ON THE RANK OF MULTI-GRADED DIFFERENTIAL MODULES

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ABSTRACT. A $\mathbb{Z}^d$-graded differential R-module is a $\mathbb{Z}^d$-graded R-module D with a morphism $\delta : D \to D$ such that $\delta^2 = 0$. For $R = k[x_1, \dots, x_d]$, this paper establishes a lower bound on the rank of such a differential module when the underlying R-module is free. We define the Betti number of a differential module and use it to show that when the homology $H(D) = \ker \delta / \operatorname{im} \delta$ of D is non-zero and finite dimensional over k then there is an inequality $\operatorname{rank}_R D \geq 2^d$.

1. Introduction

Let k be a field and set $R = k[x_1, \dots, x_d]$. A *differential R-module* D is an R-module with a square-zero homomorphism $\delta : D \to D$ called the *differential*. The homology of D is defined in the usual way: $H(D) = \ker \delta / \operatorname{im} \delta$. Differential modules have played an important role in the work of Avramov, Buchweitz, Iyengar, and Miller on the homology of finite free complexes [1, 2]. In this context, differential modules arise naturally when working with DG-modules: some constructions with desirable properties do not respect the grading but do preserve the differential (see [2, 10] for some instances of this phenomenon).

Motivated by a conjecture of Avramov, Buchweitz and Iyengar, this paper is concerned with establishing bounds on the ranks of $\mathbb{Z}^d$-graded differential modules. They conjecture:

Conjecture 1.1 ([1, Conjecture 5.3])**.** Let R be a regular local ring of dimension d, and F a differential R-module admitting a finite free flag. If $H(F)$ has non-zero finite length, then

$$\operatorname{rank}_R F \geq 2^d.$$

In this conjecture, a *free flag* on a differential module is a filtration compatible with the differential that provides appropriate lifting properties in the category of differential modules (see Definition 2.7). In their paper they show that Conjecture 1.1 is true for $d \leq 2$ or for $d = 3$ when R is a unique factorization domain [1, Theorem 5.2].

The main theorem (Theorem 6.4) of this paper establishes a lower bound on the rank of a differential module by finding a lower bound on the *Betti number* of a differential module. A consequence of this result is the following theorem that partially answers Conjecture 1.1 in the multi-graded case.

Theorem 1.2. *Let F be a finitely generated $\mathbb{Z}^d$-graded differential R-module with differential $\delta : F \to F$ that is homogeneous of degree zero, such that F is free as an R-module. If $H(F)$ has non-zero finite length then* $\operatorname{rank}_R F \geq 2^d$.

This result is new even for complexes of R-modules. Given a complex of $\mathbb{Z}^d$-graded free R-modules

$$F = \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \cdots$$

the module $\bigoplus_i F_i$ with differential $\delta = \bigoplus_i \partial^i$ forms a differential module. When $H(F)$ has non-zero finite length as an R-module then Theorem 1.2 implies that

$$\sum_i \operatorname{rank}_R F_i \geq 2^d. \tag{1.1}$$

This inequality is already known when F is a resolution—i.e. $H(F)$ is concentrated in a single homological degree—from the work of Charalambous and Santoni on the Buchsbaum-Eisenbud-Horrocks problem [6, 11]. Recall that for a $\mathbb{Z}$-graded polynomial ring, the Buchsbaum-Eisenbud-Horrocks problem is to show that all non-zero finite length R-modules M satisfy $\beta_i(M) \geq \binom{d}{i}$, where $\beta_i(M)$ is the i-th Betti number of M [3, 9]. Summing the binomial coefficients gives (1.1) when F is a free resolution of a non-zero finite length $\mathbb{Z}^d$-graded module M. However, when F is not acyclic it is not clear how to establish (1.1) without using differential modules.

Working with differential modules provides an advantage in that it simultaneously treats the case of free resolutions and free complexes with homology spread among several homological degrees, as well as other contexts. One such application arises in the conjectures of Carlsson and Halperin concerning a lower bound on the rank of DG-modules with non-zero finite length homology [4, 8]. For this connection between differential modules and DG-modules see [1, §5].

Some techniques available for complexes can be directly adapted to the case of differential modules, however there are some subtle difficulties that appear. For example, there may be no way to minimize a resolution in the category of differential modules which creates an obstruction to applying the usual tools of complexes (see Example 4.3, or Theorem 4.1 for some positive results). Not much is known about the techniques available for working with general differential modules. This work should be seen as a contribution in that direction.

In §2 we recall the theory of differential modules and define the Betti number. Some results and examples comparing complexes to differential modules are provided in §3 and §4. Section 5 adapts an inequality of Santoni [11] to differential modules. The main result, Theorem 6.4, is proved in §6.

2. Differential modules

Throughout, k is a field, $R = k[x_1, \ldots, x_d]$ is the standard $\mathbb{Z}^d$-graded polynomial ring and $\mathfrak{m} = (x_1, \ldots, x_d)$. To be specific, the grading on R is such that the degree $\deg(x_i) \in \mathbb{Z}^d$ of variable x_i is $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 appearing in the i-th coordinate. For $\mathbf{m} \in \mathbb{Z}^d$, we write $\mathbf{m}_i$ to denote the i-th coordinate. The group $\mathbb{Z}^d$ is equipped with the coordinate-wise partial order: $\mathbf{a} \leq \mathbf{b}$ if and only if $\mathbf{a}_i \leq \mathbf{b}_i$ for all i.

