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INVARIANT SUBALGEBRAS OF AFFINE VERTEX ALGEBRAS

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Dedicated to my father Michael A. Linshaw, M. D., on the occasion of his 70th birthday

ABSTRACT. Let $\mathfrak{g}$ be a simple Lie algebra of dimension n over $\mathbb{C}$, and let $V_k(\mathfrak{g})$ be the universal affine vertex algebra at level k. Suppose that G is a reductive group of automorphisms of $V_k(\mathfrak{g})$ for all $k \in \mathbb{C}$, and let $V_k(\mathfrak{g})^G$ denote the invariant subalgebra. Assuming a conjecture concerning a certain invariant subalgebra of the Heisenberg vertex algebra of rank n, we show that $V_k(\mathfrak{g})^G$ is strongly finitely generated for generic values of k. Our conjecture is true for $n = 3$, so the strong finite generation of $V_k(\mathfrak{g})^G$ for generic k holds in the case $\mathfrak{g} = \mathfrak{sl}_2$, for any G. This is the first result of its kind.

CONTENTS

1. INTRODUCTION

We call a vertex algebra $\mathcal{V}$ *strongly finitely generated* if there exists a finite set of generators such that the collection of iterated Wick products of the generators and their derivatives spans $\mathcal{V}$. This property has several important consequences, and in particular implies that both Zhu's associative algebra $A(\mathcal{V})$, and Zhu's commutative algebra $\mathcal{V}/C_2(\mathcal{V})$, are finitely generated. In recent work, we have investigated the strong finite generation of invariant vertex algebras $\mathcal{V}^G$, where $\mathcal{V}$ is simple and G is some reductive group of automorphisms of $\mathcal{V}$. This is a subtle and essentially "quantum" phenomenon, and is generally destroyed by passing to the classical limit before taking invariants. Often, $\mathcal{V}$ admits a G-invariant filtration for which $gr(\mathcal{V})$ is a commutative algebra with a derivation (i.e., an abelian vertex algebra), and the classical limit $gr(\mathcal{V}^G)$ is isomorphic to $(gr(\mathcal{V}))^G$ as a commutative algebra. Unlike $\mathcal{V}^G$, $gr(\mathcal{V}^G)$ is generally not finitely generated as a vertex algebra, and a presentation will require both infinitely many generators and infinitely many relations.

Key words and phrases. Invariant theory, vertex algebra, current algebra, orbifold construction, strong finite generation, $\mathcal{W}$-algebra.

Isolated examples of this phenomenon have been known for some years (see for example [BFH][EFH][DN][KWY]), although the first general results of this kind were obtained by the author in [LII], in the case where $\mathcal{V}$ is the $\beta\gamma$-system $\mathcal{S}(V)$ associated to the vector space $V = \mathbb{C}^n$. The full automorphism group of $\mathcal{S}(V)$ preserving a natural conformal structure is GL_n. Our approach was to first establish the strong finite generation of $\mathcal{S}(V)^{GL_n}$, which was shown by Kac-Radul in [KR] to be isomorphic to the vertex algebra $\mathcal{W}_{1+\infty}$ with central charge $-n$. In [LI] we showed that $\mathcal{W}_{1+\infty,-n}$ has a minimal strong generating set consisting of n^2+2n elements, and in particular is a $\mathcal{W}$-algebra of type $\mathcal{W}(1,2,3,\dots,n^2+2n)$. For an arbitrary reductive group G, $\mathcal{S}(V)^G$ decomposes as a direct sum of irreducible, highest-weight $\mathcal{W}_{1+\infty,-n}$-modules. The strong finite generation of $\mathcal{W}_{1+\infty,-n}$ implies a certain finiteness property of the $\mathcal{W}_{1+\infty,-n}$-submodules appearing in $\mathcal{S}(V)^G$. This property, together with some ideas from classical invariant theory, yielded the strong finite generation of $\mathcal{S}(V)^G$. Using the same approach, we also proved in [LII] that invariant subalgebras of bc-systems and $bc\beta\gamma$-systems are strongly finitely generated.

In [LIII] we attempted to carry out a similar study of the invariant subalgebras of the rank n Heisenberg vertex algebra $\mathcal{H}(n)$. The full automorphism group of $\mathcal{H}(n)$ preserving a natural conformal structure is the orthogonal group $O(n)$. Motivated by classical invariant theory, we conjectured that $\mathcal{H}(n)^{O(n)}$ has a minimal strong generating set consisting of a vertex operator in each even weight $2,4,6,\dots,n^2+3n$. In other words, it is a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,n^2+3n)$. For $n=1$, this was already known to Dong-Nagatomo [DN], and we proved it by direct computation for $n=2$ and $n=3$, but we were unable to prove it in general. However, using the same approach as [LII], we showed that the strong finite generation of $\mathcal{H}(n)^{O(n)}$ implies the strong finite generation of $\mathcal{H}(n)^G$ for an arbitrary reductive group G.

In this paper, we study invariant subalgebras of the universal affine vertex algebra $V_k(\mathfrak{g})$ of some simple, finite-dimensional complex Lie algebra $\mathfrak{g}$ at level k. Recall that $V_k(\mathfrak{g})$ has generators X^ξ, which are linear in $\xi \in \mathfrak{g}$, and satisfy the OPE relations

$$X^\xi(z)X^\eta(w) \sim k\langle \xi,\eta\rangle (z-w)^{-2} + X^{[\xi,\eta]}(w)(z-w)^{-1},$$

where $\langle , \rangle$ is the (normalized) Killing form. Let G be a reductive group of automorphisms of $V_k(\mathfrak{g})$ for all $k \in \mathbb{C}$. In particular, G acts on the weight-one subspace $V_k(\mathfrak{g})[1] \cong \mathfrak{g}$, and G preserves both the bracket and the bilinear form on $\mathfrak{g}$. Therefore G lies in the orthogonal group $O(n)$ for $n = \dim(\mathfrak{g})$, so G also acts on the Heisenberg algebra $\mathcal{H}(n)$. A priori, invariant subalgebras of $V_k(\mathfrak{g})$ are much more complicated and difficult to study than the examples considered above. However, we will see that the structure of $V_k(\mathfrak{g})^G$ is governed to a certain extent by the structure of $\mathcal{H}(n)^G$. Our main result is

Theorem 1.1. *Assume that $\mathcal{H}(n)^{O(n)}$ is strongly finitely generated for $n = dim(\mathfrak{g})$. Let G be a reductive group of automorphims of $V_k(\mathfrak{g})$ for all $k \in \mathbb{C}$. Then $V_k(\mathfrak{g})^G$ is strongly finitely generated for generic values of k, i.e., for $k \in \mathbb{C}\setminus K$ where K is at most countable.*

Since $\mathcal{H}(3)^{O(3)}$ is strongly finitely generated, and in particular is a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,18)$, an immediate consequence of Theorem 1.1 is

Corollary 1.2. *Let $\mathfrak{g} = \mathfrak{sl}_2$ and let G be a reductive group of automorphisms of $V_k(\mathfrak{sl}_2)$ for all $k \in \mathbb{C}$. Then $V_k(\mathfrak{sl}_2)^G$ is strongly finitely generated for generic values of k.*

The key idea behind the proof of Theorem 1.1 is that both $V_k(\mathfrak{g})^G$ and $\mathcal{H}(n)^G$ admit G-invariant filtrations, and we have linear isomorphisms

$$\mathcal{H}(n)^G \cong gr(\mathcal{H}(n)^G) \cong gr(\mathcal{H}(n))^G \cong gr(V_k(\mathfrak{g})^G) \cong gr(V_k(\mathfrak{g}))^G \cong V_k(\mathfrak{g})^G, \tag{1}$$

and isomorphisms of graded commutative rings

$$gr(V_k(\mathfrak{g}))^G \cong (Sym \bigoplus_{j\geq 0} V_j)^G \cong gr(\mathcal{H}(n))^G. \tag{2}$$

Here $V_j \cong \mathbb{C}^n \cong \mathfrak{g}$ as G-modules. By a theorem of Weyl, there is a natural (infinite) generating set S for $(Sym \bigoplus_{j\geq 0} V_j)^G$ consisting of a finite set of generators for $(Sym \bigoplus_{j=0}^{n-1} V_j)^G$ together with their polarizations. Under (1) and (2), S corresponds to (infinite) strong generating sets T and U for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g})^G$, respectively.

Since we are assuming the strong finite generation of $\mathcal{H}(n)^{O(n)}$, there is a finite subset $T' = \{p_1, \dots, p_s\} \subset T$, which strongly generates $\mathcal{H}(n)^G$. In other words, $\mathcal{H}(n)^G = \langle T' \rangle$, where $\langle T' \rangle$ denotes the space of normally ordered polynomials in $p_1, \dots, p_s$ and their derivatives. This implies that any $q \in T$ admits a "decoupling relation" of the form

$$q = P(p_1, \dots, p_s), \tag{3}$$

where P lies in $\langle T' \rangle$. The relations in $(Sym \bigoplus_{j\geq 0} V_j)^G$ among the generators in S are given by the *second fundamental theorem of invariant theory* for (G, V). We may view the decoupling relation (3) as a deformation of such a classical relation.

Let $U' \subset U \subset V_k(\mathfrak{g})^G$ be the set corresponding to T' under (1), which we denote by $\{\tilde{p}_1, \dots, \tilde{p}_s\}$, and let $\langle U' \rangle$ denote the space of normally ordered polynomials in $\tilde{p}_1, \dots, \tilde{p}_s$ and their derivatives. Given $q \in T$, let $\tilde{q} \in U$ be the element corresponding to q under (1). We show that for generic values of k, the decoupling relations (3) can be used to construct analogous (but more complicated) decoupling relations

$$\tilde{q} = Q(\tilde{p}_1, \dots, \tilde{p}_s), \tag{4}$$

in $V_k(\mathfrak{g})^G$ for all $\tilde{q} \in U$, where Q lies in $\langle U' \rangle$. This implies that U' is a strong generating set for $V_k(\mathfrak{g})^G$ for generic values of k.

