

REDUCING THE ERDŐS-MOSER EQUATION

$$1^n + 2^n + \cdots + k^n = (k+1)^n \\ \text{MODULO } k \text{ AND } k^2$$

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ABSTRACT. An open conjecture of Erdős and Moser is that the only solution of the Diophantine equation in the title is the trivial solution $1+2=3$. Reducing the equation modulo k and k^2 , we give necessary and sufficient conditions on solutions to the resulting congruence and supercongruence. A corollary is a new proof of Moser's result that the conjecture is true for odd exponents n . We also connect solutions k of the congruence to primary pseudoperfect numbers and to a result of Zagier. The proofs use divisibility properties of power sums as well as Lerch's relation between Fermat and Wilson quotients.

1. INTRODUCTION

Around 1953, Erdős and Moser studied the Diophantine equation

$$(1.1) \quad 1^n + 2^n + \cdots + k^n = (k+1)^n$$

and made the following conjecture.

Conjecture 1. *The only solution of (1.1) in positive integers is the trivial solution $1+2=3$.*

Moser [13] proved the statement when n is odd or $k < 10^{10^6}$. In 1987 Schinzel showed that in any solution, k is even [12, p. 800]. An extension of Schinzel's theorem to a generalization of equation (1.1) was given in 1996 by Moree [10, Proposition 9]. For a recent elementary proof of a special case, see MacMillan and Sondow [9, Proof of Proposition 2].

Many other results on the *Erdős-Moser equation* (1.1) are known, but it has not even been established that there are only finitely many solutions. For surveys of work on this and related problems, see Butske, Jaje, and Mayernik [1], Guy [4, D7], and Moree [11].

In the present paper, we first approximate equation (1.1) by the congruence

$$(1.2) \quad 1^n + 2^n + \cdots + k^n \equiv (k+1)^n \pmod{k}.$$

In Section 2, we give necessary and sufficient conditions on n and k (Theorem 1), and we show that if a solution k factors into a product of 1, 2, 3, or 4 primes, then $k = 2, 6, 42$, or 1806, respectively (Proposition 1). In Section 3, we relate solutions k to primary pseudoperfect numbers and to a result of Zagier. In the final section, Theorem 1 is extended to the supercongruence (Theorem 2)

$$1^n + 2^n + \cdots + k^n \equiv (k+1)^n \pmod{k^2}.$$

Our methods involve divisibility properties of power sums, as well as Lerch's formula relating Fermat and Wilson quotients.

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As applications of Theorems 1 and 2, we reprove Moser's result that Conjecture 1 is true for odd exponents n (Corollary 1), and for even n we show that a solution k to (1.1) cannot be a primary pseudoperfect number with 8 or fewer prime factors (Corollary 5). In a paper in preparation, we will give other applications of our results to the Erdős-Moser equation.

2. THE CONGRUENCE $1^n + 2^n + \cdots + k^n \equiv (k+1)^n \pmod{k}$

We will need a classical lemma on power sums. (An empty sum will represent 0, as usual.)

Definition 1. For integers $n \geq 0$ and $a \geq 1$, let $\Sigma_n(a)$ denote the *power sum*

$$\Sigma_n(a) := 1^n + 2^n + \cdots + a^n.$$

Set $\Sigma_n(0) := 0$.

Lemma 1. *If n is a positive integer and p is a prime, then*

$$\Sigma_n(p) \equiv \begin{cases} -1 \pmod{p}, & (p-1) \mid n, \\ 0 \pmod{p}, & (p-1) \nmid n. \end{cases}$$

Proof. Hardy and Wright's proof [5, Theorem 119] uses primitive roots. For a very elementary proof, see MacMillan and Sondow [8]. \square

We now give necessary and sufficient conditions on solutions to (1.2).

Theorem 1. *Given positive integers n and k , the congruence*

$$(2.1) \quad 1^n + 2^n + \cdots + k^n \equiv (k+1)^n \pmod{k}$$

holds if and only if prime $p \mid k$ implies

- (i). $n \equiv 0 \pmod{p-1}$, and
- (ii). $\frac{k}{p} + 1 \equiv 0 \pmod{p}$.