Recall that a $\mathbb{Z}^d$-graded module M over R is an R-module that has a decomposition $\bigoplus_{\mathbf{n} \in \mathbb{Z}^d} M_{\mathbf{n}}$ as abelian groups such that multiplication by an element of R of degree $\mathbf{m}$ takes $M_{\mathbf{n}}$ to $M_{\mathbf{m}+\mathbf{n}}$. An R-linear map ϕ between $\mathbb{Z}^d$-graded modules M and N is a *morphism* if $\phi(M_{\mathbf{m}}) \subseteq N_{\mathbf{m}}$. In particular, a complex of $\mathbb{Z}^d$-graded modules is required to have morphisms for its differentials.

For $\mathbf{n} \in \mathbb{Z}^d$ the shifted (or twisted) module $M(\mathbf{n})$ is defined to be $M_{\mathbf{m}+\mathbf{n}}$ in degree $\mathbf{m}$ for each $\mathbf{m} \in \mathbb{Z}^d$ with the same R-module structure as M. Given a morphism $\phi : M \to N$ the shifted morphism $M(\mathbf{n}) \to N(\mathbf{n})$ defined by $x \mapsto \phi(x)$ is denoted $\phi(\mathbf{n})$.

We will primarily work with $\mathbb{Z}^d$-graded modules and $\mathbb{Z}^d$-graded differential modules, so definitions will be given only in that context for simplicity; see [1, 5, 7] for details concerning arbitrary differential modules.

Definition 2.1. A *$\mathbb{Z}^d$-graded differential R-module with differential degree* $\mathbf{d} \in \mathbb{Z}^d$ is a $\mathbb{Z}^d$-graded R-module D with a morphism $\delta : D \to D(\mathbf{d})$ such that the composition

$$D(-\mathbf{d}) \xrightarrow{\delta(-\mathbf{d})} D \xrightarrow{\delta} D(\mathbf{d})$$

is zero. We say that D has *differential* δ.

The homology of a differential module D is the $\mathbb{Z}^d$-graded R-module

$$H(D) = \ker \delta / \operatorname{im}(\delta(-\mathbf{d})).$$

Any $\mathbb{Z}^d$-graded R-module, in particular $H(D)$, will be considered as a differential module with zero differential.

A *morphism* $\phi : D \to E$ between $\mathbb{Z}^d$-graded differential modules is a morphism of $\mathbb{Z}^d$-graded modules satisfying $\delta^E \circ \phi = \phi \circ \delta^D$. In particular, for there to be a non-zero morphism, D and E must have the same differential degree.

In the usual way, a morphism $\phi : D \to E$ induces a map in homology $H(\phi) : H(D) \to H(E)$. If $H(\phi)$ is an isomorphism we say that ϕ is a *quasi-isomorphism* and write $D \simeq E$ or $\phi : D \xrightarrow{\simeq} E$. Given an exact sequence of differential modules

$$0 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow D_3 \longrightarrow 0$$

there is an induced long exact sequence in homology:

$$\cdots \longrightarrow H(D_3) \longrightarrow H(D_1) \longrightarrow H(D_2) \longrightarrow H(D_3) \longrightarrow H(D_1) \longrightarrow \cdots$$

This long exact sequence is often written as a triangle:

$$\begin{tikzcd} H(D_1) \arrow[rr] & & H(D_2) \arrow[dl] \\ & H(D_3) \arrow[ul] & \end{tikzcd} \tag{2.1}$$

See [5, Chap. IV §1] for a proof.

Bounds on the rank of a differential module will be obtained by comparing the rank and the Betti number. To define the Betti number we will need a notion of a tensor product of differential modules. However adapting the usual definition of the tensor product between complexes fails to produce a differential module when applied to two differential modules. To work around this we recall the construction of a tensor product of a complex and a differential module, along with some of its properties [1, §1].

Definition 2.2. For a complex C of $\mathbb{Z}^d$-graded R-modules and a $\mathbb{Z}^d$-graded differential R-module D, define a $\mathbb{Z}^d$-graded differential module $C \boxtimes_R D$ by setting

$$C \boxtimes_R D = \bigoplus_{n \in \mathbb{Z}} (C_n \otimes_R D),$$

with differential defined by

$$\delta^{C\boxtimes_R D}(c \otimes d) = \partial^C(c) \otimes d + (-1)^n c \otimes \delta^D(d),$$

for $c \otimes d \in C_n \otimes_R D$.

We will need the following facts concerning this product.

Proposition 2.3 ([1, 1.9.3])**.** *Let X and Y be complexes and let D be a differential module. Then there is a natural isomorphism of differential modules:*

$$(X \otimes_R Y) \boxtimes_R D = X \boxtimes_R (Y \boxtimes_R D).$$

Proposition 2.4 ([1, Proposition 1.10])**.** *Let X and Y be bounded below complexes of flat R-modules, i.e. $X_i = Y_i = 0$ for sufficiently small i. Then*

(1) *the functor $X \boxtimes_R -$ preserves exact sequences and quasi-isomorphisms,*
(2) *a quasi-isomorphism $\phi : X \to Y$ induces a quasi-isomorphism*

$$\phi \boxtimes_R D : X \boxtimes_R D \to Y \boxtimes_R D$$

for all differential R-modules D.

Using this tensor product, we can define the Tor functor between R-modules and differential R-modules, and hence define a Betti number.

Definition 2.5. For a differential R-module D and an R-module M set

$$\mathrm{Tor}^R(M, D) = H(P \boxtimes_R D)$$

where P is a free resolution of M. This is well-defined as different choices of free resolution produce quasi-isomorphic differential modules by Proposition 2.4.

Definition 2.6. We define the *Betti number* $\beta^R(D)$ of a differential R-module D to be

$$\beta^R(D) = \mathrm{rank}_k \mathrm{Tor}^R(k, D).$$

The connection between ranks of differential modules and Betti numbers is provided by free flags, a notion of a free resolution for differential modules [1, §2].