The idea of studying $V_k(\mathfrak{g})^G$ via the simpler object $\mathcal{H}(n)^G$ is analogous to studying classical invariant rings of the form $U(\mathfrak{g})^G$ via the Poincare-Birkhoff-Witt abelianization $Sym(\mathfrak{g})^G \cong gr(U(\mathfrak{g})^G) \cong gr(U(\mathfrak{g}))^G$. Here $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, and G is a group of automorphisms of $U(\mathfrak{g})$. A generating set for $U(\mathfrak{g})^G$ can be obtained from a generating set for $Sym(\mathfrak{g})^G$, and the leading term of a relation among the generators of $U(\mathfrak{g})^G$ corresponds to a relation among the generators of $Sym(\mathfrak{g})^G$. However, more subtle information about the structure and representation theory of $U(\mathfrak{g})^G$ cannot be reconstructed in this way. Likewise, we can view $\mathcal{H}(n)$ as a kind of "partial abelianization" of $V_k(\mathfrak{g})$. By passing to the full abelianization $gr(V_k(\mathfrak{g})) \cong Sym \bigoplus_{j\geq 0} V_j$ of $V_k(\mathfrak{g})$, we destroy too much structure, and $gr(V_k(\mathfrak{g}))^G$ generally fails to be finitely generated. By contrast, $\mathcal{H}(n)$ retains enough structure (notably, it is still a simple vertex algebra) so that strong generating sets for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g})^G$ are closely related.

A *deformable $\mathcal{W}$-algebra* is a family of conformal vertex algebras $\mathcal{W}_c$ where c is the Virasoro central charge, equipped with strong generating sets $A_c = \{a_1^c, \dots, a_r^c\}$, whose structure constants are continuous functions of c with isolated singularities. In this terminology, the structure constants are the coefficients of each normally ordered monomial in the elements of A_c and their derivatives, which appear in the OPE of $a_i^c(z)a_j^c(w)$, for $i, j = 1, \dots, r$.

For arbitrary k, let $\mathcal{W}(\mathfrak{g}, G, k)$ denote the vertex subalgebra of $V_k(\mathfrak{g})^G$ generated by U'. Clearly $\langle U' \rangle \subset \mathcal{W}(\mathfrak{g}, G, k)$, and the equality $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, G, k)$ holds precisely when $\mathcal{W}(\mathfrak{g}, G, k)$ is *strongly* generated by U'. Assuming the strong finite generation of $\mathcal{H}(n)^{O(n)}$, an immediate consequence of Theorem 1.1 is that $\mathcal{W}(\mathfrak{g}, G, k)$ is a deformable $\mathcal{W}$-algebra. In fact, the structure constants of $\mathcal{W}(\mathfrak{g}, G, k)$ are rational functions of k, so the equality $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, G, k)$ holds for all but finitely many values of k.

Many open questions remain, the most important of which is to establish the strong finite generation of $\mathcal{H}(n)^{O(n)}$ for $n \geq 4$. Even in the case $n = 3$ where this is known, it would be useful to obtain more precise quantitative information about $\mathcal{H}(3)^G$ and $V_k(\mathfrak{sl}_2)^G$, such as *minimal* strong generating sets, or at least good bounds on the number and conformal weights of the generators. The proof in [LIII] that the strong finite generation of $\mathcal{H}(n)^{O(n)}$ implies that of $\mathcal{H}(n)^G$ for any reductive group G is essentially constructive, but it is likely that the generating sets it produces are far from minimal. Moreover, since the relations in $V_k(\mathfrak{g})^G$ are more complicated than those in $\mathcal{H}(n)^G$, it is likely that a minimal strong generating set for $V_k(\mathfrak{g})^G$ is smaller than a minimal set for $\mathcal{H}(n)^G$.

Finally, we expect that the vertex algebras $\mathcal{W}(\mathfrak{g}, G, k)$ form a new and rich class of deformable $\mathcal{W}$-algebras, and it is important to study their representation theory as well as their structure. For generic values of k, $\mathcal{W}(\mathfrak{g}, G, k)$ will be simple, but for certain special values of k these algebras should possess nontrivial ideals. It is possible that there exist new rational vertex algebras arising as quotients of vertex algebras of this kind.

2. Vertex algebras

In this section, we define vertex algebras, which have been discussed from various different points of view in the literature (see for example [B][FLM][K][FBZ]). We will follow the formalism developed in [LZI] and partly in [Li]. Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$, and let z, w be formal variables. By $QO(V)$, we mean the space of all linear maps

$$V \to V((z)) := \{\sum_{n\in\mathbb{Z}} v(n)z^{-n-1} | v(n) \in V,\ v(n) = 0 \text{ for } n >> 0\}.$$

Each element $a \in QO(V)$ can be uniquely represented as a power series

$$a = a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in (End\, V)[[z, z^{-1}]].$$

We refer to $a(n)$ as the nth Fourier mode of $a(z)$. Each $a \in QO(V)$ is assumed to be of the shape $a = a_0 + a_1$ where $a_i : V_j \to V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$.

On $QO(V)$ there is a set of nonassociative bilinear operations $\circ_n$, indexed by $n \in \mathbb{Z}$, which we call the nth circle products. For homogeneous $a, b \in QO(V)$, they are defined by

$$a(w) \circ_n b(w) = Res_z a(z)b(w)\ \iota_{|z|>|w|}(z - w)^n - (-1)^{|a||b|} Res_z b(w)a(z)\ \iota_{|w|>|z|}(z - w)^n.$$

Here $\iota_{|z|>|w|} f(z, w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$. We usually omit the symbol $\iota_{|z|>|w|}$ and just write $(z - w)^{-1}$ to mean the expansion in the region $|z| > |w|$, and write $-(w - z)^{-1}$ to mean the expansion in $|w| > |z|$. It is easy to check that $a(w) \circ_n b(w)$ above is a well-defined element of $QO(V)$.

The non-negative circle products are connected through the *operator product expansion* (OPE) formula. For $a, b \in QO(V)$, we have

$$a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w)\ (z-w)^{-n-1} + :a(z)b(w):\ , \tag{5}$$

which is often written as $a(z)b(w) \sim \sum_{n\geq 0} a(w) \circ_n b(w)\ (z-w)^{-n-1}$, where $\sim$ means equal modulo the term

$$:a(z)b(w): = a(z)_- b(w)\ +\ (-1)^{|a||b|} b(w) a(z)_+.$$

Here $a(z)_- = \sum_{n<0} a(n) z^{-n-1}$ and $a(z)_+ = \sum_{n\geq 0} a(n) z^{-n-1}$. Note that $:a(w)b(w):$ is a well-defined element of $QO(V)$. It is called the *Wick product* of a and b, and it coincides with $a \circ_{-1} b$. The other negative circle products are related to this by

$$n!\ a(z) \circ_{-n-1} b(z) = \ :(\partial^n a(z)) b(z):\ ,$$

where ∂ denotes the formal differentiation operator $\frac{d}{dz}$. For $a_1(z), ..., a_k(z) \in QO(V)$, the k-fold iterated Wick product is defined to be

$$:a_1(z) a_2(z) \cdots a_k(z): \ = \ :a_1(z) b(z):\ , \tag{6}$$

where $b(z) = \ :a_2(z) \cdots a_k(z):$. We often omit the formal variable z when no confusion will arise.

The set $QO(V)$ is a nonassociative algebra with the operations $\circ_n$ and a unit 1. We have $1 \circ_n a = \delta_{n,-1} a$ for all n, and $a \circ_n 1 = \delta_{n,-1} a$ for $n \geq -1$. A linear subspace $\mathcal{A} \subset QO(V)$ containing 1 which is closed under the circle products will be called a quantum operator algebra (QOA). In particular $\mathcal{A}$ is closed under ∂ since $\partial a = a \circ_{-2} 1$. Many formal algebraic notions are immediately clear: a homomorphism is just a linear map that sends 1 to 1 and preserves all circle products; a module over $\mathcal{A}$ is a vector space M equipped with a homomorphism $\mathcal{A} \to QO(M)$, etc. A subset $S = \{a_i |\ i \in I\}$ of $\mathcal{A}$ is said to generate $\mathcal{A}$ if any element $a \in \mathcal{A}$ can be written as a linear combination of nonassociative words in the letters a_i, $\circ_n$, for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* $\mathcal{A}$ if any $a \in \mathcal{A}$ can be written as a linear combination of words in the letters a_i, $\circ_n$ for $n < 0$. Equivalently, $\mathcal{A}$ is spanned by the collection $\{:\partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z): |\ i_1, \dots, i_m \in I,\ k_1, \dots, k_m \geq 0\}$.

We say that $a, b \in QO(V)$ *quantum commute* if $(z-w)^N [a(z), b(w)] = 0$ for some $N \geq 0$. Here $[,]$ denotes the super bracket. This condition implies that $a \circ_n b = 0$ for $n \geq N$, so (5) becomes a finite sum. If N can be chosen to be 0, we say that a, b commute. A commutative quantum operator algebra (CQOA) is a QOA whose elements pairwise quantum commute. Finally, the notion of a CQOA is equivalent to the notion of a vertex algebra. Every CQOA $\mathcal{A}$ is itself a faithful $\mathcal{A}$-module, called the *left regular module*. Define

$$\rho : \mathcal{A} \to QO(\mathcal{A}), \quad a \mapsto \hat{a}, \quad \hat{a}(\zeta) b = \sum_{n \in \mathbb{Z}} (a \circ_n b)\ \zeta^{-n-1}.$$

Then ρ is an injective QOA homomorphism, and the quadruple of structures $(\mathcal{A}, \rho, 1, \partial)$ is a vertex algebra in the sense of [FLM]. Conversely, if $(V, Y, \mathbf{1}, D)$ is a vertex algebra, the collection $Y(V) \subset QO(V)$ is a CQOA. *We will refer to a CQOA simply as a vertex algebra throughout the rest of this paper*.

The main examples we consider are the *universal affine vertex algebras* and their invariant subalgebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, equipped with a symmetric, invariant bilinear form B. The loop algebra $\mathfrak{g}[t,t^{-1}] = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ has a one-dimensional central extension $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}\kappa$ determined by B, with bracket

$$[\xi t^n, \eta t^m] = [\xi,\eta]t^{n+m} + nB(\xi,\eta)\delta_{n+m,0}\kappa,$$

and $\mathbb{Z}$-gradation $deg(\xi t^n) = n$, $deg(\kappa) = 0$. Let $\hat{\mathfrak{g}}_{\geq 0} = \bigoplus_{n\geq 0}\hat{\mathfrak{g}}_n$ where $\hat{\mathfrak{g}}_n$ denotes the subspace of degree n, and let C be the one-dimensional $\hat{\mathfrak{g}}_{\geq 0}$-module on which ξt^n acts trivially for $n\geq 0$, and κ acts by the identity. Define $V = U(\hat{\mathfrak{g}})\otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} C$, and let $X^\xi(n)\in End(V)$ be the linear operator representing ξt^n on V. Define $X^\xi(z) = \sum_{n\in\mathbb{Z}} X^\xi(n)z^{-n-1}$, which is easily seen to lie in $QO(V)$ and satisfy the OPE relation

$$X^\xi(z)X^\eta(w)\sim B(\xi,\eta)(z-w)^{-2} + X^{[\xi,\eta]}(w)(z-w)^{-1}.$$

The vertex algebra $O(\mathfrak{g},B)$ generated by $\{X^\xi|\ \xi\in\mathfrak{g}\}$ is known as the universal affine vertex algebra, or current algebra, associated to $\mathfrak{g}$ and B.

