POWERS OF LEXSEGMENT IDEALS WITH LINEAR RESOLUTION

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Abstract. All powers of lexsegment ideals with linear resolution (equivalently, with linear quotients) have linear quotients with respect to suitable orders of the minimal monomial generators. For a large subclass of the lexsegment ideals the corresponding Rees algebra has a quadratic Gröbner basis, thus it is Koszul. We also find other classes of monomial ideals with linear quotients whose powers have linear quotients too.

Keywords: Lexsegment ideals, linear resolution, linear quotients, Rees ring, Koszul algebra, ideals of fiber type, Gröbner bases.

MSC: Primary 13D02; Secondary 13C15, 13H10, 13P10.

Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. For an integer $d \geq 2$, we denote by $\mathcal{M}_d$ the set of all the monomials of S of degree d. A *lexsegment ideal* of S is a monomial ideal generated by a *lexsegment set*, that is a set of the form $L(u, v) = \{w \in \mathcal{M}_d : u \geq_{\text{lex}} w \geq_{\text{lex}} v\}$ where $u \geq_{\text{lex}} v$ are two given monomials of $\mathcal{M}_d$.

Lexsegment ideals were introduced in [8]. Their homological properties and invariants have been studied in several papers. We refer the reader to [1], [2], [4], [5], [6], [9], [10].

In [1], lexsegment ideals with linear resolution are characterized in numerical terms on the ends of the generating lexsegment set. In [6] it is shown that, for a lexsegment ideal, having a linear resolution is equivalent to having linear quotients with respect to a suitable order of the elements in the generating lexsegment set. There are known examples [3] which show that, in general, powers of monomial ideals with linear quotients may have no longer linear quotients, or even more, they do not have a linear resolution.

In this paper we show that the lexsegment ideals have a nice behavior with respect to taking powers, namely all powers of a lexsegment ideal with linear quotients (equivalently, with linear resolution) have linear quotients too (Theorem 2.11 and

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Corollary 3.9). Therefore, by collecting all the known results, we may now state the following

Theorem 1. *Let* $u = x_1^{a_1} \cdots x_n^{a_n}$ *with* $a_1 > 0$ *and* $v = x_1^{b_1} \cdots x_n^{b_n}$ *be monomials of degree* d *with* $u \geq_{lex} v$ *and let* $I = (L(u,v))$ *be a lexsegment ideal. Then the following statements are equivalent;*

(1) *I has a linear resolution.*
(2) *I has linear quotients.*
(3) *All the powers of I have linear quotients.*
(4) *All the powers of I have a linear resolution.*

In order to prove (2) ⇒ (3) in the above theorem, we are going to study in the first place (Section 2) the completely lexsegment ideals, that is, those whose generating lexsegment set has the property that its shadows are again lexsegment sets, and, secondly (Section 3), those which are not completely lexsegment ideals. For the first class of ideals we need to use and develop some of the techniques introduced in [4]. For the second class, we extend some results of [7].

It will turn out that the Rees algebras of the lexsegment ideals which are not completely have quadratic Gröbner bases, therefore they are Koszul (Corollary 3.11). For showing this property we need to slightly extend the notion of ℓ-exchange property which was defined in [7] to the notion of σ-exchange property. By exploiting this extension, we show in the last section that one may find larger classes of monomial ideals for which the Gröbner basis of the relation ideal of the Rees algebra $\mathcal{R}(I)$ can be determined (Theorem 3.4). Moreover, any monomial ideal $I \subset S$ whose minimal monomial generating set satisfies a σ-exchange property is of fiber type, that is the relations of its Rees algebra $\mathcal{R}(I)$ consist of the relations of the symmetric algebra $\mathcal{S}(I)$ and of the fiber relations (Corollary 3.5). We also show that the equigenerated monomial ideals whose minimal monomial generating set satisfies a σ-exchange property have the nice property that all their powers have linear quotients (Theorem 3.6).

1. PRELIMINARIES

In this section we recall the basic definitions and known results needed for the other sections.

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. For an integer $d \geq 2$, we denote by $\mathcal{M}_d$ the set of the monomials of degree d in S ordered lexicographically with $x_1 > x_2 > \cdots > x_n$. For two monomials $u, v \in \mathcal{M}_d$ such that $u \geq_{\text{lex}} v$, we denote by $L(u,v)$ the lexsegment set bounded by u and v, that is,

$$L(u,v) = \{w \in \mathcal{M}_d \colon u \geq_{\text{lex}} w \geq_{\text{lex}} v\}.$$

If $u = x_1^d$, then $L(u,v)$ is denoted $L^i(v)$ and is called the *initial lexsegment* determined by v. Similarly, if $v = x_n^d$, then $L(u,v)$ is denoted by $L^f(u)$ and is called the *final lexsegment* determined by u. An (*initial, final*) *lexsegment ideal* of S is a monomial ideal generated by an (initial, final) lexsegment set. According to [4], we denote by $\mathcal{L}_{u,v}$ the K-subalgebra of S generated by the monomials of $L(u,v)$. In

[4] it is proved that $\mathcal{L}_{u,v}$ is a Koszul algebra. More precisely, it is shown that the presentation ideal of $\mathcal{L}_{u,v}$ has a Gröbner basis of quadratic binomials. We briefly recall the basic tools used in [4] in proving this result, since they will be also useful in the next section.

Let $V_{n,d}$ be the Veronese subring of S, that is, $V_{n,d} = K[\mathcal{M}_d]$. Let w be a monomial in $\mathcal{M}_d$. One can write $w = x_{\mathbf{a}} = x_{a_1} \cdots x_{a_d}$, where $1 \leq a_1 \leq \cdots \leq a_d \leq n$. Consider the set of variables

$$\mathbb{T} = \{T_{\mathbf{a}} \colon \mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d, 1 \leq a_1 \leq \cdots \leq a_d \leq n\},$$

and let $\varphi \colon K[\mathbb{T}] \to V_{n,d}$ be the K-algebra homomorphism defined by

$$\varphi(T_{\mathbf{a}}) = x_{\mathbf{a}} = x_{a_1} \cdots x_{a_d}.$$

Then $V_{n,d} \cong K[\mathbb{T}]/\ker \varphi$ and $P = \ker \varphi$ is called the *toric* or the *presentation ideal* of $V_{n,d}$.

If $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d)$ are vectors with $1 \leq a_1 \leq \cdots \leq a_d \leq n$ and $1 \leq b_1 \leq \cdots \leq b_d \leq n$, we say that $\mathbf{a} > \mathbf{b}$ if $x_{\mathbf{a}} >_{\text{lex}} x_{\mathbf{b}}$, that is, if there exists $s \geq 1$ such that $a_i = b_i$ for $i \leq s-1$ and $a_s < b_s$. In this way, one gets a total order on the variables of $\mathbb{T}$ by setting $T_{\mathbf{a}} > T_{\mathbf{b}}$ if $\mathbf{a} > \mathbf{b}$. Let $>_{\text{lex}}$ be the lexicographic order on $K[\mathbb{T}]$ induced by this order of the variables of $\mathbb{T}$. Namely, we have $T_{\mathbf{a}(1)} \cdots T_{\mathbf{a}(N)} >_{\text{lex}} T_{\mathbf{b}(1)} \cdots T_{\mathbf{b}(N)}$ if there exists $1 \leq t \leq N$ such that $T_{\mathbf{a}(i)} = T_{\mathbf{b}(i)}$ for $i \leq t-1$ and $T_{\mathbf{a}(t)} > T_{\mathbf{b}(t)}$.

