INDEX THEOREMS ON MANIFOLDS WITH STRAIGHT ENDS.

WERNER BALLMANN, JOCHEN BRÜNING, AND GILLES CARRON

Abstract. We study Fredholm properties and index formulas for Dirac operators over complete Riemannian manifolds with straight ends. An important class of examples of such manifolds are complete Riemannian manifolds with pinched negative sectional curvature and finite volume.

Date: November 11, 2010.
1991 *Mathematics Subject Classification.* 53C20, 58J20.
Key words and phrases. Dirac operator, Fredholm type, non-parabolic, index.

Contents

1. Introduction

The celebrated Atiyah-Singer index theorem establishes a connection between analysis, geometry, and topology of closed manifolds. It contains the Gauss-Bonnet formula, Hirzebruch's signature theorem, and the Hirzebruch-Riemann-Roch formula as special cases. Later, Atiyah, Patodi, and Singer found a generalization of the index theorem for certain first order differential operators on compact manifolds with boundary [APS1]. In this article, they also discuss index theory for their class of operators on non-compact manifolds with cylindrical ends, and our work builds on that part of their work.

It is obvious that the structure of the underlying manifold and of the differential operator close to infinity plays an important role in this theory. Without restrictions on these data, not much can be expected.

Motivated by previous work of Barbasch-Moscovici [BaMo], Lott [Lo1, Lo2], and the first two authors [BB1, BB2], our main objective are Dirac operators on complete manifolds with pinched negative sectional curvature and finite volume. The structure of the ends of such manifolds has been determined by Patrick Eberlein and is related to the existence of so-called strictly invariant horospheres, see [Eb].

To set the stage, let M be a complete and connected Riemannian manifold of dimension m with Levi-Civita connection ∇ and curvature tensor R. Let $E \to M$ be a complex Dirac bundle[1] with Hermitian connection ∇^E, curvature tensor R^E, and Dirac operator D. For convenience, we assume throughout that R and R^E are uniformly bounded,

$$|R(X,Y)Z| \le C_R |X||Y||Z|, \quad |R^E(X,Y)\sigma| \le C_R^E |X||Y||\sigma|, \tag{1.1}$$

for all vector fields X, Y, Z on M and sections σ of E. The bound on R is equivalent to assuming a uniform bound on the modulus of the sectional curvature K_M of M.

Recall that D is an elliptic differential operator of first order. Consider D as an unbounded operator on $L^2(M,E)$ with domain $C_c^\infty(M,E)$, where $L^2(M,E)$ denotes the space of square-integrable sections of E and $C_c^\infty(M,E)$ the space of smooth sections of E with compact support, and note that D is symmetric on the latter. The closure of D has domain $H^1(M,E)$, by (1.1) and the general Bochner identity, see (2.13) and (2.14). Furthermore, $D : H^1(M,E) \to L^2(M,E)$ is self-adjoint, see [Wo] or Theorem II.5.7 in [LaMi].

We may ask under which conditions $D : H^1(M,E) \to L^2(M,E)$ is a Fredholm operator. By self-adjointness, this is the case if and only if 0 is not in the essential spectrum of D; according to a result of Nicolae

[1] in the sense of Gromov and Lawson, compare Section 2.1

Anghel, this holds if and only if there is a compact subset L in M and a constant $C = C(L)$ such that

$$\|\sigma\|_{L^2(M,E)} \leq C\|D\sigma\|_{L^2(M,E)}, \tag{1.2}$$

for all smooth sections σ of E with compact support in $M \setminus L$, see [An]. If such an estimate holds, we say that D is of *Fredholm type*.

Better adapted to our investigations and more flexible is a somewhat weaker notion, introduced by the third named author in [Ca1]:

Definition 1.3. We say that D is *non-parabolic at infinity* if there is a compact subset L in M such that, for any relatively compact open subset K of M, there is a constant $C = C(K, L)$ such that

$$\|\sigma\|_{L^2(K,E)} \leq C\|D\sigma\|_{L^2(M,E)}, \tag{1.4}$$

for all smooth sections σ of E with compact support in $M \setminus L$.

It follows from [Ca1, Théorème 1.2] that D is non-parabolic at infinity if and only if there is a Hilbert space W of sections of E which are locally H^1, such that $H^1(M,E)$ is a dense subspace of W, such that the inclusions

$$H^1(M,E) \subseteq W \subseteq H^1_{\mathrm{loc}}(M,E) \tag{1.5}$$

are continuous, and such that the extension

$$D_{\mathrm{ext}} : W \to L^2(M,E) \tag{1.6}$$

of D to W is a Fredholm operator. Here we note that, by the second inclusion in (1.5), D defines a continuous operator on W. It then follows that $H^1(M,E) = W$ if and only if D is of Fredholm type[2].

If D is non-parabolic at infinity, with associated Hilbert space W, then elements of $\ker D_{\mathrm{ext}}$ will be called *extended solutions* of D. In the case of cylindrical ends, they correpond exactly to the extended solutions in [APS1]. By the density of $C^\infty_c(M,E)$ in W, the orthogonal complement of the image of D_{ext} in $L^2(M,E)$ is equal to the space of L^2-solutions of D. Since D_{ext} is a Fredholm operator, the spaces of extended solutions and L^2-solutions of D are of finite dimension, and their difference, $\operatorname{ind} D_{\mathrm{ext}}$, is called the *extended index* of D. As a consequence of one of our main results concerning non-parabolicity, Theorem 1.14 below, we obtain the following assertion:

Theorem 1.7. *If the sectional curvature of M is negatively pinched and the volume of M is finite, then D is non-parabolic at infinity. In particular, the space of L^2-solutions of D is finite-dimensional.*

---

[2] Compare Section 5.

Under a more general assumption on the geometry of the ends of M, similar to Condition (1) in Theorem 1.13 below, John Lott showed that the space of square-integrable harmonic differential forms is finite dimensional, see Theorem 1 in [Lo1].

For manifolds with ends as in the case of finite volume manifolds of pinched negative sectional curvature, Lott also discusses the essential spectrum of $(d + d^*)^2$ on the space of differential forms, see Theorem 2 in [Lo1]. Under the same assumption on the geometry of the ends and for Dirac bundles as in Condition (2) of Theorem 1.13 below, he investigates the essential spectrum of the associated Dirac operator, see Theorem 5 in [Lo2]. Similar results have been obtained in [BB2]. In this article, we do not concentrate on the essential spectrum, but would like to mention that our investigations lead to extensions of these results.

It is clear from the definition of non-parabolicity that it only depends on the structure of D at infinity[3]. To state our results in that context, we need to introduce a further notion.

DEFINITION 1.8. We say that the ends of M are *straight* if M can be decomposed into a compact part M_0 and an unbounded part U_0 with common boundary N such that there is an open set $U \supseteq U_0$ and a C^2 distance function[4] $f : U \to \mathbb{R}$ whose gradient flow establishes a C^1 diffeomorphism

$$F : (-r, \infty) \times N \to U, \tag{1.9}$$

where $r > 0$, $U_0 = f^{-1}([0, \infty))$, $N = f^{-1}(0)$, and $f(F(t, x)) = t$. In this situation, we say that the ends of M are *smooth* if f is smooth.

If the ends of M are straight, then M is diffeomorphic to the interior of the compact manifold M_0, and the connected components of N correspond to the different ends of M. Furthermore, the induced Riemannian metric on $\mathbb{R}_+ \times N$ is of the form

$$dt^2 + g_t, \tag{1.10}$$

where $(g_t)_{t \geq 0}$ is a family of Riemannian metrics on N. The regularity of this family is a technical problem which we address in Section 3.2 and which motivated our previous work [BBC2] on Dirac systems with Lipschitz coefficients. *Cylindrical ends* as mentioned above correspond to the case of Riemannian products, that is, f is smooth and $g_t = g_0$, for all $t \in (-r, \infty)$.

[3] The same applies to the essential spectrum of D.

[4] Compare Section 3.2.

If the ends of M are straight, we fix the setup as in Definition 1.8, identify $(-r,\infty)\times N\simeq U$ via F, and call the hypersurfaces $N_t = f^{-1}(t)$, endowed with the Riemannian metric g_t, the *cross sections* of U. For convenience, we will always assume in this situation that the second fundamental forms $W = W_t$ of the cross sections are uniformly bounded,

$$|WX| \le C_W |X|, \tag{1.11}$$

for all vector fields X on U.

DEFINITION 1.12. Let $\varepsilon > 0$. We say that the ends of M are *ε-thin* if they are straight and the connected components of the cross sections N_t have diameter at most ε, for all sufficiently large t. We say that the ends of M are *cuspidal* if they are straight and there are positive constants c and C such that the metrics g_t as in (1.10) satisfy $g_t \le Ce^{c(s-t)} g_s$, for all sufficiently large $s < t$.

For example, if M has finite volume and pinched negative sectional curvature, say $-b^2 \le K_M \le -a^2 < 0$, then the ends of M are cuspidal with $c = 2a$ and $C = 1$. We note that, in this example, the distance function arises from Busemann functions on the universal covering space of M and that such Busemann functions are C^2, see [HeIH] or Proposition IV.3.2 in [Ba]. Better regularity is, in general, not expected and, at least for non-positively curved manifolds, better regularity does not hold, see [BBB].

THEOREM 1.13. *There is a positive constant $\varepsilon = \varepsilon(m, C_R, C_W)$ such that D is non-parabolic at infinity if the following two conditions hold:*

1. *All ends of M are ε-thin, for all sufficiently large t.*
2. *E is a Hermitian vector bundle associated to M via a unitary representation of* $\mathrm{O}(m)$, $\mathrm{SO}(m)$ *(if M is oriented), or* $\mathrm{Spin}(m)$ *(with respect to a spin structure of M), respectively.*

Extending Theorem 1.7 above, we also have:

THEOREM 1.14. *If the ends of M are cuspidal, then D is non-parabolic at infinity.*

Suppose from now on that D is non-parabolic at infinity so that we have the corresponding Fredholm operator $D_{\mathrm{ext}} : W \to L^2(M, E)$ as above. If, in addition, the dimension m of M is even and $E = E^+ \oplus E^-$ is a super-symmetry[5], then $W = W^+ \oplus W^-$, where $W^\pm$ consists of those

[5] See Section 2.1.

sections in W which take values in $E^{\pm}$. Restricting D_{ext} to W^+, we obtain a Fredholm operator

$$D_{\text{ext}}^+ : W^+ \to L^2(E^-). \tag{1.15}$$

In the case of closed manifolds, this is the operator the index theorem is concerned with. The local index theorem associates an index form ω_{D^+} to the differential operator D^+, determined by local data of D^+, whose evaluation is equal to the index of D^+. In the following result we introduce the notation which we use in the statements of our results on explicit index formulas.

Proposition 1.16. *If M has at most finitely many ends, D is non-parabolic at infinity, and ω_{D^+} is integrable, then*

$$\operatorname{ind} D_{\text{ext}}^+ = \int_M \omega_{D^+} + \sum\nolimits_{\mathcal{C}} \operatorname{Corr}(\mathcal{C}),$$

where ω_{D^+} is the index form associated to D^+, $\mathcal{C}$ runs over the ends of M, and $\operatorname{Corr}(\mathcal{C})$ *is a correction term determined by the end $\mathcal{C}$.*

Proposition 1.16 is a kind of relative index theorem and, assuming the non-parabolicity of D_{ext}^+, can also be proved along the lines of relative index formulas as in Theorem 4.18 in [GrLa] (see also Proposition 4.33), Theorem 6.2 in [Do], or Theorem 0.5 in [Ca1].

Clearly, the assumptions of Proposition 1.16 are satisfied if the ends of M are cuspidal. We assume the latter in the following discussion.

In dimension $m = 2$, the correction terms are known explicitly in terms of the type of E along the ends, see [BB1]. In higher dimensions and under strong pinching assumptions on the sectional curvature of M, they are known explicitly for the Gauss-Bonnet operator, see [BB2].

The most important class of examples to which our results apply are finite volume quotients of symmetric spaces of negative sectional curvature, that is, of real, complex, or quaternionic hyperbolic spaces or of the Cayley hyperbolic plane. The work of Barbasch-Moscovici [BaMo] is a milestone in the index theory of Dirac operators of homogeneous Dirac bundles over such spaces. Their arguments rely on harmonic analysis on symmetric spaces, notably the Selberg trace formula. Our approach is different in nature. Applying our results from [BBC2], we are able to discuss the contribution of each end individually. This leads to a more general setting and more transparent index formulas. Note, in particular, that our results also apply in the case where D is not of Fredholm type.

In this article, we concentrate on complex hyperbolic cusps, more precisely, cusps as they arise for quotients of complex hyperbolic space of dimension $m = 2n$,

$$\mathbb{C}H^n = \mathrm{SU}(1,n)/\,\mathrm{U}(n), \tag{1.17}$$

by neat lattices [6]. To that end, we also write $\mathbb{C}H^n \simeq S = \mathbb{R} \ltimes G_{n-1}$, where G_{n-1} is the Heisenberg group of dimension $2n-1$ and S is the solvable extension of G_{n-1} induced by the automorphism of G_{n-1} which is equal to multiplication by 2 on the center of the Lie algebra of G_{n-1}, where S is endowed with an appropriate left-invariant Riemannian metric. We assume that the cusp is given as

$$\mathcal{C} = \Gamma\backslash((-r,\infty) \times G_{n-1}) \subseteq \Gamma\backslash S, \tag{1.18}$$

where Γ is a uniform lattice in G_{n-1}, and this holds for cusps of quotients of $\mathbb{C}H^n$ by neat lattices, see lines 4 and 5 on page 193 in [BaMo]. We consider Dirac bundles E over $\mathcal{C}$ which are associated to $\Sigma \otimes V$, where $\Sigma = \Sigma_{2n}$ is the spin representation and V is an irreducible unitary representation of $\mathfrak{u}(n)$. By Theorems 7.27, 10.47, 10.72, and Corollary 9.24 we have that, for odd n,

$$\mathrm{Corr}(\mathcal{C}) = \frac{1}{2} \sum_{0 \le k \le n-1} \varepsilon_k b_k. \tag{1.19}$$

Here

$$b_k := (n-1)! \dim V \prod_{\substack{1 \le j \le m/2 \\ j \ne k+1}} |\lambda_j - \lambda_{k+1} + k + 1 - j|^{-1} \in \mathbb{N}_0, \tag{1.20}$$

where $\lambda_1 \ge \ldots \ge \lambda_n$ denotes the heighest weight of the representation V. Furthermore,

$$\varepsilon_k := \begin{cases} (-1)^k & \text{if } n-1-2k+2\lambda_{k+1} > 0, \\ (-1)^{k+1} & \text{if } n-1-2k+2\lambda_{k+1} < 0, \\ \quad 1 & \text{if } n-1-2k+2\lambda_{k+1} = 0. \end{cases} \tag{1.21}$$

For even n, we have

$$\mathrm{Corr}(\mathcal{C}) = \dim V |\Gamma| \zeta(1-n) + \frac{1}{2} \sum_{0 \le k \le n-1} \varepsilon_k b_k, \tag{1.22}$$

where $|\Gamma| \in \mathbb{N}$ is an invariant of the fundamental group $\Gamma = \Gamma_{\mathcal{C}}$ of the cusp, compare (9.1).

A specific case where these formulas apply is the *Dolbeault operator* on forms of type $(0,q)$, $0 \le q \le n$, on a finite volume quotient X of complex hyperbolic space $\mathbb{C}H^n$ by a neat lattice. Here V is of dimension

[6] Here *neat* means that the group generated by the eigenvalues of any non-identity element of the given lattice contains no roots of unity. Neat lattices are torsion-free.

1 with highest weight $\lambda_j = (n+1)/2$, $1 \leq j \leq n$. In Example 2 of Section 10.3 we explain that the Dolbeault operator is of Fredholm type and that (for each end)

$$b_k = \binom{n-1}{k}. \tag{1.23}$$

In particular, $\sum \varepsilon_k b_k = 0$. Using the Hirzebruch proportionality principle, Theorem 1.16, (1.19), and (1.22), we obtain the following result about the L^2-arithmetic genus.