Two special cases will be important to us. First, suppose that $\mathfrak{g}$ is simple, so that B is a scalar multiple of the (normalized) Killing form $\langle,\rangle$, for some scalar k. In this case, it is customary to denote $O(\mathfrak{g},B)$ by $V_k(\mathfrak{g})$. Let $n = \dim(\mathfrak{g})$, and fix an orthonormal basis $\{\xi_1,\dots,\xi_n\}$ for $\mathfrak{g}$ relative to $\langle,\rangle$. For $k\neq -h^\vee$ where $h^\vee$ is the dual Coxeter number, $V_k(\mathfrak{g})$ is a conformal vertex algebra with Virasoro element

$$L(z) = \frac{1}{2(k+h^\vee)}\sum_{i=1}^n :X^{\xi_i}(z)X^{\xi_i}(z):,$$

of central charge $\frac{k\cdot\dim(\mathfrak{g})}{k+h^\vee}$, such that each X^{ξ_i} is primary of weight one. This Virasoro element is known as the *Sugawara conformal vector*. At the critical level $k\neq -h^\vee$, $L(z)$ does not exist, but $V_k(\mathfrak{g})$ still possesses a *quasi-conformal structure*; there is an action of the Lie subalgebra $\{L_n|\ n\geq -1\}$ of the Virasoro algebra, such that L_{-1} acts by translation, L_0 acts diagonalizably, and each X^{ξ_i} has weight one. We will always assume that $V_k(\mathfrak{g})$ is equipped with this conformal weight grading; note that the weight-one subspace $V_k(\mathfrak{g})[1]$ is then linearly isomorphic to $\mathfrak{g}$. Any group G of automorphisms of $V_k(\mathfrak{g})$ preserving this grading must act on $\mathfrak{g}$ and preserve both the bracket and the bilinear form.

Next, suppose that $\mathfrak{g}$ is abelian, and that B is nondegenerate. Then $O(\mathfrak{g},B)$ is just the rank n Heisenberg vertex algebra $\mathcal{H}(n)$. If we choose an orthonormal basis $\{\xi_1,\dots,\xi_n\}$ for $\mathfrak{g}$, $\mathcal{H}(n)$ is generated by $\{\alpha^i = X^{\xi_i}|\ i=1,\dots,n\}$, satisfying the OPE relations

$$\alpha^i(z)\alpha^j(w)\sim\delta_{i,j}(z-w)^{-2}.$$

There is a conformal structure of central charge n on $\mathcal{H}(n)$, with Virasoro element

$$L(z) = \frac{1}{2}\sum_{i=1}^n :\alpha^i(z)\alpha^i(z):,$$

under which each α^i is primary of weight one.

3. Category $\mathcal{R}$

Let $\mathcal{R}$ be the category of vertex algebras $\mathcal{A}$ equipped with a $\mathbb{Z}_{\geq 0}$-filtration

$$\mathcal{A}_{(0)}\subset\mathcal{A}_{(1)}\subset\mathcal{A}_{(2)}\subset\cdots,\qquad \mathcal{A} = \bigcup_{k\geq 0}\mathcal{A}_{(k)} \tag{7}$$

such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(k)}$, $b \in \mathcal{A}_{(l)}$, we have

$$a \circ_n b \in \mathcal{A}_{(k+l)}, \quad \text{for } n < 0, \tag{8}$$

$$a \circ_n b \in \mathcal{A}_{(k+l-1)}, \quad \text{for } n \geq 0. \tag{9}$$

Elements $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ are said to have degree d.

Filtrations on vertex algebras satisfying (8)-(9) were introduced in [Li], and are known as *good increasing filtrations*. If $\mathcal{A}$ possesses such a filtration, the associated graded object $gr(\mathcal{A}) = \bigoplus_{k>0} \mathcal{A}_{(k)}/\mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$-graded associative, supercommutative algebra with a unit 1 under a product induced by the Wick product on $\mathcal{A}$. In general, there is no natural linear map $\mathcal{A} \to gr(\mathcal{A})$, but for each $r \geq 1$ we have the projection

$$\phi_r : \mathcal{A}_{(r)} \to \mathcal{A}_{(r)}/\mathcal{A}_{(r-1)} \subset gr(\mathcal{A}). \tag{10}$$

Moreover, $gr(\mathcal{A})$ has a derivation ∂ of degree zero (induced by the operator $\partial = \frac{d}{dz}$ on $\mathcal{A}$), and for each $a \in \mathcal{A}_{(d)}$ and $n \geq 0$, the operator $a\circ_n$ on $\mathcal{A}$ induces a derivation of degree $d - k$ on $gr(\mathcal{A})$, which we denote by $a(n)$. Here

$$k = sup\{j \geq 1 |\ \mathcal{A}_{(r)} \circ_n \mathcal{A}_{(s)} \subset \mathcal{A}_{(r+s-j)}\ \forall r, s, n \geq 0\},$$

as in [LL]. Finally, these derivations give $gr(\mathcal{A})$ the structure of a vertex Poisson algebra.

The assignment $\mathcal{A} \mapsto gr(\mathcal{A})$ is a functor from $\mathcal{R}$ to the category of $\mathbb{Z}_{\geq 0}$-graded supercommutative rings with a differential ∂ of degree 0, which we will call ∂-rings. A ∂-ring is the same thing as an *abelian* vertex algebra, that is, a vertex algebra $\mathcal{V}$ in which $[a(z), b(w)] = 0$ for all $a, b \in \mathcal{V}$. A ∂-ring A is said to be generated by a subset $\{a_i |\ i \in I\}$ if $\{\partial^k a_i |\ i \in I, k \geq 0\}$ generates A as a graded ring. The key feature of $\mathcal{R}$ is the following reconstruction property [LL]:

Lemma 3.1. *Let $\mathcal{A}$ be a vertex algebra in $\mathcal{R}$ and let $\{a_i |\ i \in I\}$ be a set of generators for $gr(\mathcal{A})$ as a ∂-ring, where a_i is homogeneous of degree d_i. If $a_i(z) \in \mathcal{A}_{(d_i)}$ are vertex operators such that $\phi_{d_i}(a_i(z)) = a_i$, then $\mathcal{A}$ is strongly generated as a vertex algebra by $\{a_i(z) |\ i \in I\}$.*

As shown in [LI], there is a similar reconstruction property for kernels of surjective morphisms in $\mathcal{R}$. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathcal{R}$ with kernel $\mathcal{J}$, such that f maps $\mathcal{A}_{(k)}$ onto $\mathcal{B}_{(k)}$ for all $k \geq 0$. The kernel J of the induced map $gr(f) : gr(\mathcal{A}) \to gr(\mathcal{B})$ is a homogeneous ∂-ideal (i.e., $\partial J \subset J$). A set $\{a_i |\ i \in I\}$ such that a_i is homogeneous of degree d_i is said to generate J as a ∂-ideal if $\{\partial^k a_i |\ i \in I,\ k \geq 0\}$ generates J as an ideal.

Lemma 3.2. *Let $\{a_i | i \in I\}$ be a generating set for J as a ∂-ideal, where a_i is homogeneous of degree d_i. Then there exist vertex operators $a_i(z) \in \mathcal{A}_{(d_i)}$ with $\phi_{d_i}(a_i(z)) = a_i$, such that $\{a_i(z) |\ i \in I\}$ generates $\mathcal{J}$ as a vertex algebra ideal.*

For any Lie algebra $\mathfrak{g}$ and bilinear form B, $O(\mathfrak{g}, B)$ admits a good increasing filtration

$$O(\mathfrak{g}, B)_{(0)} \subset O(\mathfrak{g}, B)_{(1)} \subset \cdots, \qquad O(\mathfrak{g}, B) = \bigcup_{j \geq 0} O(\mathfrak{g}, B)_{(j)}, \tag{11}$$

where $O(\mathfrak{g}, B)_{(j)}$ is defined to be the vector space spanned by iterated Wick products of the generators X^{ξ_i} and their derivatives, of length at most j. We say that elements of

$O(\mathfrak{g},B)_{(j)} \setminus O(\mathfrak{g},B)_{(j-1)}$ have degree j. For any group G of weight-preserving automorphisms of $O(\mathfrak{g},B)$, this filtration is G-invariant, and we have an isomorphism of commutative algebras

$$gr(O(\mathfrak{g},B)) \cong Sym \bigoplus_{j\geq 0} V_j, \tag{12}$$

were $V_j \cong \mathfrak{g}$ as G-modules. The generators of $gr(O(\mathfrak{g},B))$ are $X^{\xi_i}_j$, which correspond to the vertex operators $\partial^j X^{\xi_i}$. Since $O(\mathfrak{g},B)$ has a basis consisting of iterated Wick products of the generators and their derivatives, $O(\mathfrak{g},B) \cong gr(O(\mathfrak{g},B))$ as vector spaces, although not canonically. The filtration (11) is inherited by $O(\mathfrak{g},B)^G$, and

$$gr(O(\mathfrak{g},B))^G \cong gr(O(\mathfrak{g},B)^G) \cong (Sym \bigoplus_{j\geq 0} V_j)^G \tag{13}$$

as commutative rings. Finally, note that $Sym \bigoplus_{j\geq 0} V_j$ has a natural ∂-ring structure, where the derivation ∂ acts on the generators $x_j \in V_j$ by $\partial(x_j) = x_{j+1}$. Clearly ∂ commutes with the action of G, so ∂ acts on $(Sym \bigoplus_{j\geq 0} V_j)^G$, and (13) is an isomorphism of ∂-rings.

