A NEW LAX-OLEINIK TYPE SEMIGROUP FOR TIME-PERIODIC POSITIVE DEFINITE LAGRANGIAN SYSTEMS

KAIZHI WANG[1, 2] AND JUN YAN[1]

Abstract. In this paper we introduce a new Lax-Oleinik type semigroup associated with positive definite Lagrangian systems for both the time-independent case and the time-periodic case. We show that the new Lax-Oleinik type semigroup can take the place of the Lax-Oleinik semigroup in the weak KAM theory. More than that, the new Lax-Oleinik type semigroup converges to a backward weak KAM solution faster than the Lax-Oleinik semigroup in the time-independent case, and the new Lax-Oleinik type semigroup converges to a backward weak KAM solution in the time-periodic case, while it is shown by Fathi and Mather that there is no such convergence of the Lax-Oleinik semigroup.

1. Introduction

Let M be a compact and connected smooth manifold. Denote by TM its tangent bundle and T^*M the cotangent one. Consider a C^∞ Lagrangian $L : TM \times \mathbb{R}^1 \to \mathbb{R}^1$, $(x, v, t) \mapsto L(x, v, t)$. We suppose that L satisfies the following conditions introduced by Mather [17]:

(H1) **Periodicity**. L is 1-periodic in the $\mathbb{R}^1$ factor, i.e., $L(x, v, t) = L(x, v, t+1)$ for all $(x, v, t) \in TM \times \mathbb{R}^1$.

(H2) **Positive Definiteness**. For each $x \in M$ and each $t \in \mathbb{R}^1$, the restriction of L to $T_xM \times t$ is strictly convex in the sense that its Hessian second derivative is everywhere positive definite.

(H3) **Superlinear Growth**. $\lim_{\|v\|_x \to +\infty} \frac{L(x,v,t)}{\|v\|_x} = +\infty$ uniformly on $x \in M$, $t \in \mathbb{R}^1$, where $\|\cdot\|_x$ denotes the norm induced by a Riemannian metric on T_xM. By the compactness of M, this condition is independent of the choice of the Riemannian metric.

(H4) **Completeness of the Euler-Lagrange Flow**. The maximal solutions of the Euler-Lagrange equation, which in local coordinates is:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t),$$

are defined on all of $\mathbb{R}^1$.

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The Euler-Lagrange equation is a second order periodic differential equation on M and generates a flow of diffeomorphisms $\phi_t^L : TM \times \mathbb{S}^1 \to TM \times \mathbb{S}^1$, $t \in \mathbb{R}^1$, where $\mathbb{S}^1$ denotes the circle $\mathbb{R}^1/\mathbb{Z}$, defined by

$$\phi_t^L(x_0, v_0, t_0) = (x(t+t_0), \dot{x}(t+t_0), (t+t_0) \bmod 1),$$

where $x : \mathbb{R}^1 \to M$ is the maximal solution of the Euler-Lagrange equation with initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$. The completeness and periodicity conditions grant that this correctly defines a flow on $TM \times \mathbb{S}^1$.

We can associate with L a Hamiltonian, as a function on $T^*M \times \mathbb{R}^1$: $H(x,p,t) = \sup_{v \in T_xM}\{\langle p, v\rangle_x - L(x,v,t)\}$, where $\langle \cdot, \cdot \rangle_x$ represents the canonical pairing between the tangent and cotangent space. The corresponding Hamilton-Jacobi equation is

$$u_t + H(x, u_x, t) = c(L), \tag{1.1}$$

where $c(L)$ is the Mañé critical value [16] of the Lagrangian L. In terms of Mather's α function $c(L) = \alpha(0)$.

In this paper we also consider time-independent Lagrangians on M. Let $L_a : TM \to \mathbb{R}^1$, $(x,v) \mapsto L_a(x,v)$ be a C^2 Lagrangian satisfying the following two conditions:

(H2') **Positive Definiteness**. For each $(x,v) \in TM$, the Hessian second derivative $\frac{\partial^2 L_a}{\partial v^2}(x,v)$ is positive definite.

(H3') **Superlinear Growth**. $\lim_{\|v\|_x \to +\infty} \frac{L_a(x,v)}{\|v\|_x} = +\infty$ uniformly on $x \in M$.

It is well known that the Euler-Lagrange flow $\phi_t^{L_a}$ is complete under the assumptions (H2') and (H3'). See, for example, [2] or [9].

For $x \in M$, $p \in T_x^*M$, the conjugated Hamiltonian H_a of L_a is defined by: $H_a(x,p) = \sup_{v \in T_xM}\{\langle p, v\rangle_x - L(x,v)\}$. The corresponding Hamilton-Jacobi equation is

$$H_a(x, u_x) = c(L_a). \tag{1.2}$$

The Lax-Oleinik semigroup (hereinafter referred to as L-O semigroup) ([10, 13, 19]) is well known in several domains, such as PDE, Optimization and Control Theory, Calculus of Variations and Dynamical systems. In particular, it plays an essential role in the weak KAM theory [9].

Let us first recall the definitions of the L-O semigroups associated with L_a (time-independent case) and L (time-periodic case), respectively. For each $u \in C(M, \mathbb{R}^1)$ and each $t \geq 0$, let

$$T_t^a u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^t L_a(\gamma(s), \dot{\gamma}(s)) ds \right\} \tag{1.3}$$

for all $x \in M$, and

$$T_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), s) ds \right\} \tag{1.4}$$

for all $x \in M$, where the infimums are taken among the continuous and piecewise C^1 paths $\gamma : [0,t] \to M$ with $\gamma(t) = x$. In view of (1.3) and (1.4), for each $t \geq 0$,

T_t^a and T_t are operators from $C(M,\mathbb{R}^1)$ to itself. It is not difficult to check that $\{T_t^a\}_{t\geq 0}$ and $\{T_n\}_{n\in\mathbb{N}}$ are one-parameter semigroups of operators, which means $T_0^a = I$ (unit operator), $T_{t+s}^a = T_t^a \circ T_s^a$, $\forall t,\ s\geq 0$, and $T_0 = I$, $T_{n+m} = T_n \circ T_m$, $\forall n,\ m\in\mathbb{N}$, where $\mathbb{N} = \{0,1,2,\cdots\}$. $\{T_t^a\}_{t\geq 0}$ and $\{T_n\}_{n\in\mathbb{N}}$ are called the L-O semigroup associated with L_a and L, respectively.

The L-O semigroup is used to obtain backward weak KAM solutions (viscosity solutions) first by Lions, Papanicolaou and Varadhan [15] on the n-torus $\mathbb{T}^n$ and later by Fathi [6] for arbitrary compact manifolds. More precisely, for the time-independent case, Fathi [6] proves that there exists a unique $c_0\in\mathbb{R}^1$ ($c_0 = c(L_a)$), such that the semigroup $\hat{T}_t^a : u\to T_t^a u + c_0 t$, $t\geq 0$ has a fixed point $u^*\in C(M,\mathbb{R}^1)$ and that any fixed point is a backward weak KAM solution of (1.2). In the particular case $M = \mathbb{T}^n$, the backward weak KAM solution obtained by Fathi is just the viscosity solution obtained earlier by Lions, Papanicolaou and Varadhan. Moreover, Fathi points out that the above results for the time-independent case are still correct for the time-periodic dependent case [9]. Furthermore, for the time-independent case, he shows in [7] that for every $u\in C(M,\mathbb{R}^1)$, the uniform limit $\lim_{t\to+\infty}\hat{T}_t^a u = \bar{u}$ exists and is a fixed point of $\{\hat{T}_t^a\}_{t\geq 0}$, i.e., $\bar{u}$ is a backward weak KAM solution of (1.2). In the same paper Fathi raises the question as to whether the analogous result holds in the time-periodic case. This would be the convergence of $T_n u + nc(L)$, $\forall u\in C(M,\mathbb{R}^1)$, as $n\to+\infty$, $n\in\mathbb{N}$. In view of the relation between T_n and the Peierls barrier h (see [18] or [8, 1, 4]), if the liminf in the definition of the Peierls barrier is not a limit, then the L-O semigroup in the time-periodic case does not converge. Fathi and Mather [8] construct examples where the liminf in the definition of the Peierls barrier is not a limit, thus answering the above question negatively.

The main aim of the present paper is to introduce a new Lax-Oleinik type semigroup (hereinafter referred to as new L-O semigroup) associated with positive definite Lagrangian systems for both the time-independent case and the time-periodic dependent case. The new L-O semigroup can take the place of the L-O semigroup in the weak KAM theory. More significantly, the new L-O semigroup with an arbitrary $u\in C(M,\mathbb{R}^1)$ as initial condition converges to a backward weak KAM solution of (1.2) faster than the L-O semigroup in the time-independent case, and the new L-O semigroup with an arbitrary L-dominated function $u\in C(M\times\mathbb{S}^1,\mathbb{R}^1)$ as initial condition converges to a backward weak KAM solution of (1.1) in the time-periodic case.

Without loss of generality, we will from now on always assume $c(L_a) = c(L) = 0$.

We are now in a position to introduce the new L-O semigroups mentioned above associated with L_a (time-independent case) and L (time-periodic case), respectively.

1.1. **Time-independent case.**

Definition 1.1. For each $u\in C(M,\mathbb{R}^1)$ and each $t\geq 0$, let

$$\tilde{T}_t^a u(x) = \inf_{t\leq s\leq 2t}\inf_{\gamma}\Big\{u(\gamma(0)) + \int_0^s L_a(\gamma(\tau),\dot{\gamma}(\tau))d\tau\Big\}$$

for all $x\in M$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma:[0,s]\to M$ with $\gamma(s) = x$.

It is easy to check that $\{\tilde{T}_t^a\}_{t\geq 0} : C(M,\mathbb{R}^1)\to C(M,\mathbb{R}^1)$ is an one-parameter semigroup of operators. We call it *the new L-O semigroup associated with* L_a. We show that $u\in C(M,\mathbb{R}^1)$ is a fixed point of $\{\tilde{T}_t^a\}_{t\geq 0}$ if and only if it is a fixed point of $\{T_t^a\}_{t\geq 0}$, and that for each $u \in C(M,\mathbb{R}^1)$, the uniform limit $\lim_{t\to+\infty}\tilde{T}_t^a u = \lim_{t\to+\infty} T_t^a u = \bar{u}$. For more properties of $\tilde{T}_t^a$ we refer to Section 3.

How fast does the L-O semigroup converge? It is an interesting question which is well worth discussing. We believe that there is a deep relation between dynamical properties of Mather sets (or Aubry sets) and the rate of convergence of the L-O semigroup. To the best of our knowledge there are now two relative results: In [11], Iturriaga and Sánchez-Morgado prove that if the Aubry set consists in a finite number of hyperbolic periodic orbits or hyperbolic fixed points, the L-O semigroup converges exponentially. Recently, in [21] the authors deal with the rate of convergence problem when the Mather set consists of degenerate fixed points. More precisely, consider the standard Lagrangian in classical mechanics $L_a^0(x,v) = \frac{1}{2}v^2 + U(x)$, $x \in \mathbb{S}^1$, $v \in \mathbb{R}^1$, where U is a real analytic function on $\mathbb{S}^1$ and has a unique global minimum point x_0. Without loss of generality, one may assume $x_0 = 0$, $U(0) = 0$. Then $c(L_a^0) = 0$ and the Mather set $\tilde{\mathcal{M}}_0 = \{(0,0)\}$. An upper bound estimate of the rate of convergence of the L-O semigroup is provided in [21] under the assumption that $\{(0,0)\}$ is a degenerate fixed point: for every $u \in C(\mathbb{S}^1,\mathbb{R}^1)$, there exists a constant $C > 0$ such that

$$\|T_t^a u - \bar{u}\|_\infty \leq \frac{C}{\sqrt[k-1]{t}}, \quad \forall t > 0,$$

where $k \in \mathbb{N}$, $k \geq 2$ depends only on the degree of degeneracy of the minimum point of the potential function U.

Naturally, we also care the problem of the rate of convergence of the new L-O semigroup. We compare the rate of convergence of the new L-O semigroup to the rate for the L-O semigroup as follows. First, we show that for each $u \in C(M,\mathbb{R}^1)$, $\|\tilde{T}_t^a u - \bar{u}\|_\infty \leq \|T_t^a u - \bar{u}\|_\infty$, $\forall t \geq 0$. It means that the new L-O semigroup converges faster than the L-O semigroup for L_a.

Then, in particular, we consider a class of C^2 positive definite and superlinear Lagrangians on $\mathbb{T}^n$

$$L_a^1(x,v) = \frac{1}{2}\langle A(x)(v-\omega),(v-\omega)\rangle + f(x,v-\omega), \quad x \in \mathbb{T}^n,\ v \in \mathbb{R}^n, \tag{1.5}$$

where $A(x)$ is an $n \times n$ matrix, $\omega \in \mathbb{S}^{n-1}$ is a given vector, and $f(x,v-\omega) = O(\|v-\omega\|^3)$ as $v - \omega \to 0$. It is clear that $c(L_a^1) = 0$ and the Mather set $\tilde{\mathcal{M}}_0 =$ the Aubry set $\tilde{\mathcal{A}}_0 =$ the Mañé set $\tilde{\mathcal{N}}_0 = \cup_{x\in\mathbb{T}^n}(x,\omega)$, which is a quasi-periodic invariant torus with frequency vector ω of the Euler-Lagrange flow associated to L_a^1. For the Lagrangian system (1.5), we obtain the following two results on the rates of convergence of the L-O semigroup and the new L-O semigroup, respectively.