THEOREM 1.24. *If X is a quotient of complex hyperbolic space $\mathbb{C}H^n$ by a neat lattice, then the Dolbeault operator on X is of Fredholm type and its index $\chi_{L^2}(X, \mathcal{O})$ is given by*

$$\chi_{L^2}(X,\mathcal{O}) = (-1)^n \frac{\operatorname{vol} X}{\operatorname{vol} \mathbb{C}P^n} + \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \zeta(1-n)\sum_{\mathcal{C}} |\Gamma_{\mathcal{C}}| & \text{if } n \text{ is even.} \end{cases}$$

Another basic example is the *signature operator* on X when n is even, that is, when m is a multiple of 4. In this case, V is actually a non-trivial sum of irreducible representations of $\mathfrak{u}(n)$, namely $V = V_0 \oplus \cdots \oplus V_n$, where V_l is the irreducible representation of $\mathfrak{u}(n)$ with highest weight $\lambda_j = l - (n-1)/2$ for $1 \leq j \leq l$ and $\lambda_j = l - (n+1)/2$ for $l < j \leq n$. From Example 3 of Section 10.3, Theorem 1.16, and (1.22), we obtain the following result.

THEOREM 1.25. *If X is a quotient of complex hyperbolic space $\mathbb{C}H^n$ by a neat lattice, where n is even, then the signature operator on X is of Fredholm type and its index $\sigma(X)$ is given by*

$$\sigma(X) = \frac{\operatorname{vol} X}{\operatorname{vol} \mathbb{C}P^n} + 2^n \zeta(1-n) \sum_{\mathcal{C}} |\Gamma_{\mathcal{C}}| + \nu(-1)^{n/2}\left(\binom{n-2}{n/2} - \binom{n-2}{n/2-1}\right),$$

where ν is equal to the number of ends of X.

Formulas for $\sigma(X)$ are also stated in Theorem 7.6 of [BaMo] and Stern's article [St] (compare Formula 6.4 there). Our correction terms consist of two terms: What we call the *high energy η-invariant*[7] can be identified with a zeta contribution in [St] and with the unipotent contribution in the Arthur-Selberg trace formula in [BaMo]. Our *low energy η-invariant* corresponds to the eta term in [St] and the weighted unipotent contribution in [BaMo]. Since our corrections terms are obtained by different methods, we obtain, in particular, different interpretations of the corresponding terms in [BaMo] and [St].

[7] Our usage of the notion *high energy* follows the terminology introduced in [Lo1].

The formulas in Theorems 1.24 and 1.25 show that the volume of the quotient X of $\mathbb{C}H^n$ in question is a rational multiple of the volume of $\mathbb{C}P^n$. This was already known by Harder's Gauss-Bonnet theorem which says that $(n+1)\operatorname{vol}(X)/\operatorname{vol}\mathbb{C}P^n = (-1)^n\chi(X)$, where $\chi(X) \in \mathbb{Z}$ denotes the Euler characteristic of X, see [Ha]. Theorem 1.24 implies that $\operatorname{vol}(X)/\operatorname{vol}\mathbb{C}P^n$ is integral for odd n. The question of the integrality of $\operatorname{vol}(X)/\operatorname{vol}\mathbb{C}P^n$ has been brought to our attention by Martin Olbrich: The half-integrality of $\operatorname{vol}(X)/\operatorname{vol}\mathbb{C}P^n$ implies that certain Selberg type zeta functions are meromorphic.

As another example of our applications we mention the Dirac operator D on the spinor bundle, supposing that M admits a spin structure. The case $n = 1$, that is, of surfaces of finite area with cusps of constant negative curvature, has been dealt with in [Bä], see also [BB1]. In particular, D is of Fredholm type if and only if the spin structure is not periodic (along the cross sections) of any of the cusps, see Theorem 2 in [Bä] or Theorem 0.1 in [BB1]. In the case of (our type of) complex hyperbolic cusps, the spin structure along such a cusp is determined by a *twist homomorphism* τ from Γ to the multiplicative group $\{\pm 1\}$. The periodic spin structure corresponds to the trivial twist $\tau = 1$. As we show in Examples 9.25 and 1 of Section 10.3, the contribution of the cusp in the periodic case is

$$\operatorname{Corr}(\mathcal{C}) = \begin{cases} \frac{1}{2}\binom{n-1}{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n-2}{2}}\binom{n-2}{\frac{n-2}{2}} + \zeta(1-n)|\Gamma_{\mathcal{C}}|, & \text{if } n \text{ is even.} \end{cases} \tag{1.26}$$

If the twist is non-trivial and ζ denotes a generator of the center of Γ, then $\operatorname{Corr}(\mathcal{C})$ is equal to

$$\begin{cases} 0, & \text{if } n \text{ is odd,} \\ \zeta(1-n)|\Gamma_{\mathcal{C}}|, & \text{if } n \text{ is even and } \tau(\zeta) = 1, \\ \zeta(1-n)(2^{1-n}-1)|\Gamma_{\mathcal{C}}|, & \text{if } n \text{ is even and } \tau(\zeta) = -1. \end{cases} \tag{1.27}$$

It is clear that there is an index formula for the Dirac operator on spinors over a quotient of a complex hyperbolic space similar to the ones in Theorems 1.24 and 1.25 above. However, because of the case distinctions in (1.26) and (1.27), we prefer to refrain from stating it.

Our formulas for complex hyperbolic cusps apply to more examples, but we refer the reader to Theorems 9.7, 10.47, and 10.72 for the full scope of our results.

In Chapter 2 we discuss some notions and results which are basic for our later investigations. Chapter 3 is devoted to distance functions and their relation to Dirac systems. In particular, Section 3.2 contains a detailed study of C^2 distance functions as we need it in our application to Busemann functions. In this section, we clarify and correct some of the statements from [BB2]. Some essential parts of our later analysis depend on our previous results in [BBC2][8]. That the applications of these results are justified is the topic of Section 3.3. In Chapter 4, we discuss boundary value problems and Fredholm properties of Dirac systems which are associated to Dirac operators over straight ends. Proposition 4.45 is one of the corner stones of our later discussion. Chapter 5 contains the first applications to index formulas and a proof of Proposition 1.16. Chapter 6 and the first part of Chapter 7 contain the proofs of Theorems 1.13 and 1.14. In the last part of Chapter 7, we derive explicit index formulas under an assumption which is satisfied for natural vector bundles over cusps as they arise for finite volume quotients of hyperbolic spaces. The last three chapters are devoted to a discussion of the index contributions of such cusps. Ideas from the work of Deninger-Singhof [DeSi] are basic in our computation of high energy η-invariants of Dirac operators on compact quotients of Heisenberg groups. Following the discussion of Gordon-Wilson in [GoWi], we compute in Appendix A the spectrum of twisted Laplacians on compact quotients of Heisenberg groups. This is needed in our computation of high energy η-invariants in Chapter 9. In Chapter 10, we discuss the low energy η-invariants of Dirac bundles over complex hyperbolic cusps. One of the main ingredients in this latter discussion is a theorem of Kostant concerning Lie algebra cohomology (Theorem 4.139 in [KnVo]).

We would like to thank Patrick Ghanaat, Jean Louis Milhorat, Henri Moscovici, and Martin Olbrich for helpful discussions. We enjoyed the hospitality of the ESI in Vienna (W.B.), the University of Nantes (W.B. and J.B.), the MPIM in Bonn (J.B. and G.C.), the University of Kyoto (J.B.), and the MSRI in Berkeley (G.C.). W.B. would like to thank the MPIM and HCM in Bonn for their continuous support. J.B. appreciates the support by SFB 647. G.C. acknowledges the support by the grant GeomEinstein 06-BLAN-0154 gratefully.

[8] In some cases, the work of Marius Mitrea could also be used: In Section 5 of [Mi], Mitrea investigates the regularity of the Calderón projector for Dirac operators on Lipschitz domains with $C^{1,1}$ symbol and metric tensor, using paradifferential calculus.

2. Preliminaries

Let M be a Riemannian manifold of dimension m with Levi-Civita connection ∇ and curvature tensor R. Let $E \to M$ be a Hermitian vector bundle over M, endowed with a Hermitian connection ∇^E and associated curvature R^E. Recall that we assume that the norms of R and R^E are uniformly bounded, compare (1.1).

We denote by $C^\infty(M,E)$ and $L^2(M,E)$ the spaces of smooth and square-integrable sections of E, respectively. We let $H^1(M,E)$ be the closure of $C^\infty(M,E)$ with respect to the H^1-norm, that is, the norm associated to the inner product

$$(\sigma,\tau)_{H^1(M,E)} := (\sigma,\tau)_{L^2(M,E)} + (\nabla^E\sigma,\nabla^E\tau)_{L^2(M,E\otimes T^*M)}. \tag{2.1}$$

We denote by $C_c^\infty(M,E)$, $L_c^2(M,E)$, and $H_c^1(M,E)$ the subspaces of corresponding sections with compact support and by $L^2_{\mathrm{loc}}(M,E)$ and $H^1_{\mathrm{loc}}(M,E)$ the spaces of measurable sections σ of E such that $\varphi\sigma$ belongs to $L^2(M,E)$ and $H^1(M,E)$, respectively, for any smooth function φ on M with compact support. In the case where the boundary of M is non-empty, we use a double index cc to indicate compact support in the interior of M and an index 0 to indicate vanishing along the boundary.

For better readability, we have arranged the rest of the preliminaries into sections. In Section 2.1 we introduce Dirac bundles and operators, in Section 2.2 we collect some generalities about spinors, and in Section 2.3 we introduce complex hyperbolic spaces.

2.1. **Dirac Bundles.**

We say that E is a *Dirac bundle* over M if E is endowed with a compatible *Clifford multiplication*, that is, a field

$$TM \times E \to E, \quad (x,v) \mapsto x \cdot v, \tag{2.2}$$

of bilinear maps such that

$$XX\sigma = -|X|^2\sigma, \tag{2.3}$$

$$|X\sigma| = |X||\sigma|, \tag{2.4}$$

$$\nabla^E_X(Y\sigma) = (\nabla_X Y)\sigma + Y\nabla^E_X\sigma, \tag{2.5}$$

for all vector fields X, Y on M and sections σ of E, where we use $X\sigma$ as a shorthand for $X \cdot \sigma$.

Suppose now that E is a Dirac bundle over M. Then the *Dirac operator* D associated to E is given by

$$D\sigma = \sum_{1\le i\le m} X_i \nabla^E_{X_i}\sigma, \tag{2.6}$$

where $(X_1, \dots, X_m)$ is a local orthonormal frame of M and σ is a section of E. For any function φ on M and section σ of E,

$$D(\varphi\sigma) = \operatorname{grad}\varphi \cdot \sigma + \varphi D\sigma. \tag{2.7}$$

In particular, the principal symbol of D at $\xi \in T^*M$ is given by Clifford multiplication with the dual vector $\xi^\sharp \in TM$, and hence D is elliptic. Note also that D is formally self-adjoint, that is, D is symmetric on $C^\infty_{cc}(M, E)$.

Suppose now that M has boundary, $N := \partial M$, let T be the inward normal field along N, and set $W := \nabla T$, the *Weingarten map* of N with respect to T. We assume that the operator norm of W is uniformly bounded by a constant C_W. Change Clifford multiplication and connection of E along N by

$$X * \sigma := TX\sigma, \tag{2.8}$$

$$\nabla^T_X\sigma := \nabla^E_X\sigma - \frac{1}{2}(WX) * \sigma = \nabla^E_X\sigma - \frac{1}{2}(T\nabla_X T)\sigma. \tag{2.9}$$

It is well known that, with these new data, the restriction of E to N is again a Dirac bundle such that Clifford multiplication by T is ∇^T-parallel, see for example Section 3.10.1 in [Gil2]. The associated Dirac operator is given by

$$D^T\sigma = \sum_{2\le i\le m} X_i * \nabla^T_{X_i}\sigma = \sum_{2\le i\le m} TX_i\nabla^E_{X_i}\sigma + \frac{\kappa}{2}\sigma, \tag{2.10}$$

where $(X_1, X_2, \dots, X_m)$ is a local orthonormal frame of M along N with $X_1 = T$ and

$$\kappa = \operatorname{tr} W \tag{2.11}$$

is the *mean curvature* of N with respect to T. The curvature of ∇^T is

$$R^T(X,Y)\sigma = R^E(X,Y)\sigma - \frac{1}{2}(R(X,Y)T) * \sigma - \frac{1}{4}[WX, WY]\sigma. \tag{2.12}$$

The general Bochner identity [LaMi, Theorem II.8.2] implies that

$$\begin{aligned} &(\nabla^E\sigma_1, \nabla^E\sigma_2)_{L^2(M, E\otimes T^*M)} + (K^E\sigma_1, \sigma_2)_{L^2(M,E)} \\ &\qquad = (D\sigma_1, D\sigma_2)_{L^2(M,E)} + (D^T\sigma_1 - \frac{\kappa}{2}\sigma_1, \sigma_2)_{L^2(N,E)}, \end{aligned} \tag{2.13}$$

for all $\sigma_1, \sigma_2 \in C^\infty_c(M, E)$, where K^E is a curvature term,

$$K^E\sigma = \sum_{1\le i<j\le m} X_iX_jR^E(X_i, X_j)\sigma. \tag{2.14}$$

We see that the operator norm of K^E is bounded by $m(m-1)C^E_R/2$ and conclude that the graph norm of D is equivalent to the H^1-norm.

Since N has no boundary, (2.13) applied to N turns into

$$(\nabla^T\sigma_1, \nabla^T\sigma_2)_{L^2(N,E\otimes T^*N)} + (K^T\sigma_1, \sigma_2)_{L^2(N,E)} = (D^T\sigma_1, D^T\sigma_2)_{L^2(N,E)}, \tag{2.15}$$

where K^T denotes the curvature term built from R^T as K^E is built from R^E in (2.14). We see that the operator norm of K^T is bounded here by

$$\frac{(m-1)(m-2)}{2}\big(C_R^E + \frac{1}{2}C_R + \frac{1}{4}C_W^2\big), \tag{2.16}$$

where C_W is a uniform bound for the operator norm of W (compare (1.11)), and conclude now that, along N, the graph norm of D^T is equivalent to the H^1-norm.

Let E be a Dirac bundle over M. A *super-symmetry* of E is an orthogonal decomposition $E = E^+ \oplus E^-$, where $E^\pm$ are parallel Hermitian subbundles of E such that $XE^+ \subseteq E^-$ and $XE^- \subseteq E^+$, for all vector fields X of M. In particular, E^+ and E^- are of the same dimension. If $E = E^+ \oplus E^-$ is a super-symmetry, then the Dirac operator D of E maps sections of E^+ into sections of E^- and conversely and therefore can be written as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{2.17}$$

with respect to the super-symmetry. We can also think of a super-symmetry as a parallel field of unitary involutions of E which anti-commute with Clifford multiplication, where $E^\pm$ is the subbundle of eigenspaces of the involutions for the eigenvalue ± 1, respectively.

If M is oriented and $m = \dim M$ is even, then the *complex volume form* of M is defined to be

$$\omega_{\mathbb{C}} := i^{m/2} X_1 \cdots X_m \in \mathbb{C}\mathrm{l}(M), \tag{2.18}$$

where $(X_1, \dots, X_m)$ is an oriented local orthonormal frame of M. For any Dirac bundle E over M, multiplication by $\omega_{\mathbb{C}}$ is a parallel field of unitary involutions of E which anti-commutes with Clifford multiplication with vector fields, and hence it defines a super-symmetry $E = E^+ \oplus E^-$.

Suppose now M is complete and that the boundary of M is empty, and consider D as an unbounded operator in $L^2(M, E)$ with domain $C_c^\infty(M, E)$. Since D is symmetric on $C_c^\infty(M, E)$, it is closable in $L^2(M, E)$. Since the graph norm of D is equivalent to the H^1-norm, $H^1(M, E)$ is the domain of the closure of D. By [Wo] or Theorem II.5.7 in [LaMi], D on $H^1(M, E)$ is self-adjoint in $L^2(M, E)$.