4. Some classical invariant theory

We briefly recall some terminology and results about invariant rings of the form

$$R = (Sym \bigoplus_{j\geq 0} V_j)^G,$$

where each V_j is isomorphic to some fixed G-module V. In the terminology of Weyl, a *first fundamental theorem of invariant theory* for the pair (G,V) is a set of generators for R. In some treatments, a first fundamental theorem is defined as a set of generators for the larger ring $(Sym \bigoplus_{j\geq 0}(V_j \oplus V_j^*))^G$, where $V_j^* \cong V^*$ as G-modules, but here we only need to consider $(Sym \bigoplus_{j\geq 0} V_j)^G$. A *second fundamental theorem of invariant theory* for (G,V) is a set of generators for the ideal of relations among the generators of R.

First and second fundamental theorems of invariant theory are known for the standard representations of the classical groups [We] and for the adjoint representations of the classical groups [P], but in general it is quite difficult to describe these rings explicitly. However, there is a certain *qualitative* result of Weyl (II.5 Theorem 2.5A of [We]) that will be useful. Note that for all $p \geq 0$, there is an action of GL_p on $\bigoplus_{j=0}^{p-1} V_j$ which commutes with the action of G. The natural inclusions $GL_p \hookrightarrow GL_q$ for $p<q$ sending

$$M \to \begin{bmatrix} M & 0 \\ 0 & I_{q-p} \end{bmatrix}$$

induces an action of $GL_\infty = \lim_{p\to\infty} GL_p$ on $\bigoplus_{j\geq 0} V_j$. We obtain an action of GL_∞ on $Sym \bigoplus_{j\geq 0} V_j$ by algebra automorphisms, which commutes with the action of G. Hence GL_∞ acts on R as well. The elements of $\sigma \in GL_\infty$ are known as *polarization operators*, and given $f \in R$, σf is known as a polarization of f.

Theorem 4.1. *(Weyl) R is generated by the set of polarizations of any set of generators for $(Sym \bigoplus_{j=0}^{n-1} V_j)^G$, where $n = dim(V)$. Since G is reductive, $(Sym \bigoplus_{j=0}^{n-1} V_j)^G$ is finitely generated. Hence there exists a finite set $\{f_1,\dots,f_r\}$ of homogeneous elements of R, whose polarizations generate R.*

Note that taking the complete set of polarizations is unnecessary. We can always find a *minimal* set S of polarizations of $\{f_1, \dots, f_r\}$ which generates R, such that no element of S is a nontrivial polynomial in other elements of S. Clearly for all $p \geq n$, $S \cap (Sym \bigoplus_{j=0}^p)^G$ will be a finite generating set for $(Sym \bigoplus_{j=0}^p)^G$. Next, recall that R has the structure of a ∂-ring. For our purposes, we will need a minimal set of generators for R as a ∂-ring, and our set S may not be minimal in this sense. Any element of S of the form $\partial^k s$ for $s \in S$ and $k > 0$, and any nontrivial polynomial in the elements of S and their derivatives can be removed, and the resulting set will still generate R as a ∂-ring.

Let $\mathcal{V}$ be a vertex algebra with a good increasing filtration such that $\mathcal{V}^G \cong gr(\mathcal{V}^G)$ as linear spaces, and $gr(\mathcal{V})^G \cong gr(\mathcal{V}^G) \cong (Sym \bigoplus_{j \geq 0} V_j)^G = R$ as ∂-rings, where V_j is isomorphic to some finite-dimensional G-module V, for all $j \geq 0$. Of course, our main example of such a vertex algebra is $O(\mathfrak{g}, B)$ where $V \cong \mathfrak{g}$. Choose homogeneous elements $\{f_1, \dots, f_r\} \subset R$ whose polarizations generate R, and choose a minimal set S of polarizations which generates R as a ∂-ring. Let $T = \{s(z)|\ s \in S\} \subset \mathcal{V}^G$ be the set of vertex operators corresponding to S under the linear isomorphism $\mathcal{V}^G \cong R$. By Lemma 3.1, T is a strong generating set for $\mathcal{V}^G$.

Given a homogeneous polynomial $p \in gr(\mathcal{V}^G) \cong R$ of degree d, a *normal ordering* of p will be a choice of normally ordered polynomial $P \in (\mathcal{V}^G)_{(d)}$, obtained by replacing each $s \in S$ by $s(z) \in T$, and replacing ordinary products with iterated Wick products of the form (6). Of course P is not unique, but for any choice of P we have $\phi_d(P) = p$, where

$$\phi_d : (\mathcal{V}^G)_{(d)} \to (\mathcal{V}^G)_{(d)}/(\mathcal{V}^G)_{(d-1)} \subset gr(\mathcal{V}^G)$$

is the usual projection.

Suppose that p is a relation among the generators of R coming from the second fundamental theorem for (G, V), which we may assume to be homogeneous of degree d. Let $P^d \in \mathcal{V}^G$ be some normal ordering of p. Since $gr(\mathcal{V}^G) \cong R$ as graded rings, it follows that P^d lies in $\mathcal{V}^G_{(d-1)}$. The polynomial $\phi_{d-1}(P^d) \in R$ is homogeneous of degree $d-1$; if it is nonzero, it can be expressed as a polynomial in the variables $s \in S$ and their derivatives. Choose some normal ordering of this polynomial, and call this vertex operator $-P^{d-1}$. Then $P^d + P^{d-1}$ has the property that

$$\phi_d(P^d + P^{d-1}) = p, \qquad P^d + P^{d-1} \in (\mathcal{V}^G)_{(d-2)}.$$

Continuing this process, we arrive at a vertex operator

$$P = \sum_{k=1}^{d} P^k \in \mathcal{V}^G \tag{14}$$

which is identically zero. We view P as a quantum correction of the relation p, and it is easy to see by induction on degree that all normally ordered polynomial relations in $\mathcal{V}^G$ among the elements of T and their derivatives, are consequences of relations of this kind.

In general, R will not be finitely generated as a ∂-ring. However, since the relations (14) are more complicated than their classical counterparts, it is still possible for $\mathcal{V}^G$ to be strongly generated as a vertex algebra by a finite subset $T' \subset T$. For this to happen, each element $t \in T \setminus T'$ must admit a "decoupling relation" expressing it as a normally ordered polynomial in the elements of T' and their derivatives. Given a relation in $\mathcal{V}^G$ of the form (14), suppose that some $t \in T \setminus T'$ appears in P^k for some $k < d$, with nonzero coefficient. If the remaining terms in (14) only depend on the elements of T' and their derivatives, we can solve for t to obtain such a decoupling relation. The existence of a complete set of decoupling relations for all $t \in T \setminus T'$ is clearly a very subtle and nonclassical phenomenon, since it depends on precise properties of the quantum corrections appearing in relations of the form (14).

5. Recollections on invariants of Heisenberg algebras

In [LIII] we studied invariant subalgebras of the rank n Heisenberg vertex algebra $\mathcal{H}(n)$, and we briefly recall our main results. Recall that $\mathcal{H}(n) \cong gr(\mathcal{H}(n))$ as linear spaces, and $gr(\mathcal{H}(n)) \cong Sym \bigoplus_{j\geq 0} V_j$ as commutative rings. Here V_j is spanned by $\{\alpha^i_j |\, i = 1,\dots,n\}$, where α^i_j is the image of $\partial^j \alpha^i(z)$ under the projection $\phi_1 : \mathcal{H}(n)_{(1)} \to \mathcal{H}(n)_{(1)}/\mathcal{H}(n)_{(0)} \subset gr(\mathcal{H}(n))$. The action of $O(n)$ on $\mathcal{H}(n)$ preserves the filtration and induces an action of $O(n)$ on $gr(\mathcal{H}(n))$, and $V_j \cong \mathbb{C}^n$ as $O(n)$-modules for all $j \geq 0$. We have a linear isomorphism $\mathcal{H}(n)^{O(n)} \cong gr(\mathcal{H}(n)^{O(n)})$, and isomorphisms of commutative rings

$$gr(\mathcal{H}(n)^{O(n)}) \cong (gr(\mathcal{H}(n))^{O(n)} \cong (Sym \bigoplus_{j\geq 0} V_j)^{O(n)}. \tag{15}$$

The following classical theorem of Weyl [We] describes the generators and relations of the ring $(Sym \bigoplus_{j\geq 0} V_j)^{O(n)}$:

Theorem 5.1. *For $j \geq 0$, let V_j be the copy of the standard $O(n)$-module $\mathbb{C}^n$ with orthonormal basis $\{x_{i,j}|\, i = 1,\dots,n\}$. The invariant ring $(Sym \bigoplus_{j\geq 0} V_j)^{O(n)}$ is generated by the quadratics*

$$q_{a,b} = \sum_{i=1}^n x_{i,a}x_{i,b}, \qquad 0 \leq a \leq b. \tag{16}$$

The kernel I_n of the homomorphism $\mathbb{C}[Q_{a,b}] \to (Sym \bigoplus_{j\geq 0} V_j)^{O(n)}$ sending $Q_{a,b} \mapsto q_{a,b}$ is generated by the $(n+1)\times(n+1)$ determinants

$$d_{I,J} = \begin{bmatrix} Q_{i_0,j_0} & \cdots & Q_{i_0,j_n} \\ \vdots & & \vdots \\ Q_{i_n,j_0} & \cdots & Q_{i_n,j_n} \end{bmatrix}. \tag{17}$$

In this notation, $q_{a,b}$ for $a > b$ is defined to be $q_{b,a}$, and $I = (i_0,\dots,i_n)$ and $J = (j_0,\dots,j_n)$ are lists of integers satisfying

$$0 \leq i_0 < \cdots < i_n, \qquad 0 \leq j_0 < \cdots < j_n. \tag{18}$$

Under the projection

$$\phi_2 : (\mathcal{H}(n)^{O(n)})_{(2)} \to (\mathcal{H}(n)^{O(n)})_{(2)}/(\mathcal{H}(n)^{O(n)})_{(1)} \subset gr(\mathcal{H}(n)^{O(n)}) \cong (Sym \bigoplus_{j\geq 0} V_j)^{O(n)},$$

the generators $q_{a,b}$ of $(Sym \bigoplus_{j\geq 0} V_j)^{O(n)}$ correspond to vertex operators $\omega_{a,b}$ given by

$$\omega_{a,b} = \sum_{i=1}^n :\partial^a \alpha^i \partial^b \alpha^i:, \qquad 0 \leq a \leq b. \tag{19}$$

By Lemma 3.1, the set $\{\omega_{a,b}|\, 0 \leq a \leq b\}$ is a strong generating set for $\mathcal{H}(n)^{O(n)}$. In fact, there is a somewhat more economical set of strong generators which is more natural from the point of view of vertex algebras. For each $m \geq 0$, let A_m denote the vector space spanned by $\{\omega_{a,b}|\ a+b=m\}$, which is homogeneous of weight $m+2$. We have $\text{dim}(A_{2m}) = m = \text{dim}(A_{2m+1})$ and

$$\text{dim}\big(A_{2m}/\partial(A_{2m-1})\big) = 1, \qquad \text{dim}\big(A_{2m+1}/\partial(A_{2m})\big) = 0.$$

For $m \geq 0$, define

$$j^{2m} = \omega_{0,2m}. \tag{20}$$

Clearly j^{2m} is not a total derivative, so A_{2m} has a decomposition

$$A_{2m} = \partial(A_{2m-1}) \oplus \langle j^{2m}\rangle = \partial^2(A_{2m-2}) \oplus \langle j^{2m}\rangle, \tag{21}$$

where $\langle j^{2m}\rangle$ is the linear span of j^{2m}. Similarly,

$$A_{2m+1} = \partial^2(A_{2m-1}) \oplus \langle \partial j^{2m}\rangle = \partial^3(A_{2m-2}) \oplus \langle \partial j^{2m}\rangle, \tag{22}$$

and it is easy to see that $\{\partial^{2i} j^{2m-2i}|\ 0 \leq i \leq m\}$ and $\{\partial^{2i+1} j^{2m-2i}|0 \leq i \leq m\}$ are bases of A_{2m} and A_{2m+1}, respectively. It follows that $\{j^{2m}|\ m \geq 0\}$ is an alternative strong generating set for $\mathcal{H}(n)^{O(n)}$.