Theorem 1.2. *For any $u \in C(\mathbb{T}^n,\mathbb{R}^1)$, there is a constant $K > 0$ such that*

$$\|T_t^a u - \bar{u}\|_\infty \leq \frac{K}{t}, \quad \forall t > 0,$$

where K depends only on n and u.

We recall the notations for Diophantine vectors: for $\rho > n-1$ and $\alpha > 0$, let

$$\mathcal{D}(\rho,\alpha) = \left\{\beta \in \mathbb{S}^{n-1} \mid |\langle\beta,k\rangle| \geq \frac{\alpha}{|k|^\rho},\ \forall k \in \mathbb{Z}^n\backslash\{0\}\right\},$$

where $|k| = \sum_{i=1}^n |k_i|$.

Theorem 1.3. *Given any frequency vector $\omega \in \mathcal{D}(\rho,\alpha)$, for each $u \in C(\mathbb{T}^n, \mathbb{R}^1)$, there is a constant $\tilde{K} > 0$ such that*

$$\|\tilde{T}_t^a u - \bar{u}\|_\infty \leq \tilde{K} t^{-(1+\frac{4}{2\rho+n})}, \quad \forall t > 0,$$

where $\tilde{K}$ depends only on n, ρ, α and u.

Finally, we construct an example (Example 3.9) to show that the result of Theorem 1.2 is sharp in the sense of order. Therefore, in view of Theorems 1.2, 1.3 and Example 1, we conclude that the new L-O semigroup converges faster than the L-O semigroup in the sense of order when the Aubry set $\tilde{\mathcal{A}}_0$ of the Lagrangian system (1.5) is a quasi-periodic invariant torus with Diophantine frequency vector $\omega \in \mathcal{D}(\rho,\alpha)$.

1.2. **Time-periodic case.** For each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ and each $n \in \mathbb{N}$, let

$$\tilde{T}_n u(x,t) = \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{\gamma} \Big\{ u(\gamma(t-k), t \bmod 1) + \int_{t-k}^{t} L(\gamma(s), \dot{\gamma}(s), s) ds \Big\}$$

for all $(x,t) \in M \times \mathbb{R}^1$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [t-k, t] \to M$ with $\gamma(t) = x$.

From the compactness of M, the periodicity and the superlinearity of L, it is easy to see that for each $n \in \mathbb{N}$, $\tilde{T}_n u(x,t) \in \mathbb{R}^1$ and $\tilde{T}_n u(x,t+1) = \tilde{T}_n u(x,t)$ for all $u \in C(M\times\mathbb{S}^1, \mathbb{R}^1)$ and all $(x,t) \in M\times\mathbb{R}^1$. It follows that for each $u \in C(M\times\mathbb{S}^1, \mathbb{R}^1)$ and each $n \in \mathbb{N}$, $\tilde{T}_n u$ is 1-periodic in the $\mathbb{R}^1$ factor. And, consequently, we can define $\tilde{T}_n u$ as a function on $M \times \mathbb{S}^1$ as follows:

Definition 1.4. For each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ and each $n \in \mathbb{N}$, let

$$\tilde{T}_n u(x,\tau) = \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{\gamma} \Big\{ u(\gamma(\tau-k), \tau) + \int_{\tau-k}^{\tau} L(\gamma(s), \dot{\gamma}(s), s) ds \Big\}$$

for all $(x,\tau) \in M \times \mathbb{S}^1$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [\tau-k, \tau] \to M$ with $\gamma(\tau) = x$.

$\tilde{T}_n$ is an operator from $C(M \times \mathbb{S}^1, \mathbb{R}^1)$ to itself for each $n \in \mathbb{N}$ (see Proposition 4.1). And in view of the periodicity of L, $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ is a semigroup of operators. We call it *the new L-O semigroup associated with L*.

The main result of the paper is the following.

Theorem 1.5. *For any $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$,*

$$\lim_{n\to\infty} \tilde{T}_n u(x,\tau) = \inf_{y\in M} \{u(y,\tau) + h_{\tau,\tau}(y,x)\}$$

for all $(x,\tau) \in M \times \mathbb{S}^1$. Let $\bar{u} = \lim_{n\to+\infty} \tilde{T}_n u$. Furthermore, if u is dominated by L, then $\bar{u}$ is a backward weak KAM solution of the Hamilton-Jacobi equation

$$u_\tau + H(x, u_x, \tau) = 0. \tag{1.6}$$

Remark 1.6. For the definition of the (extended) Peierls barrier h, see [18] or [8, 1, 4]. For completeness' sake, we recall the definition in Section 4. See (1) in Definition 4.6 for the definition of L-dominated functions, which are denoted by $u \prec L$.

In addition, we show that

$$\mathcal{S}_- = \{\bar{u} \in C(M \times \mathbb{S}^1, \mathbb{R}^1) |\ \exists u \in C(M \times \mathbb{S}^1, \mathbb{R}^1),\ u \prec L,\ \bar{u} = \lim_{n \to +\infty} \tilde{T}_n u\}, \tag{1.7}$$

where $\mathcal{S}_-$ denotes the set of backward weak KAM solutions of (1.6).

The rest of the paper is organized as follows. In Section 2 we introduce the basic language and notation used in the sequel. In Section 3 we first study the properties of the new L-O semigroup for the time-independent case and then give the proofs of Theorems 1.2 and 1.3. At last, we construct the example mentioned above (Example 3.9). In Section 4 we discuss the properties of the new L-O semigroup for the time-periodic case and prove Theorem 1.5.

2. Notation and terminology

Consider the flat n-torus $\mathbb{T}^n$, whose universal cover is the Euclidean space $\mathbb{R}^n$. We view the torus as a fundamental domain in $\mathbb{R}^n$

$$\overline{A} = \underbrace{[0,1] \times \cdots \times [0,1]}_{n \text{ times}}$$

with opposite faces identified. The unique coordinates $x = (x_1, \dots, x_n)$ of a point in $\mathbb{T}^n$ will belong to the half-open cube

$$A = \underbrace{[0,1) \times \cdots \times [0,1)}_{n \text{ times}}.$$

In these coordinates the standard universal covering projection $\pi : \mathbb{R}^n \to \mathbb{T}^n$ takes the form

$$\pi(\tilde{x}) = ([\tilde{x}_1], \dots, [\tilde{x}_n]),$$

where $[\tilde{x}_i] = \tilde{x}_i \mod 1$, denotes the fractional part of $\tilde{x}_i$ ($\tilde{x}_i = \{\tilde{x}_i\} + [\tilde{x}_i]$, where $\{\tilde{x}_i\}$ is the greatest integer not greater than $\tilde{x}_i$). We can now define operations on $\mathbb{T}^n$ using the covering projection: each operation is simply the projection of the usual operation with coordinates in $\mathbb{R}^n$. Thus the flat metric $d_{\mathbb{T}^n}$ may be defined for any pair of points $x,\ y \in \mathbb{T}^n$ by $d_{\mathbb{T}^n}(x,y) = \|x - y\|$, where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^n$. And the distance between points on the torus is at most $\frac{\sqrt{n}}{2}$. For $x \in \mathbb{T}^n$ and $R > 0$, $B_R(x) = \{y \in \mathbb{T}^n |\ d_{\mathbb{T}^n}(x,y) < R\}$ denotes the open ball of the radius R centered on x in $\mathbb{T}^n$.

We choose, once and for all, a C^∞ Riemannian metric on M. It is classical that there is a canonical way to associate to it a Riemannian metric on TM. We use the same symbol "d" to denote the distance function defined by the Riemannian metric on M and the distance function defined by the Riemannian metric on TM. Denote by $\|\cdot\|_x$ the norm induced by the Riemannian metric on the fiber T_xM for $x \in M$, and by $\langle \cdot, \cdot \rangle_x$ the canonical pairing between T_xM and T_x^*M. In particular, for $M = \mathbb{T}^n$, we denote $\langle \cdot, \cdot \rangle_x$ by $\langle \cdot, \cdot \rangle$ for brevity. We use the same notation $\langle \cdot, \cdot \rangle$ for the standard inner product on $\mathbb{R}^n$. However, this should not create any ambiguity.

We equip $C(M, \mathbb{R}^1)$ and $C(M \times \mathbb{S}^1, \mathbb{R}^1)$ with the usual uniform topology (the compact-open topology, or the C^0-topology) defined by the supremum norm $\|\cdot\|_\infty$. We use $u \equiv const.$ to denote a constant function whose values do not vary.

3. The new L-O semigroup: time-independent case

As mentioned in the Introduction, in this section we first discuss the main properties of the new L-O semigroup for the time-independent case, and then give the proofs of Theorems 1.2 and 1.3. Finally, we construct an example to show that the new L-O semigroup converges faster than the L-O semigroup in the sense of order when the Aubry set $\tilde{\mathcal{A}}_0$ of the Lagrangian system (1.5) is a quasi-periodic invariant torus with Diophantine frequency vector $\omega \in \mathcal{D}(\rho, \alpha)$.

3.1. **Properties of the new L-O semigroup.** Let us recall the definition (Definition 1.1) of the new L-O semigroup $\{\tilde{T}^a_t\}_{t\geq 0}$ associated with L_a. For each $u \in C(M, \mathbb{R}^1)$ and each $t \geq 0$,

$$\tilde{T}^a_t u(x) = \inf_{t\leq s\leq 2t} \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^s L_a(\gamma(\tau), \dot{\gamma}(\tau)) d\tau \right\}$$

for all $x \in M$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [0, s] \to M$ with $\gamma(s) = x$.

Obviously $\tilde{T}^a_t u(x) = \inf_{t\leq s\leq 2t} T^a_s u(x)$. It follows that $-\infty < \tilde{T}^a_t u(x) \leq T^a_t u(x)$ which yields $\tilde{T}^a_t u(x) \in \mathbb{R}^1$, $\forall x \in M$. Moreover, for each $t \geq 0$, $\tilde{T}^a_t$ is an operator from $C(M, \mathbb{R}^1)$ to itself. In fact, for any $u \in C(M, \mathbb{R}^1)$, from [9] the function $(s, x) \mapsto T^a_s u(x)$ is continuous on $[0, +\infty) \times M$ and thus $\tilde{T}^a_t u(\cdot) = \inf_{t\leq s\leq 2t} T^a_s u(\cdot)$ is a continuous function on M since the infimum of continuous functions over a compact set is also continuous. Furthermore, from the definition it is not difficult to check that $\{\tilde{T}^a_t\}_{t\geq 0}$ is a semigroup of operators.

Proposition 3.1. For given $t > 0$, $u \in C(M, \mathbb{R}^1)$ and $x \in M$, there exist $s \in [t, 2t]$ and an extremal curve $\gamma : [0, s] \to M$ such that $\gamma(t) = x$ and

$$\tilde{T}^a_t u(x) = u(\gamma(0)) + \int_0^s L_a(\gamma, \dot{\gamma}) d\tau.$$

Proof. Since $s \mapsto T^a_s u(x)$ is continuous on $[t, 2t]$ and $\tilde{T}^a_t u(x) = \inf_{t\leq s\leq 2t} T^a_s u(x)$, then there is $s_0 \in [t, 2t]$ such that $\tilde{T}^a_t u(x) = T^a_{s_0} u(x)$. From the property of the operator $T^a_{s_0}$ (see Lemma 4.4.1 in [9]), there exists an extremal curve $\gamma : [0, s_0] \to M$ such that $\gamma(s_0) = x$ and

$$\tilde{T}^a_t u(x) = T^a_{s_0} u(x) = u(\gamma(0)) + \int_0^{s_0} L_a(\gamma, \dot{\gamma}) d\tau.$$

□

Some fundamental properties of $\tilde{T}^a_t$ are discussed in the following proposition.

Proposition 3.2.

(a) For u, $v \in C(M, \mathbb{R}^1)$, if $u \leq v$, then $\tilde{T}^a_t u \leq \tilde{T}^a_t v$, $\forall t \geq 0$.
(b) If c is a constant and $u \in C(M, \mathbb{R}^1)$, then $\tilde{T}^a_t (u + c) = \tilde{T}^a_t u + c$, $\forall t \geq 0$.
(c) For each u, $v \in C(M, \mathbb{R}^1)$ and each $t \geq 0$, $\|\tilde{T}^a_t u - \tilde{T}^a_t v\|_\infty \leq \|u - v\|_\infty$.
(d) For each $u \in C(M, \mathbb{R}^1)$, $\lim_{t\to 0^+} \tilde{T}^a_t u = u$.
(e) For each $u \in C(M, \mathbb{R}^1)$, $(t, x) \mapsto \tilde{T}^a_t u(x)$ is continuous on $[0, +\infty) \times M$.

Remark 3.3. The property (a) is monotonicity. The property (c) says that the maps $\tilde{T}^a_t$ are non-expansive. The property (d) means that the semigroup $\{\tilde{T}^a_t\}_{t\geq 0}$ is continuous at the origin or of class C_0 [12].