A *tableau* is an $N \times d$-matrix $A = [\mathbf{a}(1), \ldots, \mathbf{a}(N)]$ with entries in $\{1, \ldots, n\}$, with the property that in every row $\mathbf{a}(i) = (a_{i1}, \ldots, a_{id})$ we have $a_{i1} \leq \cdots \leq a_{id}$ and the row vectors are in decreasing lexicographic order, that is $\mathbf{a}(1) > \mathbf{a}(2) > \cdots > \mathbf{a}(N)$ or, equivalently, $T_{\mathbf{a}(1)} > T_{\mathbf{a}(2)} > \cdots > T_{\mathbf{a}(N)}$. The *support* of A is the collection $\text{supp}(A)$ of the integers which appear in the tableau with their occurrences. It is clear that one may associate to each tableau A its corresponding monomial $T_A := T_{\mathbf{a}(1)} \cdots T_{\mathbf{a}(N)}$ in $K[\mathbb{T}]$. A tableau $A = [\mathbf{a}(1), \ldots, \mathbf{a}(N)]$ is *standard* if, for every tableau $B = [\mathbf{b}(1), \ldots, \mathbf{b}(N)]$ of same support, $B \neq A$, one has

$$T_A = T_{\mathbf{a}(1)} \cdots T_{\mathbf{a}(N)} <_{\text{lex}} T_{\mathbf{b}(1)} \cdots T_{\mathbf{b}(N)} = T_B.$$

As follows from [4, Proposition 2.10], this is equivalent to saying that for any $1 \leq i < j \leq N$, the quadratic monomial $T_{\mathbf{a}(i)} T_{\mathbf{a}(j)}$ is standard. In [4, Lemma 2.9] it is shown that a quadratic monomial $T_{\mathbf{a}} T_{\mathbf{b}}$ it is standard if and only if $\mathbf{a} = \mathbf{b}$ or there exists $1 \leq i \leq d$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$, $a_i < b_i$, and, if $i < d$, then $b_{i+1} \leq \cdots \leq b_d \leq a_{i+1} \leq \cdots \leq a_d$. If A is a standard tableau, then the monomial $T_A = T_{\mathbf{a}(1)} \cdots T_{\mathbf{a}(N)}$ is called *standard*. Given a set $\mathcal{A}$ of Nd indices in the set $\{1, \ldots, n\}$, then there exists a unique standard tableau A of size $N \times d$ with $\text{supp}(A) = \mathcal{A}$.

We recall the recursive procedure given in [4] to construct a standard tableau A with a given support $\mathcal{A} = \{b_1, \ldots, b_{Nd}\}$ where $1 \leq b_1 \leq \cdots \leq b_{Nd} \leq n$. Namely, if $A = [\mathbf{a}(1), \ldots, \mathbf{a}(N)]$, where $\mathbf{a}(i) = (a_{i1}, \ldots, a_{id})$ for $1 \leq i \leq N$, then we proceed as follows. We put $b_1, \ldots, b_N$ on the first column of A, that is,

$$a_{11} = b_1, a_{21} = b_2, \ldots, a_{N1} = b_N.$$

Now we consider the decomposition of $(b_1, \ldots, b_N)$ in blocks of equal integers and fill in each sub-tableau determined by each block from the bottom to the top in an inductive way. We illustrate this procedure by a concrete example.

Let $N = 5$, $d = 3$, $n = 8$ and

$$\mathcal{A} = \{1, 1, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, 7, 8\}.$$

We indicate each step of the standard tableau of support $\mathcal{A}$.

$$\begin{array}{ccc} 1 & & \\ 1 & & \\ 2 & & \\ 3 & & \\ 3 & & \end{array} \longrightarrow \begin{array}{ccc} 1 & & \\ 1 & & \\ 2 & & \\ 3 & 4 & 4 \\ 3 & 4 & 5 \end{array} \longrightarrow \begin{array}{ccc} 1 & & \\ 1 & & \\ 2 & 5 & 6 \\ 3 & 4 & 4 \\ 3 & 4 & 5 \end{array} \longrightarrow \begin{array}{ccc} 1 & 6 & 7 \\ 1 & 6 & 8 \\ 2 & 5 & 6 \\ 3 & 4 & 4 \\ 3 & 4 & 5 \end{array}$$

We have

Proposition 1.1. [4, Proposition 2.11] *The set* $\mathcal{G} = \{T_{\mathbf{q}}T_{\mathbf{r}} - T_{\mathbf{a}}T_{\mathbf{b}} : T_{\mathbf{a}}T_{\mathbf{b}}$ *is a standard monomial and* $\operatorname{supp}[\mathbf{a}, \mathbf{b}] = \operatorname{supp}[\mathbf{q}, \mathbf{r}]\}$ *is a Gröbner basis of the presentation ideal of* $V_{n,d}$ *with respect to* $<_{\text{lex}}$.

Moreover, in [4, Lemma 2.12], it was proved that if $[\mathbf{a}, \mathbf{b}]$ is a standard tableau and $[\mathbf{q}, \mathbf{r}]$ is a non-standard tableau such that $\operatorname{supp}[\mathbf{a}, \mathbf{b}] = \operatorname{supp}[\mathbf{q}, \mathbf{r}]$, then $\mathbf{q} > \mathbf{a} \geq \mathbf{b} > \mathbf{r}$. Consequently, if $T_{\mathbf{a}}T_{\mathbf{b}}$ is a standard monomial and $T_{\mathbf{q}}T_{\mathbf{r}}$ is such that $\operatorname{supp}[\mathbf{a}, \mathbf{b}] = \operatorname{supp}[\mathbf{q}, \mathbf{r}]$, then $x_{\mathbf{a}}, x_{\mathbf{b}} \in L(u, v)$ if $x_{\mathbf{q}}$ and $x_{\mathbf{r}}$ belong to $L(u, v)$. Therefore, the set

$$\mathcal{G}' = \{T_{\mathbf{q}}T_{\mathbf{r}} - T_{\mathbf{a}}T_{\mathbf{b}} : T_{\mathbf{a}}T_{\mathbf{b}} \text{ is a standard monomial, } \operatorname{supp}[\mathbf{a}, \mathbf{b}] = \operatorname{supp}[\mathbf{q}, \mathbf{r}],$$
$$\text{and } x_{\mathbf{q}}, x_{\mathbf{r}} \in L(u, v)\}.$$

is a Gröbner basis of the presentation ideal $J_{\mathcal{L}_{u,v}}$ of the toric ring $\mathcal{L}_{u,v}$.

2. POWERS OF COMPLETELY LEXSEGMENT IDEALS WITH LINEAR RESOLUTION

In order to study the powers of the completely lexsegment ideals with linear quotients, we need to prove some preparatory results.

Definition 2.1. Let $w_1, \ldots, w_N$ be monomials in $\mathcal{M}_d$, $N \geq 2$. We call the product $w_1 \cdots w_N$ *standard* if $T_{w_1} \cdots T_{w_N}$ is a standard monomial, that is, the corresponding tableau is standard.

Definition 2.2. If $w_1, \ldots, w_N$ are monomials in $\mathcal{M}_d$, and $w_1 \cdots w_N = w'_1 \cdots w'_N$, where $w'_1, \ldots, w'_n \in \mathcal{M}_d$ and $w'_1 \cdots w'_N$ is a standard product, we call $w'_1 \cdots w'_N$ the *standard representation* of $w_1 \cdots w_N$.

Remark 2.3. Let $u_1, \ldots, u_N \in L(u, v)$, where $L(u, v) \subset \mathcal{M}_d$ is a lexsegment set. If $w_1 \cdots w_N$ is a standard representation of $u_1 \cdots u_N$, then $w_1, \ldots, w_N \in L(u, v)$. Indeed, let us assume that $T_{u_1} \cdots T_{u_N}$ is not a standard monomial, that is $T_{u_1} \cdots T_{u_N} \in \operatorname{in}_{<}(P)$, where $P \subset K[\mathbb{T}]$ is the presentation ideal of $V_{n,d}$. Then there exists $1 \leq i < j \leq N$ and $v_i, v_j \in L(u, v)$ with $u_i > v_i \geq v_j > u_j$ such that $T_{u_i}T_{u_j} - T_{v_i}T_{v_j} \in \mathcal{G}'$. Note that $T_{u_1} \cdots T_{u_N} >_{\text{lex}} T_{u_1} \cdots T_{u_{i-1}}T_{v_i}T_{u_{i+1}} \cdots T_{u_{j-1}}T_{v_j}T_{u_{j+1}} \cdots T_{u_N}$. If the product $T_{u_1} \cdots T_{u_{i-1}}T_{v_i}T_{u_{i+1}} \cdots T_{u_{j-1}}T_{v_j}T_{u_{j+1}} \cdots T_{u_N}$ is standard, we finished. Otherwise we continue the reduction. After a finite number of steps, we reach a standard product whose factors belong to the lexsegment set $L(u, v)$.

