A direct proof of the functional Santaló inequality

Joseph Lehec *

June 2008

Abstract

We give a simple proof of a functional version of the Blaschke-Santaló inequality due to Artstein, Klartag and Milman. The proof is by induction on the dimension and does not use the Blaschke-Santaló inequality.

Published in C. R. Acad. Sci. Paris, Sér. I 347 (2009) 55–58.

1 Introduction

For $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y\rangle$ and the Euclidean norm of x by $|x|$. If A is a subset of $\mathbb{R}^n$, we let $A^\circ = \{x \in \mathbb{R}^n \,|\, \forall y \in A,\ \langle x, y\rangle \le 1\}$ be its polar body. The Blaschke-Santaló inequality states that any convex body K in $\mathbb{R}^n$ with center of mass at 0 satisfies

$$\operatorname{vol}_n(K)\operatorname{vol}_n(K^\circ) \le \operatorname{vol}_n(D)\operatorname{vol}_n(D^\circ) = v_n^2, \tag{1}$$

where vol_n stands for the volume, D for the Euclidean ball and v_n for its volume. Let g be a non-negative Borel function on $\mathbb{R}^n$ satisfying $0 < \int g < \infty$ and $\int |x| g(x)\,dx < \infty$, then $\operatorname{bar}(g) = \big(\int g\big)^{-1}\big(\int g(x)x\,dx\big)$ denotes its center of mass (or barycenter). The center of mass (or centroid) of a measurable subset of $\mathbb{R}^n$ is by definition the barycenter of its indicator function.

Let us state a functional form of (1) due to Artstein, Klartag and Milman [1]. If f is a non-negative Borel function on $\mathbb{R}^n$, the polar function of f is the log-concave function defined by

$$f^\circ(x) = \inf_{y\in\mathbb{R}^n}\big(\mathrm{e}^{-\langle x,y\rangle} f(y)^{-1}\big)$$

Theorem 1 (Artstein, Klartag, Milman). *If f is a non-negative integrable function on $\mathbb{R}^n$ such that f° has its barycenter at 0, then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} f^\circ(y)\,dy \le \Big(\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}|x|^2}\,dx\Big)^2 = (2\pi)^n.$$

In the special case where the function f is even, this result follows from an earlier inequality of Keith Ball [2]; and in [4], Fradelizi and Meyer prove something more general (see also [5]). In the present note we prove the following:

*LAMA (UMR CNRS 8050) Université Paris-Est.

Theorem 2. *Let f and g be non-negative Borel functions on $\mathbb{R}^n$ satisfying the duality relation*

$$\forall x, y \in \mathbb{R}^n, \qquad f(x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}. \tag{2}$$

If f (or g) has its barycenter at 0 then

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\,dy \leq (2\pi)^n. \tag{3}$$

This is slightly stronger than Theorem 1 in which the function that has its barycenter at 0 should be log-concave. The point of this note is not really this improvement, but rather to present a simple proof of Theorem 1. Theorem 2 yields an improved Blaschke-Santaló inequality, obtained by Lutwak in [6], with a completely different approach.

Corollary 3. *Let S be a star-shaped (with respect to 0) body in $\mathbb{R}^n$ having its centroid at 0. Then*

$$\mathrm{vol}_n(S)\,\mathrm{vol}_n(S^\circ) \leq v_n^2. \tag{4}$$

Proof. Let $N_S(x) = \inf\{r > 0 \,|\, x \in rS\}$ be the gauge of S and $\phi_S = \exp\bigl(-\frac{1}{2}N_S^2\bigr)$. Integrating ϕ_S and the indicator function of S on level sets of N_S, it is easy to see that $\int_{\mathbb{R}^n} \phi_S = c_n \,\mathrm{vol}_n(S)$ for some constant c_n depending only on the dimension. Replacing S by the Euclidean ball in this equality yields $c_n = (2\pi)^{n/2} v_n^{-1}$. Therefore it is enough to prove that

$$\int \phi_S \int \phi_{S^\circ} \leq (2\pi)^n. \tag{5}$$

Similarly, it is easy to see that $\mathrm{bar}(\phi_S) = c_n' \,\mathrm{bar}(S) = 0$. Besides, we have $\langle x, y\rangle \leq N_S(x)N_{S^\circ}(y) \leq \frac{1}{2}N_S(x)^2 + \frac{1}{2}N_{S^\circ}(y)^2$, for all $x, y \in \mathbb{R}^n$. Thus ϕ_S and ϕ_{S° satisfy (2), then by Theorem 2 we get (5). □

2 Main results

Theorem 4. *Let f be a non-negative Borel function on $\mathbb{R}^n$ having a barycenter. Let H be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces H_+ and H_-. Define $\lambda \in [0, 1]$ by $\lambda \int_{\mathbb{R}^n} f = \int_{H_+} f$. Then there exists $z \in \mathbb{R}^n$ such that for every non-negative Borel function g*

$$\bigl(\forall x, y \in \mathbb{R}^n,\ f(z + x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}\bigr) \quad \Rightarrow \quad \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq \frac{1}{4\lambda(1-\lambda)}(2\pi)^n. \tag{6}$$

In particular, in every median H ($\lambda = \frac{1}{2}$) there is a point z such that for all g

$$\bigl(\forall x, y \in \mathbb{R}^n,\ f(z + x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}\bigr) \quad \Rightarrow \quad \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq (2\pi)^n. \tag{7}$$

A similar result concerning convex bodies (instead of functions) was obtained by Meyer and Pajor in [7].