2.2. **Decomposition of Spinors.** Let m be even, $m = 2n$, and consider the complex Clifford algebra $\mathbb{C}l(2n) = \mathbb{C}l(\mathbb{R}^{2n})$, where we denote the complex structure on $\mathbb{C}l(2n)$ by $\sqrt{-1}$. Fix an orthonormal basis $(e_1, \dots, e_{2n})$ of $\mathbb{R}^{2n}$ and set[9]

$$\omega_j := \sqrt{-1} e_{2j-1} e_{2j} \in \mathbb{C}l(2n), \quad 1 \leq j \leq n. \tag{2.19}$$

Then

$$\omega_j^2 = 1 \quad \text{and} \quad \omega_j \omega_k = \omega_k \omega_j, \tag{2.20}$$

for all $1 \leq j, k \leq n$, and the complex volume form is given by

$$\omega_{\mathbb{C}} = \omega_1 \cdots \omega_n, \tag{2.21}$$

compare (2.18). Let $\Sigma = \Sigma_{2n}$ be the spinor representation. Then Clifford multiplication by the ω_j defines unitary involutions of Σ. By (2.20), there is an orthogonal decomposition of Σ into simultaneous eigenspaces Σ_ε, where ε runs over all n-tuples in $\{1, -1\}^n$ and where ω_j acts by multiplication with ε_j on Σ_ε, $1 \leq j \leq n$. Because Clifford multiplication with e_{2j-1} or e_{2j} anti-commutes with ω_j and commutes with ω_k, for $1 \leq k \neq j \leq n$, we have

$$e_{2j-1} \Sigma_\varepsilon = e_{2j} \Sigma_\varepsilon = \Sigma_\delta, \tag{2.22}$$

where $\delta_k = \varepsilon_k$ for all $1 \leq k \neq j \leq n$ and $\delta_j = -\varepsilon_j$. In particular, all the subspaces Σ_ε have the same dimension, which is, for that reason, equal to $\dim \Sigma / 2^n = 1$. Clifford multiplication by the complex volume form acts by $\varepsilon_1 \cdots \varepsilon_n$ on Σ_ε, by (2.21), and hence the summands of the usual super-symmetry

$$\Sigma = \Sigma^+ \oplus \Sigma^- \tag{2.23}$$

are given by

$$\Sigma^+ = \oplus_{\varepsilon_1 \cdots \varepsilon_n = 1} \Sigma_\varepsilon \quad \text{and} \quad \Sigma^- = \oplus_{\varepsilon_1 \cdots \varepsilon_n = -1} \Sigma_\varepsilon. \tag{2.24}$$

2.3. **Complex Hyperbolic Spaces.** We represent complex hyperbolic space $\mathbb{C}H^n$ by the symmetric pair $(\mathrm{SU}(1,n), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ and endow the Lie algebra $\mathfrak{su}(1,n)$ of $\mathrm{SU}(1,n)$ with the non-degenerate symmetric bilinear from

$$(X, Y) := \frac{1}{2} \operatorname{Re} \operatorname{tr} XY, \tag{2.25}$$

a multiple of the Killing form of $\mathfrak{su}(1,n)$. We identify

$$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) = \left\{ \begin{pmatrix} \det A^{-1} & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in \mathrm{U}(n) \right\} \cong \mathrm{U}(n) \tag{2.26}$$

[9]Note that the sign convention is opposite to the one in [LaMi], page 43.

and, correspondingly,

$$\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n)) = \left\{ \begin{pmatrix} -\operatorname{tr} A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in \mathfrak{u}(n) \right\} \cong \mathfrak{u}(n). \tag{2.27}$$

The orthogonal complement $\mathfrak{p}$ of $\mathfrak{u}(n)$ in $\mathfrak{su}(1,n)$ is

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{C}^n \right\} \cong \mathbb{C}^n, \tag{2.28}$$

where we note that the latter isomorphism corresponds to the standard complex structure and Riemannian metric of $\mathbb{C}H^n$. With respect to the identifictions (2.26) – (2.28), we get

$$[A,B] = AB - BA,\ [A,x] = Ax + x \cdot \operatorname{tr} A,\ [x,y] = xy^* - yx^* \tag{2.29}$$

for the different Lie brackets and

$$\alpha(A)x := \operatorname{Ad}_A x = Ax \det A \tag{2.30}$$

for the adjoint representation α of $\mathrm{U}(n)$ on $\mathfrak{p}$. We note that α is an $n+1$ to 1 immersion. If n is odd, then α lifts to $\hat{\alpha} : \mathrm{U}(n) \to \mathrm{Spin}(\mathfrak{p})$. If n is even, then α does not lift.

We note that the coefficients of the matrix $xy^* - yx^* \in \mathfrak{u}(n)$ in (2.29) are $x_j\bar{y}_k - y_j\bar{x}_k$. In particular, for the standard unit vectors e_j and e_k in $\mathbb{C}^n$ and complex numbers x, y, we have

$$[xe_j, ye_k] = x\bar{y}E_{jk} - y\bar{x}E_{kj} \in \mathfrak{u}(n), \tag{2.31}$$

where E_{jk} denotes the matrix with entries δ_{jk}.

Let $T = e_1 \in \mathbb{C}^n \cong \mathfrak{p}$ and set $\mathfrak{a} := \mathbb{R}T$. The orthogonal complement of $\mathfrak{a}$ in $\mathbb{C}^n$ consists of all $x \in \mathbb{C}^n$ with $x_1 \in \operatorname{Im}\mathbb{C}$, that is, x_1 is purely imaginary. Let $\mathfrak{z} := \mathbb{R}Z$ with

$$Z := ie_1 - iE_{11} \in \mathfrak{p} \oplus \mathfrak{u}(n). \tag{2.32}$$

We have $[\mathfrak{z}, \mathfrak{z}] = 0$ and

$$[T, Z] = 2Z. \tag{2.33}$$

Let $\mathfrak{x}$ be the space of all

$$X_x := x + \bar{x}_2E_{12} - x_2E_{21} + \ldots + \bar{x}_nE_{1n} - x_nE_{n1} \in \mathfrak{p} \oplus \mathfrak{u}(n), \tag{2.34}$$

where $x \in \mathbb{C}^{n-1} = \{x \in \mathbb{C}^n \mid x_1 = 0\}$. Then $[\mathfrak{z}, \mathfrak{x}] = 0$ and

$$[T, X_x] = X_x, \tag{2.35}$$

$$[X_x, X_y] = 2Z_{\operatorname{Im}\bar{x}y}. \tag{2.36}$$

Set $\mathfrak{n} := \mathfrak{z} \oplus \mathfrak{x}$ and $\mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n}$. By the above, $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{su}(1,n)$ of rank two and $\mathfrak{s}$ is a solvable extension of $\mathfrak{n}$. The subgroups A, N, and S of $\mathrm{SU}(1,n)$ corresponding to $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{s}$ satisfy $S = AN$ and $\mathrm{SU}(1,n) = \mathrm{U}(n)AN$ (Iwasawa decomposition of $\mathrm{SU}(1,n)$).

Let $p \in \mathbb{C}H^n$ be the point fixed by $\mathrm{U}(n)$. Then the orbit map

$$\Phi : S \to \mathbb{C}H^n, \quad \Phi(s) = sp, \tag{2.37}$$

is a diffeomorphism, that is, S acts simply transitively on $\mathbb{C}H^n$. Endow S with the left-invariant Riemannian metric such that the differential $d\Phi : T_eS \to T_p\mathbb{C}H^n$ is isometric. Since S acts isometrically on $\mathbb{C}H^n$, we then get that Φ is an S-equivariant isometry. That is, we can think of $\mathbb{C}H^n$ as S, endowed with the chosen left-invariant metric. With respect to this metric, we get that $\mathfrak{a}$, $\mathfrak{z}$, and $\mathfrak{x}$ are perpendicular and that

$$|T| = 1, \quad |Z| = 1, \quad \langle X_x, X_y \rangle = \mathrm{Re}\, \bar{x}y. \tag{2.38}$$

Define

$$JX_x = J_Z X_x := X_{ix}. \tag{2.39}$$

Then J is skew-symmetric with $J^2 = -1$, hence the Clifford relations 10.9 are satisfied. Moreover, by (2.36) and (2.38),

$$\langle [X_x, X_y], Z \rangle = 2\langle JX_x, X_y \rangle, \tag{2.40}$$

which is (10.10) with $c = 1$. As a preferred basis of $\mathfrak{s}$, we choose the $2n$-tuple of vectors $X_1 := T$, $Y_1 = Z$,

$$X_j := e_j + E_{1j} - E_{j1} \quad \text{and} \quad Y_j := JX_j = ie_j - iE_{1j} - iE_{j1}, \tag{2.41}$$

where $2 \le j \le n$. By (2.36) and (2.39),

$$[X_j, Y_k] = 2\delta_{jk} Z. \tag{2.42}$$

In conclusion, N is isomorphic to the standard Heisenberg group of dimension $2n - 1$. By (2.33) and (2.35), the Weingarten map of N in S with respect to the unit normal field T has eigenvalues -1 and -2 on $\mathfrak{x}$ and $\mathfrak{z}$ as required.

3. Dirac Systems and Distance Functions

3.1. **Dirac Systems.** The setup and the results from [BBC2] are fundamental for the discussion of this section. Let $I \subseteq \mathbb{R}$ be an interval and H be a separable complex Hilbert space. Fix an origin $t_0 \in I$.

For each $t \in I$, let $(.,.)_t$ be a scalar product on H which is compatible with the Hilbert space structure of H and such that $(.,.)_{t_0}$ coincides with the original scalar product of H. Let $\|.\|_t$ be the norm associated to $(.,.)_t$. Let H_t be H, but equipped with $(.,.)_t$, and denote by $\mathcal{H}$ the family of Hilbert spaces H_t, $t \in I$. Assume that, for all $a < b$ in I, there is a constant $C = C(a,b)$ such that

$$|(\sigma_1, \sigma_2)_s - (\sigma_1, \sigma_2)_t| \le C\|\sigma_1\|_s \, \|\sigma_2\|_s |s - t|, \tag{3.1}$$

for all $s, t \in [a, b]$ and $\sigma_1, \sigma_2 \in H$. In other words, if $G_t \in \mathcal{L}(H)$ denotes the positive definite and symmetric operator of $H = H_{t_0}$ with

$$(G_t\sigma_1, \sigma_2)_{t_0} = (\sigma_1, \sigma_2)_t, \tag{3.2}$$

for all $\sigma_1, \sigma_2 \in H$, then the map

$$G : I \to \mathcal{L}(H), \quad G(t) := G_t,$$

is in $\mathrm{Lip}_{\mathrm{loc}}(I, \mathcal{L}(H))$. In particular, G is weakly differentiable almost everywhere in I with weak derivative G' in $L^\infty_{\mathrm{loc}}(I, \mathcal{L}(H))$. Moreover, G'_t is symmetric on H_{t_0} (for almost all $t \in I$) and we have

$$\Gamma := \frac{1}{2} G^{-1} G' \in L^\infty_{\mathrm{loc}}(I, \mathcal{L}(H)), \tag{3.3}$$

and

$$\partial := \Big(\frac{d}{dt} + \frac{1}{2}\Gamma\Big) : \mathrm{Lip}_{\mathrm{loc}}(I, H) \to L^\infty_{\mathrm{loc}}(I, H). \tag{3.4}$$

By the definition of ∂, the function $(\sigma_1, \sigma_2) = (\sigma_1(t), \sigma_2(t))_t$ satisfies

$$(\sigma_1, \sigma_2)' = (\partial\sigma_1, \sigma_2) + (\sigma_1, \partial\sigma_2), \tag{3.5}$$

for all $\sigma_1, \sigma_2 \in \mathrm{Lip}_{\mathrm{loc}}(I, H)$, where the prime indicates differentiation with respect to t.

As a second data, let $\mathcal{A}$ be a family of operators A_t, $t \in I$, on H with common dense domain H_A such that A_t is self-adjoint in H_t and such that the inclusion $H_A \hookrightarrow H$ is compact with respect to the graph norms of the A_t. Assume that, for all $a < b$ in I, there is a constant $C = C(a, b)$ such that

$$|(A_s\sigma_1, \sigma_2)_s - (A_t\sigma_1, \sigma_2)_t| \le C(\|\sigma_1\|_s + \|A_s\sigma_1\|_s)\|\sigma_2\|_s|s - t|, \tag{3.6}$$

for all $s, t \in [a, b]$ and $\sigma_1, \sigma_2 \in H_A$.

As a final data, let

$$T \in \mathrm{Lip}_{\mathrm{loc}}(I, \mathcal{L}(H)) \cap L^\infty_{\mathrm{loc}}(I, \mathcal{L}(H_A)), \tag{3.7}$$

and suppose that

$$T_t^* = T_t^{-1} = -T_t \qquad \text{on } H_t,\ \forall t \in I, \tag{3.8}$$

$$A_t T_t = -T_t A_t \quad \text{on } H_A,\ \forall t \in I, \tag{3.9}$$

$$\partial T = T\partial \qquad \text{on } \mathrm{Lip}_{\mathrm{loc}}(I, H). \tag{3.10}$$

Following [BBC2], a *Dirac system* over I consists of data $\mathcal{H}$, $\mathcal{A}$, and T as above.

Let $\mathcal{D} := (\mathcal{H}, \mathcal{A}, T)$ be a Dirac system over I. Set

$$\mathcal{L}_{\mathrm{loc}}(\mathcal{D}) := \mathrm{Lip}_{\mathrm{loc}}(I, H) \cap L^\infty_{\mathrm{loc}}(I, H_A), \tag{3.11}$$

and denote by $\mathcal{L}_c(\mathcal{D})$ and $\mathcal{L}_{cc}(\mathcal{D})$ the subspaces of $\mathcal{L}_{\mathrm{loc}}(\mathcal{D})$ of maps with compact support in I and the interior of I, respectively. On $\mathcal{L}_c(\mathcal{D})$, we define the inner product

$$(\sigma_1, \sigma_2) := \int_I (\sigma_1, \sigma_2) = \int_I (\sigma_1(t), \sigma_2(t))_t dt, \tag{3.12}$$

and let $L^2(\mathcal{D})$ be the corresponding Hilbert space of square-integrable maps, also denoted by $L^2(\mathcal{H})$.

The *Dirac operator* of $\mathcal{D}$ is the operator

$$D := T(\partial + A) : \mathcal{L}_{\mathrm{loc}}(\mathcal{D}) \to L^\infty_{\mathrm{loc}}(I, H). \tag{3.13}$$

By (3.5) and (3.8)–(3.10),

$$\int_{[a,b]} (D\sigma_1, \sigma_2) = \int_{[a,b]} (\sigma_1, D\sigma_2) - (\sigma_1, T\sigma_2)\Big|_a^b, \tag{3.14}$$

for all $\sigma, \tau \in \mathcal{L}_{\mathrm{loc}}(\mathcal{D})$ and $a < b$ in I.

A *super-symmetry* for a Dirac system $\mathcal{D}$ as above is a decomposition $H = H^+ \oplus H^-$ such that, with $H^\pm_A := H_A \cap H^\pm$,

$$H^+ \perp H^- \text{ in } H_t \quad \text{and} \quad T_t H^\pm = H^\mp, \tag{3.15}$$

$$H_A = H^+_A \oplus H^-_A \quad \text{and} \quad A_t H^\pm_A \subseteq H^\pm. \tag{3.16}$$

We write $H^\pm_t$ for $H^\pm$ endowed with the inner product $(.,.)_t$. By (3.15), $H_t = H^+_t \oplus H^-_t$ as a Hilbert space. By (3.15),

$$A_t = \begin{pmatrix} A^+_t & 0 \\ 0 & A^-_t \end{pmatrix}, \tag{3.17}$$

where $A^\pm_t$ is a self-adjoint operators in $H^\pm_t$ with domain $H^\pm_A$ and where

$$A^-_t = T_t A^+_t T_t = T_t(-A^+_t)T^{-1}_t, \tag{3.18}$$

by (3.9). We can decompose

$$L^2(\mathcal{D}) = L^{2+}(\mathcal{D}) \oplus L^{2-}(\mathcal{D}) = L^2(\mathcal{H}^+) \oplus L^2(\mathcal{H}^-), \tag{3.19}$$

where $L^2(\mathcal{H}^\pm)$ consists of the subspace of sections in $L^2(\mathcal{H})$ with image in H^+. Similar notation will be employed for other spaces.