In that case k is square-free, and if n is odd, then $k = 1$ or 2 .

Proof. First note that if n, k, p are any positive integers with $p \mid k$, then

$$(2.2) \quad \Sigma_n(k) = \sum_{h=1}^{k/p} \sum_{j=1}^p ((h-1)p + j)^n \equiv \frac{k}{p} \Sigma_n(p) \pmod{p}.$$

Now assume that (i) and (ii) hold whenever prime $p \mid k$. Then, using Lemma 1, both $\Sigma_n(p)$ and k/p are congruent to -1 modulo p , and so $\Sigma_n(k) \equiv 1 \pmod{p}$, by (2.2). Thus, as (ii) implies k is square-free, k is a product of distinct primes each of which divides $\Sigma_n(k) - 1$. It follows that $\Sigma_n(k) \equiv 1 \pmod{k}$, implying (2.1).

Conversely, assume that (2.1) holds, so that $\Sigma_n(k) \equiv 1 \pmod{k}$. If prime $p \mid k$, then (2.2) gives $\frac{k}{p} \Sigma_n(p) \equiv 1 \pmod{p}$, and so $\Sigma_n(p) \not\equiv 0 \pmod{p}$. Now Lemma 1 yields both $(p-1) \mid n$, proving (i), and $\Sigma_n(p) \equiv -1 \pmod{p}$, implying (ii).

If n is odd, then by (i) no odd prime divides k . As k is square-free, $k = 1$ or 2 . \square

Corollary 1. *The only solution of the Erdős-Moser equation with odd exponent n is $1 + 2 = 3$.*

Proof. Given a solution with n odd, Theorem 1 implies $k = 1$ or 2 . But $k = 1$ is clearly impossible, and $k = 2$ evidently forces $n = 1$. \square

Recall that, when x and y are real numbers, $x \equiv y \pmod{1}$ means that $x - y$ is an integer.

Corollary 2. *A given positive integer k satisfies the congruence (2.1), for some exponent n , if and only if the Egyptian fraction congruence*

$$(2.3) \quad \frac{1}{k} + \sum_{p|k} \frac{1}{p} \equiv 1 \pmod{1}$$

holds, where the summation is over all primes p dividing k . In that case, k is square-free, and n is any number divisible by the least common multiple $\text{LCM}\{p-1 : \text{prime } p | k\}$.

Proof. Condition (2.3) is equivalent to the congruence

$$(2.4) \quad 1 + \sum_{p|k} \frac{k}{p} \equiv 0 \pmod{k},$$

which in turn is equivalent to condition (ii) in Theorem 1, since each implies k is square-free. The theorem now implies the corollary. \square

Remark 1. In (2.3) we write $\equiv 1 \pmod{1}$, rather than the equivalent $\equiv 0 \pmod{1}$, in order to contrast the condition with that in Definition 2 of the next section.

For solutions to the congruence (2.1), we determine the possible values of k with at most four (distinct) prime factors. First we prove a lemma. (An empty product will represent 1, as usual.)

Lemma 2. *Let $k = p_1 p_2 \dots p_r$, where the p_i are primes. If k satisfies the integrality condition (2.3), then for any subset $S \subset \{p_1, \dots, p_r\}$, there exists an integer $q = q(r, S)$ such that*

$$(2.5) \quad q \prod_{p \in S} p = 1 + \sum_{p \in S} \frac{k}{p}.$$

Proof. This follows from Corollary 2 and Theorem 1 condition (ii), using the Chinese Remainder Theorem. For an alternate proof, denote the summation in (2.5) by Σ , and note that (2.3) implies $1 + \Sigma \equiv 1 + \frac{k}{p} \equiv 0 \pmod{p}$, for each $p \in S$. Then, since k is square-free, $\prod_{p \in S} p$ divides $1 + \Sigma$, and the lemma follows. \square

Proposition 1. *Let k be a product of r primes. Suppose $1^n + 2^n + \dots + k^n \equiv (k+1)^n \pmod{k}$, for some exponent $n > 0$; equivalently, suppose (2.3) holds. If $r = 0, 1, 2, 3, 4$, then $k = 1, 2, 6, 42, 1806$, respectively.*

Proof. Theorem 1 implies $k = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$ are primes.