Let us derive Theorem 2 from the latter. Let f, g satisfy (2). Assume for example that $\mathrm{bar}(g) = 0$, then 0 cannot be separated from the support of g by a hyperplane, so there exists $x_1, \ldots, x_{n+1} \in \mathbb{R}^n$ such that 0 belongs to the interior of $\mathrm{conv}\{x_1 \ldots x_{n+1}\}$ and $g(x_i) > 0$ for $i = 1 \ldots n+1$. Then (2) implies that $f(x) \leq C\mathrm{e}^{-\|x\|}$, for some $C > 0$, where $\|x\| =$

$\max\big(\langle x, x_i\rangle \,|\, i \leq n+1\big)$. Assume also that $\int f > 0$, then f has a barycenter. Apply the "$\lambda = 1/2$" part of Theorem 4 to f. There exists $z \in \mathbb{R}^n$ such that (7) holds. On the other hand, by (2)

$$f(z+x)g(y)\mathrm{e}^{\langle y,z\rangle} \leq \mathrm{e}^{-\langle z+x,y\rangle}\mathrm{e}^{\langle y,z\rangle} = \mathrm{e}^{-\langle x,y\rangle}$$

for all $x, y \in \mathbb{R}^n$. Therefore

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\mathrm{e}^{\langle y,z\rangle}\,dy \leq (2\pi)^n. \tag{8}$$

Integrating with respect to $g(y)dy$ the inequality $1 \leq \mathrm{e}^{\langle y,z\rangle} - \langle y,z\rangle$ we get

$$\int_{\mathbb{R}^n} g(y)\,dy \leq \int_{\mathbb{R}^n} g(y)\mathrm{e}^{\langle y,z\rangle}\,dy - \int_{\mathbb{R}^n} \langle y,z\rangle g(y)\,dy.$$

Since $\mathrm{bar}(g) = 0$, the latter integral is 0 and together with (8) we obtain (3). Observe also that this proof shows that Theorem 4 in dimension n implies Theorem 2 in dimension n.

In order to prove Theorem 4, we need the following logarithmic form of the Prékopa-Leindler inequality. For details on Prékopa-Leindler, we refer to [3].

Lemma 5. *Let ϕ_1, ϕ_2 be non-negative Borel functions on $\mathbb{R}_+$. If $\phi_1(s)\phi_2(t) \leq \mathrm{e}^{-st}$ for every s, t in $\mathbb{R}_+$, then*

$$\int_{\mathbb{R}_+} \phi_1(s)\,ds \int_{\mathbb{R}_+} \phi_2(t)\,dt \leq \frac{\pi}{2}. \tag{9}$$

Proof. Let $f(s) = \phi_1(\mathrm{e}^s)\mathrm{e}^s$, $g(t) = \phi_2(\mathrm{e}^t)\mathrm{e}^t$ and $h(r) = \exp(-\mathrm{e}^{2r}/2)\mathrm{e}^r$. For all $s, t \in \mathbb{R}$ we have $\sqrt{f(s)g(t)} \leq h(\frac{t+s}{2})$, hence by Prékopa-Leindler $\int_{\mathbb{R}} f \int_{\mathbb{R}} g \leq \big(\int_{\mathbb{R}} h\big)^2$. By change of variable, this is the same as $\int_{\mathbb{R}_+} \phi_1 \int_{\mathbb{R}_+} \phi_2 \leq \big(\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\,du\big)^2$ which is the result. □

3 Proof of Theorem 4

Clearly we can assume that $\int f = 1$. Let μ be the measure with density f. In the sequel we let $f_z(x) = f(z+x)$ for all x, z.