We can realize $\mathcal{H}(n)^{O(n)}$ as the quotient of a vertex algebra $\mathcal{V}_n$ which is *freely* generated by vertex operators $\{J^{2m}(z)|\ m \geq 0\}$, under a map $\pi_n : \mathcal{V}_n \to \mathcal{H}(n)^{O(n)}$ sending $J^{2m} \mapsto j^{2m}$. In other words, there are no normally ordered polynomial relations in $\mathcal{V}_n$ among the generators and their derivatives. There is an alternative strong generating set

$$\{\Omega_{a,b}|\ 0 \leq a \leq b\}$$

for $\mathcal{V}_n$ such that $\pi_n(\Omega_{a,b}) = \omega_{a,b}$. Recall the variables $Q_{a,b}$ and $q_{a,b}$ appearing in Theorem 5.1. Since $\mathcal{V}_n$ is freely generated by $\{J^{2m}|\ m \geq 0\}$, and the sets $\{\Omega_{a,b}|\ 0 \leq a \leq b\}$ and $\{\partial^k J^{2m}|\ k, m \geq 0\}$ form bases for the same vector space, we may identify $gr(\mathcal{V}_n)$ with $\mathbb{C}[Q_{a,b}]$, and we identify $gr(\mathcal{H}(n)^{O(n)})$ with $\mathbb{C}[q_{a,b}]/I_n$. Under this identification, $gr(\pi_n) : gr(\mathcal{V}_n) \to gr(\mathcal{H}(n)^{O(n)})$ is just the quotient map sending $Q_{a,b} \mapsto q_{a,b}$.

There is a good increasing filtration on $\mathcal{V}_n$ such that $(\mathcal{V}_n)_{(2k)}$ is spanned by iterated Wick products of the generators $\Omega_{a,b}$, of length at most k, and $(\mathcal{V}_n)_{(2k+1)} = (\mathcal{V}_n)_{(2k)}$. Equipped with this filtration, $\mathcal{V}_n$ lies in the category $\mathcal{R}$, and π_n is a morphism in $\mathcal{R}$. Clearly π_n maps each filtered piece $(\mathcal{V}_n)_{(k)}$ onto $(\mathcal{H}(n)^{O(n)})_{(k)}$, so the hypotheses of Lemma 3.2 are satisfied. Since $I_n = Ker(gr(\pi_n))$ is generated by the determinants $d_{I,J}$, we can apply Lemma 3.2 to find vertex operators $D_{I,J} \in (\mathcal{V}_n)_{(2n+2)}$ satisfying $\phi_{2n+2}(D_{I,J}) = d_{I,J}$, such that $\{D_{I,J}\}$ generates $\mathcal{I}_n$. Here

$$\phi_{2n+2} : (\mathcal{V}_n)_{(2n+2)} \to (\mathcal{V}_n)_{(2n+2)}/(\mathcal{V}_n)_{(2n+1)} \subset gr(\mathcal{V}_n)$$

is the usual projection. Since $\Omega_{a,b}$ has weight $a+b+2$, it follows that

$$wt(D_{I,J}) = |I| + |J| + 2n + 2, \qquad |I| = \sum_{a=0}^{n} i_a, \qquad |J| = \sum_{a=0}^{n} j_a. \tag{23}$$

Since there are no relations in $\mathcal{H}(n)^{O(n)}$ of degree less than $2n+2$, $D_{I,J}$ is uniquely determined by the conditions

$$\phi_{2n+2}(D_{I,J}) = d_{I,J}, \qquad \pi_n(D_{I,J}) = 0. \tag{24}$$

Given a homogeneous polynomial $p \in gr(\mathcal{V}_n) \cong \mathbb{C}[Q_{a,b}]$ of degree k in the variables $Q_{a,b}$, recall that a *normal ordering* of p is a choice of normally ordered polynomial $P \in (\mathcal{V}_n)_{(2k)}$, obtained by replacing $Q_{a,b}$ by $\Omega_{a,b}$, and replacing ordinary products with iterated Wick products. For any choice of P we have $\phi_{2k}(P) = p$. Let $D^{2n+2}_{I,J} \in (\mathcal{V}_n)_{(2n+2)}$ be some normal ordering of $d_{I,J}$. Then $\pi_n(D^{2n+2}_{I,J}) \in (\mathcal{H}(n)^{O(n)})_{(2n)}$. Using the procedure outlined in Section 4, we can find a sequence of quantum corrections $D^{2k}_{I,J}$ for $k = 1,\dots,n$, which are homogeneous, normally ordered polynomials of degree k in the variables $\Omega_{a,b}$, such that $\sum_{k=1}^{n+1} D^{2k}_{I,J}$ lies in the kernel of π_n. We have

$$D_{I,J} = \sum_{k=1}^{n+1} D^{2k}_{I,J}, \tag{25}$$

since $D_{I,J}$ is uniquely characterized by (24). In this decomposition, the term $D^2_{I,J}$ lies in the space A_m spanned by $\{\Omega_{a,b} |\, a+b=m\}$, for $m = |I|+|J|+2n$. Recall that for m even, $A_m = \partial^2 A_{m-2} \oplus \langle J^m \rangle$, and for m odd, $A_m = \partial^3 A_{m-3} \oplus \langle \partial J^{m-1}\rangle$. For m even (respectively odd), define $pr_m : A_m \to \langle J^m\rangle$ (respectively $pr_m : A_m \to \langle \partial J^{m-1}\rangle$) to be the projection onto the second term. Define the *remainder*

$$R_{I,J} = pr_m(D^2_{I,J}). \tag{26}$$

For all I, J, the remainder $R_{I,J}$ is independent of the choice of decomposition (25). There is a distinguished element $D_0 = D_{(0,\dots,n),(0,\dots,n)}$ which is the unique element of $\mathcal{I}_n$ of minimal weight n^2+3n+2, and we denote its remainder by R_0. Clearly R_0 is a scalar multiple of j^{n^2+3n}, and the condition $R_0 \neq 0$ is equivalent to the existence of a decoupling relation of the form

$$j^{n^2+3n} = P(j^0, j^2, \dots, j^{n^2+3n-2}) \tag{27}$$

in $\mathcal{H}(n)^{O(n)}$, where P is a normally ordered polynomial in $j^0, j^2, \dots, j^{n^2+3n-2}$ and their derivatives. In [LIII] we made the following conjecture:

Conjecture 5.2. *For all $n \geq 1$, $R_0 \neq 0$.*

If $R_0 \neq 0$, starting with (27), it is easy to construct higher decoupling relations

$$j^{2r} = Q_{2r}(j^0, j^2, \dots, j^{n^2+3n-2}), \tag{28}$$

for all $r > n^2+3n$, where Q_{2r} is a normally ordered polynomial in $j^0, j^2, \dots, j^{n^2+3n-2}$ and their derivatives. This implies that $\mathcal{H}(n)^{O(n)}$ has a minimal strong generating set $\{j^0, j^2, \dots, j^{n^2+3n-2}\}$, and in particular is a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,n^2+3n)$. For $n=1$, this was proven by Dong-Nagatomo in [DN]. By computer calculation we showed that for $n=2$ we have $R_0 = \frac{149}{600} J^{10}$, and for $n=3$ we have $R_0 = -\frac{2419}{705600} J^{18}$. However, we were unable to prove our conjecture in general.

For an arbitrary reductive group G of automorphisms of $\mathcal{H}(n)$, the structure of $\mathcal{H}(n)^G$ can be understood by decomposing $\mathcal{H}(n)^G$ as a module over $\mathcal{H}(n)^{O(n)}$. By Theorem 5.1 of [LIII], the Zhu algebra of $\mathcal{H}(n)^{O(n)}$ is abelian, which implies that all its irreducible, admissible modules are highest-weight modules. Moreover, $\mathcal{H}(n)^G$ decomposes as a direct sum of irreducible, highest-weight $\mathcal{H}(n)^{O(n)}$-modules. In fact, there is a finite set of $\mathcal{H}(n)^{O(n)}$-submodules of $\mathcal{H}(n)^G$ whose direct sum contains an (infinite) strong generating for $\mathcal{H}(n)^G$. By Lemma 6.2 of [LIII], this shows that $\mathcal{H}(n)^G$ is finitely generated as a vertex algebra. If we further assume Conjecture 5.2, it follows from Theorem 6.1 of [LIII] that $\mathcal{H}(n)^G$ is strongly finitely generated. The precise description of $\mathcal{H}(n)^{O(n)}$ as a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,n^2+3n)$ is not necessary; the proof of Theorem 6.1 of [LIII] only requires a strong finite generating set for $\mathcal{H}(n)^{O(n)}$ to exist.