Proof. Since T^a_t has the monotonicity property (see Corollary 4.4.4 in [9]), then

$$\tilde{T}^a_t u(x) = \inf_{t\leq s\leq 2t} T^a_s u(x) \leq \inf_{t\leq s\leq 2t} T^a_s v(x) = \tilde{T}^a_t v(x), \quad \forall t>0,\ \forall x\in M,$$

i.e., (a) holds. (b) results from the definition of $\tilde{T}^a_t$ directly. Note that for any $x \in M$,

$$-\|u-v\|_\infty + v(x) \leq u(x) \leq \|u-v\|_\infty + v(x).$$

By the properties of T^a_s (see Corollary 4.4.4 in [9]), for each $t \geq 0$ we have

$$T^a_s v(x) - \|u-v\|_\infty \leq T^a_s u(x) \leq T^a_s v(x) + \|u-v\|_\infty, \quad \forall s\in[t,2t].$$

Taking the infimum on s over $[t, 2t]$ yields

$$\inf_{t\leq s\leq 2t} T^a_s v(x) - \|u-v\|_\infty \leq \inf_{t\leq s\leq 2t} T^a_s u(x) \leq \inf_{t\leq s\leq 2t} T^a_s v(x) + \|u-v\|_\infty, \quad \forall x\in M,$$

and thus (c) holds.

Next we prove (d). For each $u \in C(M,\mathbb{R}^1)$ and each $\varepsilon > 0$, there is $w \in C^1(M,\mathbb{R}^1)$ such that $\|u-w\|_\infty < \varepsilon$ since $C^1(M,\mathbb{R}^1)$ is a dense subset of $C(M,\mathbb{R}^1)$ in the topology of uniform convergence. Thus, we have

$$\begin{aligned} \|\tilde{T}^a_t u - u\|_\infty &\leq \|\tilde{T}^a_t u - \tilde{T}^a_t w\|_\infty + \|\tilde{T}^a_t w - w\|_\infty + \|w-u\|_\infty \\ &\leq 2\|w-u\|_\infty + \|\tilde{T}^a_t w - w\|_\infty \\ &\leq 2\varepsilon + \|\tilde{T}^a_t w - w\|_\infty, \quad \forall t\geq 0, \end{aligned} \tag{3.1}$$

where we have used (c). Since M is compact, then w is Lipschitz. Denote the Lipschitz constant of w by K_w, and by the superlinearity of L_a there exists $C_{K_w} \in \mathbb{R}^1$ such that

$$L_a(x,v) \geq K_w\|v\|_x + C_{K_w}, \quad \forall (x,v)\in TM.$$

For each $x \in M$, $t \geq 0$, and any continuous and piecewise C^1 path $\gamma : [0, s] \to M$ with $\gamma(s) = x$ and $t \leq s \leq 2t$, since

$$d(\gamma(0),\gamma(s)) \leq \int_0^s \|\dot\gamma(\tau)\|_{\gamma(\tau)} d\tau,$$

then

$$\int_0^s L_a(\gamma,\dot\gamma)d\tau \geq K_w d(\gamma(0),\gamma(s)) + C_{K_w}s \geq w(\gamma(s)) - w(\gamma(0)) + C_{K_w}s.$$

Thus, by the definition of T^a_s we have

$$T^a_s w(x) \geq w(x) + C_{K_w}s.$$

Taking the infimum on s over $[t, 2t]$ on both sides of this last inequality yields

$$\tilde{T}^a_t w(x) \geq w(x) + O(t), \quad \text{as } t\to 0^+, \tag{3.2}$$

where $O(t)$ is independent of x. Using the constant curve $\gamma_x : [0, s] \to M$, $\tau \mapsto x$, we have

$$T_s^a w(x) \leq w(x) + L_a(x,0)s.$$

Taking the infimum on s over $[t, 2t]$, we obtain

$$\tilde{T}_t^a w(x) \leq w(x) + O(t), \quad \text{as } t \to 0^+, \tag{3.3}$$

where $O(t)$ is independent of x. Combing (3.1), (3.2) and (3.3), we have

$$\lim_{t\to 0^+} \|\tilde{T}_t^a u - u\|_\infty = 0,$$

i.e., (d) holds.

Finally, we prove (e). For any $(t_0, x_0) \in [0, +\infty) \times M$, from the semigroup property and (c) we have

$$\begin{aligned} |\tilde{T}_t^a u(x) - \tilde{T}_{t_0}^a u(x_0)| &\leq |\tilde{T}_t^a u(x) - \tilde{T}_t^a u(x_0)| + |\tilde{T}_t^a u(x_0) - \tilde{T}_{t_0}^a u(x_0)| \\ &\leq |\tilde{T}_t^a u(x) - \tilde{T}_t^a u(x_0)| + \|\tilde{T}_t^a u - \tilde{T}_{t_0}^a u\|_\infty \\ &\leq |\tilde{T}_t^a u(x) - \tilde{T}_t^a u(x_0)| + \|\tilde{T}_{|t-t_0|}^a u - u\|_\infty. \end{aligned} \tag{3.4}$$

From (3.4), $\tilde{T}_t^a u(\cdot) \in C(M, \mathbb{R}^1)$ and (d), we conclude that (e) holds. □

The proposition below establishes a relationship between $\tilde{T}_t^a$ and T_t^a.

Proposition 3.4.

(f) For each $u \in C(M, \mathbb{R}^1)$, the uniform limit $\lim_{t\to+\infty} \tilde{T}_t^a u$ exists and

$$\lim_{t\to+\infty} \tilde{T}_t^a u = \lim_{t\to+\infty} T_t^a u = \bar{u}.$$

(g) For each $t \geq 0$ and each $u \in C(M, \mathbb{R}^1)$, $\|\tilde{T}_t^a u - \bar{u}\|_\infty \leq \|T_t^a u - \bar{u}\|_\infty$.

(h) $u \in C(M, \mathbb{R}^1)$ is a fixed point of $\{\tilde{T}_t^a\}_{t\geq 0}$ if and only if it is a fixed point of $\{T_t^a\}_{t\geq 0}$.

Remark 3.5. From (f) $\lim_{t\to+\infty} \tilde{T}_t^a u$ exists and is a backward weak KAM solution of the Hamilton-Jacobi equation $H_a(x, u_x) = 0$. (g) essentially says that the new L-O semigroup converges faster than the L-O semigroup. (h) implies that $u \in C(M, \mathbb{R}^1)$ is a backward weak KAM solution if and only if it is a fixed point of $\{\tilde{T}_t^a\}_{t\geq 0}$.

Proof. First we prove (f). Assume by contradiction that there exist $\varepsilon_0 > 0$, $t_n \to +\infty$ and $x_n \in M$ such that

$$|\tilde{T}_{t_n}^a u(x_n) - \bar{u}(x_n)| \geq \varepsilon_0.$$

From the compactness of M, without loss of generality we assume that $x_n \to x_0$, $n \to +\infty$. In view of the definition of $\tilde{T}_t^a$, there exist $s_n \in [t_n, 2t_n]$ such that

$$|T_{s_n}^a u(x_n) - \bar{u}(x_n)| \geq \varepsilon_0.$$

Let $n \to +\infty$. Since $(s, x) \mapsto T_s^a u(x)$ is continuous, then we have

$$\lim_{s\to+\infty} T_s^a u(x_0) \neq \bar{u}(x_0),$$

which contradicts $\lim_{s\to+\infty} T_s^a u = \bar{u}$.

Next we show (g). For each $t \geq 0$ and each $x \in M$, there exists $t \leq s_x \leq 2t$ such that

$$|\tilde{T}^a_t u(x) - \bar{u}(x)| = |T^a_{s_x} u(x) - \bar{u}(x)|.$$

Since $\bar{u}$ is a fixed point of $\{T^a_t\}_{t\geq 0}$, then we have that $|T^a_{s_x} u(x) - \bar{u}(x)| = |T^a_{s_x} u(x) - T^a_{s_x} \bar{u}(x)| \leq \|T^a_{s_x} u - T^a_{s_x} \bar{u}\|_\infty = \|T^a_{s_x - t} \circ T^a_t u - T^a_{s_x - t} \circ T^a_t \bar{u}\|_\infty \leq \|T^a_t u - T^a_t \bar{u}\|_\infty = \|T^a_t u - \bar{u}\|_\infty$, where we have used the non-expansiveness property of $T^a_{s_x - t}$ (see Corollary 4.4.4 in [9]). Hence (g) holds.

At last, we show (h). Suppose that u is a fixed point of $\{T^a_t\}_{t\geq 0}$, i.e., $T^a_t u = u$, $\forall t \geq 0$. Then $\lim_{t\to+\infty} T^a_t u = u$. From (g) we have

$$\|\tilde{T}^a_t u - u\|_\infty \leq \|T^a_t u - u\|_\infty = 0, \quad \forall t \geq 0,$$

which implies that u is a fixed point of $\{\tilde{T}^a_t\}_{t\geq 0}$. Suppose conversely that u is a fixed point of $\{\tilde{T}^a_t\}_{t\geq 0}$. Then from (f) $\lim_{t\to+\infty} \tilde{T}^a_t u = u = \lim_{t\to+\infty} T^a_t u$. Hence u is a backward weak KAM solution of $H_a(x, u_x) = 0$ and a fixed point of $\{T^a_t\}_{t\geq 0}$. $\square$

3.2. **Rates of convergence of the L-O semigroup and the new L-O semigroup.** Recall the C^2 positive definite and superlinear Lagrangian (1.5)

$$L^1_a(x, v) = \frac{1}{2}\langle A(x)(v - \omega), (v - \omega)\rangle + f(x, v - \omega), \quad x \in \mathbb{T}^n, \ v \in \mathbb{R}^n.$$

The conjugated Hamiltonian $H^1_a : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}^1$ of L^1_a has the following form

$$H^1_a(x, p) = \langle \omega, p\rangle + \frac{1}{2}\langle A^{-1}(x)p, p\rangle + g(x, p),$$

where $g(x, p) = O(\|p\|^3)$ as $p \to 0$. It is clear that $H^1_a(x, 0) = 0$ and thus $w \equiv$ *const*. is a smooth viscosity solution of the corresponding Hamilton-Jacobi equation $H^1_a(x, u_x) = 0$. In view of the Legendre transform,

$$L^1_a(x, v) = L^1_a(x, v) - \langle w_x, v\rangle \geq -H^1_a(x, w_x) = -H^1_a(x, 0) = 0, \quad \forall (x, v) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Furthermore, if $(x, v) \in \tilde{\mathcal{M}}_0 = \cup_{x\in\mathbb{T}^n}(x, \omega)$, then $w_x = \frac{\partial L}{\partial v}(x, v)$ (see Theorem 4.8.3 in [9]), from which we have

$$L^1_a(x, v) = L^1_a(x, v) - \langle w_x, v\rangle = -H^1_a(x, w_x) = -H^1_a(x, 0) = 0.$$

Hence

$$L^1_a \geq 0, \quad \forall (x, v) \in \mathbb{T}^n \times \mathbb{R}^n$$

and in particular,

$$L^1_a|_{\cup_{x\in\mathbb{T}^n}(x,\omega)} = 0.$$

For each $u \in C(\mathbb{T}^n, \mathbb{R}^1)$, because of $c(L^1_a) = 0$ we have $\lim_{t\to+\infty} T_t u = \bar{u}$. Note that both $w \equiv$ *const*. and $\bar{u}$ are viscosity solutions of $H^1_a(x, u_x) = 0$. Hence $\bar{u} \equiv$ *const*. since the viscosity solution of $H^1_a(x, u_x) = 0$ is unique up to constants when $\mathcal{A}_0 = \mathbb{T}^n$ (see [14]), where $\mathcal{A}_0$ is the projected Aubry set.

3.2.1. *Rate of convergence of the L-O semigroup.* We present here the proof of Theorem 1.2. For this, the following lemma is needed.

Lemma 3.6. *For each $u \in C(\mathbb{T}^n, \mathbb{R}^1)$, $\bar{u} \equiv \min_{x\in\mathbb{T}^n} u(x)$.*

Proof. For any $x \in \mathbb{T}^n$, from the definition of T_t^a we have

$$\bar{u}(x) = \lim_{t\to+\infty} T_t^a u(x) = \lim_{t\to+\infty} \inf_{z\in\mathbb{T}^n} \{u(z) + \int_0^t L_a^1(\gamma_z, \dot{\gamma}_z)ds\},$$

where $\gamma_z : [0,t] \to \mathbb{T}^n$ is a Tonelli minimizer (see for example, [17, Tonelli's theorem]) with $\gamma_z(0) = z$, $\gamma_z(t) = x$. Since $L_a^1 \geq 0$, then $\bar{u}(x) \geq \min_{z\in\mathbb{T}^n} u(z)$ and therefore it suffices to show that $\bar{u}(x) \leq \min_{z\in\mathbb{T}^n} u(z)$.

Take $y \in \mathbb{T}^n$ with $u(y) = \min_{z\in\mathbb{T}^n} u(z)$. Consider the following two curves

$$\gamma_\omega : [0,t] \to \mathbb{T}^n, \ s \mapsto \omega s + y$$

and

$$\gamma_{\omega'} : [0,t] \to \mathbb{T}^n, \ s \mapsto \omega' s + y$$

with $\gamma_{\omega'}(t) = x$, where $\omega' \in \mathbb{S}^{n-1}$ and $t > 0$. It is clear that $\gamma_{\omega'}$ is a curve in $\mathbb{T}^n$ connecting y and x. Let $\Delta = \gamma_{\omega'}(t) - \gamma_\omega(t) = x - (\omega t + y)$. Then $\|\Delta\| \leq \frac{\sqrt{n}}{2}$ and $\dot{\gamma}_{\omega'} \equiv \omega' = \frac{\Delta}{t} + \omega$. Therefore, we have

$$\begin{aligned}
T_t^a u(x) &\leq u(\gamma_{\omega'}(0)) + \int_0^t L_a^1(\gamma_{\omega'}, \dot{\gamma}_{\omega'})ds \\
&= u(y) + \int_0^t \Big(\frac{1}{2}\langle A(\gamma_{\omega'})(\omega' - \omega), (\omega' - \omega)\rangle + f(\gamma_{\omega'}, \omega' - \omega)\Big)ds \\
&= u(y) + \int_0^t \Big(\frac{1}{2}\Big\langle A(\gamma_{\omega'})\frac{\Delta}{t}, \frac{\Delta}{t}\Big\rangle + f(\gamma_{\omega'}, \frac{\Delta}{t})\Big)ds \\
&\leq u(y) + \frac{C}{t} + O(\frac{1}{t^2}),
\end{aligned}$$

where C is a constant, which depends only on n.