($r = 0, 1$). If $r = 0$, then $k = 1$. If $r = 1$, then $k = p_1$ is prime, and (2.4) yields $k | 2$, so that $k = 2$.

($r = 2$). For $k = p_1 p_2$, congruence (2.4) gives $p_2 | (p_1 + 1)$. Since $p_2 \geq p_1 + 1$, it follows that $p_2 = p_1 + 1$. As p_2 and p_1 are prime, $p_2 = 3$ and $p_1 = 2$, and hence $k = 6$.

($r = 3, 4$). In general, if $k = p_1 \dots p_r$, where $p_1 < \dots < p_r$ are primes, then by Lemma 2, for $i = 1, \dots, r$ there exists an integer q_i such that $q_i p_i = \frac{k}{p_i} + 1$. In particular,

$$(2.6) \quad q_r p_r = p_1 \dots p_{r-2} p_{r-1} + 1.$$

Hence if $r > 2$, so that $p_{r-1} < p_r - 1$, then $q_r < p_1 \dots p_{r-2}$. We also have

$$q_r q_{r-1} p_{r-1} = q_r (p_1 \dots p_{r-2} p_r + 1) = p_1 \dots p_{r-2} (p_1 \dots p_{r-1} + 1) + q_r,$$

and so

$$(2.7) \quad p_{r-2} < p_{r-1} \mid (p_1 \cdots p_{r-2} + q_r) < 2p_1 \cdots p_{r-2}.$$

Now take $r = 3$, so that $k = p_1 p_2 p_3$. Then $q_2 p_2 = p_1 p_3 + 1$ and $q_3 p_3 = p_1 p_2 + 1$, for some integers q_2 and q_3 . By (2.7) we have $p_1 < p_2 \mid (p_1 + q_3) < 2p_1$. Hence $p_2 = p_1 + q_3$. Substituting $q_3 = p_2 - p_1$ into $q_3 p_3 = p_1 p_2 + 1$ yields $p_1 \mid (p_2 p_3 - 1)$. As $p_1 \mid (p_2 p_3 + 1)$, we conclude that $p_1 \mid 2$. Therefore $p_1 = 2$. Then $p_3 \mid (2p_2 + 1)$. As $p_3 > p_2$, we get $p_3 = 2p_2 + 1$. Then we have $p_2 \mid (2p_3 + 1) = 4p_2 + 3$, and so $p_2 \mid 3$. Therefore $p_2 = 3$, and hence $p_3 = 7$. Thus $k = 2 \cdot 3 \cdot 7 = 42$.

Finally, take $r = 4$. Lemma 2 with $S = \{p_1, \dots, p_r\}$ and $r = 4$ gives an integer q such that

$$qp_1 p_2 p_3 p_4 = p_1(p_2 p_3 + p_2 p_4 + p_3 p_4) + p_2 p_3 p_4 + 1 < 4p_2 p_3 p_4,$$

so that $qp_1 < 4$. Hence $q = 1$, and $p_1 = 2$ or 3 . Now

$$(p_1 - 1)p_2 p_3 p_4 = p_1(p_2 p_3 + p_2 p_4 + p_3 p_4) + 1 < 3p_1 p_3 p_4,$$

and so $(p_1 - 1)p_2 < 3p_1$. This implies $p_1 = 2$ and $p_2 = 3$. Thus $k = 2 \cdot 3p_3 p_4$. Then (2.6) and (2.7) give $q_4 p_4 = 6p_3 + 1$ and $3 < p_3 \mid (6 + q_4) < 12$. The only solution is $(q_4, p_3, p_4) = (1, 7, 43)$, and so $k = 2 \cdot 3 \cdot 7 \cdot 43 = 1806$. This completes the proof. \square