By (3.16) and the definition of ∂, see (3.4),

$$\partial = \begin{pmatrix} \partial^+ & 0 \\ 0 & \partial^- \end{pmatrix}, \quad \text{where} \quad \partial^- = T\partial T^{-1}, \tag{3.20}$$

by (3.10). Hence, by (3.15),

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \tag{3.21}$$

Clearly, D^- is the formal adjoint of D^+.

3.2. **Distance Functions.** Let U be an open subset in a Riemannian manifold M. We say that a function $f : U \to \mathbb{R}$ is a *distance function* if f is C^1 and $T := \operatorname{grad} f$ is a unit vector field. There is a synthetic characterization of distance functions, compare [BGS, pp 24–25] or also Proposition IV.3.1 in [Ba]. If f is a distance function, then the solution curves of the vector field T are unit speed geodesics, called *T-geodesics*.

Busemann functions are C^2 distance functions, see Proposition 3.1 in [HeIH] or Proposition IV.3.2 in [Ba]. We assume from now on that $f : U \to \mathbb{R}$ is a C^2 distance function. Then $T := \operatorname{grad} f$ is a C^1 unit vector field and the *cross sections* $N_t = f^{-1}(t)$ are C^2 hypersurfaces. For simplicity, we assume throughout that the cross sections N_t are compact and that the flow of T induces a C^1 diffeomorphism

$$F : I \times N \to U, \tag{3.22}$$

where I is some interval and $N = N_{t_0}$ for some $t_0 \in I$. In what follows, we often identify U with $I \times N$ by identifying $(t, x) \in I \times N$ with $F(t, x) \in U$. We keep in mind that F is a C^1 diffeomorphism.

Let $c = c(s)$ be a C^1 curve in U and $T(s) := T(c(s))$ be T along c, a C^1 curve of unit vectors. Then the variation field $J = J(t) := (\partial_s \gamma)(0, t)$ of the geodesic variation $\gamma_s = \gamma(s, t) := \exp(tT(s))$ satisfies $J(0) = \dot c(0)$. A vector field which arises in this way will be called a *T-Jacobi field*.

Lemma 3.23. *A T-Jacobi field J satisfies the Jacobi equation*

$$J'' + R(J, T)T = 0.$$

Moreover, J and J' depend continuously on $J(0)$.

Proof. Let $\Phi = \Phi(t, v)$, $t \in \mathbb{R}$ and $v \in TM$, be the geodesic flow of M. Then $\gamma'(s, t) = \Phi(t, T(s))$ and hence γ and γ' are C^1. Therefore

$$J = \partial_s \gamma \quad \text{and} \quad J' = \nabla_t \partial_s \gamma = \nabla_s \partial_t \gamma = \nabla_s T \tag{3.24}$$

exist and are continuous. Moreover,

$$(\partial_s \gamma')(s, t) = \Phi_{t*}(\partial_s T(s)) = (J(s, t), J'(s, t)) \tag{3.25}$$

with respect to the standard decomposition of TTM in horizontal and vertical component, see for example Proposition IV.1.13 in [Ba]. Hence J and J' depend continuously on $\dot c(0)$ and J satisfies the asserted Jacobi equation. □

Remark 3.26. With respect to the (t, x)-coordinates, the Riemannian metric on U is of the form $g = dt^2 + g_t$, where g_t, $t \in I$, is a family of Riemannian metrics on N. In [BB2], pages 596 and 609, it is stated erroneously that g_t and $\partial_t g_t$ are C^1 on U. This is wrong in general, since it would imply that T is C^2. Clearly, since T is C^1, $g_t(x)$ is C^1 in (t,x). Lemma 3.23 implies that $g_t(x)$ is two times continuously differentiable in t. This is sufficient for the discussion in [BB2] and the arguments below.

For $t \in I$, we let $S = S_t$ and $W = W_t$ be, respectively, the *second fundamental form* and the *Weingarten map* of the C^2 submanifold N_t with respect to the normal vector field T,

$$WX = \nabla_X T, \quad S(X,Y) = \langle \nabla_X Y, T\rangle = -\langle WX, Y\rangle, \tag{3.27}$$

where X and Y are C^1 vector fields tangent to N_t. Since T is C^1, S and W are continuous tensor fields over U. By (3.24), Jacobi fields J as in Lemma 3.23 satisfy $J' = WJ$.

Let $E \to M$ be a smooth vector bundle with smooth connection ∇^E.

Lemma 3.28. *Let X be a vector field and σ be a section of E over U, respectively. Assume that the restrictions of X and σ to N are C^1 and that X and σ are parallel in the T-direction. Then X and σ are C^1. Moreover, $\nabla^E_T\nabla^E_X\sigma$ exists, is continuous, and satisfies*

$$\nabla^E_T\nabla^E_X\sigma + \nabla^E_{WX}\sigma + R^E(X,T)\sigma = 0.$$

Proof. Let $\Psi : \mathbb{R}\times(TM\oplus E) \to E$ be the smooth map which associates to $t \in \mathbb{R}$ and $(v,e) \in TM \oplus E$ (where $v \in TM$ and $e \in E$ have the same foot point) the parallel translate $\sigma(t)$ of e along the geodesic γ with $\gamma'(0) = v$. Then, with σ as in the assertion, we have $\sigma(F(t,x)) = \Psi(t-t_0, T(x), \sigma(x))$, where we recall that $N = N_{t_0}$. Hence X and σ are C^1, where X corresponds to the special case $E = TM$.

Since $\nabla^E_T\sigma = 0$ and T is C^1, the T derivatives of the coefficients of σ with respect to a smooth local frame of E are C^1. Hence $\nabla^E_T\nabla^E_X\sigma$ exists, is continuous, and is given by

$$\begin{aligned}\nabla^E_T\nabla^E_X\sigma &= \nabla^E_X\nabla^E_T\sigma - \nabla^E_{\nabla_X T}\sigma + R^E(T,X)\sigma \\ &= -\nabla^E_{WX}\sigma - R^E(X,T)\sigma.\end{aligned}$$ □

Among others, the case $E = TM$ is interesting. In this case, vector fields over N which are tangent to N can, in general, only be chosen to be C^1.

Corollary 3.29. *The tensor field W has a continuous derivative W' in the T-direction and satisfies the Riccati equation*

$$W' + W^2 + R(.,T)T = 0.$$

Proof. Choose $\sigma = T$ in Lemma 3.28 and recall that $W = \nabla T$. □

The eigenvalues $\kappa_2, \dots, \kappa_m$ of W_t are the *principal curvatures* of the cross section N_t. We let

$$\kappa := \kappa_2 + \cdots + \kappa_m = \operatorname{tr} W = \operatorname{div} T. \tag{3.30}$$

The maps

$$F_t : N = N_{t_0} \to N_t, \quad F_t(x) := F(t,x) \tag{3.31}$$

are diffeomorphisms with Jacobians $j = j(t,x)$. Since $\kappa = \operatorname{div} T$, the latter satisfy the differential equation

$$j' = \kappa j. \tag{3.32}$$

By Corollary 3.29, we also have

$$\kappa' = -\|W\|^2 - \operatorname{Ric}(T,T), \tag{3.33}$$

where $\|W\| = (\operatorname{tr} W^2)^{1/2}$ is the Euclidean norm of W.

Let C_R, C_R^E, and C_W be uniform upper bounds for the operator norms of the curvature R of M, the curvature R^E of E, and W, respectively. Then κ, the t-derivative κ' of κ, and $\|W\|$ are uniformly bounded, and as respective uniform upper bounds C_κ, C'_κ, and C_w we may take

$$C_\kappa = mC_W, \quad C'_\kappa = m(C_W^2 + C_R), \quad C_w = \sqrt{m}C_W, \tag{3.34}$$

where we use (3.33) for the second assertion. By (3.32), we have

$$e^{-C(t-s)} j(s,x) \le j(t,x) \le e^{C(t-s)} j(s,x), \tag{3.35}$$

or all $s < t$ in I and $x \in N$, where $C = C_\kappa$.

3.3. **From Distance Functions to Dirac Systems.** Let $E \to M$ be a smooth Dirac bundle. Denote the Hermitian product on E by $\langle .\,, . \rangle$. Our aim is to identify these data over U with a Dirac system over I as in Section 3.1.

For any $t \in I$ and any given Riemannian or Hermitian vector bundle over U with any given metric connection, we let P_t be parallel translation along the T-geodesics from N to N_t. For a section σ of the vector bundle over N, we define a section $P\sigma$ over U by

$$(P\sigma)(t,x) := P_t(\sigma(x)), \quad x \in N. \tag{3.36}$$

Thus $P\sigma$ is the extension of σ to U which is parallel along the T-geodesics, and this point of view is convenient in arguments and formulations below. Furthermore, time dependent sections over N correspond to the space of all sections over U,

$$(P\sigma)(t,x) := P_t(\sigma(t,x)), \quad t \in I, x \in N. \tag{3.37}$$

We also let $P_t\sigma := P\sigma|_{N_t}$.

Now let $H := L^2(N, E)$, the Hilbert space of square integrable sections of E over $N = N_{t_0}$. For $\sigma, \tau \in H$, the L^2 product of the sections $P_t\sigma, P_t\tau$ with respect to N_t is given by

$$(\sigma, \tau)_t := \int_N \langle \sigma(x), \tau(x) \rangle j(t, x) dx, \tag{3.38}$$

where dx denotes the volume element of N. Hence, for each $t \in I$, the correspondence $\sigma \leftrightarrow P_t\sigma$ identifies the Hilbert space $L^2(N_t, E)$ topologically with H. The following estimate settles the requirement on the family $\mathcal{H}$ formulated in (3.1).

LEMMA 3.39. *For all $s < t$ in I and $\sigma_1, \sigma_2 \in H$,*

$$|(\sigma_1, \sigma_2)_t - (\sigma_1, \sigma_2)_s| \leq (e^{C(t-s)} - 1) \|\sigma_1\|_s \, \|\sigma_2\|_s,$$

where $C = C_\kappa$.

Proof. By (3.38) and (3.35),

$$\begin{aligned} |(\sigma_1, \sigma_2)_t - (\sigma_1, \sigma_2)_s| &\leq \int_N |\langle \sigma_1(x), \sigma_2(x) \rangle (j(t, x) - j(s, x)| dx \\ &\leq \int_N |\sigma_1(x)||\sigma_2(x)|(e^{C(t-s)} - 1) j(s, x) dx \\ &\leq (e^{C(t-s)} - 1) \|\sigma_1\|_s \|\sigma_2\|_s. \end{aligned}$$

□

LEMMA 3.40. *For all $s < t$ in I and C^1 sections σ of E over U which are parallel in the T-direction,*

$$e^{C_0(s-t)}(\|\sigma\|_s^2 + \|\nabla^E \sigma\|_s^2) \leq \|\sigma\|_t^2 + \|\nabla \sigma\|_t^2 \leq e^{C_0(t-s)}(\|\sigma\|_s^2 + \|\nabla^E \sigma\|_s^2),$$

where $C_0 = C_\kappa + mC_R^E + 2C_W$.

Proof. Using $\langle \sigma, \sigma \rangle' = 0$, we obtain

$$(\|\sigma\|_t^2 + \|\nabla^E \sigma\|_t^2)' = \int_N \big(\langle \nabla^E \sigma, \nabla^E \sigma \rangle' + (\langle \sigma, \sigma \rangle + \langle \nabla^E \sigma, \nabla^E \sigma \rangle) \kappa \big) j.$$

By Lemma 3.28,

$$\langle \nabla^E \sigma, \nabla^E \sigma \rangle' = 2 \sum_{2 \leq i \leq m} \big(\langle R^E(T, X_i)\sigma, \nabla^E_{X_i}\sigma \rangle - \langle \nabla^E_{WX_i}\sigma, \nabla^E_{X_i}\sigma \rangle \big),$$

where $(T, X_1, \ldots, X_n)$ is a local orthonormal frame of M. Hence

$$\begin{aligned} |(\|\sigma\|_t^2 + \|\nabla^E \sigma\|_t^2)'| &\leq mC_R^E \|\sigma\|_t^2 + C_R^E \|\nabla^E \sigma\|_t^2 \\ &\quad + 2C_W \|\nabla^E \sigma\|_t^2 + C_\kappa (\|\sigma\|_t^2 + \|\nabla^E \sigma\|_t^2) \\ &\leq (C_\kappa + mC_R^E + 2C_W)(\|\sigma\|_t^2 + \|\nabla^E \sigma\|_t^2). \end{aligned}$$

□

Along the cross sections N_t, we change Clifford multiplication and connection of E according to (2.8) and (2.9). Denote by ∇^t the new connection and by D_t the associated Dirac operator as in (2.10). We note that Clifford multiplication with T is ∇^t-parallel. For convenience, we will not keep the $*$ notation, but will write $TX\sigma$ instead of $X * \sigma$. With this in mind, the Dirac operators D and D_t are related by

$$D = T(\nabla^E_T + \sum TX_i \nabla^E_{X_i}) = T\big((\nabla^E_T + \frac{\kappa}{2}) - D_t\big), \tag{3.41}$$

where $(T, X_2, \dots, X_m)$ is a local orthonormal frame of M.

Lemma 3.42. *For any C^1 section σ of E over U which is parallel in the T-direction,*

$$\|\nabla^E \sigma|_{N_t} - \nabla^t \sigma\|^2 = \frac{1}{4}\|W\|^2 |\sigma|^2 \quad \textit{and} \quad |TD\sigma - D_t\sigma|^2 = \frac{1}{4}\kappa^2 |\sigma|^2.$$

Proof. The second assertion is immediate from (3.41). As for the first, let $(T, X_2, \dots, X_m)$ be an orthonormal frame of M. Then

$$\begin{aligned} 4\|\nabla^E \sigma|_{N_t} - \nabla^t \sigma\|^2 &= 4 \sum \langle \nabla^E_{X_i}\sigma - \nabla^t_{X_i}\sigma, \nabla^E_{X_i}\sigma - \nabla^t_{X_i}\sigma \rangle \\ &= \sum \langle TWX_i\sigma, TWX_i\sigma \rangle \\ &= \sum |WX_i|^2 |\sigma|^2 = \|W\|^2 |\sigma|^2. \end{aligned}$$

□

Since the cross sections N_t are C^2 submanifolds of U, the restrictions of E to them are C^2 bundles. However, because of the term involving $W = \nabla T$, the connection ∇^t is, in the generality we strive for, only continuous. If ∇^t were a C^1 connection, we would get (2.12) for its curvature, now denoted R^t. The right hand side of (2.12) makes sense in the case where W is only continuous, so that we may consider it as defining R^t. Approximating N_t by smooth submanifolds and C^1 sections by smooth sections, (2.15) implies that

$$(\nabla^t \sigma_1, \nabla^t \sigma_2)_t + (K^t \sigma_1, \sigma_2)_t = (D_t \sigma_1, D_t \sigma_2)_t \tag{3.43}$$

for all C^1 sections σ and τ of the restriction of E to N_t, where the curvature term in the Lichnerowicz formula as in (2.15) is now denoted by K^t. We recall from (2.16) that K^t is uniformly bounded.

We extend our correspondence $\sigma \leftrightarrow P\sigma$ as in (3.36) and (3.37): Since T is parallel in the T-direction, Clifford multiplication by T along N satisfies

$$TP\sigma = PT\sigma \quad \text{and} \quad \nabla_T P\sigma = P\sigma', \tag{3.44}$$

for any time dependent section σ of E over N. Finally, we define A_t to be the differential operator on sections of E over N which corresponds to the operator $-D_t$,

$$P_t(A_t\sigma) = -D_t P_t\sigma. \tag{3.45}$$

In this notation, D corresponds to the operator

$$T(\partial + A), \quad \text{where } \partial := \frac{d}{dt} + \frac{\kappa}{2}. \tag{3.46}$$

Thus we have associated the Dirac system

$$\mathcal{D} := (\mathcal{H}, \mathcal{A}, T) \tag{3.47}$$

to the distance function f on and the Dirac bundle E over U, where we recall from (3.32) that κ (which occurs in the definition of ∂) is defined by these data. We will now proceed with discussing the requirement for Dirac systems as formulated in Section 3.1. We already observed that Lemma 3.39 settles (3.1). Furthermore, Clifford multiplication by T satisfies the requirements (3.7)–(3.10), by (3.44) and since Clifford multiplication by T is ∇^t-parallel.