Definition 2.7. A *free flag* on a differential module F is a family $\{F^n\}_{n\in\mathbb{Z}}$ of R-submodules such that

(1) $F^n = 0$ for $n < 0$,
(2) $F^n \subseteq F^{n+1}$ for all n,
(3) $\delta^F(F^n) \subseteq F^{n-1}$ for all n,
(4) $\bigcup_{n\in\mathbb{Z}} F^n = F$,
(5) F^n/F^{n-1} is a free R-module for all n.

A differential module F with a free flag *resolves* D if there is a quasi-isomorphism $F \xrightarrow{\simeq} D$.

With differential modules that admit a free flag providing a resolution of a differential module, the Tor functor is balanced, which gives the connection between the rank and Betti number of a differential module.

Proposition 2.8 ([1, Proposition 2.4])**.** *Let F be a differential module with a free flag. Then the functor $- \boxtimes_R F$ preserves exact sequences and quasi-isomorphisms.*

Lemma 2.9. *Let P be a free resolution of a module M and let F be a free flag resolving a differential module D. Then $H(P \boxtimes_R D)$ is isomorphic to $H(M \boxtimes_R F)$ as R-modules.*

Proof. Let $\varepsilon : P \to M$ and $\eta : F \to D$ be quasi-isomorphisms. Then there are morphisms

$$P \boxtimes_R D \xleftarrow{P \boxtimes_R \eta} P \boxtimes_R F \xrightarrow{\varepsilon \boxtimes_R F} M \boxtimes_R F.$$

By Proposition 2.4 and Proposition 2.8 these are quasi-isomorphisms. □

Theorem 2.10. *Let F be differential module admitting a free flag. Then*

$$\beta^R(F) \leq \operatorname{rank}_R F.$$

Proof. By Lemma 2.9,

$$\beta^R(F) = \operatorname{rank}_k \operatorname{Tor}^R(k, F) = \operatorname{rank}_k H(k \boxtimes_R F).$$

Since k is an R-module, $k \boxtimes_R F = k \otimes_R F$. Since $H(k \otimes_R F)$ is a subquotient of $k \otimes_R F$, we have

$$\operatorname{rank}_k H(k \boxtimes_R F) \leq \operatorname{rank}_k k \otimes_R F = \operatorname{rank}_R F.$$ □

Remark 2.11. When $\delta(F) \subseteq \mathfrak{m}F$ we have $\beta^R(F) = \operatorname{rank}_R F$ as the differential of $k \boxtimes_R F$ is zero. In general, the inequality can be strict; see Example 4.3.

We finish this section by collecting a few properties of the Tor functor for use later.

Lemma 2.12. *Consider an exact sequence of differential R-modules*

$$0 \longrightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \longrightarrow 0\ .$$

For each R-module M there is an exact commutative diagram:

$$\begin{tikzcd} \operatorname{Tor}^R(M, D_1) \arrow[rr, "\alpha"] & & \operatorname{Tor}^R(M, D_2) \arrow[dl, "\beta"] \\ & \operatorname{Tor}^R(M, D_3) \arrow[ul, "\gamma"] & \end{tikzcd}$$

Proof. Take a free resolution P of the module M. By Proposition 2.4 the sequence of differential modules remains exact after applying $P \boxtimes_R -$:

$$0 \longrightarrow P \boxtimes_R D_1 \xrightarrow{P \boxtimes \alpha} P \boxtimes_R D_2 \xrightarrow{P \boxtimes \beta} P \boxtimes_R D_3 \longrightarrow 0\ .$$

The exact triangle (2.1) coming from this exact sequence is the desired one. □

Lemma 2.13. *Let $R \to S$ be a homomorphism of rings where S is flat over R. Let M be an S-module and D a differential R-module. Then*

$$\operatorname{Tor}^S(M, S \boxtimes_R D) \cong \operatorname{Tor}^R(M, D).$$

Proof. Let P be a free resolution of M. Then using Proposition 2.3 one gets:

$$\begin{aligned} \operatorname{Tor}^S(M, S \boxtimes_R D) &= H(P \boxtimes_S (S \boxtimes_R D)) \\ &\cong H((P \otimes_S S) \boxtimes_R D) \\ &\cong H(P \boxtimes_R D) \\ &= \operatorname{Tor}^R(M, D). \end{aligned}$$ □

3. Compression

Every complex of R-modules produces a differential module by forming its *compression*. This construction allows results about differential modules to be translated to results about complexes of modules. In fact, the differential modules produced by compressing always have differential degree $\mathbf{0}$ so it is sufficient to restrict to differential modules with differential degree $\mathbf{0}$ if one is interested in establishing results about complexes. Note that not every differential module of differential degree $\mathbf{0}$ arises this way (see Example 3.3).

Construction 3.1 ([1, 1.3])**.** If C is a complex of $\mathbb{Z}^d$-graded R-modules, then its *compression* is the $\mathbb{Z}^d$-graded differential module

$$C_\Delta = \bigoplus_{n\in\mathbb{Z}} C_n$$

with differential $\delta^{C_\Delta} = \bigoplus_{n\in\mathbb{Z}} \partial_n^C$.

We have $\deg(\delta^{C_\Delta}) = \mathbf{0}$ because the differentials of the complex C are required to have degree zero. By the definition of δ^{C_Δ}, we have $H(C_\Delta) = \bigoplus_{n\in\mathbb{Z}} H_n(C)$.