Before stating the preliminary results, we fix some notations. For a monomial u of S, we denote by $\nu_i(u)$ the exponent of the variable x_i in u, that is, $\nu_i(u) = \deg_{x_i}(u)$ for all $1 \leq i \leq n$. We denote $\mathrm{supp}(u) = \{i : \nu_i(u) > 0\}$ and set $\max(u) = \max \mathrm{supp}(u)$, $\min(u) = \min \mathrm{supp}(u)$.

Lemma 2.4. *Let $w_1 \cdots w_N$ be a standard monomial and let $x_1^d w_1 \cdots w_N = w'_1 \cdots w'_{N+1}$ be the standard representation of $x_1^d w_1 \cdots w_N$. Then $w'_1 \geq_{lex} w_1$.*

Proof. We make induction on the number of variables. The case $n = 2$ is straightforward. Let $n > 2$. One may assume, by induction on the degree d of the monomials, that $\nu_1(w_N) = 0$. If $\nu_1(w_1) = 0$, then $x_1 \mid w'_1$, hence $w'_1 >_{lex} w_1$. Let $\nu_1(w_1) = 1$. Therefore, there exists $0 < s < N$ such that $x_1 \mid w_s$ and $x_1 \nmid w_{s+1}$. If $s + d \geq N + 2$, then we finished, since $x_1^2 \mid w'_1$ by the construction of the standard monomials, and $\nu_1(w_1) = 1$. Now, let us consider $s + d \leq N + 1$. Let

$$q = \min\left(\frac{w_1}{x_1} \cdots \frac{w_s}{x_1}\right) \text{ and } q' = \min\left(\frac{w'_1}{x_1} \cdots \frac{w'_{s+d}}{x_1}\right).$$

Then, since $w_1 \cdots w_N$ and $w'_1 \cdots w'_{N+1}$ are standard products, we have $q = \min(w_1/x_1)$, $q' = \min(w'/x_1)$, $\max(w_j) \leq q \leq \min(w_i/x_1)$ for all $1 \leq i \leq s < j \leq N$, and $\max(w'_j) \leq q' \leq \min(w'_i/x_1)$ for all $1 \leq i \leq s+d < j \leq N+1$. If $q < q'$, then we get

$$\nu_{1<m\leq q}(w_1 \cdots w_N) := \sum_{1<m\leq q} \nu_m(w_1 \cdots w_N) \geq \deg(w_{s+1} \cdots w_N) =$$

$$= (N-s)d > (N+1-s-d)d \geq \nu_{1<m\leq q}(w'_1 \cdots w'_{N+1}) := \sum_{1<m\leq q} \nu_m(w'_1 \cdots w'_{N+1}),$$

which is impossible since $\nu_m(w_1 \cdots w_N) = \nu_m(w'_1 \cdots w'_{N+1})$ for all $m > 1$. Therefore, we must have $q \geq q'$. If $q > q'$, then we finished since $w'_1/x_1 >_{\mathrm{lex}} w_1/x_1$, whence $w'_1 >_{\mathrm{lex}} w_1$. What is left to consider is the case $q = q'$. In this case we have

$$\nu_{1<m\leq q}(w_1 \cdots w_N) = (N-s)d + \nu_q(w_1) + \cdots + \nu_q(w_s)$$

and

$$\nu_{1<m\leq q}(w'_1 \cdots w'_{N+1}) = (N+1-s-d)d + \nu_q(w'_1) + \cdots + \nu_q(w'_{s+d}).$$

Since $\nu_{1<m\leq q}(w_1 \cdots w_N) = \nu_{1<m\leq q}(w'_1 \cdots w'_{N+1})$, we obtain

$$\nu_q(w'_1) + \cdots + \nu_q(w'_{s+d}) = d(d-1) + \nu_q(w_1) + \cdots + \nu_q(w_s).$$

This implies that

$$\frac{w'_1}{x_1} \cdots \frac{w'_{s+d}}{x_1} = x_q^{d(d-1)}\left(\frac{w_1}{x_1} \cdots \frac{w_s}{x_1}\right).$$

Note that $\frac{w_1}{x_1} \cdots \frac{w_s}{x_1}$ is a standard product in the variables $x_q, \ldots, x_n$. Applying induction on the number n of variables, we have, after d steps, that

$$x_q^{d(d-1)}\left(\frac{w_1}{x_1} \cdots \frac{w_s}{x_1}\right) = \bar{w}_1 \cdots \bar{w}_{s+d},$$

where $\bar{w}_1 \cdots \bar{w}_{s+d}$, is a standard product and $\bar{w}_1 \geq_{lex} w_1/x_1$. But $\frac{w'_1}{x_1} \cdots \frac{w'_{s+d}}{x_1}$ is a standard product as well, hence we have $\bar{w}_1 = w'_1/x_1 \geq_{lex} w_1/x_1$, whence $w'_1 \geq_{lex} w_1$. $\square$

Lemma 2.5. *Let $u_1 \cdots u_N$ and $w_1 \cdots w_N$ be standard products and $u_1 \cdots u_N x_n = x_1 w_1 \cdots w_N$. Then we have $u_1 \geq_{lex} w_1$.*

Proof. We use induction on N. If $N = 1$, the inequality $u_1 \geq_{lex} w_1$ is obvious. Now we assume $N > 1$ and let $u_1 \cdots u_N = x_{b_1} \cdots x_{b_{Nd}}$, where $1 = b_1 \leq \cdots \leq b_{Nd} \leq n$ and $\min(u_j) = b_j$ for all $1 \leq j \leq N$. We first notice that we may assume without loss of generality that $\nu_1(u_i) \leq 1$ for all $1 \leq i \leq N$. If $b_2 > b_1$, we obviously have $w_1 \leq_{lex} u_1$ since $\min(w_1) = b_2$. Therefore, we may assume $b_1 = b_2 = 1$. If $b_1 < b_N$, let $k \leq N$ be the largest integer such that $b_{k-1} < b_k = \cdots = b_N$. We have $k \geq 3$. Since $u_1 \cdots u_N$ is a standard product, we get

$$u_1 \cdots u_{k-1} = x_{b_1} \cdots x_{b_{k-1}} x_{b_{N+(d-1)(N-k+1)+1}} \cdots x_{Nd}.$$

Similarly, since $w_1 \cdots w_N$ is a standard product, we get

$$w_1 \cdots w_{k-2} = x_{b_2} \cdots x_{b_{k-1}} x_{b_{N+(d-1)(N-k+2)+2}} \cdots x_{Nd} x_n.$$

Therefore, there exists a monomial $w \in \mathcal{M}_d$, namely

$$w = x_{b_{N+(d-1)(N-k+1)+1}} \cdots x_{b_{N+(d-1)(N-k+2)+1}}$$

such that

$$x_1 w_1 \cdots w_{k-2} w = u_1 \cdots u_{k-1} x_n.$$

One observes that $w_1 \cdots w_{k-2} w$ and $u_1 \cdots u_{k-1}$ are standard products. Then, by induction on N, it follows that $w_1 \leq_{lex} u_1$.