It follows from (3.43) that, on sections of the restriction of E to N_t, the graph norm of D_t is equivalent to the H^1 norm. In particular, D_t is self-adjoint with domain $H^1(N_t, E)$ in $L^2(N_t, E)$. Moreover, since the inclusion $H^1(N_t, E) \hookrightarrow L^2(N_t, E)$ is compact, the spectrum of D_t consists of eigenvalues with finite multiplicities. We also observe that, for any section σ of E over N, $P\sigma|_{N_t} \in H^1(N_t, E)$ if and only if $\sigma \in H^1(N, E)$, by Lemma 3.40. Thus the operators A_t are all self-adjoint with the same domain, $H_A := H^1(N, E)$, in $H = L^2(N, E)$, and the embedding $H_A \to H$ is compact with respect to the graph norm of any of the operators A_t. This settles the first part of the requirements for the A_t in Section 3.1.

Lemma 3.48. *For any C^1 section σ of E over U which is parallel in the T-direction,*

$$D_t'\sigma = \sum_{2\le i\le m} TX_i\{R^E(T, X_i)\sigma - \nabla^E_{WX_i}\sigma\} + \frac{\kappa'}{2}\sigma,$$

where $(T, X_2, \ldots, X_m)$ is a local orthonormal frame of M.

Proof. By Lemma 3.28,

$$\begin{aligned} D_t'\sigma &= \sum_{2\le i\le m} TX_i\nabla^E_T\nabla^E_{X_i}\sigma + \frac{\kappa'}{2}\sigma \\ &= \sum_{2\le i\le m} TX_i\{R^E(T, X_i)\sigma - \nabla^E_{WX_i}\sigma\} + \frac{\kappa'}{2}\sigma. \end{aligned}$$

□

Corollary 3.49. *For any C^1 section σ of E over U, which is parallel in the T-direction,*

$$\|D_t'\sigma\|_t \le C_1\|\sigma\|_t + C_w\|\nabla^E\sigma\|_t \le C_2\|\sigma\|_t + C_w\|D_t\sigma\|_t,$$

where $C_1 = mC_R^E + C_\kappa'$ and $C_2 = mC_R^E + C_\kappa' + C_w^2 + C_wC_K^{1/2}$.

Proof. By Lemmas 3.48 and 3.42, we have, at any point p of N_t,

$$\begin{aligned}|D_t'\sigma| &\le (mC_R^E + \frac{1}{2}C_\kappa')|\sigma| + \sum|\kappa_i||\nabla^E_{X_i}\sigma|\\ &\le (mC_R^E + \frac{1}{2}C_\kappa')|\sigma| + \|W\|\|\nabla^E\sigma\|\\ &\le (mC_R^E + \frac{1}{2}C_\kappa')|\sigma| + C_w\|\nabla^E\sigma\|\\ &\le (mC_R^E + \frac{1}{2}C_\kappa' + \frac{1}{2}C_w^2)|\sigma| + C_w\|\nabla^t\sigma\|,\end{aligned}$$

where $(T, X_2, \dots, X_m)$ is an orthonormal frame at p such that the X_i are eigenvectors of W with corresponding eigenvalues κ_i. By (3.43),

$$\|\nabla^t\sigma\|_t^2 \le |D_t\sigma|_t^2 + C_K|\sigma|_t^2. \qquad \square$$

Lemma 3.50. *For all $s < t$ in I and C^1 sections $\sigma_1, \sigma_2 \in H$ of E,*

$$|(A_t\sigma_1,\sigma_2)_t - (A_s\sigma_1,\sigma_2)_s| \le C(e^{C_0(t-s)/2} - 1)(\|\sigma_1\|_s + \|A_s\sigma_1\|_s)\|\sigma_2\|_s,$$

where $C = C(C_R, C_R^E, C_W, m)$.

Proof. Extend σ_1 and σ_2 by parallel translation along the T-geodesics. Then D_t corresponds to $-A_t$, and we get

$$\begin{aligned}|(D_t\sigma_1,\sigma_2)_t - (D_s\sigma_1,\sigma_2)_s| &\le \Big|\int_s^t\int_N \big(\langle D_r\sigma_1,\sigma_2\rangle j\big)'\Big|\\ &\le \int_s^t\int_N \big|\langle D_r'\sigma_1,\sigma_2\rangle + \langle D_r\sigma_1,\sigma_2\rangle\kappa\big| j\\ &\le \int_s^t\int_N \big(\|D_r'\sigma_1\| + C_\kappa\|D_r\sigma_1\|\big)\|\sigma_2\| j. \qquad (3.51)\end{aligned}$$

By Corollary 3.49 and Lemma 3.40, the first term on the right hand side of (3.51) can be estimated by

$$\begin{aligned}\int_s^t\int_N \|D_r'\sigma_1\|\|\sigma_2\|j &\le 2(C_1 + C_w)\int_s^t (\|\sigma_1\|_r^2 + \|\nabla^E\sigma_1\|_r^2)^{1/2}\|\sigma_2\|_r\\ &\le 2(C_1 + C_w)\int_s^t e^{C_0(r-s)/2}(\|\sigma_1\|_s^2 + \|\nabla^E\sigma_1\|_s^2)^{1/2}\|\sigma_2\|_s\\ &= 4\frac{C_1 + C_w}{C_0}(e^{C_0(t-s)/2} - 1)(\|\sigma_1\|_s^2 + \|\nabla^E\sigma_1\|_s^2)^{1/2}\|\sigma_2\|_s.\end{aligned}$$

Concerning the second term on the right hand side of (3.51), namely the integral of $\|D_r\sigma_1\|\|\sigma_2\|j$, we note that $\|D_r\sigma\| \le \sqrt{m-1}\|\nabla^r\sigma\|$. Hence we can estimate this term in a similar way, using Lemma 3.42. We arrive at an estimate

$$\begin{aligned}|(D_t\sigma_1, \tau)_t - (D_s\sigma_1, \sigma_2)_s|& \\ \le C'(e^{C_0(t-s)/2} - 1)(\|\sigma_1\|_s^2 + \|\nabla^E\sigma_1\|_s^2)^{1/2}\|\sigma_2\|_s,&\end{aligned}$$

where $C' = C'(C_R, C_R^E, C_W, m)$. Finally, the Bochner formula (3.43) and the ensuing lines show that

$$\begin{aligned}\|\sigma_1\|_s^2 + \|\nabla^E\sigma_1\|_s^2 &\le C(C_R^E, m)(\|\sigma_1\|_s + \|D_s\sigma_1\|_s) \\ &= C(C_R^E, m)(\|\sigma_1\|_s + \|A_s\sigma_1\|_s). \qquad \square\end{aligned}$$

Lemma 3.50 confirms the remaining requirements for the operators A_t in Section 3.1. Thus the sytem $\mathcal{D} = (\mathcal{H}, \mathcal{A}, T)$ over I from (3.47) is a Dirac system in the sense of Section 3.1 and, therefore, in the sense of Section 2.1 in [BBC2].

4. Boundary Values and Fredholm Properties

Let $\mathcal{D} = (\mathcal{H}, \mathcal{A}, T)$ be a Dirac system over

$$I = \mathbb{R}_+ := [0, \infty). \tag{4.1}$$

with origin $t_0 = 0$, where we note that an analogous discussion holds true for other intervals with non-empty boundary. By (3.5), the restriction $D_{0,c}$ of the Dirac operator D to

$$\mathcal{L}_{0,c}(\mathcal{D}) := \{\sigma \in \mathcal{L}_c(\mathcal{D}) : \sigma(0) = 0\} \tag{4.2}$$

is symmetric. The adjoint operator of $D_{0,c}$ with respect to $L^2(\mathcal{D}) \supseteq \mathcal{L}_{0,c}(\mathcal{D})$ is called the *maximal extension* of D on $\mathcal{L}_c(\mathcal{D})$. We denote it by $D_{\max}$ and let $\operatorname{dom} D_{\max}$ be the domain of $D_{\max}$, endowed with the graph norm of $D_{\max}$. The adjoint operator $D_{\min}$ of $D_{\max}$ is equal to the closure of D on $\mathcal{L}_c(\mathcal{D})$. It is called the *minimal extension* of D, and its domain is denoted by $\operatorname{dom} D_{\min}$. We also let $H^1(\mathcal{D})$ be the completion of $\mathcal{L}_c(\mathcal{D})$ with respect to the norm

$$\|\sigma\|_{H^1(\mathcal{D})}^2 := \|\sigma\|_{L^2(\mathcal{D})}^2 + \|\partial\sigma\|_{L^2(\mathcal{D})}^2 + \|A\sigma\|_{L^2(\mathcal{D})}^2. \tag{4.3}$$

Obviously,

$$\mathcal{L}_c(\mathcal{D}) \subseteq H^1(\mathcal{D}) \subseteq \operatorname{dom} D_{\max} \subseteq L^2(\mathcal{D}). \tag{4.4}$$

To formulate the main results on $\operatorname{dom} D_{\max}$ from [BBC2], we need to discuss boundary values of sections at $t = 0$. As for proofs of the corresponding assertions, we refer to the discussion in Chapters 1 and 2 of [BBC2] and, in particular, to Proposition 2.30 loc.cit.

4.1. **Boundary Values.** Recall the convention $H = H_0$. Recall also that A_0 is self-adjoint in H with domain H_A. It will be convenient, in this section, to denote elements of H by letters x, y and to call them vectors. Fix an orthonormal basis (x_i) of H which consists of eigenvectors of A_0, $A_0 x_i = \lambda_i x_i$.

For $s \geq 0$, let $H^s = H^s(A_0) \subseteq H = H_0$ be the domain of $|A_0|^s$. Then $H^0 = H$, $H^1 = H_A$, and $H^\infty = H^\infty(A_0) := \cap_{s\geq 0} H^s$ is a dense subspace of H. For $s \in \mathbb{R}$, define an inner product $\langle ., . \rangle_s$ on H^∞,

$$\langle x, y\rangle_s := ((I + A_0^2)^{s/2} x, (I + A_0^2)^{s/2} y). \tag{4.5}$$

For $s \geq 0$, the norm $\|.\|_s$ associated to $\langle ., . \rangle_s$ is equivalent to the graph norm of $|A_0|^s$, and H^s is equivalent to the completion of H^∞ with respect to $\|.\|_s$. For $s < 0$, define $H^s = H^s(A_0)$ to be the completion of H^∞ with respect to $\|.\|_s$ and set $H^{-\infty} = H^{-\infty}(A_0) := \cup_{s\in\mathbb{R}} H^s$. In terms of the above basis (x_i) of eigenvectors, H^s consists of all linear combinations $x = \sum \xi_i x_i$ with

$$\sum (1 + \lambda_i^2)^s |\xi_i|^2 < \infty. \tag{4.6}$$

The pairing

$$B_s : H^s \times H^{-s} \to \mathbb{C}, \quad B_s(x, y) := ((I + A_0^2)^{s/2} x, (I + A_0^2)^{-s/2} y), \tag{4.7}$$

is perfect, that is, identifies H^{-s} with the dual space of H^s.

For a subset $J \subset \mathbb{R}$, let $Q_J = Q_J(A_0)$ be the corresponding spectral projection of A_0 in the spaces H^s. The image of H^s under Q_J is

$$H_J^s = H_J^s(A_0) := \{x = \sum \xi_i x_i \in H^s : \xi_i = 0 \text{ if } \lambda_i \notin J\}. \tag{4.8}$$

For $x \in H^s$, we also let $x_J := Q_J(x)$. For any bounded subset J of $\mathbb{R}$, we have $H_J^s \subseteq H^\infty$. Since $T = T_0$ anti-commutes with A_0,

$$TQ_J = Q_{-J} T \quad \text{and} \quad TH_J^s = H_{-J}^s \tag{4.9}$$

As shorthand, we use, for $a \in \mathbb{R}$,

$$Q_{>a} := Q_{(a,\infty)}, \qquad Q_{\geq a} := Q_{[a,\infty)}, \tag{4.10}$$

$$Q_{<a} := Q_{(-\infty,a)}, \quad Q_{\leq a} := Q_{(-\infty,a]}, \tag{4.11}$$

and similarly for the spaces $H_J^s = Q_J(H^s)$. We also need to introduce the hybrid Sobolev space

$$\check{H} = \check{H}(A_0) := H_{\leq 0}^{1/2} \oplus H_{>0}^{-1/2}. \tag{4.12}$$

Since $H_J \subseteq H^\infty$, for any bounded $J \subseteq \mathbb{R}$,

$$\check{H} = H_{\leq a}^{1/2} \oplus H_{>a}^{-1/2} = H_{<a}^{1/2} \oplus H_{\geq a}^{-1/2}, \tag{4.13}$$

for any $a \in \mathbb{R}$. By (4.7) and (4.9),

$$\omega(x,y) := B_{1/2}(x_{\leq -a}, Ty_{\geq a}) + B_{-1/2}(x_{>-a}, Ty_{<a}) \tag{4.14}$$

is well defined for $x, y \in \check{H}$ and independent of the choice of a. We note that ω is continuous, non-degenerate, and skew-Hermitian on $\check{H}$.

Proposition 4.15. *The maximal domain* $\operatorname{dom} D_{\max}$ *satisfies:*

(1) $\mathcal{L}_c(\mathcal{D})$ *is dense in* $\operatorname{dom} D_{\max}$.
(2) *Evaluation at* $t = 0$ *on* $\mathcal{L}_c(\mathcal{D})$ *induces a continuous surjection*

$$\mathcal{R}_{\max} : \operatorname{dom} D_{\max} \to \check{H}, \quad \mathcal{R}_{\max}(\sigma) =: \sigma(0).$$

(3) $\sigma \in \operatorname{dom} D_{\max}$ *is in* $H^1_{\mathrm{loc}}(\mathcal{D})$ *iff* $\sigma(0) \in H^{1/2}$.
(4) $\sigma \in \operatorname{dom} D_{\max}$ *is in* $\operatorname{dom} D_{\min}$ *iff* $\sigma(0) = 0$.
(5) *For all* $\sigma_1, \sigma_2 \in \operatorname{dom} D_{\max}$

$$(D_{\max}\sigma_1, \sigma_2)_{L^2(\mathcal{D})} - (\sigma_1, D_{\max}\sigma_2)_{L^2(\mathcal{D})} = \omega(\sigma_1(0), \sigma_2(0)).$$

Closed extensions of D between $D_{\min}$ and $D_{\max}$ correspond precisely to closed linear subspaces B of $\check{H}$, called *boundary conditions*. For any such boundary condition B, the domain of the corresponding extension $D_{B,\max}$ is given by

$$\operatorname{dom} D_{B,\max} = \{\sigma \in \operatorname{dom} D_{\max} : \sigma(0) \in B\}. \tag{4.16}$$

We are also interested in the operator D_B with domain

$$\operatorname{dom} D_B = \operatorname{dom} D_{B,\max} \cap H^1_{\mathrm{loc}}(\mathcal{D}). \tag{4.17}$$

A boundary condition $B \subseteq \check{H}$ is called *regular* if $D_B = D_{B,\max}$. By Proposition 4.15, $\sigma \in \operatorname{dom} D_{\max}$ is in $\operatorname{dom} D_B$ if and only if $\sigma(0)$ belongs to $B \cap H^{1/2}$. In particular, B is a regular boundary condition if B is a closed subspace of $\check{H}$ that is contained in $H^{1/2} \subseteq \check{H}$.

Let $B \subseteq \check{H}$ be a boundary condition. Since ω is non-degenerate, the adjoint operator of $D_{B,\max}$ is given by $D_{B^a,\max}$, where

$$B^a = \{x \in \check{H} : \omega(x,y) = 0 \text{ for all } y \in B\}, \tag{4.18}$$

by Proposition 4.15. We say that a boundary condition B is *elliptic* if B and B^a are regular. Typical examples of elliptic boundary conditions are the Atiyah-Patodi-Singer boundary condition $B_{\mathrm{APS}} = H^{1/2}_{<0}$ and the more general $B = H^{1/2}_{<a}$ and $B = H^{1/2}_{\leq a}$. The adjoint boundary conditions for the latter are given by $B = H^{1/2}_{\leq -a}$ and $B = H^{1/2}_{<-a}$, respectively. The maximal operators corresponding to the boundary conditions $B = H^{1/2}_{<a}$ and $B = H^{1/2}_{\leq a}$ will be denoted by $D_{<a,\max}$ and $D_{\leq a,\max}$, respectively, and similarly in other cases. By ellipticity, we actually have $D_{<a,\max} = D_{<a}$ and $D_{\leq a,\max} = D_{\leq a}$.