6. INVARIANT SUBALGEBRAS OF $V_k(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra of dimension n, and let G be a reductive group of weight-preserving automorphisms of $V_k(\mathfrak{g})$, for all $k \in \mathbb{C}$. Since G acts on $\mathfrak{g}$ and preserves both the bracket and the bilinear form, G lies in $O(n)$, and therefore acts on $\mathcal{H}(n)$ as well. In this section, we will assume Conjecture 5.2, which implies the strong finite generation of $\mathcal{H}(n)^G$. We prove Theorem 1.1, which states that under this assumption, $V_k(\mathfrak{g})^G$ is strongly finitely generated for generic values of k. Since our conjecture is true for $n = 3$, and in particular $\mathcal{H}(3)^{O(3)}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, 6, \dots, 18)$, an immediate consequence is that for $\mathfrak{g} = \mathfrak{sl}_2$, $V_k(\mathfrak{sl}_2)^G$ is strongly finitely generated for generic values of k.

The key idea behind the proof of Theorem 1.1 that both $V_k(\mathfrak{g})$ and $\mathcal{H}(n)$ admit G-invariant, good increasing filtrations, and we have linear isomorphisms

$$\mathcal{H}(n)^G \cong gr(\mathcal{H}(n)^G) \cong gr(\mathcal{H}(n))^G \cong gr(V_k(\mathfrak{g})^G) \cong gr(V_k(\mathfrak{g}))^G \cong V_k(\mathfrak{g})^G$$

and isomorphisms of graded commutative rings

$$gr(V_k(\mathfrak{g}))^G \cong (Sym \bigoplus_{j\geq 0} V_j)^G \cong gr(\mathcal{H}(n))^G,$$

where $V_j \cong \mathbb{C}^n \cong \mathfrak{g}$ as G-modules, for all $j \geq 0$. Hence both $V_k(\mathfrak{g})^G$ and $\mathcal{H}(n)^G$ can be viewed as deformations of the same classical invariant ring $R = (Sym \bigoplus_{j\geq 0} V_j)^G$. By Theorem 4.1, there is a finite set $\{f_1, \dots, f_t\}$ of homogeneous generators for $(Sym \bigoplus_{j=0}^{n-1} V_j)^G$, whose polarizations generate R. Let $\{d_1, \dots, d_r\}$ be the set of distinct degrees of the elements $\{f_1, \dots, f_t\}$, with $d_1 < d_2 < \cdots < d_r$. Since the action of GL_∞ on R preserves degree, all polarizations of these elements must have degree d_j for some $j = 1, \dots, r$. As in Section 4, we may choose a minimal subset S of these polarizations, which generates R as a ∂-ring. It follows from Lemma 3.1 that the sets $T \subset \mathcal{H}(n)^G$ and $U \subset V_k(\mathfrak{g})^G$ corresponding to S under the linear isomorphisms $\mathcal{H}(n)^G \cong R \cong V_k(\mathfrak{g})^G$, are strong generating sets for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g})^G$, respectively.

Even though S is a minimal generating set for R as a ∂-ring, T and U need not be minimal strong generating sets for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g})^G$, respectively, as vertex algebras. In fact, since we are assuming the strong finite generation of $\mathcal{H}(n)^G$, there is a finite set

$$T' = \{p_1, \dots, p_s\},$$

which we may assume without loss of generality to be a subset of T, which strongly generates $\mathcal{H}(n)^G$. In other words, $\mathcal{H}(n)^G = \langle T' \rangle$, where $\langle T' \rangle$ denotes the space of normally ordered polynomials in $p_1, \dots, p_s$ and their derivatives. Let $S' \subset S \subset R$ be the subset corresponding to T' under the linear isomorphism $\mathcal{H}(n)^G \cong R$, and let $\langle S' \rangle$ denote the space of polynomials in the elements of S' and their derivatives. In general, S' will not generate R as a ∂-ring, so $\langle S' \rangle$ will be a proper subset of S. Since S is a minimal generating set for R as a ∂-ring, it follows that

$$(S \setminus S') \cap \langle S' \rangle = \emptyset. \tag{29}$$

Otherwise such elements could be removed from S and the resulting set would still generate R as a ∂-ring.

Let w_0 be the maximal weight of elements of T'. Without loss of generality, we may assume that T' contains all elements of T of weight at most w_0. Let $q \in T$ be a vertex operator of weight $w > w_0$ and degree d, which must coincide with d_j for some $j =$

$1, \dots, r$, since these are the only possible degrees of elements of T. Since T' is a strong generating set for $\mathcal{H}(n)^G$, there exists a decoupling relation

$$q = P(p_1, \dots, p_s) \tag{30}$$

where P lies in $\langle T' \rangle$. Without loss of generality, let us choose P so that its leading term has minimal degree e. We may write

$$P = \sum_{a=d_1}^{e} P^a,$$

where d_1 is the minimal degree of the elements of T, and $P^a \in \langle T' \rangle$ is a homogeneous, normally ordered polynomial of total degree a. (In other words, P^a is a normal ordering of some homogeneous polynomial of degree a in the elements of S' and their derivatives). Moreover, since there are only double contractions in the vertex algebra $\mathcal{H}(n)$, we may assume that the degree of each nontrivial term appearing in P has the same parity as d. In other words, $P^a = 0$ whenever $a - d \equiv 1$ modulo 2. In particular, e has the same parity as d. It is clear from (29) that $e > d$ and that the leading term P^e is a normal ordering of some relation in R; as is Section 4, we may view (30) as a quantum correction of this relation. In fact, we can rewrite (30) in the form

$$\sum_{a=d_1}^{e} A^a = 0, \qquad A^a = P^a \text{ for } a \neq d, \qquad A^d = P^d - q. \tag{31}$$

It is convenient to introduce the parameter k into our Heisenberg algebra $\mathcal{H}(n)$. We denote by $\mathcal{H}_k(n)$ the vertex algebra with generators $\alpha^i(z)$, $i = 1, \dots, n$, and OPE relations

$$\alpha^i(z)\alpha^j(w) \sim k\delta_{i,j}(z-w)^{-2}.$$

Of course the vertex algebras $\mathcal{H}_k(n)$ are all isomorphic for $k \neq 0$, and for any reductive $G \subset O(n)$, the invariant subalgebras $\mathcal{H}_k(n)^G$ are isomorphic as well. The generators of $\mathcal{H}_k(n)^G$ are the same for all $k \neq 0$, but since each double contraction among the generators of $\mathcal{H}_k(n)$ introduces a factor of k, the decoupling relation corresponding to (30) becomes

$$k^{\frac{1}{2}(e-d)} q = \sum_{a=d_1}^{e} k^{\frac{1}{2}(e-a)} P^a. \tag{32}$$

We use the same notation $T' = \{p_1, \dots, p_s\}$ for our strong generating set, regarded now as a subset of $\mathcal{H}_k(n)^G$. As above, we may rewrite (32) in the form

$$\sum_{a=d_1}^{e} k^{\frac{1}{2}(e-a)} A^a = 0. \tag{33}$$

Let $U' \subset U \subset V_k(\mathfrak{g})^G$ denote the subset corresponding to T' under the linear isomorphism $\mathcal{H}_k(n)^G \cong R \cong V_k(\mathfrak{g})^G$. Let $\tilde{p}_i \in U'$ be the elements corresponding to $p_i \in T'$, and let $\langle U' \rangle$ denote the space of normally ordered polynomials in $\tilde{p}_1, \dots, \tilde{p}_s$ and their derivatives. Given $\tilde{q} \in U$ corresponding to $q \in T$, we will use the modified decoupling relation (32) to construct an analogous relation

$$\lambda(k)\tilde{q} = \sum_{a=d_1}^{e} Q^a, \tag{34}$$

for $\tilde{q}$ in $V_k(\mathfrak{g})^G$. Here Q^a is a normally ordered polynomial of total degree a in the elements of U and their derivatives, and the coefficient $\lambda(k)$ is a polynomial in k whose leading term is $k^{\frac{1}{2}(e-d)}$. Unlike P^a which lies in $\langle T'\rangle$, Q^a need not lie in $\langle U'\rangle$, so (34) need not be a decoupling relation, and will require further modification. If we work in an orthonormal basis $\{X^{\xi_i}|\, i=1,\dots,n\}$ for $\mathfrak{g}$ relative to $\langle,\rangle$, the second-order poles in the OPEs of $X^{\xi_i}(z)X^{\xi_j}(w)$ and $\alpha^i(z)\alpha^j(w)$ are the same, namely $k\delta_{i,j}(z-w)^{-2}$. Therefore passing from $\mathcal{H}_k(n)$ to $V_k(\mathfrak{g})$ amounts to "turning on" the first-order polar part of the OPEs among the generators X^{ξ_i} of $V_k(\mathfrak{g})$.

Given a normally ordered polynomial $P\in\mathcal{H}_k(n)^G$ in the elements of T, let $\tilde{P}\in V_k(\mathfrak{g})^G$ denote the corresponding normally ordered polynomial in the elements of U. Since $gr(\mathcal{H}_k(n))^G\cong R\cong gr(V_k(\mathfrak{g}))^G$ as graded rings, $\tilde{P}^e$ is a normal ordering of a relation in R. As in Section 4, there exist $B^a\in V_k(\mathfrak{g})^G$ of degree a for $a=d_1,\dots,e$ with $B^e=\tilde{P}^e$, and $\sum_{a=d_1}^e B^a=0$. Note that the indices a need not be parity-homogenous because there are both single and double contractions in $V_k(\mathfrak{g})^G$. There is a lot of flexibility in the choice of these quantum corrections, and as we shall see, certain pieces have to be chosen with some care.

The important observation is that the coefficients of all monomials appearing in B^a are polynomials in k. The leading degree of such a coefficient, regarded as a polynomial in k, will be called the *k-degree*. It is clear from the OPE relations in $V_k(\mathfrak{g})$ that each double contraction among the generators X^{ξ_i}, which lowers the total degree by two, will introduce a factor of k, and each single contraction, which lowers the degree by one, will not introduce a factor of k. For each term B^a for which a has the the same parity as e, the highest k-degree that can occur is $\frac{1}{2}(e-a)$, which occurs only if every possible contraction was a double contraction. These terms are therefore the same as the corresponding terms in the relation (33) in $\mathcal{H}_k(n)^G$. More precisely, for all a of the same parity as e, we may choose B^a so that

$$B^a=k^{\frac{1}{2}(e-a)}\tilde{A}^a+\cdots\,, \tag{35}$$

where $(\cdots)$ consists of monomials whose coefficients have lower k-degree. Note also that when a has the same parity as e, the coefficient of each term appearing in B^{a-1} has k-degree at most $\frac{1}{2}(e-a)$.