It remains to consider $b_1 = \cdots = b_N = 1 < b_{N+1} \leq \cdots \leq b_{Nd}$, since, by our assumption on $u_1, \ldots, u_N$, we cannot have $b_{N+1} = 1$. If $b_{N+1} < b_{N+d+1}$, then, by the construction of standard products, we get $w_1 <_{lex} u_1$. Let $b_{N+1} = b_{N+2} = \cdots = b_{N+d+1}$. Then we obtain

$$x_n \cdot \frac{u_1}{x_1} \cdots \frac{u_N}{x_1} = \frac{w_1}{x_1} \cdots \frac{w_{N-1}}{x_1} \left(x_{b_{N+1}} \cdots x_{b_{N+d}} \right),$$

whence

$$x_n \left(\frac{u_1}{x_1} \cdots \frac{u_N}{x_1} \right) = x_{b_{N+1}} \left(x_{b_{N+1}}^{d-1} \frac{w_1}{x_1} \cdots \frac{w_{N-1}}{x_1} \right).$$

Let $w'_1 \cdots w'_N$ be the standard representation of $x_{b_{N+1}}^{d-1} \frac{w_1}{x_1} \cdots \frac{w_{N-1}}{x_1}$. By Lemma 2.4, we have $w'_1 \geq_{lex} w_1/x_1$. On the other hand, we have

$$x_n \left(\frac{u_1}{x_1} \cdots \frac{u_N}{x_1} \right) = x_{b_{N+1}} \left(w'_1 \cdots w'_N \right),$$

with $\frac{u_1}{x_1} \cdots \frac{u_N}{x_1}$ and $w'_1 \cdots w'_N$ standard monomials in a number of variables smaller than n. By induction on n we get $u_1/x_1 \geq_{lex} w'_1$ whence $u_1/x_1 \geq_{lex} w_1/x_1$, which yields $u_1 \geq_{lex} w_1$. $\square$

Lemma 2.6. *Let* $u_1 \geq_{lex} \cdots \geq_{lex} u_N \geq_{lex} u_{N+1}$ *be monomials of degree* d *with* $\nu_1(u_i) \leq 1$ *for all* $1 \leq i \leq N$, *such that* $u_1 \cdots u_N$ *is a standard product and* $\max(\text{supp}(u_1 \cdots u_N)) \leq \min(\text{supp}(u_{N+1}))$. *Let* $v_1 \cdots v_{N+1}$ *be the standard representation of* $u_1 \cdots u_N u_{N+1}$. *Then* $v_{N+1} \leq_{lex} u_N$.

Proof. We use induction on N. For $N = 1$, since $v_1 v_2 = u_1 u_2$ and $v_1 v_2$ is a standard product, then we have $u_1 >_{lex} v_1 \geq_{lex} v_2 >_{lex} u_2$.

Let $N > 1$ and assume that $u_1 \cdots u_N = x_{b_1} \cdots x_{b_{Nd}}$ and $u_{N+1} = x_{b_{Nd+1}} \cdots x_{b_{(N+1)d}}$ with

$$b_1 \leq \cdots \leq b_{Nd} \leq b_{Nd+1} \leq \cdots \leq b_{(N+1)d}.$$

Since $u_1 \cdots u_N$ is a standard product, we have $\min(u_j) = b_j$ for all $1 \leq j \leq N$. Since $v_1 \cdots v_N v_{N+1}$ is standard, we have $\min(v_j) = b_j$ for all $1 \leq j \leq N+1$. If $b_{N+1} > b_N$, we obviously have $v_{N+1} \leq_{lex} u_N$. Therefore, it remains to consider that $b_N = b_{N+1}$. Let $1 \leq k \leq N$ be the largest integer such that $b_{k-1} < b_k = \cdots = b_N$. We have $k > 1$ since otherwise $\nu_1(u_1) \geq 2$. Since $u_1 \cdots u_N$ is standard, we get that

$$u_k \cdots u_N = x_{b_k} \cdots x_{b_N} x_{b_{N+1}} \cdots x_{b_{N+(d-1)(N-k+1)}}.$$

Similarly, since $v_1 \cdots v_{N+1}$ is standard, we get

$$v_k \cdots v_{N+1} = x_{b_k} \cdots x_{b_N} x_{b_{N+1}} \cdots x_{b_{N+(d-1)(N-k+2)+1}}.$$

Therefore, there exists a monomial $w \in \mathcal{M}_d$, namely

$$w = x_{b_{N+(d-1)(N-k+1)+1}} \cdots x_{b_{N+(d-1)(N-k+2)+1}},$$

such that $v_k \cdots v_{N+1} = u_k \cdots u_N w$ and $\max(\text{supp}(u_k \cdots u_N)) \leq \min(\text{supp}(w))$. One may note that $u_k \cdots u_N$ and $v_k \cdots v_{N+1}$ are standard products as well. By the induction hypothesis, we get $v_{N+1} \leq_{lex} u_N$. □

Lemma 2.7. *Let* $u_1, \ldots, u_N, w_1, \ldots, w_N$ *be monomials of degree* d *in* S *such that* $x_n u_1 \cdots u_N = x_1 w_1 \cdots w_N$, *where* $u_1 \cdots u_N$, $w_1 \cdots w_N$ *are standard products. Then* $u_N \geq_{lex} w_N$.

Proof. We may assume that $\nu_1(w_N) = 0$ which implies that $\nu_1(u_i) \leq 1$ for all $1 \leq i \leq N$. Let $u_1 \cdots u_N = x_{b_1} \cdots x_{b_{Nd}}$ with $1 = b_1 \leq \cdots \leq b_{Nd}$ and $\min(u_j) = b_j$ for all $1 \leq j \leq N$. We have $\min(w_j) = b_{j+1}$ for all $1 \leq j \leq N$. If $b_{N+1} > b_N$, then $w_N \leq_{lex} u_N$. Let $b_{N+1} = b_N$ and $1 \leq k \leq N$ be the largest integer such that $b_{k-1} < b_k = \cdots = b_N$. If $k = 1$, then $b_1 = \cdots = b_N = b_{N+1}$. Since $w_1 \cdots w_N$ is a standard product, we get $\nu_1(w_N) > 0$, which is impossible by our assumption. Therefore, it follows that $k > 1$. Since $u_1 \cdots u_N$ is a standard product, we have

$$u_k \cdots u_N = x_{b_k} \cdots x_{b_N} x_{b_{N+1}} \cdots x_{b_{N+(d-1)(N-k+1)}}.$$

Similarly, since $w_1 \cdots w_N$ is a standard product, we get

$$w_{k-1} \cdots w_N = x_{b_k} \cdots x_{b_N} x_{b_{N+1}} \cdots x_{b_{N+(d-1)(N-k+2)+1}}.$$

Therefore, if

$$w = x_{b_{N+(d-1)(N-k+1)+1}} \cdots x_{b_{N+(d-1)(N-k+2)+1}},$$

we have

$$w_{k-1} \cdots w_N = u_k \cdots u_N w.$$

and $\max(\operatorname{supp}(u_k \cdots u_N)) \leq \min(w)$. Since $u_k \cdots u_N$ and $w_{k-1} \cdots w_N$ are also standard products, by using the previous lemma, we get $w_N \leq_{lex} u_N$. □

In order to state the main theorem of this section we need to recall the following

Theorem 2.8 ([6],[2])**.** *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, with $a_1 > 0$, and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d with $u \geq_{lex} v$ and let $I = (L(u,v))$ be a completely lexsegment ideal. The following statements are equivalent:*

1. *u and v satisfy one of the following conditions:*
   - (i) *$u = x_1^a x_2^{d-a}$, $v = x_1^a x_n^{d-a}$ for some a with $0 < a \leq d$;*
   - (ii) *$b_1 < a_1 - 1$;*
   - (iii) *$b_1 = a_1 - 1$ and, for the largest monomial w of degree d with $w <_{lex} v$, one has $x_1 w / x_{\max(w)} \leq_{lex} u$.*
2. *I has linear quotients.*
3. *I has a linear resolution.*

Remark 2.9. It is obviously that, if a completely lexsegment ideal is determined by u and v satisfying condition (i) in the above theorem, then all its powers have linear quotients. Therefore, we only need to study the powers of completely lexsegment ideals which are determined by monomials u and v satisfying condition (ii) or (iii) in Theorem 2.8.

Theorem 2.10. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 > 0$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d with $u \geq_{lex} v$ and let $I = (L(u,v))$ be a completely lexsegment ideal with linear quotients. Then all the powers of I have linear quotients.*

Proof. By using Remark 2.9, we have to consider only the cases when u and v satisfy one of the following conditions:

- (a) $b_1 < a_1 - 1$;
- (b) $b_1 = a_1 - 1$ and for the largest monomial w of degree d with $w <_{lex} v$, one has $x_1 w / x_{\max(w)} \leq_{lex} u$.