As for boundary conditions in the super-symmetric case,

$$H = H^+ \oplus H^-, \tag{4.19}$$

we may choose orthonormal bases $x_i^\pm$ of $H^\pm$ consisting of eigenvectors of $A_0^\pm$. By (3.18), we may actually choose $x_i^- = T_0 x_i^+ T_0^{-1}$. We get

$$H^s = H^{s+} \oplus H^{s-} \quad \text{and} \quad \check{H} = \check{H}^+ \oplus \check{H}^-, \tag{4.20}$$

where

$$H^{s+} = H^s(A_0^+), \quad H^{s-} = H^s(A_0^-), \quad \check{H}^+ = \check{H}(A_0^+), \tag{4.21}$$

and

$$\begin{aligned} \check{H}^- &= \check{H}(A_0^-) = T_0 \check{H}(-A_0^+) T_0^{-1} \\ &\simeq \hat{H}^+(A_0^+) = H_{\le 0}^{-1/2} \oplus H_{>0}^{1/2}. \end{aligned} \tag{4.22}$$

Furthermore, $\check{H}^+$ and $\check{H}^-$ are Lagrangian with respect to the non-degenerate skew-Hermitian form ω.

In the super-symmetric case, we consider super-symmetric boundary conditions $B \subseteq \check{H}$, that is,

$$B = B^+ \oplus B^-, \tag{4.23}$$

where $B^\pm = B \cap \check{H}^\pm$. Then the adjoint boundary condition is super-symmetric as well. Moreover, a super-symmetric boundary condition B is regular or elliptic if and only if B^+ and B^- are regular or elliptic in $\check{H}^+$ and $\check{H}^-$, respectively. For example, $B = H_{<a}^{1/2}$ and $B = H_{\le a}^{1/2}$ are elliptic super-symmetric boundary conditions. The maximal operators corresponding to these will be denoted by $D^\pm_{<a,\max}$ and $D^\pm_{\le a,\max}$, respectively, and similarly in other cases.

4.2. **More Function Spaces.** For convenience, we assume from now on that $\mathcal{D}$ is the Dirac system associated to a Dirac bundle E over a straight end U of M with distance function f and C^1 diffeomorphism $F : \mathbb{R}_+ \times N \to U$ as in Definition 1.8.

Let $H^1(U_0, E)$ be the space of sections σ in $L^2(U_0, E)$ with square integrable weak derivative, $\nabla^E \sigma \in L^2(U_0, T^*M \otimes E)$; that is, we have

$$(\nabla^E \sigma, \tau)_{L^2(U_0,E)} = (\sigma, (\nabla^E)^* \tau)_{L^2(U_0,E)}, \tag{4.24}$$

for all $\tau \in C^\infty_{cc}(U_0, T^*M \otimes E)$, where $(\nabla^E)^*$ is the formal adjoint of the operator ∇^E. Recall that $H^1(U_0, E)$ is a Hilbert space with respect to the norm defined by the inner product

$$(\sigma, \tau)_{H^1(U_0,E)} = (\sigma, \tau)_{L^2(U_0,E)} + (\nabla^E \sigma, \nabla^E \tau)_{L^2(U_0,T^*M\otimes E)}. \tag{4.25}$$

There is the corresponding space $H^1(U, E)$, and $C^\infty_c(U, E)$ is dense in $H^1(U, E)$. Any section in $H^1(U_0, E)$ is the restriction of some section from $H^1(U,E)$; see Theorem 7.25 in [GiTu], noting that the problem is local and that $H^1_{\rm loc}$ is invariant under C^1 diffeomorphisms. It follows easily that the space $C^\infty_c(U_0,E)$ of restrictions of sections in $C^\infty_c(U,E)$ to U_0 is dense in $H^1(U_0,E)$. The trace map

$$\mathcal{R} : H^1(U_0,E) \to H^{1/2}(N,E) \tag{4.26}$$

is a well defined bounded operator; see Theorem 3.10 in [Ag] or Proposition 4.4.5 in [Ta], noting again that the problem is local and that $H^1_{\rm loc}$ is invariant under C^1 diffeomorphisms. The closure of $C^1_{cc}(U_0,E)$ in $H^1(U_0,E)$, and therefore also of $C^\infty_{cc}(U_0,E)$ in $H^1(U_0,E)$, is

$$H^1_0(U_0,E) := \{\sigma \in H^1(U_0,E) : \mathcal{R}\sigma = 0\}. \tag{4.27}$$

As for partial integration,

$$(\nabla^E\sigma,\tau)_{L^2(U_0,T^*M\otimes E)} = (\sigma,(\nabla^E)^*\tau)_{L^2(U_0,E)} - (\sigma,\tau(T))_{L^2(N,E)}, \tag{4.28}$$

for all $\sigma \in H^1(U_0,E)$ and $\tau \in H^1(U_0,T^*M \otimes E)$. It follows that

$$(D\sigma,\tau)_{L^2(U_0,E)} = (\sigma,D\tau)_{L^2(U_0,E)} + (\sigma,T\tau)_{L^2(N,E)}, \tag{4.29}$$

for all $\sigma,\tau \in H^1(U_0,E)$. In particular, any $\sigma \in H^1(U_0,E)$ belongs to the domain $\operatorname{dom} D_{\max}$ of the adjoint operator $D_{\max}$ of D, the latter considered as an unbounded operator on $L^2(U_0,E)$ with domain $C^\infty_{cc}(U_0,E)$ or, equivalently, $H^1_0(U_0,E)$.

We switch now to the associated Dirac system $\mathcal{D}$ over $\mathbb{R}_+ = [0,\infty)$. With respect to the natural identifications,

$$C^\infty_c(U_0,E) \subseteq \mathcal{L}_c(\mathcal{D}) \subseteq H^1(\mathcal{D}) = H^1(U_0,E), \tag{4.30}$$

where we use (3.43) and (2.16) for the latter equality. The convenience we had in mind further up refers to the density of $C^\infty_c(U_0,E)$ in $H^1(\mathcal{D})$. Another convenience: We often write $\|.\|_I$ for the L^2-norm of maps defined on an interval I (if meaningful).

Proposition 4.31. *For all $w \in \mathbb{R}$ and $\sigma \in H^1(\mathcal{D})$,*

$$\begin{aligned}\|D\sigma - wT\sigma\|^2_{\mathbb{R}_+} = \|\partial\sigma\|^2_{\mathbb{R}_+} + \|(A-w)\sigma\|^2_{\mathbb{R}_+} \\ - \operatorname{Re}(A'\sigma,\sigma)_{\mathbb{R}_+} - (\sigma(0),(A_0-w)\sigma(0))_0.\end{aligned}$$

Remark 4.32. As for the meaning of the last term on the right, we note that the trace $\sigma(0)$ of σ is in $H^{1/2}(A_0)$, hence A_0 applied to it is in $H^{-1/2}(A_0)$, and hence $(\sigma(0),(A_0-w)\sigma(0))_0$ is well defined.

Proof of Proposition 4.31. By the density of $C^\infty_c(U_0,E)$ in $H^1(U_0,E)$, we may assume that σ is smooth with compact support. Then

$$\begin{aligned}\|(D - wT)\sigma\|^2_{\mathbb{R}_+} = \|\partial\sigma\|^2_{\mathbb{R}_+} + \|(A-w)\sigma\|^2_{\mathbb{R}_+} \\ + 2\operatorname{Re}(\partial\sigma,A\sigma)_{\mathbb{R}_+} + (\sigma(0),w\sigma(0))_0.\end{aligned}$$

Now

$$\begin{aligned}(\partial\sigma, A\sigma)_{\mathbb{R}_+} &= \int_N \int_{\mathbb{R}_+} \big(\langle\sigma, A\sigma\rangle j\big)' dt dx - (\sigma, \partial A\sigma)_{\mathbb{R}_+} \\ &= -(\sigma(0), A_0\sigma(0))_0 - (\sigma, \partial A\sigma)_{\mathbb{R}_+}.\end{aligned}$$

Since $(A\sigma)' = A'\sigma + A\sigma'$, we conclude that

$$(\sigma, \partial A\sigma)_{\mathbb{R}_+} = (\sigma, A'\sigma)_{\mathbb{R}_+} + (A\sigma, \partial\sigma)_{\mathbb{R}_+} + i\,\mathrm{Im}(\kappa\sigma, A\sigma)_{\mathbb{R}_+}. \qquad \square$$

4.3. **Fredholm Properties of $\mathcal{D}$.** We say that $\mathcal{D}$ is of *Fredholm type* if there is a constant $C > 0$ such that

$$\|\sigma\|_{\mathbb{R}_+} \le C\|D\sigma\|_{\mathbb{R}_+}, \quad \forall \sigma \in \mathcal{L}_{0,c}(\mathcal{D}), \tag{4.33}$$

and that $\mathcal{D}$ is *non-parabolic* if, for each $t > 0$, there is a constant $C > 0$ such that

$$\|\sigma\|_{[0,t]} \le C\|D\sigma\|_{\mathbb{R}_+}, \quad \forall \sigma \in \mathcal{L}_{0,c}(\mathcal{D}). \tag{4.34}$$

Obviously, if $\mathcal{D}$ is of Fredholm type, then it is non-parabolic. In Lemma 2.38 of [BBC2], we showed that $\mathcal{D}$ is non-parabolic if and only if, for each $t > 0$, there is a constant $C > 0$ such that

$$\|\sigma\|_{[0,t]} \le C\big(\|\sigma(0)\|^2_{\check{H}} + \|D\sigma\|^2_{\mathbb{R}_+}\big)^{1/2} =: \|\sigma\|_W, \tag{4.35}$$

for all $\sigma \in \mathcal{L}_c(\mathcal{D})$. If $\mathcal{D}$ is non-parabolic, we let $W \subseteq L^2_{\mathrm{loc}}(\mathcal{D})$ be the completion of $\mathcal{L}_c(\mathcal{D})$ with respect to the norm $\|.\|_W$. We note that $\|.\|_W$ is weaker than the graph norm of D, hence $\mathrm{dom}\, D_{\max} \subseteq W$, by Proposition 4.15.1, and equality holds if and only if $\mathcal{D}$ is of Fredholm type. Moreover, if $\varphi : \mathbb{R}_+ \to \mathbb{C}$ is Lipschitz continuous with compact support and $\sigma \in W$, then $\varphi\sigma \in \mathrm{dom}\, D_{\max}$. In particular, the trace $\mathcal{R}$ is well defined and continuous on W and takes values in $\check{H} = \check{H}(A_0)$.

Assume now that $\mathcal{D}$ is non-parabolic. Then, by the definition of W, D extends to a bounded operator $D_{\mathrm{ext}} : W \to L^2(\mathcal{D})$. For a boundary condition $B \subseteq \check{H}$, we define $D_{B,\mathrm{ext}}$ to be the operator in W with target $L^2(\mathcal{D})$ and domain

$$\mathrm{dom}\, D_{B,\mathrm{ext}} = \{\sigma \in W : \sigma(0) \in B\}. \tag{4.36}$$

Obviously, $D_{B,\mathrm{ext}}$ is closed and extends $D_{B,\max}$, and $D_{B,\mathrm{ext}} = D_{B,\max}$ if and only if $\mathcal{D}$ is of Fredholm type.

In Theorem 2.43 of [BBC2] we showed that, for $\mathcal{D}$ non-parabolic and B regular, $D_{B,\mathrm{ext}}$ has finite dimensional kernel and closed image with

$$(\mathrm{im}\, D_{B,\mathrm{ext}})^\perp = \ker D_{B^a,\max}. \tag{4.37}$$

Thus, if $\mathcal{D}$ is non-parabolic and B is elliptic, then $D_{B,\mathrm{ext}}$ is a Fredholm operator and the L^2-index

$$\operatorname{ind}_{L^2} D_{B,\max} := \dim\ker D_{B,\max} - \dim\ker D_{B^a,\max} \tag{4.38}$$

of $D_{B,\max}$ is well defined and finite.

Proposition 4.39. *Assume that, for some $a \geq 0$,*

$$(A_t\sigma, A_t\sigma)_t \geq \operatorname{Re}(A_t'\sigma,\sigma)_t + a\|\sigma\|_t^2,$$

for all $t \geq 0$ and $\sigma \in H_A$. Then $\mathcal{D}$ is non-parabolic and $D_{<0,\mathrm{ext}}$ is an isomorphism. Moreover, if $a > 0$, then $\mathcal{D}$ is of Fredholm type.

Proof. Recall the Hardy inequality,

$$\int_{\mathbb{R}_+} |\phi'|^2 \geq \int_{\mathbb{R}_+} \frac{|\phi|^2}{4t^2}, \tag{4.40}$$

for any C^1 function ϕ on $\mathbb{R}_+$ with $\phi(0) = 0$. By Proposition 4.31,

$$\|D\sigma\|_{\mathbb{R}_+}^2 \geq \|\partial\sigma\|_{\mathbb{R}_+}^2 + a\|\sigma\|_{\mathbb{R}_+}^2,$$

for all $\sigma \in H_0^1(U_0, E)$. Applying (3.32) and (4.40) we get

$$\begin{aligned}\|\partial\sigma\|_{\mathbb{R}_+}^2 &= \int_N \int_0^\infty \|(j^{1/2}\sigma)'\|^2 dt dx \\ &\geq \int_N \int_0^\infty \frac{1}{4t^2}\|\sigma\|^2 j dt dx = \int_0^\infty \frac{1}{4t^2}\|\sigma\|_t^2 dt.\end{aligned}$$

It follows that $\mathcal{D}$ is non-parabolic. Clearly, if $a > 0$, then $\mathcal{D}$ is of Fredholm type.

Using the density of $\mathcal{L}_c(\mathcal{D})$ in W, Proposition 4.31 together with the assumed inequality implies that

$$\|\partial\sigma\|_{\mathbb{R}_+}^2 - (\sigma(0), A_0\sigma(0))_0 \leq \|D\sigma\|_{\mathbb{R}_+},$$

for any $\sigma \in W$. Hence $D\sigma = 0$ and $\sigma(0) \in \check{H}_{<0}$ implies that $\partial\sigma = 0$ and $\sigma(0) = 0$. That is, σ solves

$$\sigma' = -\frac{\kappa}{2}\sigma, \tag{4.41}$$

with $\sigma(0) = 0$, hence $\sigma = 0$, and therefore $\ker D_{<0,\mathrm{ext}}$ is trivial.

Conversely, the cokernel of $D_{<0,\mathrm{ext}}$ is isomorphic to $\ker D_{\leq 0,\max}$, by what we said further up. Now the same argument as above shows that any $\sigma \in \ker D_{\leq 0,\max}$ with $\sigma(0) \in \check{H}_{\leq 0}$ satisfies $\partial\sigma = 0$. It follows that σ solves (4.41) and hence, by (3.32), that

$$\sigma(t,x) = j^{-1/2}(t,x)\sigma(0,x),$$

for all $t \in \mathbb{R}_+$ and $x \in N$. Since the L^2-norm of σ is finite, we conclude that $\sigma = 0$. Hence $\operatorname{coker} D_{<0,\mathrm{ext}}$ is trivial as well. □

By Corollary 3.49,

$$c_0 := \sup_{t\in\mathbb{R}_+,\sigma\in H_A\setminus\{0\}} \frac{\|A_t'\sigma\|_t}{\|\sigma\|_t + \|A_t\sigma\|_t} \le C(C_R, C_R^E, C_W, n) < \infty. \tag{4.42}$$

Corollary 4.43. *Suppose that* $\operatorname{spec} A_t \cap (-\lambda,\lambda) = \emptyset$, *for all* $t \in \mathbb{R}_+$, *where*

$$2\lambda \ge c_0 + \sqrt{4c_0 + c_0^2}.$$

Then $\mathcal{D}$ *is non-parabolic and* $D_{<0,\mathrm{ext}}$ *is an isomorphism. Moreover, if the inequality is strict, then* $\mathcal{D}$ *is of Fredholm type.*

Proof. Choose $a \ge 0$ with

$$2\lambda \ge c_0 + \sqrt{4c_0 + c_0^2 + 4a}.$$

Then we have, for all $t \ge 0$ and $\sigma \in H_A$,

$$\begin{aligned}\|A_t\sigma\|_t^2 - \operatorname{Re}(A_t'\sigma,\sigma)_t &\ge \|A_t\sigma\|_t^2 - c_0(\|A_t\sigma\|_t + \|\sigma\|_t)\|\sigma\|_t \\ &\ge (\lambda^2 - c_0\lambda - c_0)\|\sigma\|_t^2 \ge a\|\sigma\|_t^2,\end{aligned}$$

by the definition of c_0, and hence Proposition 4.39 applies. □

The following estimate relates boundary conditions to Fredholm properties of D, as we will see further on.