When the complex C is bounded below and consists of free R-modules then the compression has a free flag. Indeed, suppose $C_i = 0$ for i sufficiently small. Then setting $F^n = \bigoplus_{i\le n} C_i$ forms a free flag.

Computing the Betti number of a compression is a straight-forward application of Theorem 2.10 and Remark 2.11.

Lemma 3.2. *Let C be a bounded below complex of free modules that is minimal in the sense that $\partial_n^C(C_n) \subseteq \mathfrak{m}C_{n-1}$. Then*

$$\beta(C_\Delta) = \sum_i \operatorname{rank}_R C_i.$$

When C is a minimal free resolution of a module M we have

$$\beta(C_\Delta) = \sum_i \beta_i(M),$$

where $\beta_i(M)$ is the usual Betti number of M.

Proof. Since C is a bounded below complex of free modules, C_Δ has a free flag. We have

$$\delta(C_\Delta) = \bigoplus_{n\in\mathbb{Z}} \partial_n(C_n) \subseteq \bigoplus_{n\in\mathbb{Z}} \mathfrak{m}C_{n-1} = \mathfrak{m}C_\Delta,$$

so by Remark 2.11 we have

$$\begin{aligned}\beta(C_\Delta) &= \operatorname{rank}_R C_\Delta \\ &= \sum_i \operatorname{rank}_R C_i.\end{aligned}$$

When C is a minimal free resolution of M we have $\operatorname{rank}_R C_i = \beta_i(M)$, which completes the proof. $\square$

Obviously differential modules with non-zero differential degree do not come from compressing a complex, but the following example shows that there are also differential modules with differential degree zero that are not compressions of a complex.

Example 3.3. Let $R = k[x, y]$ and let $F = R(0, 0) \oplus R(-1, 0) \oplus R(0, -1) \oplus R(-1, -1)$. Viewing F as column vectors, define a differential δ by left-multiplication by the matrix

$$\begin{bmatrix} 0 & x & y & xy \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a differential module with $\deg \delta = \mathbf{0}$. Represented diagrammatically this has the form of a Koszul complex on x, y modified by adding an additional map:

$$\begin{tikzcd} R(-1,-1) \arrow[r, "\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]"'] \arrow[rr, bend left, "xy"] & R(-1,0) \oplus R(0,-1) \arrow[r, "{[x\ y]}"'] & R(0,0) \arrow[r] & 0 \end{tikzcd}.$$

Reading the diagram from right to left produces a free flag:

$$0 \subset R(0,0) \subset R(0,0) \oplus R(-1,0) \oplus R(0,-1) \subset \\ R(0,0) \oplus R(-1,0) \oplus R(0,-1) \oplus R(-1,-1) = F.$$

To calculate $H(F)$, consider the first differential submodule of the flag $F^0 = R(0, 0)$. It is straight-forward to see that

$$H(F^0) = R(0,0)$$
$$H(F/F^0) = (R(-1,0) \oplus R(0,-1))/R(-y \oplus x).$$

From the short exact sequence

$$0 \longrightarrow F^0 \longrightarrow F \longrightarrow F/F^0 \longrightarrow 0$$

we have the long exact sequence

$$\cdots \longrightarrow H(F) \longrightarrow H(F/F^0) \xrightarrow{\alpha} H(F^0) \xrightarrow{\beta} H(F) \longrightarrow \cdots$$

where the map α is given by the matrix $\begin{bmatrix} x & y \end{bmatrix}$. Since α is injective, β must be a surjection, giving

$$H(F) = H(F^0)/\operatorname{im} \alpha = R/(x, y) = k.$$

To compute the Betti number, note that $\delta(F) \subseteq \mathfrak{m}F$, so we have $\beta^R(F) = \operatorname{rank}_R F = 4$ by Remark 2.11.

4. Non-positive differential degree

Every differential R-module with a free flag is free as an R-module, but not conversely (see Example 4.4). Even when a differential module admits a free flag there may be no way to "minimize," unlike finite free complexes that can be decomposed into an acyclic complex and a minimal complex C with $\partial(C) \subseteq \mathfrak{m}C$ (see Example 4.3). Restricting to the case of a differential module D with $\deg \delta^D \leq \mathbf{0}$ we can avoid both of these difficulties.

Theorem 4.1. *Let F be a finitely generated $\mathbb{Z}^d$-graded differential R-module with $\deg \delta^F \leq \mathbf{0}$ that is free as an R-module. Then F has a free flag and there is a submodule F' that is a direct summand in the category of $\mathbb{Z}^d$-graded differential R-modules such that*

(1) *F' has a free flag,*

(2) $\delta(F') \subseteq \mathfrak{m}F'$,
(3) $H(F') = H(F)$.

Remark 4.2. The hypothesis that $\deg \delta \leq \mathbf{0}$ is necessary. See Examples 4.3 and 4.4.

Proof. We induce on $\operatorname{rank}_R F$: if $\operatorname{rank}_R F = 1$ then the differential of F is multiplication by an element of R. Since R is a domain, this element must be zero; hence $F^0 = F$ is a free flag. As $\delta(F^0) = 0$ we conclude that $\delta(F) \subseteq \mathfrak{m}F$ as well.