We recall (see [6, Theorem 1.2]) that in these cases, I has linear quotients with respect to the following order on $\mathcal{M}_d$. For $w, w' \in \mathcal{M}_d$ we set $w \succ w'$ if $\nu_1(w) < \nu_1(w')$ or $\nu_1(w) = \nu_1(w')$ and $w >_{lex} w'$.

Let $N > 1$. We show that I^N has linear quotients with respect to the order $\succ$ on the set $\mathcal{M}_{Nd}$. Let $u_1 \cdots u_N, v_1 \cdots v_N \in I^N$ be two standard products such that $v_1 \cdots v_N \succ u_1 \cdots u_N$. We have to show that there exists a monomial $w \in I^N$ such that $w \succ u_1 \cdots u_N$, $w / \gcd(w, u_1 \cdots u_N) = x_i$ and x_i divides the monomial $v_1 \cdots v_N / \gcd(v_1 \cdots v_N, u_1 \cdots u_N)$. We have to analyze two cases.

Case I: $\nu_1(v_1 \cdots v_N) = \nu_1(u_1 \cdots u_N)$. By the definition of the order $\succ$, we must have $v_1 \cdots v_N >_{lex} u_1 \cdots u_N$. Let $i \geq 2$ be the smallest index such that $\nu_i(v_1 \cdots v_N) > \nu_i(u_1 \cdots u_N)$. We claim that there exists $1 \leq q \leq N$ such that $i < \max(u_q)$. Indeed, otherwise we have $i \geq \max(u_1 \cdots u_N)$ and obtain

$$Nd = \deg(u_1 \cdots u_N) = \sum_{k=1}^{i} \nu_k(u_1 \cdots u_N) < \sum_{k=1}^{i} \nu_k(v_1 \cdots v_N) \leq Nd,$$

a contradiction.

Let, therefore, $1 \leq q \leq N$ be such that $i < \max(u_q)$. Then we get

$$\frac{x_i u_q}{x_1} \in L(u,v) \text{ or } \frac{x_i u_q}{x_{\max(u_q)}} \in L(u,v)$$

(see also the proof of [6, Theorem 1.2]). We recall the argument which was used in [6, Theorem 1.2] and will be also used in this proof several times. We have $x_i u_q/x_1 <_{lex} u_q \leq_{lex} u$ and $x_i u_q/x_{\max(u_q)} >_{lex} u_q \geq_{lex} v$. If we assume that $x_i u_q/x_1 <_{lex} v$ and $x_1 u_q/x_{\max(u_q)} >_{lex} u$, we get $b_1 = a_1 - 1$ and $x_i u_q/x_1 \leq_{lex} w$, where w is the largest monomial of degree d such that $w <_{lex} v$. We get

$$\frac{x_i u_q}{x_1 x_{\max(x_i u_q/x_1)}} \leq_{lex} \frac{w}{x_{\max(w)}},$$

which, by using condition (b), leads to

$$\frac{x_i u_q}{x_{\max(u_q)}} \leq_{lex} \frac{x_1 w}{x_{\max(w)}} \leq_{lex} u,$$

a contradiction. Therefore, one of the monomials $u'_q = x_i u_q/x_1$ or $u''_q = x_i u_q/x_{\max(u_q)}$ belongs to $L(u,v)$. Note that $u'_q \succ u_q$ and $u''_q \succ u_q$. Then we may take $w = u_1 \cdots u_{q-1} u'_q u_{q+1} \cdots u_N$ or $w = u_1 \cdots u_{q-1} u''_q u_{q+1} \cdots u_N$. In each case it follows that $w \succ u_1 \cdots u_N$, $w/\gcd(w, u_1 \cdots u_N) = x_i$ and $x_i | v_1 \cdots v_N / \gcd(v_1 \cdots v_N, u_1 \cdots u_N)$.

Case II: $\nu_1(u_1 \cdots u_N) > \nu_1(v_1 \cdots v_N)$. Then there exist two monomials $m, m' \in S$ of same degree, let us say p, such that $\gcd(m, m') = 1$ and

$$m u_1 \cdots u_N = m' v_1 \cdots v_N. \tag{2.1}$$

Since $\nu_1(u_1 \cdots u_N) > \nu_1(v_1 \cdots v_N)$, we get $x_1 | m'$ and $x_1 \nmid m$. Let $i = \min(\operatorname{supp}(m))$.

If there exists $1 \leq q \leq N$ such that $i < \max(u_q)$, then, as in the proof of Case (I), we may take $w = u_1 \cdots u'_q \cdots u_N$ where $u'_q = x_i u_q/x_1$ or $u'_q = x_i u_q/x_{\max(u_q)}$. Then the following conditions hold: $w \succ u_1 \cdots u_N$, $w/\gcd(w, u_1 \cdots u_N) = x_i$ and x_i divides the monomial $v_1 \cdots v_N/(\gcd(v_1 \cdots v_N, u_1 \cdots u_N))$.

Now let $\max(u_q) \leq i$ for all $1 \leq q \leq N$, that is, $\operatorname{supp}(u_1 \cdots u_N) \subset \{1, \ldots, i\}$. We show by induction on $p = \deg(m)$ that there exists $j > 1$ such that $x_j | m$ and

$$x_j u_1 \cdots u_N = x_1 w_1 \cdots w_N, \tag{2.2}$$

where $w_1, \ldots, w_N \in L(u,v)$ and $w_1 \cdots w_N$ is a standard product. If $p = 1$, there is nothing to prove. Let $p > 1$ and assume that there exists $1 < j < i$ such that $x_j | m'$. There exists $1 \leq q \leq N$ such that $j < i \leq \max(v_q)$ since $x_i | v_1 \cdots v_N$. As $j < \max(v_q)$, it follows that one of the monomials $x_j v_q/x_1 \in L(u,v)$ or $x_j v_q/x_{\max(v_q)} \in L(u,v)$. Let us consider that $v'_q = x_j v_q/x_1 \in L(u,v)$. By using (2.1), we get the relation

$$m u_1 \cdots u_N = (x_1 m'/x_j)(v_1 \cdots v'_q \cdots v_N).$$

If $v''_q = x_j v_q/x_{\max(v_q)} \in L(u,v)$, then, by using again (2.1), we get the relation

$$m u_1 \cdots u_N = (x_{\max(v_q)} m'/x_j)(v_1 \cdots v''_q \cdots v_N).$$

These last two relations show that either there exists a relation of the form $m u_1 \cdots u_N = x_1^p w_1 \cdots w_N$ where $w_1 \cdots w_N$ is a standard product of monomials of $L(u,v)$, with $\deg(m) = p$ and $x_1 \nmid m$, or we may apply induction on p and reach the desired conclusion. In the first case, let $m = x_{i_1}x_{i_2}\cdots x_{i_p}$, with $i = i_1 \leq i_2 \leq \cdots \leq i_p \leq n$. For $j = \overline{1,p}$, let $w_{j1}\cdots w_{jN}$ be the standard product such that

$$x_{i_1}u_1\cdots u_N = x_1w_{11}w_{12}\cdots w_{1N},$$
$$x_{i_2}w_{11}w_{12}\cdots w_{1N} = x_1w_{21}w_{22}\cdots w_{2N},$$
$$x_{i_3}w_{21}w_{22}\cdots w_{2N} = x_1w_{31}w_{32}\cdots w_{3N},$$
$$\vdots$$
$$x_{i_p}w_{p-1,1}w_{p-1,2}\cdots w_{p-1,N} = x_1w_{p1}w_{p2}\cdots w_{pN}.$$

Multiplying these equalities, we get

$$mu_1\cdots u_N = x_1^p w_{p1}w_{p2}\cdots w_{pN},$$

hence $w_{pi} = v_i$, for $1 \leq i \leq N$, since $w_{p1}w_{p2}\cdots w_{pN}$ and $v_1\cdots v_N$ are standard products.