Lemma 4.44. *For all* $\sigma \in H_c^1(U_0, E)$ *and* $w \in \mathbb{R}$,

$$\begin{aligned}\|\partial\sigma\|_{\mathbb{R}_+}^2 &+ \frac{1}{2}\|(A-w)\sigma\|_{\mathbb{R}_+}^2 \\ &\le \|(D-wT)\sigma\|_{\mathbb{R}_+}^2 + c_1\|\sigma\|_{\mathbb{R}_+}^2 + (\sigma_0,(A_0-w)\sigma_0)_0,\end{aligned}$$

where $2c_1 = c_0(c_0 + 2 + 2|w|)$.

Proof. By Proposition 4.31 and the definition of c_0,

$$\begin{aligned}\|\partial\sigma\|_{\mathbb{R}_+}^2 + \|(A-w)\sigma\|_{\mathbb{R}_+}^2 &- \|(D-wT)\sigma\|_{\mathbb{R}_+}^2 - (\sigma_0,(A_0-w)\sigma_0)_0 \\ &= \operatorname{Re}(A'\sigma,\sigma)_{\mathbb{R}_+} \\ &\le c_0(\|A\sigma\|_{\mathbb{R}_+} + \|\sigma\|_{\mathbb{R}_+})\|\sigma\|_{\mathbb{R}_+} \\ &\le c_0\big(\|(A-w)\sigma\|_{\mathbb{R}_+} + (1+|w|)\|\sigma\|_{\mathbb{R}_+}\big)\|\sigma\|_{\mathbb{R}_+} \\ &\le \frac{1}{2}\|(A-w)\sigma\|_{\mathbb{R}_+}^2 + c_0(\frac{c_0}{2} + 1 + |w|)\|\sigma\|_{\mathbb{R}_+}^2.\end{aligned}$$ □

Proposition 4.45. *Assume that there are* $\Lambda > \lambda \ge 0$ *such that*

$$(\Lambda - \lambda)^2 > 4c_0(c_0 + 2 + \lambda + \Lambda) \quad \textit{and} \quad \operatorname{spec} A_t \cap (\lambda, \Lambda) = \emptyset, \tag{4.46}$$

for all $t \in \mathbb{R}_+$. *Suppose* $w \in (\lambda, \Lambda)$ *satisfies*

$$|\lambda - w|^2, |\Lambda - w|^2 > 2c_1 = c_0(c_0 + 2 + 2w). \tag{4.47}$$

Then we have:

(1) *If* $\sigma \in e^{-wt}L^2(\mathcal{D})$ *solves* $D\sigma = 0$ *in the sense of distributions with* $\sigma(0) \in \check{H}_{<\Lambda}$, *then* $\sigma = 0$.
(2) *If* $\sigma \in e^{wt}L^2(\mathcal{D})$ *solves* $D\sigma = 0$ *in the sense of distributions with* $\sigma(0) \in \check{H}_{<-\lambda}$, *then* $\sigma = 0$.
(3) $\mathcal{D}$ *is non-parabolic and* $D_{<-\lambda,\mathrm{ext}}$ *is injective.*

The first assumption in Proposition 4.45 implies that the set of w in (λ, Λ) satisfying the required inequalities is non-empty. We also recall from (3.9) that the spectrum of A_t, $t \in \mathbb{R}$, is symmetric about 0 so that the second assumption implies that spec A_t has empty intersection with $-(\lambda, \Lambda)$ as well. We get that $H^s_{\le\lambda} = H^s_{<\Lambda}$ and that $H^s_{<-\lambda} = H^s_{\le-\Lambda}$, for all $s \in \mathbb{R}$.

Proof of Proposition 4.45. Let $\sigma \in H^1_c(U_0, E)$, $v \in \mathbb{R}$, and set $\tau = e^{vt}\sigma$. Then

$$\begin{aligned}\|e^{vt}D\sigma\|^2_{\mathbb{R}_+} &+ (\tau_0, (A_0 - v)\tau_0)_0 - \|\partial\tau\|^2_{\mathbb{R}_+} \\ &= \|(D - vT)\tau\|^2_{\mathbb{R}_+} + (\tau_0, (A_0 - v)\tau_0)_0 - \|\partial\tau\|^2_{\mathbb{R}_+} \\ &\ge \frac{1}{2}\|(A - v)\tau\|^2_{\mathbb{R}_+} - \frac{c_0}{2}(c_0 + 2 + 2|v|)\|\tau\|^2_{\mathbb{R}_+},\end{aligned}$$

by Lemma 4.44. Suppose now that $w \in (\lambda, \Lambda)$ satisfies the required inequalities, and choose $\varepsilon > 0$ such that

$$|\Lambda - w|^2, |\lambda - w|^2 \ge c_0(c_0 + 2 + 2w) + 2\varepsilon. \tag{4.48}$$

Then, with $v = \pm w$, we continue the above computation and get

$$\|(D - vT)\tau\|^2_{\mathbb{R}_+} + (\tau_0, (A_0 - v)\tau_0)_0 \ge \|\partial\tau\|^2_{\mathbb{R}_+} + \varepsilon\|\tau\|^2_{\mathbb{R}_+}. \tag{4.49}$$

By the density of $H^1_c(U_0, E)$ in dom $D_{\max}$, any $\tau \in$ dom $D_{\max}$ satisfies

$$\|(D - vT)\tau\|^2_{\mathbb{R}_+} + (\tau_0, (A_0 - v)\tau_0)_0 \ge \varepsilon\|\tau\|^2_{\mathbb{R}_+}, \tag{4.50}$$

where $v = \pm w$ and ε are as above. Now with σ as in the first two assertions and $v = w$ and $v = -w$, respectively, $\tau = e^{vt}\sigma$ is in dom $D_{\max}$ and satisfies $D_{\max}\tau = vT\tau$. The boundary condition for σ implies that the boundary term in (4.50) is non-positive, hence $\tau = 0$, and hence $\sigma = 0$. This shows the two first assertions. As for the last assertion, we note that

$$\begin{aligned}\|D\sigma\|^2_{\mathbb{R}_+} &+ (\sigma_0, (A_0 - w)\sigma_0)_0 \\ &\ge \|e^{-wt}D\sigma\|^2_{\mathbb{R}_+} + (\sigma_0, (A_0 - w)\sigma_0)_0 \\ &\ge \varepsilon\|e^{-wt}\sigma\|^2_{\mathbb{R}_+},\end{aligned} \tag{4.51}$$

for any $\sigma \in H^1_c(U_0, E)$. □

For later purposes we note that the computations in the above proof also show that

$$\|e^{wt} D\sigma\|^2_{\mathbb{R}_+} + (\sigma_0, (A_0 - w)\sigma_0)_0 \ge \varepsilon \|e^{wt}\sigma\|^2_{\mathbb{R}_+}, \tag{4.52}$$

for any $\sigma \in H^1_c(U_0, E)$, where $w \in (\lambda, \Lambda)$ and ε is as in (4.48).

Suppose now that the assumptions of Proposition 4.45 are satisfied and that $w \in (\lambda, \Lambda)$ satisfies the corresponding inequalities. Then (4.51) and (4.52) lead us to consider the *weighted Lebesgue spaces* $L^2_{\pm w}(\mathcal{D}) := e^{\mp wt} L^2(\mathcal{D})$, with norm associated to the inner product

$$(\sigma, \tau)_{\pm w} := (e^{\pm wt}\sigma, e^{\pm wt}\tau)_{\mathbb{R}_+}, \tag{4.53}$$

and the *weighted Sobolev spaces* $H^1_{w,<\mu}(\mathcal{D})$, the completions of $H^1_{<\mu,c}(\mathcal{D})$ with respect to the norms

$$\|\sigma\|_{H^1_w(\mathcal{D})} := \|\sigma\|_w + \|D\sigma\|_w. \tag{4.54}$$

Corollary 4.55. *If the assumptions of Proposition 4.45 hold and $w \in (\lambda, \Lambda)$ satisfies the corresponding inequalities, then the operators*

$$D_{w,<\Lambda} : H^1_{w,<\Lambda}(\mathcal{D}) \to L^2_w(\mathcal{D}) \quad \textit{and}$$
$$D_{-w,<-\lambda} : H^1_{-w,<-\lambda}(\mathcal{D}) \to L^2_{-w}(\mathcal{D})$$

are adjoints of each other and isomorphisms.

We note that D_w on $L^2_w(\mathcal{D})$ is conjugate to the operator $D - w$ on $L^2(\mathcal{D})$, and similarly for D_{-w}. Hence the operators $D_{\pm w}$ are Dirac-Schrödinger operators in the sense of [BBC2] (compare also Remark 2.27 of loc.cit.).

Proof of Corollary 4.55. The operators are adjoints of each other since $\check{H}_{<-\lambda} = \check{H}_{\le -\Lambda}$, by the assumptions of Proposition 4.45. By (4.51) and (4.52), the images of the operators are closed. The first two assertions of Proposition 4.45 say that their kernels are trivial. By integration by parts as in (5) of Proposition 4.15, we see that $\sigma \in L^2_w(\mathcal{D})$ is in the orthogonal complement of $D(H^1_{w,<\Lambda}(\mathcal{D}))$ if $\tau := e^{2wt}\sigma$ solves $D\tau = 0$ weakly with $\tau(0) \in H_{\le -\Lambda} = H_{<-\lambda}$. Now $\tau \in e^{wt}L^2(\mathcal{D})$, hence $\tau = 0$, by the second assertion of Proposition 4.45. This shows that the first operator is an isomorphism. The claim for the second follows in a similar fashion, using the first assertion of Proposition 4.45. □

Corollary 4.56. *If the assumptions of Proposition 4.45 hold, then*

$$\operatorname{ind} D_{<0,\mathrm{ext}} = \dim H_{[-\lambda,0)} - \dim\ker D_{\le\lambda,\max}.$$

In the super-symmetric case,

$$\operatorname{ind} D^+_{<0,\mathrm{ext}} = \dim H^+_{[-\lambda,0)} - \dim\ker D^-_{\le\lambda,\max}.$$

Proof. By Theorem 3.14 of [BBC2], we have

$$\operatorname{ind} D_{<0,\mathrm{ext}} = \operatorname{ind} D_{<-\lambda,\mathrm{ext}} + \dim H_{[-\lambda,0)}.$$

By Proposition 4.45, $D_{<-\lambda,\mathrm{ext}}$ is injective. On the other hand, the orthogonal complement of $\operatorname{im} D_{<-\lambda,\mathrm{ext}}$ is given by the space of σ in $L^2(\mathcal{D})$ with $D\sigma = 0$ and $\sigma(0) \in H_{\leq\lambda}$. This shows the first claim, and the proof of the second is similar. □

5. Decomposition and Index

We assume from now on that we have a decomposition $M = M_0 \cup U_0$, where M_0 and U_0 are domains in M such that M_0 is compact and connected and such that $N := M_0 \cap U_0 \neq \emptyset$ is a level surface of a C^2 distance function f which is defined in some open neighborhood of N in M. We assume that $T := \operatorname{grad} f$ points into the direction of U_0, set $A_0 := -D_N$ as in (3.45) and get the associated Sobolev spaces $H^s = H^s(A_0)$ as in Section 4.1.

Lemma 5.1. *There is a constant $C > 1$ such that*

$$\|\sigma\|_{H^1(M_0,E)} \leq C\big(\|\sigma|_N\|_{H^{1/2}} + \|D\sigma\|_{L^2(M_0,E)}\big),$$

for all $\sigma \in H^1(M_0, E)$; that is, D is of Fredholm type over M_0.

Proof. Let $\mathcal{R}_0 : H^1(M_0, E) \to H^{1/2}$ be restriction to N, $\mathcal{R}_0\sigma := \sigma|_N$, and $\mathcal{E}_0 : H^{1/2} \to H^1(M_0, E)$ be an extension operator. Since $\mathcal{E}_0$ and $\mathcal{R}_0$ are continuous and $H^1_0(M_0, E)$ is the kernel of $\mathcal{R}_0$,

$$H^1(M_0, E) \to H^{1/2} \times H^1_0(M_0, E), \quad \sigma \mapsto (\mathcal{R}_0\sigma, \sigma - \mathcal{E}_0\mathcal{R}_0\sigma),$$

is a continuous bijection, hence an isomorphism (of topological vector spaces). Since M_0 is compact and connected with non-empty boundary N, there is a constant C such that

$$\|\sigma\|_{L^2(M_0,E)} \leq C\|D\sigma\|_{L^2(M_0,E)},$$

for any $\sigma \in H^1_0(M_0, E)$, by the unique continuation property for solutions of the Dirac equation. □

It will be convenient to write D_{M_0} for the restriction of D to M_0, and similarly in corresponding cases.

Consider the manifold $\tilde{M}$ which is the disjoint union of M_0 and U_0, endowed with the Dirac bundle $\tilde{E} \to \tilde{M}$ induced by E. We want to apply the results from Chapter 5 of [BBC2] to the Dirac operator $\tilde{D}$ of $\tilde{E}$ and, therefore, need to check whether the requirements of Axiom VI there are satisfied. The only requirement in that axiom which might be non-obvious is dealt with in the following lemma.

LEMMA 5.2. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support which is equal to* 1 *close to* 0*. Then $(1-\chi\circ f)\sigma \in \operatorname{dom} D_{U_0,\min}$ for all $\sigma \in \operatorname{dom} D_{U_0,\max}$, and similarly for M_0.*

Proof. We note first that $(1-\chi\circ f)\sigma$ is a section in $\operatorname{dom} D_{U_0,\max}$ which vanishes in a neighborhood of the boundary N of U_0. Hence the extension $\tilde\sigma$ of $(1-\chi\circ f)\sigma$ by 0 to M_0 is in $\operatorname{dom} D_{\max}$. Now we have $\operatorname{dom} D_{\max} = \operatorname{dom} D_{\min}$, by Theorem II.5.7 of [LaMi]. Hence there is a sequence of smooth sections $\sigma_k \in C_c^\infty(M,E)$ such that $\sigma_k \to \tilde\sigma$ in $\operatorname{dom} D_{\min}$. It follows that $(1-\tilde\chi\circ f)\sigma_k \to (1-\chi\circ f)\sigma$ in $\operatorname{dom} D_{U_0,\min}$, where $\tilde\chi : \mathbb{R}\to\mathbb{R}$ is a smooth function with compact support such that $(1-\tilde\chi)(1-\chi) = (1-\chi)$. □

Because T is the exterior normal to M_0, the space of boundary values of the maximal extension $D_{M_0,\max}$ of D over M_0 is the hybrid Sobolev space

$$\hat H = H^{-1/2}_{<0} \oplus H^{1/2}_{\ge 0} = \check H(-A_0). \tag{5.3}$$

LEMMA 5.4. *For any $\lambda \ge 0$, we have*

$$\operatorname{ind} D_{M_0,\ge -\lambda} = \frac{1}{2}\dim H_{[-\lambda,\lambda]}.$$

Proof. Since $D_{M_0,\ge-\lambda}$ is the adjoint operator of $D_{M_0,>\lambda}$, we have

$$\operatorname{ind} D_{M_0,\ge-\lambda} = -\operatorname{ind} D_{M_0,>\lambda}.$$

On the other hand,

$$\operatorname{ind} D_{M_0,\ge-\lambda} - \operatorname{ind} D_{M_0,>\lambda} = \dim H_{[-\lambda,\lambda]},$$

by Theorem 5.16 in [BBC2]. □

The same argument applies to $D_{U_0,\le\lambda,\max}$ if D is of Fredholm type.