Example 1. Take $k = 1806$ in Corollary 2. Since $1806 = 2 \cdot 3 \cdot 7 \cdot 43$ and

$$\frac{1}{1806} + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} = 1,$$

and since $\text{LCM}(1, 2, 6, 42) = 42$, one solution of (2.1) is

$$1^{42} + 2^{42} + \cdots + 1806^{42} \equiv 1807^{42} \pmod{1806}.$$

3. PRIMARY PSEUDOPERFECT NUMBERS

Recall that a positive integer is called *perfect* if it is the sum of *all* its proper divisors, and *pseudoperfect* if it is the sum of *some* of its proper divisors [4, B2].

Definition 2. (From [1].) A *primary pseudoperfect number* is an integer $K > 1$ that satisfies the Egyptian fraction equation

$$(3.1) \quad \frac{1}{K} + \sum_{p \mid K} \frac{1}{p} = 1,$$

where the summation is over all primes dividing K .

Multiplying (3.1) by K , we see that K is square-free, and that every primary pseudoperfect number, except 2, is pseudoperfect.

Corollary 3. *Every primary pseudoperfect number K is a solution to the congruence (2.1), for some exponent n .*

Proof. This is immediate from Definition 2 and Corollary 2. \square

A priori, the equality (3.1) is a stronger condition than the congruence (2.3) in Corollary 2. However, (3.1) and (2.3) may in fact be equivalent, because all the known solutions of (2.3) also satisfy (3.1) — see [1]. In other words, primary pseudoperfect numbers may be the only solutions k to the congruence (2.1), except for $k = 1$.

According to [1], Table 1 contains all primary pseudoperfect numbers K with $r \leq 8$ (distinct) prime factors. In particular, for each $r = 1, 2, \dots, 8$, there exists exactly one such K (as conjectured

by Ke and Sun [6] and by Cao, Liu, and Zhang [2]). No K with $r > 8$ prime factors is known. As with perfect numbers, no odd primary pseudoperfect number has been discovered.

TABLE 1. (from [1]) The primary pseudoperfect numbers K with $r \leq 8$ prime factors

r	K	prime factors
1	2	2
2	6	2, 3
3	42	2, 3, 7
4	1806	2, 3, 7, 43
5	47058	2, 3, 11, 23, 31
6	2214502422	2, 3, 11, 23, 31, 47059
7	52495396602	2, 3, 11, 17, 101, 149, 3109
8	8490421583559688410706771261086	2, 3, 11, 23, 31, 47059, 2217342227, 1729101023519

Table 1 was obtained in [1] using computation and computer search techniques. Note that the cases $r = 1, 2, 3, 4$ follow a fortiori from our Proposition 1.

TABLE 2. The known solutions to $1^n + 2^n + \dots + k^n \equiv (k+1)^n \pmod{k}$

k	n is any multiple of
1	1
2	1
6	2
42	6
1806	42
47058	330
2214502422	235290
52495396602	310800
8490421583559688410706771261086	1863851053628494074457830

Table 2 was calculated from Table 1, using Corollaries 2 and 3.

Example 2. The simplest case of the congruence (2.1) in which k has 8 prime factors is

$$\sum_{j=1}^{8490421583559688410706771261086} j^{1863851053628494074457830} \equiv 8490421583559688410706771261087^{1863851053628494074457830} \pmod{8490421583559688410706771261086}.$$

Zagier gave three characterizations of the numbers 1, 2, 6, 42, 1806. We add two more.

Proposition 2. Each of the following five conditions is equivalent to $k \in \{1, 2, 6, 42, 1806\}$.