It is easily seen that $\mathrm{supp}(w_{j1}\cdots w_{jN}) \subset \{1,\ldots,i_j\}$ for all $1 \leq j \leq p$. Therefore, we may apply Lemma 2.5 and Lemma 2.7 and get

$$u \geq_{lex} u_1 \geq_{lex} w_{11} \geq_{lex} w_{21} \geq_{lex} \cdots \geq_{lex} w_{p1} = v_1 \geq_{lex} v$$

and

$$u \geq_{lex} u_N \geq_{lex} w_{1N} \geq_{lex} w_{2N} \geq_{lex} \cdots \geq_{lex} w_{pN} = v_N \geq_{lex} v.$$

In particular, we have

$$u \geq_{lex} w_{11} \geq_{lex} \cdots \geq_{lex} w_{1N} \geq_{lex} v,$$

whence

$$x_{i_1}u_1\cdots u_N = x_1w_{11}\cdots w_{1N},$$

and $w_{11},\ldots,w_{1N} \in L(u,v)$. Therefore, we have an equality of the form $x_ju_1\cdots u_N = x_1w_1\cdots w_N$, where $w_1\cdots w_N \in I^N$ is a standard product and $j \geq 2$. Let $w = (x_ju_1\cdots u_N)/x_1$. Then $w \succ u_1\cdots u_N, w/\gcd(w,u_1\cdots u_N) = x_j$ and x_j divides the monomial $v_1\cdots v_N/\gcd(v_1\cdots v_N, u_1\cdots u_N)$, which ends our proof. $\square$

Combining the above theorem with [1, Theorem 1.3] and [6, Theorem 1.2], we get the following equivalent statements.

Theorem 2.11. *Let $u = x_1^{a_1}\cdots x_n^{a_n}$ with $a_1 > 0$ and $v = x_1^{b_1}\cdots x_n^{b_n}$ be monomials of degree d with $u \geq_{lex} v$ and let $I = (L(u,v))$ be a completely lexsegment ideal with linear quotients. The the following statements are equivalent;*

1. *u and v satisfy one of the following conditions:*
   - (i) *$u = x_1^a x_2^{d-a}$, $v = x_1^a x_n^{d-a}$ for some a with $0 < a \leq d$;*
   - (ii) *$b_1 < a_1 - 1$;*
   - (iii) *$b_1 = a_1 - 1$ and, for the largest monomial of degree d with $w <_{lex} v$, one has $x_1w/x_{\max(w)} \leq_{lex} u$.*
2. *I has a linear resolution.*
3. *I has linear quotients.*
4. *All the powers of I have linear quotients.*
5. *All the powers of I have a linear resolution.*

3. Exchange properties and applications

We first fix some notations. As in the previous section, let $S = K[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over a field K and $\mathcal{M}_d$ the set of all monomials of degree d in S. If $B \subset \mathcal{M}_d$ is a nonempty set, we denote by $K[B]$ the K-subalgebra of S generated by the monomials of B.

Let $R = K[\{T_u\}_{u \in B}]$ be the polynomial ring in a set of variables indexed over B and $\pi : R \to K[B]$ the surjective K-algebra homomorphism defined by $\pi(T_u) = u$, for all $u \in B$. $J_{K[B]} := \ker \pi$ is called the *toric ideal* of $K[B]$.

Let $<$ be a monomial order on R and $\mathrm{in}_<(J_{K[B]})$ the initial ideal of $J_{K[B]}$ with respect to $<$. A monomial $T_{u_1} \cdots T_{u_N} \in R$ is a *standard monomial* of $J_{K[B]}$ with respect to $<$ if $T_{u_1} \cdots T_{u_N} \notin \mathrm{in}_<(J_{K[B]})$. We recall the following definition which was given in [7].

Definition 3.1. [7, Definition 4.1] We say that a nonempty set $B \subset \mathcal{M}_d$ satisfies the *ℓ-exchange property* with respect to a monomial order $<$ on R if B posseses the following property: if $T_{u_1} \cdots T_{u_N}$ and $T_{v_1} \cdots T_{v_N}$ are standard monomials of $J_{K[B]}$ with respect to $<$ such that

(a) $\nu_i(u_1 \cdots u_N) = \nu_i(v_1 \cdots v_N)$ for $1 \leq i \leq q-1$ (with $q \leq n-1$),
(b) $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$,

then there exist $1 \leq \delta \leq N$, and $q < j \leq n$ with $j \in \mathrm{supp}(u_\delta)$ and $x_q u_\delta / x_j \in B$.

Inspired by this definition we consider the following slight generalization. Let $<_\sigma$ be a monomial order on S.

Definition 3.2. We say that B satisfies the *σ-exchange property* with respect to $<$ if B has the following property: if $T_{u_1} \cdots T_{u_N}$ and $T_{v_1} \cdots T_{v_N}$ are standard monomials of $J_{K[B]}$ with respect to $<$ such that $u_1 \cdots u_N <_\sigma v_1 \cdots v_N$, then there exist $1 \leq \delta \leq N$, $q \in \mathrm{supp}(v_1 \cdots v_N)$, and $j \in \mathrm{supp}(u_\delta)$ such that

(i) $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$,
(ii) $x_j <_\sigma x_q$,
(iii) $x_q u_\delta / x_j \in B$.

It is straightforward to show that if B satisfies the ℓ-exchange property with respect to a monomial order $<$ on R, then B satisfies the σ-exchange property with respect to $<$ for $<_\sigma = <_{lex}$ on S with $x_1 >_{lex} \cdots >_{lex} x_n$.

Example 3.3. Let $<_\sigma$ be a monomial order on S defined as follows. For m, m' monomials in S, we set $m <_\sigma m'$ if $\deg(m) < \deg(m')$ or $\deg(m) = \deg(m')$ and $m >_{revlex} m'$, that is, if $m = x_1^{a_1} \cdots x_n^{a_n}$, $m' = x_1^{b_1} \cdots x_n^{b_n}$, then there exists some $1 \leq s \leq n$ such that $a_n = b_n$, $a_{n-1} = b_{n-1}, \ldots, a_{s+1} = b_{s+1}$, and $a_s < b_s$. In particular, we have $x_n >_\sigma x_{n-1} >_\sigma \cdots >_\sigma x_1$. We call this monomial order the decreasing revlexicographical order on S.

Any final lexsegment set $L^f(v)$, $v \in \mathcal{M}_d$, satisfies the σ-exchange property for $<_\sigma$ as above, with respect to any monomial order $<$ on $R = K[\{T_w : w \in L^f(v)\}]$. In order to prove this claim, let $T_{u_1} \cdots T_{u_N}$ and $T_{v_1} \cdots T_{v_N}$ be two standard monomials of $J_{K[B]}$ with respect to $<$ such that $u_1 \cdots u_N <_\sigma v_1 \cdots v_N$, that is

$$u_1 \cdots u_N >_{revlex} v_1 \cdots v_N.$$

Then there exists $1 \leq q \leq n$ such that $\nu_i(u_1 \cdots u_N) = \nu_i(v_1 \cdots v_N)$ for all $i \geq q+1$ and $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$. Since $\deg(u_1 \cdots u_N) = \deg(v_1 \cdots v_N)$, we must have at least an index $j < q$ such that $\nu_j(u_1 \cdots u_N) > \nu_j(v_1 \cdots v_N)$. Let $1 \leq \delta \leq N$ be such that $j \in \operatorname{supp}(u_\delta)$. Then the following conditions hold: $x_j >_{revlex} x_q$, that is $x_j <_\sigma x_q$ and $x_q u_\delta / x_j <_{lex} u_\delta$, whence $x_q u_\delta / x_j \in L^f(v)$.