From the arguments above we know that for any $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$ there exists $\gamma_{\omega'} : [0,t] \to \mathbb{T}^n$ with $\gamma_{\omega'}(t) = x$, and

$$T_t^a u(x) \leq u(\gamma_{\omega'}(0)) + \int_0^t L_a^1(\gamma_{\omega'}, \dot{\gamma}_{\omega'})ds \leq \min_{z\in\mathbb{T}^n} u(z) + \varepsilon.$$

Hence $\bar{u}(x) = \lim_{t\to+\infty} T_t^a u(x) \leq \min_{z\in\mathbb{T}^n} u(z)$. □

Proof of Theorem 1.2. In order to prove our result, it is sufficient to show that for each $u \in C(\mathbb{T}^n, \mathbb{R}^1)$, there exists a constant $K > 0$ such that the following two inequalities hold.

$$T_t^a u(x) - \bar{u}(x) \leq \frac{K}{t}, \quad \forall t > 0, \ \forall x \in \mathbb{T}^n; \tag{I1}$$

$$\bar{u}(x) - T_t^a u(x) \leq \frac{K}{t}, \quad \forall t > 0, \ \forall x \in \mathbb{T}^n. \tag{I2}$$

Obviously, (I2) holds. In fact, for each $t > 0$ and each $x \in \mathbb{T}^n$, from the definition of T_t^a we have

$$T_t^a u(x) = \inf_{z\in\mathbb{T}^n}\{u(z) + \int_0^t L_a^1(\gamma_z,\dot\gamma_z)ds\},$$

where $\gamma_z : [0,t] \to \mathbb{T}^n$ is a Tonelli minimizer with $\gamma_z(0) = z$, $\gamma_z(t) = x$. In view of $L_a^1 \geq 0$ and Lemma 3.6, we have

$$T_t^a u(x) = \inf_{z\in\mathbb{T}^n}\{u(z) + \int_0^t L_a^1(\gamma_z,\dot\gamma_z)ds\} \geq \min_{z\in\mathbb{T}^n} u(z) = \bar u(x).$$

Thus $\bar u(x) - T_t^a u(x) \leq 0$, $\forall t > 0$, $\forall x \in \mathbb{T}^n$ and (I2) holds.

Next we prove (I1). It suffices to show that there exists a constant $C > 0$ such that for sufficiently large $t > 0$,

$$T_t^a u(x) - \bar u(x) \leq \frac{C}{t}, \quad \forall x \in \mathbb{T}^n, \tag{3.5}$$

where C depends only on n. In deed, since $(s,z) \mapsto T_s u(z)$ is continuous on $[0,\infty)\times\mathbb{T}^n$, if (3.5) holds, then there exists a constant $K > 0$ such that

$$T_t^a u(x) - \bar u(x) \leq \frac{K}{t}, \quad \forall t > 0,\ \forall x \in \mathbb{T}^n,$$

where K depends only on n and u.

Take $y \in \mathbb{T}^n$ with $u(y) = \min_{z\in\mathbb{T}^n} u(z)$. Let us consider the following curve in $\mathbb{T}^n$

$$\gamma_\omega : [0,t] \to \mathbb{T}^n,\ s \mapsto \omega s + y,$$

where $t > 0$. Then for each $x \in \mathbb{T}^n$, let

$$\gamma_{\omega'} : [0,t] \to \mathbb{T}^n,\ s \mapsto \omega' s + y$$

be a curve in $\mathbb{T}^n$ connecting y and x, where $\omega' \in \mathbb{S}^{n-1}$. Let $\Delta = \gamma_{\omega'}(t) - \gamma_\omega(t) = x - (\omega t + y)$. Then $\|\Delta\| \leq \frac{\sqrt{n}}{2}$ and $\dot\gamma_{\omega'} \equiv \omega' = \frac{\Delta}{t} + \omega$. Hence,

$$\begin{aligned}
T_t^a u(x) &\leq u(\gamma_{\omega'}(0)) + \int_0^t L_a^1(\gamma_{\omega'},\dot\gamma_{\omega'})ds \\
&= u(y) + \int_0^t \Big(\frac{1}{2}\langle A(\gamma_{\omega'})(\omega'-\omega),(\omega'-\omega)\rangle + f(\gamma_{\omega'},\omega'-\omega)\Big)ds \\
&= u(y) + \int_0^t \Big(\frac{1}{2}\Big\langle A(\gamma_{\omega'})\frac{\Delta}{t},\frac{\Delta}{t}\Big\rangle + f(\gamma_{\omega'},\frac{\Delta}{t})\Big)ds \\
&\leq u(y) + \frac{C_1}{t} + O(\frac{1}{t^2}),
\end{aligned}$$

where C_1 is a constant which depends only on n. From Lemma 3.6, we have $T_t^a u(x) - \bar u(x) \leq \frac{C}{t}$ for $t > 0$ large enough, where C is a constant which still depends only on n, i.e., (3.5) holds. □

3.2.2. *Rate of convergence of the new L-O semigroup.* To complete the proof of Theorem 1.3, we review preliminaries on the ergodization rate for linear flows on the torus $\mathbb{T}^n$, i.e., the rate at which the image of a point fills the torus when subjected to linear flows. There is a direct relationship between the rate of convergence of the new L-O semigroup and the ergodization rate for linear flows on the torus $\mathbb{T}^n$. Let us recall the following result of Dumas' [5] concerning the estimate of ergodization time.

For each $t \in \mathbb{R}^1$ and each $\omega \in \mathbb{S}^{n-1}$, consider the one-parameter family of translation maps $\omega_t : \mathbb{T}^n \to \mathbb{T}^n$, $x \mapsto x + \omega t$. A rectilinear orbit of $\mathbb{T}^n$ with direction vector ω and initial condition x is defined as the image of x under the linear flow ω_t over some closed interval $[t_0, t_1] \subset \mathbb{R}^1$, i.e.,

$$\bigcup_{t_0 \le t \le t_1} \omega_t(x).$$

Given $R > 0$, the direction vector $\omega \in \mathbb{S}^{n-1}$ is said to ergodize $\mathbb{T}^n$ to within R after time T if

$$\bigcup_{0 \le t \le T} \omega_t(B_R(x)) = \mathbb{T}^n \tag{3.6}$$

for all $x \in \mathbb{T}^n$.

As defined in the Introduction, for $\rho > n-1$ and $\alpha > 0$,

$$\mathcal{D}(\rho, \alpha) = \Big\{\beta \in \mathbb{S}^{n-1} \mid |\langle \beta, k \rangle| > \frac{\alpha}{|k|^\rho},\ \forall k \in \mathbb{Z}^n \backslash \{0\}\Big\},$$

whose elements can not be approximated by rationals too rapidly.

Theorem 3.7 (Dumas [5])**.** *Let $0 < R \le 1$. Given any highly nonresonant direction vector $\omega \in \mathcal{D}(\rho, \alpha)$, rectilinear orbits of $\mathbb{T}^n$ with direction vector ω will ergodize $\mathbb{T}^n$ to within R after time T, where*

$$T = \frac{2\|V_*\|_\triangle}{\alpha \pi R^{\rho + n/2}}$$

is independent of ω.

Remark 3.8. The constant $\|V_*\|_\triangle$ is a Sobolev norm of a certain "smoothest test function" and it depends only on n and ρ. See [5] for complete details.

We are now in a position to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Our purpose is to show that for each $u \in C(\mathbb{T}^n, \mathbb{R}^1)$, there exists a constant $\tilde{K} > 0$ such that the following two inequalities hold.

$$\tilde{T}^a_t u(x) - \bar{u}(x) \le \tilde{K} t^{-(1 + \frac{4}{2\rho + n})}, \quad \forall t > 0,\ \forall x \in \mathbb{T}^n; \tag{I3}$$

$$\bar{u}(x) - \tilde{T}^a_t u(x) \le \tilde{K} t^{-(1 + \frac{4}{2\rho + n})}, \quad \forall t > 0,\ \forall x \in \mathbb{T}^n. \tag{I4}$$

First we show (I4). For each $t > 0$ and each $x \in \mathbb{T}^n$, by the definition of $\tilde{T}^a_t$ we have

$$\tilde{T}^a_t u(x) = \inf_{t \le s \le 2t} \inf_{z \in \mathbb{T}^n} \{u(z) + \int_0^s L^1_a(\gamma_z, \dot{\gamma}_z) d\tau\},$$

where $\gamma_z : [0, s] \to \mathbb{T}^n$ is a Tonelli minimizer with $\gamma_z(0) = z$, $\gamma_z(s) = x$. In view of $L^1_a \geq 0$ and Lemma 3.6, we have

$$\tilde{T}^a_t u(x) = \inf_{t \leq s \leq 2t} \inf_{z \in \mathbb{T}^n} \{u(z) + \int_0^s L^1_a(\gamma_z, \dot{\gamma}_z) d\tau\} \geq \min_{z \in \mathbb{T}^n} u(z) = \bar{u}(x).$$

Thus $\bar{u}(x) - \tilde{T}^a_t u(x) \leq 0$, $\forall t > 0$, $\forall x \in \mathbb{T}^n$, i.e., (I4) holds.

Then it remains to show (I3). When $R = 1$, according to Theorem 3.7 the ergodization time $T = \frac{2\|V_*\|_\triangle}{\alpha\pi}$. For any $t \geq T$, let $R_t = \sqrt[\rho+n/2]{\frac{2\|V_*\|_\triangle}{\alpha\pi t}}$. Then $0 < R_t \leq 1$.

Take $y \in \mathbb{T}^n$ with $u(y) = \min_{z \in \mathbb{T}^n} u(z)$. Let $y_t = \omega_t(y) = \omega t + y$. For R_t defined above, since $\omega \in \mathcal{D}(\rho, \alpha)$, then from Theorem 3.7 and (3.6) we have

$$\bigcup_{0 \leq \sigma \leq t} \omega_\sigma(B_{R_t}(y_t)) = \mathbb{T}^n.$$

Therefore, for each $x \in \mathbb{T}^n$, there exists $0 \leq \sigma' \leq t$ such that $d_{\mathbb{T}^n}(\omega_{\sigma'}(y_t), x) \leq R_t$, i.e., $d_{\mathbb{T}^n}(\omega(t + \sigma') + y, x) \leq R_t$. Equivalently this means that there exists $t \leq s' \leq 2t$ such that

$$d_{\mathbb{T}^n}(\omega s' + y, x) \leq R_t,$$

where $s' = t + \sigma'$. Consider the following curve in $\mathbb{T}^n$

$$\gamma_{\omega'} : [0, s'] \to \mathbb{T}^n, \ \tau \mapsto \omega'\tau + y$$

with $\gamma_{\omega'}(s') = x$, where $\omega' \in \mathbb{S}^{n-1}$. It is clear that $\gamma_{\omega'}$ connects y and x. Let $\Delta = \gamma_{\omega'}(s') - \omega_{s'}(y) = x - (\omega s' + y)$. Then $\|\Delta\| = d_{\mathbb{T}^n}(x, \omega s' + y) \leq R_t$ and $\dot{\gamma}_{\omega'} \equiv \omega' = \frac{\Delta}{s'} + \omega$. Hence we have

$$\begin{aligned}
\tilde{T}^a_t u(x) - \bar{u}(x) &\leq u(\gamma_{\omega'}(0)) + \int_0^{s'} L^1_a(\gamma_{\omega'}, \dot{\gamma}_{\omega'}) d\tau - \bar{u}(x) \\
&= \int_0^{s'} \Big(\frac{1}{2}\langle A(\gamma_{\omega'})(\omega' - \omega), (\omega' - \omega)\rangle + f(\gamma_{\omega'}, \omega' - \omega)\Big) d\tau \\
&\leq \frac{CR_t^2}{t}
\end{aligned}$$

for sufficiently large $t > 0$ and some constant $C > 0$. Since $R_t^2 = (\frac{2\|V_*\|_\triangle}{\alpha\pi t})^{\frac{2}{\rho+n/2}}$, then for $t > 0$ large enough we have

$$\tilde{T}^a_t u(x) - \bar{u}(x) \leq C_1 t^{-(1+\frac{4}{2\rho+n})}, \quad \forall x \in \mathbb{T}^n,$$

where C_1 is a constant which depends only on n, ρ and α. From (e) of Proposition 3.2, $(\tau, z) \mapsto \tilde{T}^a_\tau u(z)$ is continuous on $[0, \infty) \times \mathbb{T}^n$. Hence there exists a constant $\tilde{K} > 0$ such that

$$\tilde{T}^a_t u(x) - \bar{u}(x) \leq \tilde{K} t^{-(1+\frac{4}{2\rho+n})}, \quad \forall t > 0, \ \forall x \in \mathbb{T}^n,$$

where $\tilde{K}$ depends only on n, ρ, α and u, i.e., (I3) holds. □

3.2.3. *An example.*

Example 3.9. Consider the following integrable C^2 Lagrangian

$$\bar{L}_a^1(x,v)=\frac{1}{2}\langle v-\omega,v-\omega\rangle,\quad x\in\mathbb{T}^n,\ v\in\mathbb{R}^n,\ \omega\in\mathbb{S}^{n-1}.$$

It is easy to see that $\bar{L}_a^1$ is a special case of L_a^1. For $\bar{L}_a^1$, we show that there exist $u\in C(\mathbb{T}^n,\mathbb{R}^1)$, $x^0\in\mathbb{T}^n$ and $t_m\to+\infty$ as $m\to+\infty$ such that

$$|T^a_{t_m}u(x^0)-\bar{u}(x^0)|=O(\frac{1}{t_m}),\quad m\to+\infty,$$

which implies that the result of Theorem 1.2 is sharp in the sense of order.