Now suppose $\operatorname{rank}_R F > 1$. If $\delta^F(F) \not\subseteq \mathfrak{m}F$ then there is some homogeneous basis element e with $\delta^F(e) \notin \mathfrak{m}F$. We first show that e and $\delta(e)$ are linearly independent. Let $e, v_1, \ldots, v_n$ be a basis for F and write

$$\delta(e) = re + s_1 v_1 + \cdots + s_n v_n. \tag{4.1}$$

Suppose that $ae + b\delta(e) = 0$ with $a, b \in R$. In terms of the basis we have

$$(a + br)e + bs_1 v_1 + \cdots + bs_n v_n = 0.$$

As this is a basis we conclude that $a + br = 0$ and $bs_i = 0$ for all i. Since $\delta(e) \notin \mathfrak{m}F$ one of $r, s_1, \ldots, s_n$ is not in $\mathfrak{m}$. If $s_i \notin \mathfrak{m}$ for some i we conclude that $b = 0$ and therefore e and $\delta(e)$ are linearly independent. If $s_i \in \mathfrak{m}$ for all i then we have $r \notin \mathfrak{m}$. Condensing (4.1) gives $\delta(e) = re + v$ with $v \in \mathfrak{m}F$ and $re \notin \mathfrak{m}F$. Because δ is a differential,

$$0 = \delta^2(e) = r^2 e + rv + \delta(v).$$

We have $\delta(v) \in \mathfrak{m}F$ and $rv \in \mathfrak{m}F$ since $v \in \mathfrak{m}F$, and therefore $r^2 e \in \mathfrak{m}F$. But r is a unit, so $re \in \mathfrak{m}F$, a contradiction.

So e and $\delta^F(e)$ are linearly independent. By Nakayama's lemma we can take $\{e, \delta^F(e)\}$ to be part of a basis of F. Let $G = Re \oplus R\delta^F(e)$. Then G is a differential sub-module. So we have an exact sequence of differential modules:

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0\,. \tag{4.2}$$

Since $H(G) = 0$, the long exact sequence in homology coming from (4.2) shows that $H(F/G) = H(F)$. The module F/G is a free summand of F since G is generated by basis elements of F. So by induction F/G has a free flag $\{G^n\}_{n \in \mathbb{Z}}$. Setting

$$\begin{aligned} F^0 &= R\delta^F(e), \\ F^1 &= R\delta^F(e) \oplus Re, \\ F^n &= R\delta^F(e) \oplus Re \oplus G^{n-2}, \quad n \geq 2 \end{aligned}$$

gives a free flag on F. The induction hypothesis also shows that F/G has a direct summand F' with a free flag such that $\delta(F') \subseteq \mathfrak{m}F'$ and such that $H(F') = H(F/G) = H(F)$. This completes the proof when $\delta^F(F) \not\subseteq \mathfrak{m}F$.

Now suppose that $\delta^F(F) \subseteq \mathfrak{m}F$. In this case it suffices to show that F has a free flag. Let $e_1, \ldots, e_n$ be a homogeneous basis for F. Let $\mathbf{e}$ be a minimal element of $\{\deg(e_1), \ldots, \deg(e_n)\}$ under the partial order on $\mathbb{Z}^d$. Set

$$G = \bigoplus_{\deg(e_i) = \mathbf{e}} Re_i.$$

Then $\delta^F(G) \subseteq G$ since $\deg(\delta^F(e_i)) \leq \deg(e_i)$ for all i as the degree of δ^F is non-positive in each coordinate. So G is a differential sub-module.

We claim that $\delta^F|_G = 0$. When $\deg \delta^F < \mathbf{0}$, we have $\delta^F|_G = 0$ as $\deg(\delta^F(e_i)) < \deg(e_i)$ and all the generators e_i of G have the same degree. When $\deg \delta^F = \mathbf{0}$ the matrix representing $\delta^F|_G$ has entries in k since all generators of G are in the same degree. So $\delta^F|_G = 0$, otherwise there would be an element of $\delta^F(G)$ that is not in $\mathfrak{m}F$, contrary to assumption.

Since $\delta^F|_G = 0$ we get $\delta^F(F^0) = 0$ by setting $F^0 = G$. As F^0 is generated by basis elements of F, the quotient F/F^0 is a free R-module, so the induction hypothesis produces a free flag $\{G^n\}_{n\in\mathbb{Z}}$ for F/F^0. Setting $F^n = F^0 \oplus G^{n-1}$ for $n \geq 0$ and $F^n = 0$ for $n < 0$ gives a free flag on F. □

The next example illustrates many of the difficulties in dealing with differential modules with non-zero differential degree. It provides an obstruction to extending Theorem 6.4 and Lemma 4.1 to differential modules with $\deg\delta > \mathbf{0}$. By [1, Theorem 5.2], a differential module over $k[x, y]$ with a free flag must have rank at least 4. From this we conclude that the following example also shows that Lemma 4.1 cannot be extended to differential modules with $\deg\delta > \mathbf{0}$ as no summand can have a free flag.

Example 4.3. Let $R = k[x, y]$ and let $F = R(0, 0) \oplus R(0, 1) \oplus R(1, 0) \oplus R(1, 1)$ have differential given by the matrix,

$$\delta = \begin{bmatrix} 0 & x & y & 1 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a differential module with differential degree $(1, 1)$. Represented diagrammatically:

$$\begin{tikzcd} R(1,1) \arrow[r, "{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]}"'] \arrow[rr, bend left, "1"] & R(0,1) \oplus R(1,0) \arrow[r, "{[\,x\ y\,]}"'] & R(0,0) \arrow[r] & 0 . \end{tikzcd} \tag{4.3}$$

As in Example 3.3, reading the diagram from right to left gives a free flag. The same computation from Example 3.3 shows that $H(F) = k$. As F has a free flag, we can compute $\beta^R(F)$ by $\operatorname{rank}_k H(k \boxtimes_R F)$. Applying $k \boxtimes_R -$ to (4.3) we have the vector space k^4 (suppressing the grading) with differential given by the diagram:

$$\begin{tikzcd} k \arrow[r, "0"'] \arrow[rr, bend left, "1"] & k^2 \arrow[r, "0"'] & k \arrow[r] & 0 . \end{tikzcd}$$

The homology is k^2, so $\beta^R(F) = 2$.