We also notice that, if we choose $<$ on R to be the monomial order given in the previous section, that is the lexicographical order on the monomials $\{T_w \,:\, w \in L^f(v)\}$ induced by $T_{w_1} > T_{w_2}$ if $w_1 >_{lex} w_2$, then $L^f(v)$ does not satisfy the ℓ-exchange property with respect to $<$. For example, let $v = x_1x_3x_4 \in K[x_1, x_2, x_3, x_4]$. Let $u_1 = x_2^3$ and $v_1 = x_1x_3x_4$, $u_1, v_1 \in L^f(v)$. Then $(T_{u_1})^2$ and $(T_{v_1})^2$ are standard monomials with respect to $<$ on $R = K[\{T_w : w \in L^f(v)\}]$ and $u_1^2 <_{lex} v_1^2$. In the ℓ-exchange property, we have to take $q = 1$. Since $\operatorname{supp}(u_1) = \{2\}$, we should have $x_1 u_1 / x_2 = x_1 x_2^2 \in L^f(v)$, which is not possible.

Following closely the ideas from the last section in [7], we may prove a slight generalization of [7, Theorem 5.1].

Let $I \subset S$ be a monomial ideal generated in degree d and let $B = G(I)$ its minimal monomial generating set. Let $\mathcal{T} = S[\{T_u\}_{u \in B}] = K[x_1, \ldots, x_n, T_u : u \in B]$ be the polynomial ring over K. $\mathcal{T}$ is bigraded by $\deg(x_i) = (1, 0)$ for all $1 \leq i \leq n$ and $\deg(T_u) = (0, 1)$ for all $u \in B$.

Let $\mathcal{R}(I) = \bigoplus_{j \geq 0} I^j t^j = S[\{ut\}_{u \in B}] \subset S[t]$ be the Rees ring of I. $\mathcal{R}(I)$ is also naturally bigraded by $\deg(x_i) = (1, 0)$ for $1 \leq i \leq n$ and $\deg(ut) = (0, 1)$ for all $u \in B$. There exists a canonical bigraded surjective K-algebra homomorphism $\varphi : \mathcal{T} \to \mathcal{R}(I)$ defined b $\varphi(x_i) = x_i$ for $1 \leq i \leq n$ and $\varphi(T_u) = ut$ for all $u \in B$. Let $P_{\mathcal{R}(I)} := \ker \varphi$ be the toric ideal of $\mathcal{R}(I)$. $P_{\mathcal{R}(I)}$ is bihomogeneous and generated by irreducible bihomogeneous binomials of $\mathcal{T}$. Let $<^\#$ be an arbitrary monomial order on R and $<_\sigma$ be an arbitrary monomial order on S. By $<_\sigma^\#$ we will denote the product of these two orders which is a monomial order on $\mathcal{T}$. More precisely, for $mT_{u_1} \cdots T_{u_N}$, $m'T_{v_1} \cdots T_{v_N}$, monomials in $\mathcal{T}$, with m, m' monomials in S, we have $mT_{u_1} \cdots T_{u_N} <_\sigma^\# m'T_{v_1} \cdots T_{v_N}$ if $m <_\sigma m'$ or $m = m'$ and $T_{u_1} \cdots T_{u_N} <^\# T_{v_1} \cdots T_{v_N}$.

The following theorem generalizes [7, Theorem 5.1].

Theorem 3.4. *Let $I \subset S$ be a monomial ideal generated in degree d, $B = G(I)$, $<^\#$ a monomial order on R and $<_\sigma$ a monomial order on S. Let $G_{<^\#}(J_{K[B]})$ be the reduced Gröbner basis of the toric ideal $J_{K[B]}$ with respect to $<^\#$. Suppose that B satisfies the σ-exchange property with respect to $<^\#$. Then the reduced Gröbner basis of the toric ideal $P_{\mathcal{R}(I)}$ with respect to $<_\sigma^\#$ consists of all binomials belonging to $G_{<^\#}(J_{K[B]})$ together with the binomials of the form*

$$x_i T_u - x_j T_v \in P_{\mathcal{R}(I)}$$

where x_j is the smallest variable with respect to $<_\sigma$ such that $x_i >_\sigma x_j$ and $x_i u / x_j \in B$.

Proof. We closely follow the ideas from the proof of [7, Theorem 5.1].

We first show that the set

$$G = G_{<^\#}(J_{K[B]}) \cup \{x_i T_u - x_j T_v \in P_{\mathcal{R}(I)} \;:\; x_i >_\sigma x_j\}$$

is a Gröbner basis of $P_{\mathcal{R}(I)}$ with respect to $<_\sigma^\#$.

Let $f \in P_{\mathcal{R}(I)} \subset \mathcal{T}$ be an irreducible binomial. If $\mathrm{in}_{<_\sigma^\#}(f) \in R$, then $f \in P_{\mathcal{R}(I)} \cap R = J_{K[B]}$, hence there is a binomial belonging to $G_{<^\#}(J_{K[B]})$ which divides $\mathrm{in}_{<_\sigma^\#}(f)$.

Let $\mathrm{in}_{<_\sigma^\#}(f) \notin R$, that is, we may write

$$f = x_{i_1} \cdots x_{i_t} T_{u_1} \cdots T_{u_N} - x_{j_1} \cdots x_{j_t} T_{v_1} \cdots T_{v_N}$$

with $\{i_1, \ldots, i_t\} \cap \{j_1, \ldots, j_t\} = \emptyset$ and where we assume that $x_{i_1} \geq_\sigma \cdots \geq_\sigma x_{i_t}$ and $x_{j_1} \geq_\sigma \ldots \geq_\sigma x_{j_t}$. By successively reductions modulo the binomials from $G_{<^\#}(J_{K[B]})$ we may assume that $T_{u_1} \cdots T_{u_N}$ and $T_{v_1} \cdots T_{v_N}$ are standard monomials with respect to $<^\#$. Let $\mathrm{in}_{<_\sigma^\#}(f) = x_{i_1} \cdots x_{i_t} T_{u_1} \cdots T_{u_N}$. Then $x_{i_1} \cdots x_{i_t} >_\sigma x_{j_1} \cdots x_{j_t}$. By using the equality

$$x_{i_1} \cdots x_{i_t} u_1 \cdots u_N = x_{j_1} \cdots x_{j_t} v_1 \cdots v_N,$$

we obtain $u_1 \cdots u_N <_\sigma v_1 \cdots v_N$, $\nu_{i_s}(u_1 \cdots u_N) < \nu_{i_s}(v_1 \cdots v_N)$ for $1 \leq s \leq t$, and $\nu_k(u_1 \cdots u_N) \geq \nu_k(v_1 \cdots v_N)$ for all $k \notin \{i_1, \ldots, i_t\}$. Since B satisfies the σ-exchange property, we have that there exist $1 \leq \delta \leq N$, $j \in \mathrm{supp}(u_\delta)$ and $q \in \mathrm{supp}(v_1 \cdots v_N)$ such that $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$, $x_j <_\sigma x_q$, and $x_q u_\delta / x_j \in B$.

The first above condition on q shows that $q = i_s$, for some $1 \leq s \leq t$. Therefore we have $x_{i_s} u_\delta = x_j v$ for some $v \in B$ and the proof of our claim is finished.

To end the proof, let us take some binomial $x_i T_u - x_j T_v$, where $u, v \in B$, $x_i u = x_j v$ and $x_j <_\sigma x_i$ is the smallest variable with respect to $<_\sigma$ such that $x_i u / x_j \in B$. Assume that $x_j T_v$ is not reduced, hence there exists some binomial $x_j T_v - x_l T_w$ with $x_l <_\sigma x_j$, which belongs to $P_{\mathcal{R}(I)}$. Then $x_i T_u - x_l T_w \in P_{\mathcal{R}(I)}$ and $x_l <_\sigma x_j <_\sigma x_i$, a contradiction. □

Corollary 3.5. *Let $I \subset S$ be a monomial ideal generated in degree d and $B = G(I)$. Let $<^\#$ be a monomial order on R and $<_\sigma$ a monomial order on S. If B satisfies the σ-exchange property with respect to $<^\#$, then I is of fiber type.*

We recall (see [7]) that an ideal $I \subset S$ is called *of fiber type* if the fiber relations together with the relations of the symmetric algebra of I generate all the relations of the Rees algebra of I.