- (i) The congruence $a^{k+1} \equiv a \pmod{k}$ holds, for all a .
- (ii) $k = p_1 p_2 \dots p_r$, where $r \geq 0$, the p_i are distinct primes, and $(p_i - 1) \mid k$.
- (iii) $k = p_1 p_2 \dots p_r$, where $r \geq 0$ and $p_i = p_1 \dots p_{i-1} + 1$ is prime, $i = 1, \dots, r$.
- (iv) k is a product of at most 4 primes, and $1^n + 2^n + \dots + k^n \equiv (k+1)^n \pmod{k}$, for some exponent n .
- (v) Either $k = 1$ or k is a primary pseudoperfect number with 4 or fewer prime factors.

Proof. For (i), (ii), (iii), see the solution to the first problem of Zagier [14]. Proposition 1 yields (iv). Corollary 3 and (iv) give (v). \square

4. SUPERCONGRUENCES

If the conditions in Theorem 1 are satisfied, the following corollary shows that the congruence (2.2) can be replaced with a “supercongruence” (compare Zudilin [15]).

Corollary 4. *If $1^n + 2^n + \dots + k^n \equiv (k+1)^n \pmod{k}$ and prime $p \mid k$, then*

$$(4.1) \quad \Sigma_n(k) \equiv \frac{k}{p} \Sigma_n(p) \pmod{p^2}.$$

Proof. By Theorem 1, it suffices to prove the more general statement that, if prime $p \mid k$ and $(p-1) \mid n$, and if either $k=2$ or n is even, then (4.1) holds. Set $a = k/p$ in the equation (2.2). Expanding and summing, we see that

$$\Sigma_n(k) \equiv a \Sigma_n(p) + \frac{1}{2} a(a-1) n p \Sigma_{n-1}(p) \pmod{p^2}.$$

If $p > 2$, then $(p-1) \mid n$ implies $(p-1) \nmid (n-1)$, and Lemma 1 gives $p \mid \Sigma_{n-1}(p)$. In case $p=2$, either $a = k/2 = 1$ or $2 \mid n$, and each implies $2 \mid (1/2)a(a-1)n$. In all cases, (4.1) follows. \square

For an extension of Theorem 1 itself to a supercongruence, we need a definition and a lemma.

Definition 3. By Fermat’s and Wilson’s theorems, for any prime p the *Fermat quotient*

$$(4.2) \quad q_p(j) := \frac{j^{p-1} - 1}{p}, \quad p \nmid j,$$

and the *Wilson quotient*

$$W_p := \frac{(p-1)! + 1}{p}$$

are integers.

Lemma 3 (Lerch [7]). *If p is an odd prime, then the Fermat and Wilson quotients are related by Lerch’s formula*

$$\sum_{j=1}^{p-1} q_p(j) \equiv W_p \pmod{p}.$$

Proof. Given a and b with $p \nmid ab$, set $j = ab$ in (4.2). Substituting $a^{p-1} = pq_p(a) + 1$ and $b^{p-1} = pq_p(b) + 1$, we deduce Eisenstein’s relation [3]

$$q_p(ab) \equiv q_p(a) + q_p(b) \pmod{p},$$

which implies

$$q_p((p-1)!) \equiv \sum_{j=1}^{p-1} q_p(j) \pmod{p}.$$

On the other hand, setting $j = (p-1)! = pW_p - 1$ in (4.2) and expanding, the hypothesis $p-1 \geq 2$ leads to $q_p((p-1)!) \equiv W_p \pmod{p}$. This proves the lemma. \square

Theorem 2. For $n = 1$, the supercongruence

$$(4.3) \quad 1^n + 2^n + \dots + k^n \equiv (k+1)^n \pmod{k^2}$$

holds if and only if $k = 1$ or 2 . For $n \geq 3$ odd, (4.3) holds if and only if $k = 1$. Finally, for $n \geq 2$ even, (4.3) holds if and only if prime $p \mid k$ implies

- (i). $n \equiv 0 \pmod{(p-1)}$, and
- (ii). $\frac{k}{p} + 1 \equiv p(n(W_p + 1) - 1) \pmod{p^2}$.