This final example shows that a differential module that is free as an R-module need not have a free flag; thus Lemma 4.1 cannot be strengthened to apply to differential modules with $\deg\delta > \mathbf{0}$.

Example 4.4. Let F be as in Example 4.3. Let e be the basis element in degree $(-1, -1)$ and set $G = Re \oplus R\delta^F(e)$. Then calculation shows that F/G is the differential module $D = R(0, 1) \oplus R(1, 0)$ with

$$\delta = \begin{bmatrix} xy & -y^2 \\ x^2 & -xy \end{bmatrix}.$$

This is a differential module with $\deg\delta = (1, 1)$. Since $H(G) = 0$, an exact sequence argument shows that the map $F \to F/G$ is a quasi-isomorphism; hence $H(D) =$

$H(F) = k$. As F admits a free flag, it is a resolution of D. So we have $\beta^R(D) = \beta^R(F) = 2$.

The differential module D itself cannot have a free flag since $\operatorname{rank}_R D = 2 < 4$, as noted before Example 4.3.

5. High-low decompositions

The main tool, Theorem 5.5, we use for finding a bound on the Betti number comes from an inequality of Santoni [11] formulated to apply to differential modules. The modified statements and proofs are provided in this section for completeness. The essential idea is to use information about the "top" and "bottom" degree parts to derive information about the entire module. The meaning of "top" and "bottom" is made precise by a *high-low decomposition*, Definition 5.4.

For this section let S be an arbitrary commutative ring, and let $\mathcal{C}$ be a class of differential S-modules which is closed under taking submodules and quotients. Take λ to be a superadditive function from $\mathcal{C}$ to an ordered commutative monoid such that $\lambda(C) \geq 0$ for all $C \in \mathcal{C}$. Recall that λ is superadditive if an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of differential modules in $\mathcal{C}$ gives an inequality

$$\lambda(B) \geq \lambda(A) + \lambda(C).$$

Example 5.1. For our purposes, S will be a $\mathbb{Z}^d$-graded polynomial ring over a field, $\mathcal{C}$ will be the collection of differential S-modules with non-zero homology in finitely many degrees and λ will be the length of the homology module.

Lemma 5.2. *Let B be a differential S-module and suppose we have the following commutative diagrams in $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, two heads, "\psi_A"'] & B \arrow[d, "\psi_B"] \\ A'' \arrow[r, hook, "\iota''"'] & B'' \end{tikzcd} \qquad \begin{tikzcd} B' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\phi_B"'] & C' \arrow[d, hook, "\phi_C"] \\ B \arrow[r, "\varepsilon"'] & C \end{tikzcd}$$

Then the following inequalities hold:

$$\lambda(\operatorname{im}\iota) \geq \lambda(\operatorname{im}\iota''),$$
$$\lambda(\operatorname{im}\varepsilon) \geq \lambda(\operatorname{im}\varepsilon').$$

Furthermore, if $\varepsilon\iota = 0$ then

$$\lambda(B) \geq \lambda(\operatorname{im}\iota'') + \lambda(\operatorname{im}\varepsilon').$$

Proof. For the first inequality, there is a surjection $\operatorname{im}\iota \twoheadrightarrow \operatorname{im}\psi_B\iota$, so

$$\lambda(\operatorname{im}\iota) \geq \lambda(\operatorname{im}\psi_B\iota) = \lambda(\operatorname{im}\iota''\psi_A).$$

Because ψ_A is surjective there is also a surjection $\operatorname{im}\iota''\psi_A \twoheadrightarrow \operatorname{im}\iota''$. This gives the desired inequality, $\lambda(\operatorname{im}\iota) \geq \lambda(\operatorname{im}\iota'')$.

For the second inequality, there is an inclusion $\operatorname{im}\varepsilon' \hookrightarrow \operatorname{im}\varepsilon$ since ϕ_C is injective. By superadditivity, $\lambda(\operatorname{im}\varepsilon) \geq \lambda(\operatorname{im}\varepsilon')$.

For the final inequality, note that $\varepsilon\iota = 0$ implies that $\operatorname{im}\iota \subseteq \ker\varepsilon$. The exact sequence

$$0 \longrightarrow \ker\varepsilon \longrightarrow B \longrightarrow \operatorname{im}\varepsilon \longrightarrow 0\ ,$$

then implies $\lambda(B) \geq \lambda(\operatorname{im}\varepsilon) + \lambda(\operatorname{im}\iota) \geq \lambda(\operatorname{im}\varepsilon') + \lambda(\operatorname{im}\iota'')$ using the first two inequalities. $\square$

Lemma 5.3. *Let D be a differential $S[x]$-module and consider it as a differential S-module via the inclusion $S \hookrightarrow S[x]$. Viewing $S[x] \boxtimes_S D$ as a $S[x]$-module via the action $r(s \otimes d) = (rs) \otimes d$, there is a sequence of differential $S[x]$-modules*

$$0 \longrightarrow S[x] \boxtimes_S D \xrightarrow{\sigma} S[x] \boxtimes_S D \xrightarrow{\varepsilon} D \longrightarrow 0 \, ,$$

with $\sigma(1 \otimes d) = x \otimes d - 1 \otimes xd$ and $\varepsilon(a \otimes d) = ad$. This sequence is exact and functorial in D. The map σ is given by multiplication by x if and only if $xD = 0$.

Proof. It is straight-forward to check that σ and ε are morphisms and that the sequence is exact and functorial. Evidently σ is multiplication by x when $xD = 0$. The exactness of the sequence shows that the converse holds. $\square$

The following definition and theorem are the differential module versions of Santoni's results for R-modules [11].