The above corollary may be used to find equigenerated monomial ideals of fiber type. Let $<_\sigma$ be an arbitrary graded monomial order on S, $u \in \mathcal{M}_d$ and $I = (L^i_{<_\sigma}(u))$, where $L^i_{<_\sigma}(u) = \{w \in \mathcal{M}_d : w >_\sigma u\}$. Then it is easily seen that $(L^i_{<_\sigma}(u))$ satisfies the σ-exchange property for any monomial order on $R = K[\{T_w : w \in L^i_{<_\sigma}(u)\}]$, hence I is of fiber type.

We prove now a significant property of the monomial ideals whose minimal monomial generating system satisfies a σ-exchange property.

Theorem 3.6. *Let $I \subset S$ be a monomial ideal generated in degree d and $B = G(I)$. Let $<^\#$ be a monomial order on $R = K[\{T_u : u \in B\}]$ and $<_\sigma$ a monomial order on S. If B satisfies the σ-exchange property with respect to $<^\#$, then I^N has linear quotients with respect to $>_\sigma$ for $N \geq 1$.*

Proof. Let $G\left(I^N\right) = \{w_1, \cdots, w_r\}$, where $w_1 >_\sigma \cdots >_\sigma w_r$ and let $T_{w_1}, \ldots, T_{w_r}$ be standard monomials of $J_{K[B]}$ with respect to $<^\#$. Let $1 \leq j < i \leq r$ be two integers and assume that $w_j = v_1 \cdots v_N$ and $w_i = u_1 \cdots u_N$ for $u_1, \ldots, u_N, v_1, \ldots, v_N \in G(I)$,

$u_1 \geq_\sigma \cdots \geq_\sigma u_N$, $v_1 \geq_\sigma \cdots \geq_\sigma v_N$. We have to prove that there exist $1 \leq k < i$ and $1 \leq q \leq n$ such that

$$\frac{w_k}{\gcd(w_k, w_i)} = x_q \text{ and } x_q \mid \frac{w_j}{\gcd(w_j, w_i)}.$$

Since $w_j >_\sigma w_i$, by using the σ-exchange property of B, there exist $1 \leq \delta \leq N$, $l \in \operatorname{supp}(u_\delta)$, and $q \in \operatorname{supp}(v_1 \cdots v_N)$ such that $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$, $x_l <_\sigma x_q$, and $x_q u_\delta / x_l \in B$. Let

$$w_k = u_1 \cdots u_{\delta-1} \frac{x_q u_\delta}{x_l} u_{\delta+1} \cdots u_N = \frac{x_q w_i}{x_l}.$$

Then w_k satisfies the required conditions. □

In the sequel we show that the lexsegment ideals with a linear resolution which are not completely satisfies an exchange property.

We first recall the following

Theorem 3.7 ([1]). *Let $I = (L(u, v))$ be a lexsegment ideal with $x_1 \mid u$ and $x_1 \nmid v$ which is not a completely lexsegment ideal. Then I has a linear resolution if and only if u and v have the following form:*

$$u = x_1 x_{l+1}^{a_{l+1}} \cdots x_n^{a_n} \text{ and } v = x_l x_n^{d-1}$$

for some l, $2 \leq l \leq n-1$.

Theorem 3.8. *Let $<_\sigma$ be the decreasing revlexicographical order on S and $I = (L(u, v))$ a lexsegment ideal with linear resolution which is not a completely lexsegment ideal. Then $L(u, v)$ satisfies the σ-exchange property with respect to any monomial order on $R = K[\{T_w : w \in L(u, v)\}]$.*

Proof. Let $u = x_1 x_{l+1}^{a_{l+1}} \cdots x_n^{a_n}$, $v = x_l x_n^{d-1}$ for some $2 \leq l \leq n-1$. Let us assume that there exists a monomial order $<$ on R such that $L(u, v)$ does not satisfy the σ-exchange property with respect to $<$. Then there exist two standard monomials $T_{u_1} \cdots T_{u_N}$ and $T_{v_1} \cdots T_{v_N}$ such that $u_1 \cdots u_N >_{revlex} v_1 \cdots v_N$ and with the property that for all $1 \leq \delta \leq N$, $j \in \operatorname{supp}(u_\delta)$ and $q \in \operatorname{supp}(v_1 \cdots v_N)$ such that $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$ and $x_j >_{revlex} x_q$, we have $x_q u_\delta / x_j \notin L(u, v)$. Since $u_1 \cdots u_N >_{revlex} v_1 \cdots v_N$ there exists some q, $1 \leq q \leq n$, such that

$$\nu_i(u_1 \cdots u_N) = \nu_i(v_1 \cdots v_N) \text{ for all } i \geq q+1$$

and $\nu_q(u_1 \cdots u_N) < \nu_q(v_1 \cdots v_N)$. Since $\deg(u_1 \cdots u_N) = \deg(v_1 \cdots v_N)$ there exists some $s < q$ such that $\nu_s(u_1 \cdots u_N) > \nu_s(v_1 \cdots v_N)$. Let u_δ be such that $s \in \operatorname{supp}(u_\delta)$. By our assumption, we must have $x_q u_\delta / x_s <_{lex} v$, that is $x_q u_\delta / x_s \leq_{lex} x_{l+1}^d$. This implies, in particular, that $q \geq l+1$, and that for all δ, $1 \leq \delta \leq N$, there exists a unique $j_\delta \leq l$ such that $u_\delta = x_{j_\delta} w_\delta$ where $\min(w_\delta) \geq l+1$.

Therefore we have $u_1 \cdots u_N = x_{j_1} \cdots x_{j_N} x_t^{a_t} \cdots x_n^{a_n}$, where $j_1, \cdots, j_N < q$ and $t \geq q$. We have

$$a_t + \cdots + a_n = \deg(x_t^{a_t} \cdots x_n^{a_n}) = Nd - N = N(d-1).$$

Let $v_1 \cdots v_N = x_1^{b_1} \cdots x_n^{b_n}$. By hypothesis, we have $a_q < b_q$ and $a_i = b_i$ for all $i \geq q+1$. Since each monomial $v_\gamma \in L(u,v)$ it is divisible by some variable x_i with $i \leq l < q$, we have $b_1 + \cdots + b_{q-1} \geq N$. Then we have

$$Nd = b_1 + \cdots + b_{q-1} + b_q + \cdots + b_n > b_1 + \cdots + b_{q-1} + a_q + \cdots + a_t + \cdots + a_n \geq$$

$$\geq N + N(d-1) = Nd,$$

a contradiction. □

Corollary 3.9. *All powers of a lexsegment ideal with a linear resolution which is not a completely lexsegment ideal have linear quotients with respect to the increasing revlexicographic order.*

Corollary 3.10. *Any lexsegment ideal with a linear resolution which is not a completely lexsegment ideal is of fiber type.*

Corollary 3.11. *Let $I = (L(u,v))$ be a lexsegment ideal with a linear resolution which is not a completely lexsegment ideal. Then the Rees algebra $\mathcal{R}(I)$ is Koszul.*

Proof. Let $<^{\#}$ be the lexicographical monomial order on $R = K[\{T_w : w \in L(u,v)\}]$ induced by $T_{w_1} > T_{w_2}$ if $w_1 >_{lex} w_2$ and $<_\sigma$ be the decreasing revlexicographic order on S. By Theorem 3.4, the reduced Gröbner basis of $P_{\mathcal{R}(I)}$ with respect to the product order $<_\sigma^{\#}$ on $\mathcal{T}$ is formed by the binomials from $G_{<^{\#}}\left(J_{K[L(u,v)]}\right)$, the reduced Gröbner basis of $J_{K[L(u,v)]}$, and by the binomials of the form

$$x_i T_{u'} - x_j T_{v'},$$

where $x_i >_\sigma x_j$, $x_i u' = x_j v'$ and j is the smallest integer with $x_i u'/x_j \in L(u,v)$. Since $G_{<^{\#}}\left(J_{K[L(u,v)]}\right)$ is quadratic ([4, Proposition 2.13]), the statement follows. □

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