Recall the universal covering projection $\pi:\mathbb{R}^n\to\mathbb{T}^n$. Let $x^0\in\mathbb{T}^n$ such that each point $\tilde{x}^0\in\mathbb{R}^n$ in the fiber over x^0 ($\pi\tilde{x}^0=x^0$) is the center of each fundamental domain in $\mathbb{R}^n$. Define a continuous function on $\mathbb{R}^n$ as follows: for $\tilde{x}\in\mathbb{R}^n$

$$\tilde{u}(\tilde{x})=\begin{cases}\delta-\|\tilde{x}-\tilde{x}^0\|, & \|\tilde{x}-\tilde{x}^0\|\le\delta,\\ 0, & \text{otherwise,}\end{cases}$$

where $0<\delta<\frac{1}{2}$. We then define a continuous function on $\mathbb{T}^n$ as $u(x)=\tilde{u}(\tilde{x})$ for all $x\in\mathbb{T}^n$, where $\tilde{x}$ is an arbitrary point in the fiber over x. Thus, from Lemma 3.6, $\bar{u}\equiv\min_{x\in\mathbb{T}^n}u(x)=0$.

Now fix a point $\tilde{x}^0_0$ in the fiber over x^0. Then there exist $\{\tilde{x}^0_m\}_{m=1}^{+\infty}$ in the fiber over x^0 and $t_m\to+\infty$ as $m\to+\infty$ such that $\|(\tilde{x}^0_m-\omega t_m)-\tilde{x}^0_0\|\le\frac{\delta}{2}$. Let $\tilde{z}_m=\tilde{x}^0_m-\omega t_m$. Then $\|\tilde{z}_m-\tilde{x}^0_0\|\le\frac{\delta}{2}$. For each t_m there exists $y_m\in\mathbb{T}^n$ such that

$$T^a_{t_m}u(x^0)=u(y_m)+\int_0^{t_m}\bar{L}_a^1(\gamma_{y_m},\dot{\gamma}_{y_m})ds,$$

where $\gamma_{y_m}:[0,t_m]\to\mathbb{T}^n$ is a Tonelli minimizer with $\gamma_{y_m}(0)=y_m$, $\gamma_{y_m}(t_m)=x^0$. In view of the lifting property of the covering projection, there is a unique curve $\tilde{\gamma}_{y_m}:[0,t_m]\to\mathbb{R}^n$ with $\pi\tilde{\gamma}_{y_m}=\gamma_{y_m}$ and $\tilde{\gamma}_{y_m}(t_m)=\tilde{x}^0_m$. Set $\tilde{y}_m=\tilde{\gamma}_{y_m}(0)$. Then $\pi\tilde{y}_m=y_m$. Moreover, $\tilde{\gamma}_{y_m}$ has the following form

$$\tilde{\gamma}_{y_m}(s)=\omega's+\tilde{y}_m,\quad s\in[0,t_m],$$

where $\omega'\in\mathbb{S}^{n-1}$. It is clear that $\tilde{\gamma}_{y_m}(0)=\tilde{y}_m$ and $\tilde{y}_m=\tilde{x}^0_m-\omega't_m$.

If $\|\tilde{y}_m-\tilde{z}_m\|\le\frac{\delta}{4}$, then from $\|\tilde{z}_m-\tilde{x}^0_0\|\le\frac{\delta}{2}$ we have $\|\tilde{y}_m-\tilde{x}^0_0\|\le\frac{3\delta}{4}$. Hence,

$$\begin{aligned}T^a_{t_m}u(x^0)&=u(y_m)+\int_0^{t_m}\bar{L}_a^1(\gamma_{y_m},\dot{\gamma}_{y_m})ds\\&\ge\tilde{u}(\tilde{y}_m)\ge\delta-\frac{3\delta}{4}=\frac{\delta}{4}.\end{aligned}\tag{3.7}$$

From (3.7), we may deduce that there can only be a finite number of $\tilde{y}_m$'s such that $\|\tilde{y}_m-\tilde{z}_m\|\le\frac{\delta}{4}$. For, otherwise, there would be $\{t_{m_i}\}_{i=1}^{+\infty}$ and $\{\tilde{y}_{m_i}\}_{i=1}^{+\infty}$ such that

$$T^a_{t_{m_i}}u(x^0)\ge\frac{\delta}{4},\quad i=1,2,\cdots,$$

which contradicts $\lim_{i\to+\infty}T^a_{t_{m_i}}u(x^0)=\bar{u}(x^0)=0$.

For $\tilde{y}_m$ with $\|\tilde{y}_m-\tilde{z}_m\|>\frac{\delta}{4}$, we have

$$\frac{\delta}{4} < \|\tilde{y}_m - \tilde{z}_m\| = \|\tilde{x}_m^0 - \omega' t_m - (\tilde{x}_m^0 - \omega t_m)\| = \|\omega - \omega'\| t_m.$$

Thus,

$$\begin{aligned} T_{t_m}^a u(x^0) &= u(y_m) + \int_0^{t_m} \bar{L}_a^1(\gamma_{y_m}, \dot{\gamma}_{y_m}) ds \\ &\geq \frac{1}{2} t_m \|\omega - \omega'\|^2 = \frac{1}{2} \frac{t_m^2 \|\omega - \omega'\|^2}{t_m} \geq \frac{\delta^2}{32 t_m}. \end{aligned} \tag{3.8}$$

Therefore, from (3.8) and Theorem 1.2 we have

$$|T_{t_m}^a u(x^0) - \bar{u}(x^0)| = |T_{t_m}^a u(x^0)| = O(\frac{1}{t_m}), \quad m \to +\infty.$$

4. The new L-O semigroup: time-periodic case

In this section we first discuss some basic properties of the new L-O semigroup for the time-periodic case, i.e., $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ associated with L, and then study the convergence of $\tilde{T}_n u$ as $n \to +\infty$. As last, we discuss the relation between the limit $\lim_{n\to+\infty} \tilde{T}_n u$ and the backward weak KAM solution.

4.1. Basic properties of the new L-O semigroup.

Recall the definition (Definition 1.4) of the new L-O semigroup $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ associated with L. For each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ and each $n \in \mathbb{N}$,

$$\tilde{T}_n u(x, \tau) = \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{\gamma} \Big\{ u(\gamma(\tau - k), \tau) + \int_{\tau-k}^{\tau} L(\gamma(s), \dot{\gamma}(s), s) ds \Big\}$$

for all $(x, \tau) \in M \times \mathbb{S}^1$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [\tau - k, \tau] \to M$ with $\gamma(\tau) = x$.

First of all, we show that for each $n \in \mathbb{N}$, $\tilde{T}_n$ is an operator from $C(M \times \mathbb{S}^1, \mathbb{R}^1)$ to itself. For this, it suffices to prove the following result.

Proposition 4.1. For each $n \in \mathbb{N}$ and each $u \in C(M\times\mathbb{S}^1, \mathbb{R}^1)$, $\tilde{T}_n u$ is a continuous function on $M \times \mathbb{S}^1$.

Proof. Following Mather ([18], also see [1]), it is convenient to introduce, for $t' \geq t$ and x, $y \in M$, the following quantity:

$$F_{t,t'}(x, y) = \inf_{\gamma} \int_t^{t'} L(\gamma(s), \dot{\gamma}(s), s) ds,$$

where the infimum is taken over the continuous and piecewise C^1 paths $\gamma : [t, t'] \to M$ such that $\gamma(t) = x$ and $\gamma(t') = y$.

By the definition of $\tilde{T}_n$, for each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ and each $(x, \tau) \in M \times \mathbb{S}^1$, we have

$$\tilde{T}_n u(x, \tau) = \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{y\in M} \Big\{ u(y, \tau) + F_{\tau-k,\tau}(y, x) \Big\}.$$

Since the function $(y, x, \tau) \mapsto F_{\tau-k,\tau}(y, x)$ is continuous for each $n \leq k \leq 2n$, $k \in \mathbb{N}$ (see [1]), then from the compactness of M the function $(x, \tau) \mapsto \inf_{y\in M}\{u(y, \tau) +$

$F_{\tau-k,\tau}(y,x)\}$ is also continuous. Therefore, the function $(x,\tau) \mapsto \tilde{T}_n u(x,\tau)$ is continuous on $M \times \mathbb{S}^1$. □

By the periodicity of L and the above arguments, it is not difficult to check that $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ is a semigroup of operators from $C(M \times \mathbb{S}^1, \mathbb{R}^1)$ to itself.

Proposition 4.2. For given $n \in \mathbb{N}$, $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ and $(x,\tau) \in M \times \mathbb{S}^1$, there exist $n \leq k_0 \leq 2n$, $k_0 \in \mathbb{N}$ and an extremal curve $\gamma : [\tau - k_0, \tau] \to M$ such that $\gamma(\tau) = x$ and

$$\tilde{T}_n u(x,\tau) = u(\gamma(\tau - k_0), \tau) + \int_{\tau-k_0}^{\tau} L(\gamma(s), \dot{\gamma}(s), s)ds.$$

Proof. Recall that

$$\tilde{T}_n u(x,\tau) = \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{y\in M} \Big\{ u(y,\tau) + F_{\tau-k,\tau}(y,x) \Big\}.$$

For given x, τ and k, the function $y \mapsto u(y,\tau) + F_{\tau-k,\tau}(y,x)$ is continuous on M. Thus, from the compactness of M there exists $y^k \in M$ such that

$$\inf_{y\in M} \Big\{ u(y,\tau) + F_{\tau-k,\tau}(y,x) \Big\} = u(y^k,\tau) + F_{\tau-k,\tau}(y^k,x).$$

Then it is clear that there is $n \leq k_0 \leq 2n$, $k_0 \in \mathbb{N}$ such that

$$\tilde{T}_n u(x,\tau) = u(y^{k_0},\tau) + F_{\tau-k_0,\tau}(y^{k_0},x).$$

It follows from Tonelli's theorem (see, for example, [17]) that there exists a minimizing extremal curve $\gamma : [\tau - k_0, \tau] \to M$ such that $\gamma(\tau - k_0) = y^{k_0}$, $\gamma(\tau) = x$ and

$$F_{\tau-k_0,\tau}(y^{k_0},x) = \int_{\tau-k_0}^{\tau} L(\gamma(s), \dot{\gamma}(s), s)ds.$$

Hence,

$$\tilde{T}_n u(x,\tau) = u(\gamma(\tau - k_0), \tau) + \int_{\tau-k_0}^{\tau} L(\gamma(s), \dot{\gamma}(s), s)ds.$$

□

Proposition 4.3.

(a') For u, $v \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$, if $u \leq v$, then $\tilde{T}_n u \leq \tilde{T}_n v$, $\forall n \in \mathbb{N}$.

(b') If c is a constant and $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$, then $\tilde{T}_n(u + c) = \tilde{T}_n u + c$, $\forall n \in \mathbb{N}$.

(c') For each u, $v \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ and each $n \in \mathbb{N}$, $\|\tilde{T}_n u - \tilde{T}_n v\|_\infty \leq \|u - v\|_\infty$.

Proof. For each $n \in \mathbb{N}$ and each $(x,\tau) \in M \times \mathbb{S}^1$,

$$\begin{aligned}\tilde{T}_n u(x,\tau) &= \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{y\in M} \Big\{ u(y,\tau) + F_{\tau-k,\tau}(y,x) \Big\} \\ &\leq \inf_{\substack{k\in\mathbb{N} \\ n\leq k\leq 2n}} \inf_{y\in M} \Big\{ v(y,\tau) + F_{\tau-k,\tau}(y,x) \Big\} \\ &= \tilde{T}_n v(x,\tau),\end{aligned}$$

which proves (a'). (b') results from the definition of $\tilde{T}_n$ directly. To prove (c'), we notice that for each $(x,\tau) \in M \times \mathbb{S}^1$,

$$-\|u-v\|_\infty + v(x,\tau) \le u(x,\tau) \le \|u-v\|_\infty + v(x,\tau).$$

From (a') and (b') we have

$$\tilde{T}_n v(x,\tau) - \|u-v\|_\infty \le \tilde{T}_n u(x,\tau) \le \tilde{T}_n v(x,\tau) + \|u-v\|_\infty, \quad \forall (x,\tau) \in M\times\mathbb{S}^1, \forall n \in \mathbb{N}.$$

Hence, $\|\tilde{T}_n u - \tilde{T}_n v\|_\infty \le \|u-v\|_\infty$. □

4.2. **Convergence of the new L-O semigroup.** Here we deal with the convergence of the new L-O semigroup associated with L. We show that $\lim_{n\to+\infty} \tilde{T}_n u(x,\tau)$ exists for each $u \in C(M\times\mathbb{S}^1, \mathbb{R}^1)$ and each $(x,\tau) \in M\times\mathbb{S}^1$. But this is an immediate consequence of Proposition 4.5 below.