Definition 5.4. A differential $S[x]$-module D admits a *high-low decomposition* if there are non-zero differential $S[x]$-modules D_h and D_ℓ each annihilated by x and there are morphisms of differential $S[x]$-modules $D_h \hookrightarrow D$ and $D \twoheadrightarrow D_\ell$ that split in the category of differential S-modules.

Theorem 5.5. *Let K be an $S[x]$-module such that $xK = 0$, and assume $\mathcal{C}$ is closed under $\operatorname{Tor}^{S[x]}(K, -)$. Let $D \in \mathcal{C}$ be a differential module which admits a high-low decomposition. Then*

$$\lambda(\operatorname{Tor}^{S[x]}(K, D)) \geq \lambda(\operatorname{Tor}^{S}(K, D_\ell)) + \lambda(\operatorname{Tor}^{S}(K, D_h)).$$

Proof. Applying the functoriality of Lemma 5.3 to the high-low decomposition $D_h \hookrightarrow D$ and $D \twoheadrightarrow D_\ell$ gives two exact commutative diagrams:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S[x] \boxtimes_S D_h \arrow[r, "\sigma'"] \arrow[d] & S[x] \boxtimes_S D_h \arrow[r, "\varepsilon'"] \arrow[d] & D_h \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S[x] \boxtimes_S D \arrow[r, "\sigma"] & S[x] \boxtimes_S D \arrow[r, "\varepsilon"] & D \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & S[x] \boxtimes_S D \arrow[r, "\sigma"] \arrow[d] & S[x] \boxtimes_S D \arrow[r, "\varepsilon"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S[x] \boxtimes_S D_\ell \arrow[r, "\sigma''"] \arrow[d] & S[x] \boxtimes_S D_\ell \arrow[r, "\varepsilon''"] \arrow[d] & D_\ell \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In both diagrams the first two columns are split exact over $S[x]$ due to the high-low decomposition. Because D_h and D_ℓ are annihilated by x, Lemma 5.3 implies that σ' and σ'' are multiplication by x. The $S[x]$-action on $\operatorname{Tor}^{S[x]}(K, -)$ is via K and

$xK = 0$, so after applying $\operatorname{Tor}^{S[x]}(K, -)$ and using Lemma 2.13 the maps σ' and σ'' become zero, leaving

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Tor}^S(K, D_h) \arrow[r, "\varepsilon'"] \arrow[d] & \operatorname{Tor}^{S[x]}(K, D_h) \arrow[r, "\gamma'"] \arrow[d] \arrow[dr, phantom, "(\dagger)"] & \operatorname{Tor}^S(K, D_h) \arrow[r] \arrow[d] & 0 \\
\operatorname{Tor}^S(K, D) \arrow[r, "\sigma"] & \operatorname{Tor}^S(K, D) \arrow[r, "\varepsilon"] & \operatorname{Tor}^{S[x]}(K, D) \arrow[r, "\gamma"] & \operatorname{Tor}^S(K, D) \arrow[r, "\sigma"] & \operatorname{Tor}^S(K, D)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{Tor}^S(K, D) \arrow[r, "\sigma"] & \operatorname{Tor}^S(K, D) \arrow[r, "\varepsilon"] \arrow[d] \arrow[dr, phantom, "(\ddagger)"] & \operatorname{Tor}^{S[x]}(K, D) \arrow[r, "\gamma"] \arrow[d] & \operatorname{Tor}^S(K, D) \arrow[r, "\sigma"] \arrow[d] & \operatorname{Tor}^S(K, D) \\
0 \arrow[r] & \operatorname{Tor}^S(K, D_\ell) \arrow[r, "\varepsilon''"] \arrow[d] & \operatorname{Tor}^{S[x]}(K, D_\ell) \arrow[r, "\gamma''"] & \operatorname{Tor}^S(K, D_\ell) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Using Lemma 5.2 on the commutative squares (†) and (‡) gives the desired inequality. □

6. Lower bound on the Betti number

In order to apply the results for high-low decompositions we need to establish some results on the existence of high-low decompositions D_h and D_ℓ with $H(D_h) \neq 0$ and $H(D_\ell) \neq 0$.

Recall that $\mathbf{m}_i$ denotes the i-th coordinate of a d-tuple $\mathbf{m} \in \mathbb{Z}^d$.

Definition 6.1. Let D be a $\mathbb{Z}^d$-graded differential module and let $1 \leq i \leq d$. We say that D is *bounded in the i-th direction* if there are $a, b \in \mathbb{Z}$ such that $\mathbf{m}_i \notin [a, b]$ implies $D_{\mathbf{m}} = 0$.

Remark 6.2. When D is finitely generated the condition that D is bounded in the i-th direction for all i is equivalent to the condition that $\operatorname{rank}_k D < \infty$.