Proof. To prove the first two statements, use Theorem 1 together with the fact that the congruences $1^n + 2^n \equiv 1 \pmod{4}$ and $3^n \equiv (-1)^n \equiv -1 \pmod{4}$ all hold when $n \geq 3$ is odd.

Now assume $n \geq 2$ is even. Let p denote a prime. By Theorem 1, we may assume that (i) holds if $p \mid k$, and that k is square-free. It follows that the supercongruence (4.3) is equivalent to the system

$$\Sigma_n(k) \equiv (k+1)^n \pmod{p^2}, \quad p \mid k.$$

Corollary 4 and expansion of $(k+1)^n$ allow us to write the system as

$$\frac{k}{p} \Sigma_n(p) \equiv 1 + nk \pmod{p^2}, \quad p \mid k.$$

Since n is at least 2 and $(p-1) \mid n$, we have

$$\begin{aligned} \Sigma_n(p) &\equiv \Sigma_n(p-1) \pmod{p^2} \\ &= \sum_{j=1}^{p-1} (j^{p-1})^{n/(p-1)}. \end{aligned}$$

Substituting $j^{p-1} = 1 + pq_p(j)$ and expanding, the result is

$$(4.4) \quad \Sigma_n(p) \equiv \sum_{j=1}^{p-1} \left(1 + \frac{n}{p-1} pq_p(j) \right) \equiv p-1 - np \sum_{j=1}^{p-1} q_p(j) \pmod{p^2},$$

since $n/(p-1) \equiv -n \pmod{p}$. Now Lerch's formula (if p is odd), together with the equality $q_2(1) = 0$ and the evenness of n (if $p = 2$), yield

$$\Sigma_n(p) \equiv p-1 - npW_p \pmod{p^2}.$$

Summarizing, the supercongruence (4.3) is equivalent to the system

$$\frac{k}{p} (p-1 - npW_p) \equiv 1 + nk \pmod{p^2}, \quad p \mid k.$$

It in turn can be written as

$$(4.5) \quad \frac{k}{p} + 1 \equiv -k(n(W_p + 1) - 1) \pmod{p^2}, \quad p \mid k.$$

On the right-hand side, we substitute $k \equiv -p \pmod{p^2}$ (deduced from (4.5) multiplied by p), and arrive at (ii). This completes the proof. \square

Corollary 5. *Let $n \geq 2$ be even and let K be a primary pseudoperfect number with $r \leq 8$ prime factors.*

- (i). *Then (n, K) is a solution of the supercongruence (4.3) if and only if either $K = 2$, or $K = 42$ and $n \equiv 12 \pmod{42}$.*
(ii). *The supercongruence*

$$(4.6) \quad 1^n + 2^n + \cdots + K^n \equiv (K + 1)^n \pmod{K^3}$$

holds if and only if $K = 2$ and $n \geq 4$.

- (iii). *The Erdős-Moser equation has no solution (n, k) with $k = K$.*

Proof. (i). We use Table 1.

($r = 1$). Theorem 2 with $k = p = 2$ implies $(n, 2)$ is a solution to (4.3). (This can also be seen directly from (4.3): both sides are congruent to 1 modulo 4.)

($r = 2$). Suppose $k = 2 \cdot 3$ is a solution to (4.3). Since $2 \mid n$, condition (ii) in Theorem 2 with $p = 2$ gives $3 + 1 = \frac{k}{p} + 1 \equiv -2 \pmod{4}$, a contradiction. Therefore, there is no solution with $k = 6$.

($r = 3$). For $k = 2 \cdot 3 \cdot 7$, condition (i) in Theorem 2 requires $6 \mid n$. Then (ii) is satisfied for $p = 2$ and 3. For $p = 7$, we need $6 + 1 \equiv 7((103 + 1)n - 1) \pmod{49}$, which reduces to $3n \equiv 1 \pmod{7}$. Since also $6 \mid n$, only $n \equiv 12 \pmod{42}$ gives a solution with $k = 42$.