We prove the theorem by induction on the dimension. Let f be a non-negative Borel function on the line, let $r \in \mathbb{R}$ and $\lambda = \mu\big([r,\infty)\big) \in [0,1]$. Let g satisfy $f(r+s)g(t) \leq \mathrm{e}^{-st}$, for all s, t. Apply Lemma 5 twice: first to $\phi_1(s) = f(r+s)$ and $\phi_2(t) = g(t)$ then to $\phi_1(s) = f(r-s)$ and $\phi_2(t) = g(-t)$. Then

$$\int_{\mathbb{R}_+} f_r \int_{\mathbb{R}_+} g \leq \frac{\pi}{2} \qquad \text{and} \qquad \int_{\mathbb{R}_-} f_r \int_{\mathbb{R}_-} g \leq \frac{\pi}{2}.$$

Therefore $\int_{\mathbb{R}_+} g \leq \frac{\pi}{2\lambda}$ and $\int_{\mathbb{R}_-} g \leq \frac{\pi}{2(1-\lambda)}$, which yields the result in dimension 1.
Assume the theorem to be true in dimension $n-1$. Let H be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces H_+ and H_- and let $\lambda = \mu(H_+)$. Provided that $\lambda \neq 0, 1$ we can define b_+ and b_- to be the barycenters of $\mu_{|H_+}$ and $\mu_{|H_-}$, respectively. Since $\mu(H) = 0$, the point b_+ belongs to the interior of H_+, and similarly for b_-. Hence the line passing through b_+ and b_- intersects H at one point, which we call z. Let us prove that z satisfies (6), for all g. Clearly, replacing f by f_z and H by $H - z$, we can assume that $z = 0$. Let g satisfy

$$\forall x, y \in \mathbb{R}^n, \qquad f(x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}. \tag{10}$$

Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $H = e_n^\perp$ and $\langle b_+, e_n \rangle > 0$. Let $v = b_+ / \langle b_+, e_n \rangle$ and A be the linear operator on $\mathbb{R}^n$ that maps e_n to v and e_i to itself for $i = 1 \dots n-1$ and let $B = (A^{-1})^t$. Define

$$F_+ : y \in H \mapsto \int_{\mathbb{R}_+} f(y + sv)\, ds \qquad \text{and} \qquad G_+ : y' \in H \mapsto \int_{\mathbb{R}_+} g(By' + te_n)\, dt.$$

By Fubini, and since A has determinant 1, $\int_H F_+ = \int_{H_+} f \circ A = \mu(H_+) = \lambda$. Also, letting P be the projection with range H and kernel $\mathbb{R}v$, we have

$$\mathrm{bar}(F_+) = \frac{1}{\lambda} \int_{H_+} P(Ax) f(Ax)\, dx = \frac{1}{\lambda} P\Big(\int_{H_+} x f(x)\, dx \Big) = P(b_+),$$

and this is 0 by definition of P. Since $\langle Ax, Bx' \rangle = \langle x, x' \rangle$ for all $x, x' \in \mathbb{R}^n$, we have $\langle y + sv, By' + te_n \rangle = \langle y, y' \rangle + st$ for all $s, t \in \mathbb{R}$ and $y, y' \in H$. So (10) implies

$$f(y + sv) g(By' + te_n) \leq \mathrm{e}^{-st - \langle y, y' \rangle}.$$

Applying Lemma 5 to $\phi_1(s) = f(y + sv)$ and $\phi_2(t) = g(By' + te_n)$ we get $F_+(y) G_+(y') \leq \frac{\pi}{2} \mathrm{e}^{-\langle y, y' \rangle}$ for every $y, y' \in H$. Recall that $\mathrm{bar}(F_+) = 0$, then by the induction assumption (which implies Theorem 2 in dimension $n-1$)

$$\int_H F_+ \int_H G_+ \leq \frac{\pi}{2} (2\pi)^{n-1}. \tag{11}$$

hence $\int_{H_+} g(Bx)\, dx \leq \frac{1}{4\lambda} (2\pi)^n$. In the same way $\int_{H_-} g(Bx)\, dx \leq \frac{1}{4(1-\lambda)} (2\pi)^n$, adding these two inequalities, we obtain

$$\int_{\mathbb{R}^n} g(Bx)\, dx \leq \frac{1}{4\lambda(1-\lambda)} (2\pi)^n$$

which is the result since B has determinant 1.

References

[1] S. Artstein-Avidan, B. Klartag, and V. Milman, *The Santaló point of a function, and a functional form of Santaló inequality*, Mathematika **51** (2005) 33–48.

[2] K. Ball, *Isometric problems in ℓ_p and sections of convex sets*, doctoral thesis, University of Cambridge, 1986.

[3] K. Ball, An elementary introduction to modern convex geometry, in *Flavors of geometry*, edited by S. Levy, Cambridge University Press, 1997.

[4] M. Fradelizi and M. Meyer, *Some functional forms of Blaschke-Santaló inequality*, Math. Z. **256** (2007) (2) 379–395.

[5] J. Lehec, *Partitions and functional Santaló inequalities*, Arch. Math. 92 (2009) (1) 89–94.

[6] E. Lutwak, *Extended affine surface area*, Adv. Math. **85** (1991) (1) 39–68.

[7] M. Meyer and A. Pajor, *On the Blaschke Santaló inequality*, Arch. Math. (Basel) **55** (1990) 82–93.