Lemma 6.3. *Let D be a $\mathbb{Z}^d$-graded differential module with $H(D) \neq 0$. Fix an index $1 \leq i \leq d$ and suppose that $(\deg \delta^D)_i = 0$. If $H(D)$ is bounded in the i-th direction then there is a $\mathbb{Z}^d$-graded differential module D' that is quasi-isomorphic to D such that D' has a high-low decomposition D'_h and D'_ℓ with $H(D'_h)$ and $H(D'_\ell)$ both non-zero.*

Proof. Let $a \in \mathbb{Z}$ be the largest integer such that $H(D)_{\mathbf{m}} = 0$ whenever $\mathbf{m}_i < a$. Such an integer exists because $H(D)$ is non-zero and bounded in the i-th direction. Set

$$E = \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}^d \\ a \leq \mathbf{m}_i}} D_{\mathbf{m}}.$$

This is an R-submodule. Since $(\deg \delta^D)_i = 0$ it is closed under δ^D as well. So E is a differential submodule of D. By the definition of E, we have

$$D/E = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ \mathbf{m}_i<a}} D_{\mathbf{m}}.$$

Let z be a cycle in $(D/E)_{\mathbf{m}}$. If $\mathbf{m}_i \geq a$ then $z = 0$ as $(D/E)_{\mathbf{m}} = 0$. If $\mathbf{m}_i < a$ then $z \in (D/E)_{\mathbf{m}} = D_{\mathbf{m}}$ so there is a $z' \in D$ with $\delta^D(z') = z$ as $H(D)_{\mathbf{m}} = 0$. So $\delta^{D/E}(z' + E) = z$. Therefore $H(D/E)_{\mathbf{m}} = 0$ for all $\mathbf{m} \in \mathbb{Z}^d$, and so $H(D/E) = 0$. From a short exact sequence we conclude that $E \simeq D$.

Let $b \in \mathbb{Z}$ be the smallest integer such that $H(E)_{\mathbf{m}} = 0$ when $\mathbf{m}_i > b$. Again, such an integer exists because $H(E) \cong H(D)$ is non-zero and bounded in the i-th direction. Set

$$E' = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ b+1\leq\mathbf{m}_i}} E_{\mathbf{m}}.$$

Then E' is a differential submodule of E with $H(E') = 0$ by the definition of b. Set $D' = E/E'$. From a short exact sequence we conclude that $H(E/E') \cong H(E)$ so that $D' = E/E' \simeq E \simeq D$.

By construction, $D'_{\mathbf{m}} = 0$ for $\mathbf{m}_i < a$ and for $\mathbf{m}_i > b$. Also, by the definitions of a and b, there are $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ with $\mathbf{m}_i = a$ and $\mathbf{n}_i = b$ such that $H(D')_{\mathbf{m}} \neq 0$ and $H(D')_{\mathbf{n}} \neq 0$; hence $D'_{\mathbf{m}} \neq 0$ and $D'_{\mathbf{n}} \neq 0$ as well.

Set

$$D'_\ell := \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ \mathbf{m}_i=a}} D'_{\mathbf{m}}$$

and set

$$D'_h := \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ \mathbf{m}_i=b}} D'_{\mathbf{m}}.$$

Then D'_ℓ and D'_h are both non-zero and annihilated by x_i. The two morphisms $D'_h \hookrightarrow D'$ and $D' \twoheadrightarrow D'_\ell$ split in the category of differential modules because $(\deg \delta^D)_i = 0$. So D'_ℓ and D'_h form a high-low decomposition. As noted above $H(D'_\ell)$ and $H(D'_h)$ are both non-zero, so D' is the desired differential module. $\square$

The proof of the main theorem proceeds by using Theorem 5.5 inductively, after first using Lemma 6.3 to find a differential module with a high-low decomposition.

Note that $H(D)$ is not required to be finitely generated in the following theorem. If $H(D)$ is finitely generated then the hypothesis on $H(D)$ is equivalent to $0 < \operatorname{rank}_k H(D) < \infty$; see Remark 6.2.

Theorem 6.4. *If D is a $\mathbb{Z}^d$-graded differential module with $\deg \delta^D = \mathbf{0}$ and such that $H(D) \neq 0$ is bounded in the i-th direction for all i, then*

$$\beta^R(D) \geq 2^d.$$

Proof. Use induction on d. For $d = 0$, so that $R = k$, we have

$$\operatorname{Tor}^k(k, D) = H(k \boxtimes_k D) \cong H(D) \neq 0.$$

So $\beta^k(D) \geq 1$.

Now suppose $d > 1$. Then $H(D)$ is bounded in the d-th direction by assumption. By Proposition 2.8 the Betti number is preserved under quasi-isomorphisms, so

Lemma 6.3 allows us to assume that D has a high-low decomposition D_h and D_ℓ with $H(D_h) \neq 0$ and $H(D_\ell) \neq 0$. By definition of a high-low decomposition, $H(D_h)$ and $H(D_\ell)$ are submodules of $H(D)$ since the splitting happens in the category of differential modules. In particular, $H(D_h)$ and $H(D_\ell)$ are bounded in the i-th direction for all i.

Thus the induction hypothesis applies to D_h and D_ℓ. From Theorem 5.5 we have inequalities:

$$\begin{aligned}\beta^R(D) &\geq \beta^{k[x_1,\ldots,x_{d-1}]}(D_\ell) + \beta^{k[x_1,\ldots,x_{d-1}]}(D_h)\\ &\geq 2^{d-1} + 2^{d-1}\\ &= 2^d.\end{aligned}$$

□

Remark 6.5. Example 4.3 shows that Theorem 6.4 cannot be extended to differential modules D with $\deg \delta^D > \mathbf{0}$.

Via Theorem 2.10 this result provides an affirmative answer to Conjecture 1.1 for $\mathbb{Z}^d$-graded differential modules with $\deg \delta = \mathbf{0}$.

Corollary 6.6. *If F is a finitely generated $\mathbb{Z}^d$-graded differential module that is free as an R-module such that $\deg \delta^F = \mathbf{0}$ and such that $H(F)$ has non-zero finite length then*

$$\operatorname{rank}_R F \geq 2^d.$$

Proof. By Lemma 4.1, F has a free flag. So Theorem 2.10 implies that $\beta^R(F) \leq \operatorname{rank}_R F$. Applying Theorem 6.4 gives the desired inequality. □

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