Following Mañé [16] and Mather [18], define the action potential and the extended Peierls barrier as follows.

Action Potential: for each $(\tau,\tau') \in \mathbb{S}^2$, let

$$\Phi_{\tau,\tau'}(x,x') = \inf F_{t,t'}(x,x')$$

for all $(x,x') \in M \times M$, where the infimum is taken on the set of $(t,t') \in \mathbb{R}^2$ such that $\tau = [t]$, $\tau' = [t']$ and $t' \ge t+1$.

Extended Peierls Barrier: for each $(\tau,\tau') \in \mathbb{S}^2$, let

$$h_{\tau,\tau'}(x,x') = \liminf_{t'-t\to+\infty} F_{t,t'}(x,x') \tag{4.1}$$

for all $(x,x') \in M \times M$, where the liminf is restricted to the set of $(t,t') \in \mathbb{R}^2$ such that $\tau = [t]$, $\tau' = [t']$.

From the above definitions, it is not hard to see that

$$\Phi_{\tau,\tau'}(x,x') \le h_{\tau,\tau'}(x,x'), \quad \forall (x,\tau),\ (x',\tau') \in M \times \mathbb{S}^1 \tag{4.2}$$

and

$$h_{\tau,t}(x,y) \le h_{\tau,s}(x,z) + \Phi_{s,t}(z,y), \quad \forall (x,\tau),\ (y,t),\ (z,s) \in M \times \mathbb{S}^1. \tag{4.3}$$

It can be shown that the extended Peierls barrier $h_{\tau,\tau'}$ is Lipschitz and that, the liminf in (4.1) can not always be replaced with a limit, which leads to the non-convergence of the L-O semigroup associated with L (see [8]). See [20] for more details about the action potential and the extended Peierls barrier. Before stating Proposition 4.5, we introduce the following lemma.

Lemma 4.4 (A Priori Compactness)**.** *If $t > 0$ is fixed, there exists a compact subset $\mathcal{C}_t \subset TM \times \mathbb{S}^1$ such that for each minimizing extremal curve $\gamma : [a,b] \to M$ with $b - a \ge t$, we have*

$$(\gamma(s), \dot{\gamma}(s), [s]) \in \mathcal{C}_t, \quad \forall s \in [a,b].$$

The lemma may be proved by small modifications of the proof found in [9, Corollary 4.3.2].

Proposition 4.5.

$$\lim_{n\to+\infty}\inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}} F_{\tau-k,\tau}(y,x)=h_{\tau,\tau}(y,x),\quad \forall x,y\in M,\ \forall\tau\in\mathbb{S}^1.$$

Proof. Throughout this proof we use C to denote a generic positive constant not necessarily the same in any two places. For each x, $y\in M$ and each $\tau\in\mathbb{S}^1$, by the definition of $h_{\tau,\tau}$, we have $\liminf_{k\to+\infty}F_{\tau-k,\tau}(y,x)=h_{\tau,\tau}(y,x)$. Then there exist $\{k_i\}_{i=1}^{+\infty}$ such that $k_i\to+\infty$ and $F_{\tau-k_i,\tau}(y,x)\to h_{\tau,\tau}(y,x)$ as $i\to+\infty$. Tonelli's theorem guarantees the existence of the minimizing extremal curves $\gamma_{k_i}:[\tau-k_i,\tau]\to M$ with $\gamma_{k_i}(\tau-k_i)=y$, $\gamma_{k_i}(\tau)=x$ and $A(\gamma_{k_i})=F_{\tau-k_i,\tau}(y,x)$, where

$$A(\gamma_{k_i})=\int_{\tau-k_i}^{\tau}L(\gamma_{k_i},\dot\gamma_{k_i},s)ds.$$

Thus, we have $A(\gamma_{k_i})\to h_{\tau,\tau}(y,x)$ as $i\to+\infty$. Then for every $\varepsilon>0$, there exists $I\in\mathbb{N}$ such that $|A(\gamma_{k_i})-h_{\tau,\tau}(y,x)|<\varepsilon$ if $i\ge I$, $i\in\mathbb{N}$. And it is clear that for each k_i, $(\gamma_{k_i}(s),\dot\gamma_{k_i}(s),[s]):[\tau-k_i,\tau]\to TM\times\mathbb{S}^1$ is a trajectory of the Euler-Lagrange flow.

To prove our result, it suffices to show that for $n\in\mathbb{N}$ large enough, we can find a curve $\tilde\gamma:[\tau-k_0,\tau]\to M$ with $\tilde\gamma(\tau-k_0)=y$, $\tilde\gamma(\tau)=x$, where $n\le k_0\le 2n$, $k_0\in\mathbb{N}$, such that

$$|A(\tilde\gamma)-A(\gamma_{k_I})|\le C\varepsilon$$

for some constant $C>0$. In fact, if such a curve exists, then

$$\inf_{\substack{k\in\mathbb{N}\\ n\le k}}F_{\tau-k,\tau}(y,x)\le\inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}}F_{\tau-k,\tau}(y,x)\le A(\tilde\gamma)\le A(\gamma_{k_I})+C\varepsilon\le h_{\tau,\tau}(y,x)+C\varepsilon.$$

By letting $n\to+\infty$, from the arbitrariness of $\varepsilon>0$, we have

$$\begin{aligned}h_{\tau,\tau}(y,x)&=\liminf_{k\to+\infty}F_{\tau-k,\tau}(y,x)\\&=\lim_{n\to+\infty}\inf_{\substack{k\in\mathbb{N}\\ n\le k}}F_{\tau-k,\tau}(y,x)\\&\le\lim_{n\to+\infty}\inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}}F_{\tau-k,\tau}(y,x)\\&\le h_{\tau,\tau}(y,x),\end{aligned}$$

which implies that

$$\lim_{n\to+\infty}\inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}}F_{\tau-k,\tau}(y,x)=h_{\tau,\tau}(y,x).$$

Our task is now to construct the curve mentioned above. Note that for the above $\varepsilon>0$, there exists $I'\in\mathbb{N}$ such that there exists

$$(z_{k_i},v_{z_{k_i}},t_{z_{k_i}})\in O_i:=\{(\gamma_{k_i}(s),\dot\gamma_{k_i}(s),[s])\mid\tau-k_i\le s\le\tau\}\subset TM\times\mathbb{S}^1$$

such that

$$d((z_{k_i},v_{z_{k_i}},t_{z_{k_i}}),\tilde{\mathcal{M}}_0)<\varepsilon,$$

if $i \geq I'$, where $\tilde{\mathcal{M}}_0$ is the Mather set of cohomology class 0. As usual, distance is measured with respect to smooth Riemannian metrics. Since $\tilde{\mathcal{M}}_0$ is compact and by the a priori compactness given by Lemma 4.4, O_i is contained in the compact subset $\mathcal{C}_{k_{I'}}$ of $TM \times \mathbb{S}^1$ for each $i \geq I'$, then it doesn't matter which Riemannian metrics we choose to measure distance.

Let $I = \max\{I, I'\}$. Then $|A(\gamma_{k_I}) - h_{\tau,\tau}(y,x)| < \varepsilon$ and there exists $(z_0, v_{z_0}, t_{z_0}) \in O_I = \{(\gamma_{k_I}(s), \dot{\gamma}_{k_I}(s), [s]) \mid \tau - k_I \leq s \leq \tau\}$ such that

$$d((z_0, v_{z_0}, t_{z_0}), \tilde{\mathcal{M}}_0) < \varepsilon. \tag{4.4}$$

In view of (4.4), there exists an ergodic minimal measure μ_e on $TM \times \mathbb{S}^1$ [17] such that $\mu_e(\mathrm{supp}\mu_e \cap B_{2\varepsilon}(z_0, v_{z_0}, t_{z_0})) = \Delta > 0$, where $B_{2\varepsilon}(z_0, v_{z_0}, t_{z_0})$ denotes the open ball of radius 2ε centered on (z_0, v_{z_0}, t_{z_0}) in $TM \times \mathbb{S}^1$. Set $A_{2\varepsilon} = \mathrm{supp}\mu_e \cap B_{2\varepsilon}(z_0, v_{z_0}, t_{z_0})$. Since μ_e is an ergodic measure, then

$$\mu_e(\bigcup_{t=1}^{+\infty} \phi^L_{-t}(A_{2\varepsilon})) = 1.$$

Thus, for any $0 < \Delta' < \Delta$, there exists $T > 0$ such that

$$\mu_e(\bigcup_{t=1}^{T'} \phi^L_{-t}(A_{2\varepsilon})) \geq 1 - \Delta',$$

if $T' \geq T$. From this, we may deduce that for each $n \in \mathbb{N}$,

$$\Big(\bigcup_{t=1}^{T} \phi^L_{-t}(A_{2\varepsilon})\Big) \cap \phi^L_n(A_{2\varepsilon}) \neq \emptyset. \tag{4.5}$$

For, otherwise, there would be $n_0 \in \mathbb{N}$ such that

$$0 = \mu_e\Big(\bigcup_{t=1}^{T} \phi^L_{-t}(A_{2\varepsilon})\Big) + \mu_e(\phi^L_{n_0}(A_{2\varepsilon})) \geq 1 - \Delta' + \Delta > 1,$$

which is a contradiction.

For a given $n \in \mathbb{N}$ large enough with $\max\{k_I, T\} \leq \{\frac{n}{2}\}$, from (4.5) there exist (e_0, v_{e_0}, t_{e_0}), $(\bar{e}_0, v_{\bar{e}_0}, t_{\bar{e}_0}) \in A_{2\varepsilon}$ and $1 \leq t \leq T$ such that

$$\phi^L_{-t}(e_0, v_{e_0}, t_{e_0}) = (e, v_e, t_e) = \phi^L_n(\bar{e}_0, v_{\bar{e}_0}, t_{\bar{e}_0}) \tag{4.6}$$

for some $(e, v_e, t_e) \in \tilde{\mathcal{M}}_0$. Since $(e_0, v_{e_0}, t_{e_0}) \in A_{2\varepsilon}$, then

$$d((e_0, v_{e_0}, t_{e_0}), (z_0, v_{z_0}, t_{z_0})) < 2\varepsilon. \tag{4.7}$$

Thus, $|t_{e_0} - t_{z_0}| < 2\varepsilon$. Without loss of generality, we assume $t_{e_0} \geq t_{z_0}$ (As mentioned in Section 2, we view the unit circle $\mathbb{S}^1$ as a fundamental domain in $\mathbb{R}^1$, $[0,1]$ with opposite faces identified). The case $t_{e_0} < t_{z_0}$ can be treated similarly. Then $0 \leq t_{e_0} - t_{z_0} < 2\varepsilon$. Set $(z_1, v_{z_1}, t_{z_1}) = \phi^L_{t_{e_0} - t_{z_0}}(z_0, v_{z_0}, t_{z_0})$. Then $t_{z_1} = t_{e_0}$ and from (4.7) we have

$$d((e_0, v_{e_0}, t_{e_0}), (z_1, v_{z_1}, t_{z_1})) < C\varepsilon \tag{4.8}$$

for some constant $C > 0$. Note that t_{e_0}, $\tau \in \mathbb{S}^1$ and thus $|\tau - t_{e_0}| \leq 1$. Without loss of generality, we assume $t_{e_0} \leq \tau$. Let $(z_2, v_{z_2}, \tau) = \phi^L_{\tau - t_{e_0}}(z_1, v_{z_1}, t_{e_0})$ and $(e_1, v_{e_1}, \tau) = \phi^L_{\tau - t_{e_0}}(e_0, v_{e_0}, t_{e_0})$. Then by the differentiability of the solutions of the Euler-Lagrange equation with respect to initial values, we have

$$d((e_1, v_{e_1}, \tau), (z_2, v_{z_2}, \tau)) < C\varepsilon \tag{4.9}$$

for some constant $C > 0$.

Since (e_0, v_{e_0}, t_{e_0}), $(\bar{e}_0, v_{\bar{e}_0}, t_{\bar{e}_0}) \in A_{2\varepsilon}$, then

$$d((e_0, v_{e_0}, t_{e_0}), (\bar{e}_0, v_{\bar{e}_0}, t_{\bar{e}_0})) < 4\varepsilon, \tag{4.10}$$

from which, without loss of generality, we assume $0 \leq t_{e_0} - t_{\bar{e}_0} < 4\varepsilon$. Set $(\bar{e}_1, v_{\bar{e}_1}, t_{e_0}) = \phi^L_{t_{e_0} - t_{\bar{e}_0}}(\bar{e}_0, v_{\bar{e}_0}, t_{\bar{e}_0})$. Then from (4.10) we have

$$d((e_0, v_{e_0}, t_{e_0}), (\bar{e}_1, v_{\bar{e}_1}, t_{e_0})) < C\varepsilon \tag{4.11}$$

for some constant $C > 0$. Set $(\bar{e}_2, v_{\bar{e}_2}, \tau) = \phi^L_{\tau - t_{e_0}}(\bar{e}_1, v_{\bar{e}_1}, t_{e_0})$. Recall that $(e_1, v_{e_1}, \tau) = \phi^L_{\tau - t_{e_0}}(e_0, v_{e_0}, t_{e_0})$. Then from the differentiability of the solutions of the Euler-Lagrange equation with respect to initial values, we have

$$d((e_1, v_{e_1}, \tau), (\bar{e}_2, v_{\bar{e}_2}, \tau) < C\varepsilon \tag{4.12}$$

for some constant $C > 0$.