In degree d, since $A^d=P^d-q$ and $P^d\in\langle T'\rangle$, it follows that

$$B^d=k^{\frac{1}{2}(e-d)}\tilde{P}^d-k^{\frac{1}{2}(e-d)}\tilde{q}+(\cdots), \tag{36}$$

where the coefficient of each term appearing in $(\cdots)$ has k-degree at most $\frac{1}{2}(e-d)-1$. If $\tilde{q}$ appears in $(\cdots)$, we can rewrite (36) in the form

$$B^d=k^{\frac{1}{2}(e-d)}\tilde{P}^d-\lambda(k)\tilde{q}+(\cdots),$$

where $\tilde{q}$ does not appear in $(\cdots)$, and $\lambda(k)$ is a polynomial in k whose leading term is $k^{\frac{1}{2}(e-d)}$. Finally, letting $Q^a=B^a$ for $a\neq d$ and $Q^d=B^d+\lambda(k)\tilde{q}$, we obtain a relation $\lambda(k)\tilde{q}=\sum_{a=d_1}^e Q^a$ of the form (34). Note that Q^d is independent of $\tilde{q}$, and whenever a and e have the same parity, Q^a satisfies

$$Q^a=k^{\frac{1}{2}(e-a)}\tilde{P}^a+\cdots\,,$$

where the coefficient of each term in $(\cdots)$ has lower k-degree. For $a<e$, the term $\tilde{P}^a$ lies in $\langle U'\rangle$, but the other terms appearing in Q^a need not lie in $\langle U'\rangle$, and can depend on other

elements of U. We need to systematically eliminate these variables for generic values of k, and construct a *decoupling* relation expressing $\tilde{q}$ as an element of $\langle U'\rangle$.

We begin with the elements of weight w_0+1. Recall that $d_1,\dots,d_r$ are the distinct degrees of the elements of U, with $d_1<d_2<\cdots<d_r$. For $j=1,\dots,r$, let $\{\tilde{q}_{j,1},\dots,\tilde{q}_{j,t_j}\}$ be the set of elements of U of weight w_0+1 and degree d_j. Let

$$\{q_{j,1},\dots,q_{j,t_j}\}\subset T\subset \mathcal{H}_k(n)^G, \qquad \{s_{j,1},\dots,s_{j,t_j}\}\subset S\subset R$$

be the sets corresponding to $\{\tilde{q}_{j,1},\dots,\tilde{q}_{j,t_j}\}$ under the linear isomorphisms $V_k(\mathfrak{g})^G\cong R\cong \mathcal{H}_k(n)^G$. First we consider the elements $\tilde{q}_{1,1},\dots,\tilde{q}_{1,t_1}$ of minimal degree d_1. For each of the corresponding elements $q_{1,i}\in T$, there is a decoupling relation

$$k^{\frac{1}{2}(e_{1,i}-d_1)}q_{1,i}=\sum_{a=d_1}^{e_{1,i}}k^{\frac{1}{2}(e_{1,i}-a)}P^a_{1,i} \tag{37}$$

in $\mathcal{H}_k(n)^G$ of the form (32), where $e_{1,i}$ has the same parity as d_1, and each $P^a_{1,i}$ lies in $\langle T'\rangle$. There is a corresponding relation

$$\lambda_{1,i}(k)\tilde{q}_{1,i}=\sum_{a=d_1}^{e_{1,i}}Q^a_{1,i} \tag{38}$$

in $V_k(\mathfrak{g})^G$ of the form (34), where the coefficient $\lambda_{1,i}(k)$ has k-degree $\frac{1}{2}(e_{1,i}-d_1)$, and $\tilde{q}_{1,i}$ does not appear in $Q^{d_1}_{1,i}$. Since the term $P^{d_1}_{1,i}$ appearing in (37) lies in $\langle T'\rangle$, and $q_{1,j}$ lies in $T\setminus T'$ for $j=1,\dots,t_1$, it follows from (29) that $q_{1,j}$ does not appear in $P^{d_1}_{1,i}$. Therefore the coefficient of $\tilde{q}_{1,j}$ in $Q^{d_1}_{1,i}$ for $j\neq i$ can have k-degree at most $\frac{1}{2}(e_{1,i}-d_1)-1$. We may write $Q^{d_1}_{1,i}$ in the form

$$Q^{d_1}_{1,i}=\sum_{j=1}^{t_1}\epsilon_{i,j}(k)\tilde{q}_{1,j}+\cdots,$$

where $\epsilon_{i,i}(k)=0$, and the term $(\cdots)$ lies in $\langle U'\rangle$. Clearly each $\epsilon_{i,j}(k)$ for $j\neq i$ has k-degree at most $\frac{1}{2}(e_{1,i}-d_1)-1$. We can rewrite these relations in the form of a linear system of t_1 equations

$$\sum_{j=1}^{t_1}\mu_{i,j}(k)\tilde{q}_{1,j}=R_{1,i}, \qquad i=1,\dots,t_1. \tag{39}$$

In this notation, $\mu_{i,j}(k)=\epsilon_{i,j}(k)$ for $j\neq i$, and $\mu_{i,i}(k)=-\lambda_{1,i}(k)$. Moreover, the vertex operator $R_{1,i}=\sum_{a=d_1}^{e_{1,i}}R^a_{1,i}$ has the property that $R^{d_1}_{1,i}\in\langle U'\rangle$, since all the elements of degree d_1 that do *not* lie in $\langle U'\rangle$ now appear on the left hand side of (39). In fact, we claim that for $d_1<a<d_2$, each term $R^a_{1,i}$ lies in $\langle U'\rangle$. Since every element of U has degree d_i for some $i=1,\dots,r$, and $d_1<a<d_2$, it follows that $R^a_{1,i}$ is a normally ordered polynomial in the degree d_1 elements of U and their derivatives. Since $a>d_1$, each term in $R^a_{1,i}$ must be *nonlinear* in these generators, and in particular a must be a multiple of d_1. Since the weight of $R^a_{1,i}$ is w_0+1, each factor of degree d_1 appearing in each monomial can have weight at most w_0, and hence will lie in $\langle U'\rangle$.

Since the diagonal entries $\mu_{i,i}(k)$ appearing in (39) have k-degree $\frac{1}{2}(e_{1,i}-d_1)$, and all other entries $\mu_{i,j}(k)$ for $j\neq i$ have lower k-degree, the determinant of this system is a nonzero polynomial of leading k-degree $\sum_{i=1}^{t_1} \frac{1}{2}(e_{1,i} - d_1)$. Hence this system is invertible over the fraction field $\mathbb{C}(k)$, so we obtain new relations of the form

$$\tilde{q}_{1,i} = \sum_{a=d_1}^{e'_{1,i}} S^a_{1,i}, \qquad i = 1, \dots, t_1. \tag{40}$$

Here $S^a_{1,i}$ is a normally ordered polynomial in the elements of U and their derivatives, which is homogeneous of degree a. By construction, $S^a_{1,i}$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_2$. Note that the indices $e'_{1,i}$ need not be the same as the indices $e_{1,i}$ appearing in the old relations (38).

Clearly the higher terms $S^a_{1,i}$ for $a \geq d_2$ need not lie in $\langle U' \rangle$, so we need to iterate this procedure and further modify the relations (40). Recall that $\tilde{q}_{2,1}, \dots, \tilde{q}_{2,t_2} \in U$ are the elements of degree d_2 and weight $w_0 + 1$, and $q_{2,1}, \dots, q_{2,t_2}$ are the corresponding elements of $T \subset \mathcal{H}_k(n)^G$. There are decoupling relations in $\mathcal{H}_k(n)^G$

$$k^{\frac{1}{2}(e_{2,i} - d_2)} q_{2,i} = \sum_{a=d_1}^{e_{2,i}} k^{\frac{1}{2}(e_{2,i} - a)} P^a_{2,i} \tag{41}$$

of the form (32), where $e_{2,i}$ has the same parity as d_2, and $P^a_{2,i}$ is a homogeneous normally ordered polynomial of degree a in $\langle T' \rangle$. The corresponding relation in $V_k(\mathfrak{g})^G$ is of the form

$$\lambda_{2,i}(k) \tilde{q}_{2,i} = \sum_{a=d_1}^{e_{2,i}} Q^a_{2,i} \tag{42}$$

where $\lambda_{2,i}(k)$ has k-degree $\frac{1}{2}(e_{2,i} - d_2)$. Moreover, the leading term $Q^{e_{2,i}}_{2,i} = \tilde{P}^{e_{2,i}}_{2,i}$, and for each a of the same parity as d_2, we have $Q^a_{2,i} = k^{\frac{1}{2}(e_{2,i} - a)} \tilde{P}^a_{2,i} + \cdots$, where the coefficient of each monomial appearing in $(\cdots)$ has k-degree at most $\frac{1}{2}(e_{2,i} - a) - 1$. In degree d_2, $Q^{d_2}_{2,i}$ is independent of $\tilde{q}_{2,i}$, but may depend on the elements $\tilde{q}_{2,j}$ for $j \neq i$. However, the coefficient of $\tilde{q}_{2,j}$ in $Q^{d_2}_{2,i}$ will have k-degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$.