($r = 4$). If $k = 2 \cdot 3 \cdot 7 \cdot 43$, condition (ii) with $p = 2$ rules out any solution.

($r = 5$). For $k = 2 \cdot 3 \cdot 11 \cdot 23 \cdot 31$, condition (i) gives $3 \mid n$. As $\frac{k}{3} + 1 \equiv 0 \not\equiv -3 \pmod{9}$, by (ii) there is no solution.

($r = 6, 7, 8$). For the numbers K in Table 1 with $r = 6, 7, 8$ prime factors, conditions (i) and (ii) require $\frac{k}{p} + 1 \equiv -p \pmod{p^2}$, for $p = 2, 3, 2$, respectively. But the requirement is violated in each case, and so no solution exists.

(ii). Part (i) implies that the only *possible* solutions (n, K) of (4.6) are $K = 2$, and $K = 42$ with $n \equiv 12 \pmod{42}$.

It is easy to check that $(n, K) = (2, 2)$ is not a solution. To see that $(n, 2)$ is a solution when $n \geq 4$ is even, we need to show that $1 + 2^n \equiv 3^n \pmod{2^3}$. Since $1 + 2^n = 1 + 4^{n/2} \equiv 1 \pmod{8}$ and $3^n = 9^{n/2} = (1 + 8)^{n/2} \equiv 1 \pmod{8}$, the case $K = 2$ is proved.

Now suppose (4.6) holds with $K = 42$ and $n \equiv 12 \pmod{42}$. Since

$$(K + 1)^n \equiv 1 + nK + \frac{1}{2}n(n - 1)K^2 \pmod{K^3},$$

by setting $n = 6n_1$ we infer that

$$(4.7) \quad \Sigma_n(42) \equiv 1 - 5040n_1 + 31752n_1^2 \equiv 1 \pmod{2^3}.$$

But as $n \geq 4$ is even, each of the 42 terms in the sum $\Sigma_n(42)$ is congruent to 1 or 0 modulo 8 according as the term is odd or even, and so $\Sigma_n(42) \equiv 21 \pmod{8}$. This contradicts (4.7), proving (ii).

(iii). This follows from (ii) and the fact that if $k = 2$ in the Erdős-Moser equation, then evidently $n = 1$. \square

Example 3. The simplest cases of (i) are $1^2 + 2^2 \equiv 3^2 \pmod{2^2}$ and

$$1^{12} + 2^{12} + \dots + 42^{12} \equiv 43^{12} \pmod{42^2}.$$

An example of (ii) is $1^4 + 2^4 \equiv 3^4 \pmod{2^3}$. (More generally, one can show that

$$1^n + 2^n \equiv 3^n \pmod{2^d}, \quad \text{if } 2^{d-1} \mid n,$$

for any positive integers n and d .)

In light of Theorem 1 and Corollary 4, it is natural to ask whether Theorem 2 has an analogous corollary about supercongruences modulo p^3 .

Conjecture 2. *If $1^n + 2^n + \dots + k^n \equiv (k+1)^n \pmod{k^2}$ and prime $p \mid k$, then*

$$\Sigma_n(k) \equiv \frac{k}{p} \Sigma_n(p) \pmod{p^3}.$$

Example 4. For $p = 2, 3, 7$, one can compute that

$$1^{12} + 2^{12} + \dots + 42^{12} \equiv \frac{42}{p} (1^{12} + 2^{12} + \dots + p^{12}) \pmod{p^3}.$$

In fact, for $p = 2, 3, 7$ it appears that $\Sigma_n(42) \equiv \frac{42}{p} \Sigma_n(p) \pmod{p^3}$ holds true not only when $n \equiv 12 \pmod{42}$, but indeed for all $n \equiv 0 \pmod{6}$. One reason may be that, for $p = 7$ (but not for $p = 2$ or 3), apparently $6 \mid n$ implies $p^2 \mid \Sigma_{n-1}(p)$. (Compare $p \mid \Sigma_{n-1}(p)$ in the proof of Corollary 4.)

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