Note that since $(z_0, v_{z_0}, t_{z_0}) \in O_I = \{(\gamma_{k_I}(s), \dot{\gamma}_{k_I}(s), [s]) \mid \tau - k_I \leq s \leq \tau\}$, where O_I is an orbit of the Euler-Lagrange flow, then $(z_2, v_{z_2}, \tau) \in O_I$. And thus, there exists $k_{I_2} \in \mathbb{N}$ with $k_{I_1} + k_{I_2} = k_I$ such that

$$(z_2, v_{z_2}, \tau) = (\gamma_{k_I}(\tau - k_{I_2}), \dot{\gamma}_{k_I}(\tau - k_{I_2}), \tau).$$

We are now in a position to construct the curve we needed. We treat the case $k_{I_1} \neq 0$, $k_{I_2} \neq 0$ and the remaining cases can be treated similarly. Let $\alpha_3 : [\tau - k_{I_2}, \tau] \to M$ with $\alpha_3(\tau - k_{I_2}) = e_1$ and $\alpha_3(\tau) = x$ be a Tonelli minimizer such that $A(\alpha_3) = \int_{\tau - k_{I_2}}^{\tau} L(\alpha_3, \dot{\alpha}_3, s)ds = F_{\tau - k_{I_2}, \tau}(e_1, x)$. Since $\gamma_{k_I} : [\tau - k_I, \tau] \to M$ is a minimizing extremal curve, then $\gamma_{k_I}|_{[\tau - k_{I_2}, \tau]}$ is also a minimizing extremal curve and thus $A(\gamma_{k_I}|_{[\tau - k_{I_2}, \tau]}) = F_{\tau - k_{I_2}, \tau}(z_2, x)$. Therefore, from the Lipschtiz property of the function $F_{\tau - k_{I_2}, \tau}$ (see, for example, [1]) and (4.9) we have

$$|A(\alpha_3) - A(\gamma_{k_I}|_{[\tau - k_{I_2}, \tau]})| = |F_{\tau - k_{I_2}, \tau}(e_1, x) - F_{\tau - k_{I_2}, \tau}(z_2, x)| \leq Dd(e_1, z_2) \leq C\varepsilon \tag{4.13}$$

for some constant $C > 0$, where $D > 0$ is a Lipschitz constant of F_{t_1, t_2} which is independent of t_1, t_2 with $t_1 + 1 \leq t_2$.

Let $\beta(s) = p\phi^L_{s - (\tau - k_{I_2})}(e_1, v_{e_1}, \tau)$, $\forall s \in \mathbb{R}^1$, where $p : TM \times \mathbb{S}^1 \to M$ denotes the projection. Then $(\beta(s), \dot{\beta}(s), [s]) = \phi^L_{s - (\tau - k_{I_2})}(e_1, v_{e_1}, \tau)$, $\forall s \in \mathbb{R}^1$, and $(\beta(\tau - k_{I_2}), \dot{\beta}(\tau - k_{I_2})) = (e_1, v_{e_1})$. Hence , from (4.6) we have

$(\beta(\tau-k_{I_2}-(\tau-t_{e_0})-t),\dot{\beta}(\tau-k_{I_2}-(\tau-t_{e_0})-t),[\tau-k_{I_2}-(\tau-t_{e_0})-t])=(e,v_e,t_e)$, and

$$(\bar{e}_2,v_{\bar{e}_2},\tau)=(\beta(l),\dot{\beta}(l),[l])=(\beta(l'),\dot{\beta}(l'),[l']),$$

where $l=\tau-k_{I_2}-(\tau-t_{e_0})-t-n+(t_{e_0}-t_{\bar{e}_0})+(\tau-t_{e_0})$ and $l'=\tau-k_{I_2}-t-n+t_{e_0}-t_{\bar{e}_0}$. Then $[l']=[\tau-k_{I_2}-t-n+t_{e_0}-t_{\bar{e}_0}]=[\tau-t+t_{e_0}-t_{\bar{e}_0}]=\tau$, which means that $t-(t_{e_0}-t_{\bar{e}_0})\in\mathbb{Z}$. Hence, we have $0\le t-(t_{e_0}-t_{\bar{e}_0})\le T-(t_{e_0}-t_{\bar{e}_0})\le\{\frac{n}{2}\}$. Furthermore,

$$n\le k_I+n+t-(t_{e_0}-t_{\bar{e}_0})\le k_I+n+\{\frac{n}{2}\}\le 2n. \tag{4.14}$$

Let $m=n+t-(t_{e_0}-t_{\bar{e}_0})\in\mathbb{Z}$ and $\alpha_2=\beta|_{[\tau-k_{I_2}-m,\tau-k_{I_2}]}$. Then $\alpha_2(\tau-k_{I_2}-m)=\bar{e}_2$ and $\alpha_2(\tau-k_{I_2})=e_1$. In view of $(e_0,v_{e_0},t_{e_0})\in A_{2\varepsilon}\subset\tilde{\mathcal{M}}_0$ and the definitions of β and α_2, $(\alpha_2(s),\dot{\alpha}_2(s),[s])$ is a trajectory of the Euler-Lagrange flow in $\tilde{\mathcal{M}}_0$. Moreover, according to [17, Proposition 3] and the definition of $h_{\tau,\tau}$, we have

$$A(\alpha_2)=F_{\tau-k_{I_2}-m,\tau-k_{I_2}}(\bar{e}_2,e_1)=h_{\tau,\tau}(\bar{e}_2,e_1).$$

Hence, on account of the Lipschitz property of $h_{\tau,\tau}$ and (4.12),

$$|A(\alpha_2)-h_{\tau,\tau}(e_1,e_1)|=|h_{\tau,\tau}(\bar{e}_2,e_1)-h_{\tau,\tau}(e_1,e_1)|\le\bar{D}d(\bar{e}_2,e_1)\le C\varepsilon$$

for some constant $C>0$, where $\bar{D}$ is a Lipschitz constant of $h_{\tau,\tau}$. Since $(e_1,\tau)\in\mathcal{M}_0$, where $\mathcal{M}_0\subset M\times\mathbb{S}^1$ is the projected Mather set, then $h_{\tau,\tau}(e_1,e_1)=0$, and thus

$$|A(\alpha_2)|\le C\varepsilon. \tag{4.15}$$

Let $\alpha_1:[\tau-k_I-m,\tau-k_{I_2}-m]\to M$ with $\alpha_1(\tau-k_I-m)=y$ and $\alpha_1(\tau-k_{I_2}-m)=\bar{e}_2$ be a Tonelli minimizer such that

$$A(\alpha_1)=F_{\tau-k_I-m,\tau-k_{I_2}-m}(y,\bar{e}_2).$$

Since $\gamma_{k_I}:[\tau-k_I,\tau]\to M$ is a minimizing extremal curve, then $\gamma_{k_I}|_{[\tau-k_I,\tau-k_{I_2}]}$ is also a minimizing extremal curve and thus

$$A(\gamma_{k_I}|_{[\tau-k_I,\tau-k_{I_2}]})=F_{\tau-k_I,\tau-k_{I_2}}(y,z_2)=F_{\tau-k_I-m,\tau-k_{I_2}-m}(y,z_2).$$

Therefore, from the Lipschitz property of $F_{\tau-k_I-m,\tau-k_{I_2}-m}$, (4.9) and (4.12), we have

$$\begin{aligned}|A(\alpha_1)-A(\gamma_{k_I}|_{[\tau-k_I,\tau-k_{I_2}]})|&=|F_{\tau-k_I-m,\tau-k_{I_2}-m}(y,\bar{e}_2)-F_{\tau-k_I-m,\tau-k_{I_2}-m}(y,z_2)|\\&\le Dd(\bar{e}_2,z_2)\\&\le C\varepsilon\end{aligned} \tag{4.16}$$

for some constant $C>0$.

Consider the curve $\tilde{\gamma}:[\tau-k_I-m,\tau]\to M$ connecting y and x defined by

$$\tilde{\gamma}(s)=\begin{cases}\alpha_1(s), & s\in[\tau-k_I-m,\tau-k_{I_2}-m],\\ \alpha_2(s), & s\in[\tau-k_{I_2}-m,\tau-k_{I_2}],\\ \alpha_3(s), & s\in[\tau-k_{I_2},\tau].\end{cases}$$

By (4.14), $n\leq k_0:=k_I+m\leq 2n$. From (4.13), (4.15) and (4.16), we have

$$|A(\tilde{\gamma})-A(\gamma_{k_I})|\leq C\varepsilon$$

for some constant $C>0$. Hence, $\tilde{\gamma}$ is just the curve we needed. This completes the proof of the proposition.

□

From the definition of $\tilde{T}_n$, for each $u\in C(M\times\mathbb{S}^1,\mathbb{R}^1)$ and each $(x,\tau)\in M\times\mathbb{S}^1$, we have

$$\tilde{T}_n u(x,\tau)=\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}\inf_{y\in M}\{u(y,\tau)+F_{\tau-k,\tau}(y,x)\}.$$

Notice that $y\mapsto F_{\tau-k,\tau}(y,x)$ is a Lipschitz function on M for every $k\in\mathbb{N}$, and the Lipschitz constant D is independent of k. Hence for each $n\in\mathbb{N}$, the function $y\mapsto\inf_{n\leq k\leq 2n}F_{\tau-k,\tau}(y,x)$ is also Lipschitz with the Lipschitz constant D, and thus $\{\inf_{n\leq k\leq 2n}F_{\tau-k,\tau}(\cdot,x)\}_{n=1}^{+\infty}$ are equicontinuous. In view of Proposition 4.5, we have

$$\lim_{n\to+\infty}\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}F_{\tau-k,\tau}(y,x)=h_{\tau,\tau}(y,x),\quad\forall y\in M.$$

Then $\{\inf_{n\leq k\leq 2n}F_{\tau-k,\tau}(\cdot,x)\}_{n=1}^{+\infty}$ converges uniformly to $h_{\tau,\tau}(\cdot,x)$ as $n\to+\infty$. Let

$$E^n=\inf_{y\in M}\{u(y,\tau)+\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}F_{\tau-k,\tau}(y,x)\}.$$

Note that

$$E^n=\inf_{y\in M}\{u(y,\tau)+h_{\tau,\tau}(y,x)+\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}F_{\tau-k,\tau}(y,x)-h_{\tau,\tau}(y,x)\},$$

$$\inf_{y\in M}\{u(y,\tau)+h_{\tau,\tau}(y,x)\}-\sup_{y\in M}|\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}F_{\tau-k,\tau}(y,x)-h_{\tau,\tau}(y,x)|\leq E^n,$$

and

$$E^n\leq\inf_{y\in M}\{u(y,\tau)+h_{\tau,\tau}(y,x)\}+\sup_{y\in M}|\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}F_{\tau-k,\tau}(y,x)-h_{\tau,\tau}(y,x)|.$$

By letting $n\to+\infty$, we have

$$\inf_{y\in M}\{u(y,\tau)+h_{\tau,\tau}(y,x)\}=\lim_{n\to+\infty}E^n=\lim_{n\to+\infty}\inf_{y\in M}\{u(y,\tau)+\inf_{\substack{k\in\mathbb{N}\\ n\leq k\leq 2n}}F_{\tau-k,\tau}(y,x)\}.$$

Since

$$\inf_{y\in M}\{u(y,\tau)+\inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}} F_{\tau-k,\tau}(y,x)\} = \inf_{y\in M}\inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}}\{u(y,\tau)+F_{\tau-k,\tau}(y,x)\}$$
$$= \inf_{\substack{k\in\mathbb{N}\\ n\le k\le 2n}}\inf_{y\in M}\{u(y,\tau)+F_{\tau-k,\tau}(y,x)\}$$
$$= \tilde{T}_n u(x,\tau),$$

then we have

$$\lim_{n\to+\infty}\tilde{T}_n u(x,\tau) = \inf_{y\in M}\{u(y,\tau)+h_{\tau,\tau}(y,x)\},$$

thus proving the first assertion of Theorem 1.5.

4.3. **The limit $\lim_{n\to+\infty}\tilde{T}_n u$ and backward weak KAM solutions.** Here we discuss the relation between the backward weak KAM solution and the limit $\lim_{n\to+\infty}\tilde{T}_n u$. Following Fathi [6], as done by Contreras et al. in [4], we give the definition of the backward weak KAM solution as follows.

Definition 4.6. A backward weak KAM solution of the Hamilton-Jacobi equation (1.6) is a function $u : M\times\mathbb{S}^1\to\mathbb{R}^1$ such that

(1) u is dominated by L, i.e.,
$$u(x,\tau)-u(y,s)\le\Phi_{s,\tau}(y,x),\quad \forall (x,\tau),\ (y,s)\in M\times\mathbb{S}^1.$$
We use the notation $u\prec L$.

(2) For every $(x,\tau)\in M\times\mathbb{S}^1$ there exists a curve $\gamma:(-\infty,\tilde{\tau}]\to M$ with $\gamma(\tilde{\tau})=x$ and $[\tilde{\tau}]=\tau$ such that
$$u(x,\tau)-u(\gamma(t),[t])=\int_t^{\tilde{\tau}}L(\gamma(s),\dot{\gamma}(s),s)ds,\quad \forall t\in(-\infty,\tilde{\tau}].$$

We denote by $\mathcal{S}_-$ the set of backward weak KAM solutions. Let us recall two known results [4] on backward weak KAM solutions, which will be used later in the paper.