By degree and weight considerations, the terms $Q^a_{2,i}$ for $d_1 < a < d_2$ each lie in $\langle U' \rangle$. The argument is the same as the proof of the corresponding statement for $R^a_{1,i}$. On the other hand, the degree d_1 term $Q^{d_1}_{2,i}$ in (42) need not lie in $\langle U' \rangle$, and may depend on the elements $\tilde{q}_{1,1}, \dots, \tilde{q}_{1,t_1}$. We need to eliminate these terms using the relations (40) constructed above. Suppose first that d_1 and d_2 have the same parity. Since the degree d_1 term $P^{d_1}_{2,i} \in \mathcal{H}_k(n)^G$ in (41) lies in $\langle T' \rangle$, it follows that the coefficient of each term $\tilde{q}_{1,1}, \dots, \tilde{q}_{1,t_1}$ appearing in $Q^{d_1}_{2,i}$ can have k-degree at most $\frac{1}{2}(e_{2,i} - d_1) - 1$. If $\tilde{q}_{1,l}$ appears in $Q^{d_1}_{2,i}$ for some $l = 1, \dots, t_1$, we can use (40) in the case $i = l$ to eliminate it. Of course this can introduce a new multiple of $\tilde{q}_{2,j}$ for $j = 1, \dots, t_2$, in degree d_2. However, the maximal k-degree of terms appearing in $S^{d_2}_{1,l}$ is $-\frac{1}{2}(d_2 - d_1)$, which is a negative integer since (40) has been normalized so that the k-degree of the left hand side is zero. Hence the coefficients of these new multiples of $\tilde{q}_{2,j}$ must be rational functions of k, of k-degree at most

$$\frac{1}{2}(e_{2,i} - d_1) - 1 - \frac{1}{2}(d_2 - d_1) = \frac{1}{2}(e_{2,i} - d_2) - 1.$$

Since the coefficient $\lambda_{2,i}(k)$ of $\tilde{q}_{2,i}$ in (42) has k-degree $\frac{1}{2}(e_{2,i} - d_2)$, any multiple of $\tilde{q}_{2,i}$ introduced by our procedure of eliminating the degree d_1 terms $\tilde{q}_{1,l}$ cannot cancel the term $\lambda_{2,i}(k)\tilde{q}_{2,i}$. So we can collect terms involving $\tilde{q}_{2,i}$ and rewrite our relation in the form

$$\nu_{2,i}(k)\tilde{q}_{2,i} = \sum_{a=d_1}^{e'_{2,i}} R^a_{2,i}, \tag{43}$$

where the coefficient $\nu_{2,i}(k)$ is a rational function of k-degree $\frac{1}{2}(e_{2,i} - d_2)$, and $R^a_{2,i}$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_2$. Note that our elimination procedure may have introduced terms in degree greater than $e_{2,i}$, so it is possible that $e'_{2,i} > e_{2,i}$. Finally, the degree d_2 term $R^{d_2}_{2,i}$ is independent of $\tilde{q}_{2,i}$, and still has the property that the coefficient of any term of the form $\tilde{q}_{2,j}$ for $j \neq i$ has k-degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$.

Next, suppose that d_1 and d_2 have opposite parity. Since the parity of $e_{2,i}$ is the same as the parity of d_2 (and hence opposite to the parity of d_1), the coefficient of each term in $Q^{d_1}_{2,i}$ can have k-degree at most $\frac{1}{2}(e_{2,i} - d_1 - 1)$. Any terms appearing in $Q^{d_1}_{2,i}$ of the form $\tilde{q}_{1,l}$ can be eliminated using the relations (40). Since the maximal k-degree of terms appearing in $S^{d_2}_{1,l}$ is $-\frac{1}{2}(d_2 - d_1 + 1)$, this procedure may introduce new terms of the form $\tilde{q}_{2,j}$ in degree d_2, but their coefficients will be rational functions in k, of k-degree at most

$$\frac{1}{2}(e_{2,i} - d_1 - 1) - \frac{1}{2}(d_2 - d_1 + 1) = \frac{1}{2}(e_{2,i} - d_2) - 1.$$

Thus we can collect terms involving $\tilde{q}_{2,i}$ as above, and we obtain a similar relation of the form (43), where $R^a_{2,i}$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_2$. As above, $R^{d_2}_{2,i}$ is independent of $\tilde{q}_{2,i}$, but may depend on $\tilde{q}_{2,j}$ for $j \neq i$. However, the coefficient of such terms can have k-degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$.

Regardless of whether d_1 and d_2 have the same or opposite parity, each monomial appearing in $R^a_{2,i}$ for $d_2 < a < d_3$ must be nonlinear in the elements of U of degrees d_1 or d_2, and their derivatives. Since the weight of each factor can be at most w_0, it follows that $R^a_{2,i} \in \langle U' \rangle$ for $d_2 < a < d_3$. The degree d_2 term $R^{d_2}_{2,i}$ need not lie in $\langle U' \rangle$, but we can eliminate the terms $\tilde{q}_{2,j}$ for $j \neq i$ which do not lie in $\langle U' \rangle$ using the same procedure we used in the degree d_1 case. First, we rewrite the relations (43) in the form of a linear system of t_2 equations

$$\sum_{j=1}^{t_2} \mu_{i,j}(k)\tilde{q}_{2,j} = S_{2,i}, \qquad i = 1, \dots, t_2.$$

Here $\mu_{i,i}(k) = -\nu_{2,i}(k)$, which has k-degree $\frac{1}{2}(e_{2,i} - d_2)$, and $\mu_{i,j}(k)$ has k-degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$ for $j \neq i$. Moreover, the vertex operators $S_{2,i} = \sum_{a=d_1}^{e'_{2,i}} S^a_{2,i}$ have the property that $S^a_{2,i} \in \langle U' \rangle$ for $d_1 \leq a < d_3$. As in the case of (39), the corresponding determinant is invertible over $\mathbb{C}(k)$, so we obtain new relations

$$\tilde{q}_{2,i} = \sum_{a=d_1}^{e''_{2,i}} T^a_{2,i}$$

for $i = 1, \dots, t_2$, where by construction, $T^a_{2,i}$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_3$. Note that the upper indices $e''_{2,i}$ may be different from $e'_{2,i}$. Of course $T^a_{2,i}$ need not lie in $\langle U' \rangle$ for $a \geq d_3$.

Now, using the same procedure, for each element $\tilde{q}_{3,1}, \dots, \tilde{q}_{3,t_3} \in U$ of weight w_0+1 and degree d_3, we can construct a relation of the form

$$\tilde{q}_{3,i} = \sum_{a=d_1}^{e_{3,i}} T^a_{3,i},$$

where $T^a_{3,i} \in \langle U' \rangle$ for $d_1 \leq a < d_4$. Continuing this process, in degree d_{j-1}, each element $\tilde{q}_{j-1,i}$ for $i = 1, \dots, t_{j-1}$ satisfies a relation

$$\tilde{q}_{j-1,i} = \sum_{a=d_1}^{e_{j-1,i}} T^a_{j-1,i}, \tag{44}$$

where $T^a_{j-1,i} \in \langle U' \rangle$ for $d_1 \leq a < d_j$. Finally, in the highest degree d_r among elements of U, we have honest decoupling relations

$$\tilde{q}_{r,i} = \sum_{a=d_1}^{e_{r,i}} T^a_{r,i}$$

for $i = 1, \dots, t_r$, where each $T^a_{r,i}$ lies in $\langle U' \rangle$. Now we can substitute these relations into the relations (44) for $\tilde{q}_{r-1,i}$ in the case $j = r$, eliminating all variables of the form $\tilde{q}_{r,i}$ that occur in $T^a_{r-1,i}$ for $a \geq d_r$. This produces decoupling relations expressing each $\tilde{q}_{r-1,i}$ as an element of $\langle U' \rangle$, for $i = 1, \dots, t_{r-1}$.

Inductively, once we have constructed decoupling relation for all elements of the form $\tilde{q}_{k,l}$ with $j \leq k \leq r$ and $l = 1, \dots, t_k$, we can eliminate all variables of the form $\tilde{q}_{k,l}$ for $j \leq k \leq r$ in the relations (44) for $\tilde{q}_{j-1,i}$. We obtain decoupling relations expressing each $\tilde{q}_{j-1,i}$ for $i = 1, \dots, t_{j-1}$, as an element of $\langle U' \rangle$. At the end of this procedure, we have a decoupling relation for each element of U of weight w_0+1. We repeat this entire procedure for weights w_0+2, w_0+3, etc. This shows that if we work over the field $\mathbb{C}(k)$, we can find a complete set of decoupling relations, expressing each element of U as an element of $\langle U' \rangle$.

For arbitrary k, let $\mathcal{W}(\mathfrak{g}, G, k)$ denote the vertex subalgebra of $V_k(\mathfrak{g})^G$ generated by U'. Clearly $\langle U' \rangle \subset \mathcal{W}(\mathfrak{g}, G, k)$, and it is easy to see that $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, G, k)$ precisely when $\mathcal{W}(\mathfrak{g}, G, k)$ is *strongly* generated by U'. In each weight w, the construction of our decoupling relations required us to invert a finite set of determinants over $\mathbb{C}(k)$ which are certain rational function of k, so finitely many values of k must be excluded. Taking the union of these excluded values over all weights, it follows that the set of k for which the equality

$$\langle U' \rangle = \mathcal{W}(\mathfrak{g}, G, k) = V_k(\mathfrak{g})^G \tag{45}$$

can fail is at most countable. Therefore (45) holds for generic values of k. This completes the proof of Theorem 1.1.

A *deformable $\mathcal{W}$-algebra* is a family of conformal vertex algebras $\mathcal{W}_c$ where c is the Virasoro central charge, equipped with strong generating sets $A_c = \{a^c_1, \dots, a^c_r\}$, whose structure constants are continuous functions of c with isolated singularities. Recall that the structure constants are the coefficients of each normally ordered monomial in the elements of A_c and their derivatives, which appear in the OPE of $a^c_i(z)a^c_j(w)$, for $i, j = 1, \dots, r$. Assuming the strong finite generation of $\mathcal{H}(n)^{O(n)}$, we claim that $\mathcal{W}(\mathfrak{g}, G, k)$ is a deformable $\mathcal{W}$-algebra. Let $E \subset \mathbb{C}$ be the set of all values of k for which $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, G, k)$.

For $k \in E$, the structure constants of $\mathcal{W}(\mathfrak{g}, G, k)$ are clearly rational functions of k. Since there are only finitely many structure constants, there can be only finitely many singular points $k_1, \dots, k_t$, and we have $E = \mathbb{C} \setminus \{k_1, \dots, k_t\}$. The claim follows from the fact that the Virasoro central charge c is given by $\frac{k \cdot \dim(\mathfrak{g})}{k+h^\vee}$, so k is a rational function of c. In the case $\mathfrak{g} = \mathfrak{sl}_2$, Corollary 1.2 shows that $\mathcal{W}(\mathfrak{sl}_2, G, k)$ is indeed a deformable $\mathcal{W}$-algebra for any reductive group G of automorphisms of $V_k(\mathfrak{sl}_2)$.

To the best of our knowledge, Corollary 1.2 is the first result of its kind asserting that invariant subalgebras of $V_k(\mathfrak{g})$ for some simple $\mathfrak{g}$ are strongly finitely generated. We emphasize that this is a very preliminary result. A priori, it is possible that the set D on which (45) holds, is a proper subset of E. In other words, there may be values of k for which $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, G, k)$, but $\mathcal{W}(\mathfrak{g}, G, k)$ is a proper subset of $V_k(\mathfrak{g})^G$. Moreover, for $k \notin D$, it is possible that $V_k(\mathfrak{g})^G$ is still strongly finitely generated, even though U' will not be a strong generating set in this case. We hope to study the structure and representation theory of the vertex algebras $V_k(\mathfrak{g})^G$ in more detail in future work.

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