$ are uniformly bounded.

Indeed, to prove *b*) implies *c*), it is sufficient to assume that A does not contain an interval, otherwise statement *c*) is true regardless of *b*).

We are going to show, by a density argument, that the same function f provided by *b*) works well.

So let f and M as in *b*). Let $\xi_0 \in A$ and $n \in \mathbb{N}$. Since $S_n(f)$ is continuous in ξ_0, we can find an open neighborhood I_0 of ξ_0 such that

$$|\, S_n(f)(\xi) - S_n(f)(\xi_0)\, | < 1, \quad \text{for all } \xi \in I_0.$$

There exists at least one $\xi \in A^c \cap I_0$ (otherwise A contains an interval), and finally

$$|\, S_n(f)(\xi_0)\, | \leq 1 + |\, S_n(f)(\xi)\, | < 1 + M.$$

3. Looking for a nice function

To the best of my knowledge, the existence of a *nice* function f satisfying condition c), for a given set $A \subset [0,1]$, can be derived by the following theorem in Kislyakov's paper [3, Theorem 4], whose proof relies also upon a result by Vinogradov [6]. The latter makes use of Carleson almost everywhere convergence theorem.

We recall first some basic notations: U^∞ denotes the space of functions $f \in L^\infty(\mathbb{T})$ for which the following norm is finite

$$\|f\|_{U^\infty} = \sup\left\{\left|\sum_{n\le k\le m} \hat{f}(k)\xi^k\right|,\ n,m\in\mathbb{Z},\ n\le m,\ \xi\in\mathbb{T}\right\}.$$

Theorem 4. *For every $F \in L^\infty(\mathbb{T})$ with $\|F\|_\infty \le 1$ and every $0 < \varepsilon \le 1$ there exists a function $G \in U^\infty$ with the following properties: $|G| + |F - G| = |F|$; $|\{\xi \in \mathbb{T}, F(\xi) \ne G(\xi)\}| \le \varepsilon\|F\|_1$; $\|G\|_{U^\infty} \le \mathrm{const}(1 + \log(\varepsilon^{-1}))$.*

The application of Theorem 4 is clear: For any measurable set $A \subset [0,1]$, $|A| > 0$, $|A^c| > 0$, it is sufficient to take $F = \chi_A$ and $0 < \varepsilon < 1$. Then G provided above is not zero since otherwise $|\{\xi \in \mathbb{T}, \chi_A(\xi) \ne 0\}| = |A| \le \varepsilon|A|$; the support of G is contained in A since $|G| \le |F|$; and finally $\|G\|_{U^\infty} \le \mathrm{const}(1 + \log(\varepsilon^{-1}))$ implies that the partial sums of Fourier series of G are uniformly bounded.

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