Lemma 4.7. *Given a fixed $(y,s)\in M\times\mathbb{S}^1$, the function*

$$(x,\tau)\mapsto h_{s,\tau}(y,x),\quad (x,\tau)\in M\times\mathbb{S}^1$$

is a backward weak KAM solution.

Lemma 4.8. *If $\mathcal{U}\subset\mathcal{S}_-$, let $\underline{u}(x,\tau):=\inf_{u\in\mathcal{U}}u(x,\tau)$ then either $\underline{u}\equiv-\infty$ or $\underline{u}\in\mathcal{S}_-$.*

We define the projected Aubry set $\mathcal{A}_0$ as follows:

$$\mathcal{A}_0:=\{(x,\tau)\in M\times\mathbb{S}^1 \mid h_{\tau,\tau}(x,x)=0\}.$$

Note that $\mathcal{A}_0=\Pi\tilde{\mathcal{A}}_0$, where $\Pi: TM\times\mathbb{S}^1\to M\times\mathbb{S}^1$ denotes the projection and $\tilde{\mathcal{A}}_0$ denotes the Aubry set in $TM\times\mathbb{S}^1$, i.e., the union of global static orbits. See for instance [1] for the definition of static orbits and more details on $\tilde{\mathcal{A}}_0$.

From the definition of $\mathcal{A}_0$, (4.2) and (4.3), it is straightforward to show that if $(x,\tau)\in\mathcal{A}_0$, then

$$h_{\tau,s}(x,y)=\Phi_{\tau,s}(x,y) \tag{4.17}$$

for all $(y,s) \in M \times \mathbb{S}^1$. Define an equivalence relation on $\mathcal{A}_0$ by saying that (x,τ) and (y,s) are equivalent if and only if

$$\Phi_{\tau,s}(x,y) + \Phi_{s,\tau}(y,x) = 0. \tag{4.18}$$

By (4.17), it is simple to see that (4.18) is equivalent to

$$h_{\tau,s}(x,y) + h_{s,\tau}(y,x) = 0.$$

The equivalent classes of this relation are called static classes. Let A be the set of static classes. For each static class $\Gamma \in$ A choose a point $(x,\tau) \in \Gamma$ and let $\mathbb{A}$ be the set of such points.

Contreras et al. [4] characterize backward weak KAM solutions of the Hamilton-Jacobi equation (1.6) in terms of their values at each static class and the extended Peierls barrier. See [3] for similar results in the time-independent case.

Theorem 4.9 (Contreras et al. [4])**.** *The map* $\{f : \mathbb{A} \to \mathbb{R}^1 \mid f \prec L\} \to \mathcal{S}_-$

$$f \mapsto u_f(x,\tau) = \min_{(p,s)\in\mathbb{A}} (f(p,s) + h_{s,\tau}(p,x))$$

is a bijection.

Now we state the last proposition of this paper.

Proposition 4.10.

$$\{\bar{u} \in C(M \times \mathbb{S}^1, \mathbb{R}^1) \mid \exists u \in C(M \times \mathbb{S}^1, \mathbb{R}^1),\ u \prec L,\ \bar{u} = \lim_{n\to+\infty} \tilde{T}_n u\} = \mathcal{S}_-.$$

Remark 4.11. Proposition 4.10 tells us two things: (i) For each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ with $u \prec L$, $\bar{u} = \lim_{n\to+\infty} \tilde{T}_n u$ is a backward weak KAM solution of (1.6), which proves the second assertion of Theorem 1.5. (ii) For each $w \in \mathcal{S}_-$ there exists $\tilde{w} \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ with $\tilde{w} \prec L$ such that $w = \lim_{n\to+\infty} \tilde{T}_n \tilde{w}$. Moreover, we know from the proof of Proposition 4.10 that $\tilde{w} = w$.

For the proof of Proposition 4.10, we need the following two lemmas.

Lemma 4.12. *For each* $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ *with* $u \prec L$*, if* (p,s) *and* (q,τ) *are in the same static class, then*

$$u(p,s) + h_{s,t}(p,x) = u(q,\tau) + h_{\tau,t}(q,x), \quad \forall (x,t) \in M \times \mathbb{S}^1.$$

Proof. By (4.3) and (4.17), for each $(x,t) \in M \times \mathbb{S}^1$ we have

$$\begin{aligned} h_{\tau,t}(q,x) &\le h_{\tau,s}(q,p) + \Phi_{s,t}(p,x) \\ &= h_{\tau,s}(q,p) + h_{s,t}(p,x) \\ &\le h_{\tau,s}(q,p) + h_{s,\tau}(p,q) + \Phi_{\tau,t}(q,x) \\ &= h_{\tau,t}(q,x). \end{aligned}$$

Hence, $h_{\tau,t}(q,x) = h_{\tau,s}(q,p) + h_{s,t}(p,x)$. Since $u \prec L$, then

$$u(p,s) - u(q,\tau) \le \Phi_{\tau,s}(q,p) = h_{\tau,s}(q,p), \tag{4.19}$$

and

$$u(q,\tau) - u(p,s) \le \Phi_{s,\tau}(p,q) = h_{s,\tau}(p,q). \tag{4.20}$$

From (4.19) and (4.20) we have

$$\begin{aligned} u(p,s) &\le u(q,\tau) + h_{\tau,s}(q,p) \\ &\le u(p,s) + h_{s,\tau}(p,q) + h_{\tau,s}(q,p) \\ &= u(p,s). \end{aligned}$$

Hence, $u(p,s) = u(q,\tau) + h_{\tau,s}(q,p)$. Therefore, we have

$$\begin{aligned} u(q,\tau) + h_{\tau,t}(q,x) &= u(q,\tau) + h_{\tau,s}(q,p) + h_{s,t}(p,x) \\ &= u(p,s) + h_{s,t}(p,x). \end{aligned}$$

□

Lemma 4.13. *For each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ with $u \prec L$, if $(p,s) \in \mathcal{A}_0$, then*

$$u(p,s) = \inf_{y \in M} (u(y,\tau) + h_{\tau,s}(y,p)), \quad \forall \tau \in \mathbb{S}^1.$$

Proof. Since $u \prec L$, then for each $(y,\tau) \in M \times \mathbb{S}^1$ we have

$$u(p,s) - u(y,\tau) \le \Phi_{\tau,s}(y,p) \le h_{\tau,s}(y,p)$$

and thus

$$u(p,s) \le \inf_{y \in M} (u(y,\tau) + h_{\tau,s}(y,p)), \quad \forall \tau \in \mathbb{S}^1.$$

It suffices to prove that for each $\tau \in \mathbb{S}^1$, there exists $y_\tau \in M$ such that

$$u(p,s) = u(y_\tau,\tau) + h_{\tau,s}(y_\tau,p).$$

Let $(p, v_p, s) = \Pi^{-1}(p,s) \in \tilde{\mathcal{A}}_0$ and $(\gamma(t), \dot{\gamma}(t), [t]) = \phi^L_{t-s}(p, v_p, s)$, $t \in \mathbb{R}^1$ be a trajectory of the Euler-Lagrange flow. Set $(y_\tau, \tau) = (\gamma(\tau), \tau)$, then (p,s) and (y_τ,τ) are in the same static class, i.e.,

$$h_{s,\tau}(p,y_\tau) + h_{\tau,s}(y_\tau,p) = 0. \tag{4.21}$$

Since $u \prec L$, then

$$u(p,s) - u(y_\tau,\tau) \le \Phi_{\tau,s}(y_\tau,p) = h_{\tau,s}(y_\tau,p) \tag{4.22}$$

and

$$u(y_\tau,\tau) - u(p,s) \le \Phi_{s,\tau}(p,y_\tau) = h_{s,\tau}(p,y_\tau). \tag{4.23}$$

By (4.21) and (4.23), we have

$$u(p,s) - u(y_\tau,\tau) \ge -h_{s,\tau}(p,y_\tau) = h_{\tau,s}(y_\tau,p). \tag{4.24}$$

Combining (4.22) and (4.24) yields

$$u(p, s) = u(y_\tau, \tau) + h_{\tau,s}(y_\tau, p).$$

□

Proof of Proposition 4.10. First we show that for each $u \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ with $u \prec L$, $\bar{u} = \lim_{n\to+\infty} \tilde{T}_n u$ is a backward weak KAM solution of (1.6). For each $(x, \tau) \in M \times \mathbb{S}^1$, from Proposition 4.5 we have

$$\begin{aligned} \bar{u}(x, \tau) &= \inf_{y\in M} (u(y, \tau) + h_{\tau,\tau}(y, x)) \\ &= \inf_{y\in M} \big(u(y, \tau) + \min_{(p,s)\in\mathcal{A}_0} (h_{\tau,s}(y, p) + h_{s,\tau}(p, x))\big) \\ &= \inf_{y\in M} \min_{(p,s)\in\mathcal{A}_0} \big(u(y, \tau) + h_{\tau,s}(y, p) + h_{s,\tau}(p, x)\big) \\ &= \min_{(p,s)\in\mathcal{A}_0} \Big(\inf_{y\in M} \big(u(y, \tau) + h_{\tau,s}(y, p)\big) + h_{s,\tau}(p, x)\Big). \end{aligned}$$

In view of Lemma 4.13, we have

$$\bar{u}(x, \tau) = \min_{(p,s)\in\mathcal{A}_0} \big(u(p, s) + h_{s,\tau}(p, x)\big). \tag{4.25}$$

Combing Lemmas 4.7 and 4.8 we get that the function $(x, \tau) \mapsto \bar{u}(x, \tau)$ is a backward weak KAM solution.

Then we show that for each $w \in \mathcal{S}_-$, there exists $\tilde{w} \in C(M\times\mathbb{S}^1, \mathbb{R}^1)$ with $\tilde{w} \prec L$ such that $w = \lim_{n\to+\infty} \tilde{T}_n\tilde{w}$. From Theorem 4.9 there exists $f : \mathbb{A} \to \mathbb{R}^1$ with $f \prec L$ such that $w(x, \tau) = \min_{(p,s)\in\mathbb{A}}(f(p, s) + h_{s,\tau}(p, x))$ for all $(x, \tau) \in M \times \mathbb{S}^1$ and in fact, $f = w|_{\mathbb{A}}$ (see [4] for details). Hence, for each $(x, \tau) \in M\times\mathbb{S}^1$, $w(x, \tau) = \min_{(p,s)\in\mathbb{A}}(w(p, s) + h_{s,\tau}(p, x))$. By Lemma 4.12, we have

$$w(x, \tau) = \min_{(p,s)\in\mathcal{A}_0} (w(p, s) + h_{s,\tau}(p, x)).$$

Then according to Lemma 4.13, we obtain

$$\begin{aligned} w(x, \tau) &= \min_{(p,s)\in\mathcal{A}_0} \Big(\inf_{y\in M} (w(y, \tau) + h_{\tau,s}(y, p)) + h_{s,\tau}(p, x)\Big) \\ &= \inf_{y\in M} \min_{(p,s)\in\mathcal{A}_0} (w(y, \tau) + h_{\tau,s}(y, p) + h_{s,\tau}(p, x)) \\ &= \inf_{y\in M} \Big(w(y, \tau) + \min_{(p,s)\in\mathcal{A}_0} (h_{\tau,s}(y, p) + h_{s,\tau}(p, x))\Big) \\ &= \inf_{y\in M} (w(y, \tau) + h_{\tau,\tau}(y, x)) \\ &= \lim_{n\to+\infty} \tilde{T}_n w(x, \tau). \end{aligned}$$

The proof is now complete. □

Remark 4.14. Let $u \in C(M\times\mathbb{S}^1, \mathbb{R}^1)$ with $u \prec L$. Then we obtain a backward weak KAM solution $\bar{u} = \lim_{n\to+\infty} \tilde{T}_n u$ immediately from Proposition 4.10. Moreover, from (4.25) it is not hard to show that

$$\bar{u}|_{\mathcal{A}_0} = u|_{\mathcal{A}_0}. \tag{4.26}$$

Let $v \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$ with $v \prec L$. If $u|_{\mathcal{A}_0} \neq v|_{\mathcal{A}_0}$, then by (4.26) $\bar{u} \neq \bar{v}$. If $u|_{\mathcal{A}_0} = v|_{\mathcal{A}_0}$, then from (4.25) $\bar{u} = \bar{v}$. Based on the above arguments, we define an equivalence relation on the set $\{u \in C(M \times \mathbb{S}^1, \mathbb{R}^1) \mid u \prec L\}$ by saying that u and v are equivalent if and only if $u|_{\mathcal{A}_0} = v|_{\mathcal{A}_0}$. Let F be the set of the equivalent classes. For each equivalent class $\Lambda \in \mathrm{F}$ choose a function $u \in \Lambda$ and let $\mathbb{F}$ be the set of such functions. Then in view of Proposition 4.10 we have

$$\{\bar{u} \in C(M \times \mathbb{S}^1, \mathbb{R}^1) \mid \exists u \in \mathbb{F},\ \bar{u} = \lim_{n \to +\infty} \tilde{T}_n u\} = \mathcal{S}_-.$$

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[1] School of Mathematical Sciences and Key Lab of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, China

[2] College of Mathematics, Jilin University, Changchun 130012, China
E-mail address: kaizhiwang@163.com; yanjun@fudan.edu.cn