Specifying the data in the definition of non-parabolicity of the third named author, compare [Ca1], we say that D is *non-parabolic* with respect to some subset $L \subseteq M$ if, for any relatively compact open subset $K \subseteq M$, there exists a constant $C = C(K,L)$ such that

$$\|\sigma\|_{L^2(K,E)} \le C\|D\sigma\|_{L^2(M,E)}, \tag{5.5}$$

for any smooth section σ of E with compact support such that $\sigma|_L = 0$. Obviously, if D is of Fredholm type, then D is non-parabolic with respect to any sufficiently large compact subset, and if D is non-parabolic with respect to some subset, then also with respect to any larger subset. Furthermore, if M is connected, then D is non-parabolic with respect to any subset whose complement is relatively compact, by Lemma 5.1. If D is non-parabolic with respect to some compact subset, we say that D is *non-parabolic at infinity.*

PROPOSITION 5.6. *Suppose that the ends of M are straight in the sense of Definition 1.8 and let $\mathcal{D}$ be the Dirac system over $\mathbb{R}_+$ associated to D over U_0 as in Section 3.3. Then D is non-parabolic with respect to M_0 if and only if $\mathcal{D}$ is non-parabolic in the sense of Section 4.3. In particular, if $\mathcal{D}$ satisfies the assumptions of Proposition 4.45, then D is non-parabolic with respect to M_0.* □

Assume from now on that D is non-parabolic with respect to M_0. Let $W(M,E)$ and $W(U_0,E)$ be the completion of $H^1_c(M,E)$ and $H^1_c(U_0,E)$ with respect to the norms associated to the inner products

$$(5.7)\qquad \begin{aligned}(\sigma,\tau)_{W(M,E)} &:= (\sigma,\tau)_{H^1(M_0,E)} + (D\sigma, D\tau)_{L^2(U_0,E)},\\ (\sigma,\tau)_{W(U_0,E)} &:= (\sigma,\tau)_{\check{H}} + (D\sigma, D\tau)_{L^2(U_0,E)},\end{aligned}$$

respectively. We have

$$(5.8)\qquad \begin{aligned}W(M,E) &= \{(\sigma,\tau) \in H^1(M_0,E) \oplus W(U_0,E) : \sigma|_N = \tau|_N\}\\ &\subseteq H^1_{\mathrm{loc}}(M,E),\end{aligned}$$

since the transmission condition $\sigma|_N = \tau|_N$ is a regular boundary condition for the manifold $\tilde{M}$ as above, see Example 1.85 in [BBC2]. By definition, D induces continuous operators

$$(5.9)\qquad \begin{aligned}D_{\mathrm{ext}} &: W(M,E) \to L^2(M,E),\\ D_{U_0,\mathrm{ext}} &: W(U_0,E) \to L^2(U_0,E).\end{aligned}$$

We arrive at the following version of Théorème 0.3 of [Ca2].

THEOREM 5.10. *Suppose that D is non-parabolic with respect to M_0. Then $D_{\mathrm{ext}} : W(M,E) \to L^2(M,E)$ is a Fredholm operator with*

$$(\operatorname{im} D_{\mathrm{ext}})^\perp = \ker D_{\max} = \{\sigma \in L^2(M,E) : D\sigma = 0 \textit{ weakly}\}.$$

Proof. Theorem 5.12 in [BBC2] implies that the image of D_{ext} is closed and that $\ker D_{\mathrm{ext}}$ is of finite dimension. The last claim follows from the density of $H^1_c(M,E)$ in $W(M,E)$ and since D is formally self-adjoint. Finally, since $\ker D_{\max} \subseteq \ker D_{\mathrm{ext}}$ and the latter is of finite dimension, D_{ext} is a Fredholm operator. □

In the super-symmetric case $E = E^+ \oplus E^-$, we get operators

$$(5.11)\qquad D^\pm_{\mathrm{ext}} : W(M,E^\pm) \to L^2(M,E^\mp).$$

Since D_{ext} is a Fredholm operator, the operators $D^\pm_{\mathrm{ext}}$ are Fredholm operators as well and

$$(5.12)\qquad \operatorname{ind} D^+_{\mathrm{ext}} = \dim\ker D^+_{\mathrm{ext}} - \dim\ker D^-_{\max},$$

by Theorem 5.10 (and since D is formally self-adjoint).

The transmission condition $\sigma|_N = \tau|_N$ as above is elliptic. Therefore it can be decoupled into separate boundary conditions for M_0 and U_0, respectively, compare Theorems 3.24 and 5.12 in [BBC2]. This leads to the following index formulas.

THEOREM 5.13. *Suppose that D is non-parabolic with respect to M_0. Then we have, for any $\lambda \geq 0$,*

$$\operatorname{ind} D_{\mathrm{ext}} = \frac{1}{2} \dim H_{[-\lambda,\lambda]} + \operatorname{ind} D_{U_0,<-\lambda,\mathrm{ext}}.$$

In the super-symmetric case,

$$\operatorname{ind} D^+_{\mathrm{ext}} = \operatorname{ind} D^+_{M_0,\geq 0} + \dim H^+_{[-\lambda,0)} + \operatorname{ind} D^+_{U_0,<-\lambda,\mathrm{ext}}.$$

Proof. The assertions are immediate consequences of Theorems 3.24, 4.17, and 5.12 in [BBC2] and Lemma 5.4 above. □

Suppose now that the ends of M are straight in the sense of Definition 1.8. We may then consider weighted Lebesgue and Sobolev spaces, following the discussion just before and in Corollary 4.55. For $w \in \mathbb{R}$, let $L^2_w(M, E)$ be the space of measurable sections of E which are square integrable over M with respect to the weight which is equal to 1 over M_0 and equal to e^{2wt} over U_0. Endow $L^2_w(M, E)$ with the corresponding inner product

$$(\sigma, \tau)_{L^2_w(M,E)} := (\sigma, \tau)_{L^2(M_0,E)} + (e^{wt}\sigma, e^{wt}\tau)_{L^2(U_0,E)}. \tag{5.14}$$

Furthermore, let $H^1_w(M, E)$ be the completion of $H^1_c(M, E)$ with respect to the norm associated to the inner product

$$(\sigma, \tau)_{H^1_w(M,E)} := (\sigma, \tau)_{L^2_w(M,E)} + (D\sigma, D\tau)_{L^2_w(M,E)}. \tag{5.15}$$

Assume from now on that the assumptions of Proposition 4.45 are satisfied and that $w \in \mathbb{R}$ satisfies the corresponding inequalities. Then, by (4.51), (4.52), and Lemma 5.1, the $H^1_{\pm w}(M, E)$-norm is equivalent to the norm

$$\|\sigma\|_{\pm w} := \|\sigma|_N\|_{H^{1/2}} + \|D\sigma\|_{L^2(M_0,E)} + \|e^{\pm wt} D\sigma\|_{L^2(U_0,E)}. \tag{5.16}$$

Thus, by restriction to M_0 and U_0, respectively, $H^1_w(M, E)$ is isomorphic to the space of pairs (σ, τ) in $H^1(M_0, E) \oplus H^1_w(U_0, E)$ satisfying the transmission condition $\sigma|_N = \tau|_N$.

THEOREM 5.17. *Suppose that the Dirac system $\mathcal{D}$ over $\mathbb{R}_+$ associated to E over U_0 satisfies the assumptions of Proposition 4.45 and that $w > 0$ satisfies the corresponding inequalities. Then*

$$D_{-w} : H^1_{-w}(M, E) \to L^2_{-w}(M, E)$$

is a Fredholm operator with index

$$\operatorname{ind} D_{-w} = \frac{1}{2} \dim H_{[-\lambda,\lambda]}.$$

In the super-symmetric case,

$$\operatorname{ind} D^+_{-w} = \operatorname{ind} D^+_{M_0,\geq 0} + \dim H^+_{[-\lambda,0)}.$$

Proof. By (4.51), D_{-w} as above is a Fredholm operator. We also note that $D_{U_0,-w}$ is conjugate to the operator $D_{U_0} + w \operatorname{grad} f$, where f is the given distance function over U_0. Hence the results of Section 3 in [BBC2] apply (compare also Remark 2.27 of loc.cit.) and show that the Calderón projections associated to L^2_{-w}-solutions of the equation $D\sigma = 0$ over M_0 and U_0 are elliptic. Hence, by Theorems 3.24 and 5.12 in [BBC2], D as above has index

$$\operatorname{ind} D_{-w} = \operatorname{ind} D_{M_0,\geq-\lambda} + \operatorname{ind} D_{U_0,-w,<-\lambda}.$$

By Corollary 4.55, $\operatorname{ind} D_{U_0,-w,<-\lambda} = 0$, hence the formula for $\operatorname{ind} D_{-w}$ follows from Lemma 5.4. In the super-symmetric case,

$$\begin{aligned}\operatorname{ind} D^+_{-w} &= \operatorname{ind} D^+_{M_0,\geq-\lambda} + \operatorname{ind} D^+_{U_0,-w,<-\lambda} \\ &= \operatorname{ind} D^+_{M_0,\geq-\lambda} = \operatorname{ind} D^+_{M_0,\geq 0} + \dim H^+_{[-\lambda,0)}.\end{aligned}$$ ☐

In the case where the boundary $N = N_0$ of M_0 is smooth, Theorem 3.1 in Atiyah-Patodi-Singer [APS1] applies and gives

$$\begin{aligned}\operatorname{ind} D^+_{M_0,\geq 0} &= \int_{M_0} \omega_{D^+} + \int_{N_0} \tau_{D^+} \\ &\quad + \frac{1}{2}\big(\eta(A_0^+) + \dim\ker A_0^+\big),\end{aligned} \tag{5.18}$$

where ω_{D^+} is the index form and τ_{D^+} the transgression form. We remark that ω_{D^+} is a universal polynomial in the curvatures of M and E and that τ_{D^+} is a universal polynomial in the curvature of M and E and the second fundamental form of N; compare [Gil1] and Section 3.10 in [Gil2]. Now we may approximate M_0 by smooth domains such that the second fundamental forms of their boundaries approximate the second fundamental form of N. Then the integrals of ω_{D^+} and τ_{D^+} over the approximating domains and their boundaries converge to the integral of the corresponding forms over M_0 and N_0, respectively. On the other hand, the coefficients of A_0^+ are only C^1 in general, and therefore the η-invariant of A_0^+ may not be well defined. However, since the other terms on the right hand side of (5.18) are well defined, we may define $\eta(A_0^+)$ to be the number such that (5.18) holds. In [Hi], Michel Hilsum defined η-invariants for Lipschitz manifolds in a similar way, and he showed that they enjoy many of the properties of "smooth"

η-invariants. We do not pursue this issue any further since we apply the APS-formula only in the smooth case.

Assuming now that the ends of M are smooth, we may combine the index formula for D^+ in Theorems 5.13 and 5.17 with (5.18). To that end, we continue to assume that the assumptions of Proposition 4.45 are satisfied. Then the spectrum of A_t has two parts, the part consisting of eigenvalues of modulus at most λ and the part consisting of those of modulus at least Λ. Following a corresponding convention in [Lo2], we call the first the *low energy* and the second the *high energy* part and get the corresponding spectral projections and spaces,

$$P_t := Q_{[-\lambda,\lambda]}(A_t^+), \quad H_t^{\mathrm{le}} := P_t(H_t), \quad A_t^{\mathrm{le}} := A_t|_{H_t^{\mathrm{le}}}, \tag{5.19}$$

$$Q_t := I - P_t, \quad H_t^{\mathrm{he}} := Q_t(H_t), \quad A_t^{\mathrm{he}} := A_t|_{H_t^{\mathrm{he}}}, \tag{5.20}$$

where we note that $H_t = H_t^{\mathrm{le}} \oplus H_t^{\mathrm{he}}$ is an orthogonal decomposition which is invariant under A_t. In the super-symmetric case we get similar decompositions and set

$$\eta^{\mathrm{le}}(A_t^+) := \eta(A_t^{\mathrm{le},+}) \quad \text{and} \quad \eta^{\mathrm{he}}(A_t^+) := \eta(A_t^{\mathrm{he},+}), \tag{5.21}$$

the *low* and *high energy* η-invariant of A_t, rspectively. We have

$$\eta(A_t^+) = \eta^{\mathrm{le}}(A_t^+) + \eta^{\mathrm{he}}(A_t^+). \tag{5.22}$$

Corollary 5.23. *Assume that the ends of M are smooth and straight and that the Dirac system over $\mathbb{R}_+$ associated to E over U_0 satisfies the assumptions of Proposition 4.45. Then we have, in the super-symmetric case,*

$$\begin{aligned} \operatorname{ind} D_{-w}^+ &= \int_{M_0} \omega_{D^+} + \int_{N_0} \tau_{D^+} + \frac{1}{2}\left(\dim H^+_{[-\lambda,\lambda]} + \eta^{\mathrm{he}}(A_0^+)\right), \\ \operatorname{ind} D_{\mathrm{ext}}^+ &= \operatorname{ind} D_{-w}^+ + \operatorname{ind} D^+_{U_0,<-\lambda,\mathrm{ext}}. \quad \square \end{aligned}$$

Since $\operatorname{ind} D_{-w}^+$ does not change when replacing the parameter t along the ends by $t - t_0$, for any $t_0 > 0$, it follows that $\operatorname{ind} D^+_{U_0,<-\lambda,\mathrm{ext}}$ is an asymptotic invariant of D (for λ as in Proposition 4.45). Compare also Corollary 4.56.

The formulas in Corollary 5.23 can be used to define high energy η-invariants in the case where the ends of M are not smooth. We expect that these enjoy nice properties because the family of high energy operators A_t^{he} has no spectral flow.

We conclude this chapter by explaining the

Proof of Proposition 1.16. Since M has only finitely many ends, there is a decomposition $M = M_0 \cup U_0$, where M_0 and U_0 are domains in M such that M_0 is compact, such that the common boundary $N := M_0 \cap U_0$

of M_0 and U_0 is smooth, such that each connected component of N bounds exactly one connected component of U_0, and such that the latter are in one to one correspondence with the ends of M.

For each connected component C of N, let A_C^+ be the restriction of A_0^+ to sections of E with support on C. Then A_0^+ is the direct sum of the A_C^+ over the connected components C of N. Hence

$$\eta(A_0^+) = \sum\nolimits_C \eta(A_C^+) \quad \text{and} \quad \dim\ker A_0^+ = \sum\nolimits_C \dim\ker A_C^+.$$

For the connected component $\mathcal{C}$ of U_0 with $\partial\mathcal{C} = C$ we now set

$$\operatorname{Corr}(\mathcal{C}) := \operatorname{ind} D^+_{\mathcal{C},<0,\mathrm{ext}} - \int_{\mathcal{C}} \omega_{D^+} + \int_C \tau_{D^+} + \frac{1}{2}\left(\eta(A_C^+) + \dim\ker A_C^+\right).$$

Then, by Theorem 5.13 and (5.18),

$$\operatorname{ind} D^+_{\mathrm{ext}} = \int_M \omega_{D^+} + \sum\nolimits_{\mathcal{C}} \operatorname{Corr}(\mathcal{C}).$$

By Theorem 3.24 of [BBC2], the terms $\operatorname{Corr}(\mathcal{C})$ only depend on the ends of M and not on the chosen decomposition of M as above. □

6. Manifolds with ε-Thin Ends

Let N be a closed and connected Riemannian manifold of dimension n. We say that N is *ε-flat* if

$$\sqrt{K}\,\operatorname{diam} N \le \varepsilon, \tag{6.1}$$

where K is some upper bound of the modulus of the sectional curvature of N. By Gromov's theorem on almost flat manifolds, there is a constant $\varepsilon(n)$ such that N is an infra-nilmanifold if N is $\varepsilon(n)$-flat [Gr]. In what follows we need some details from the proof of Gromov's theorem from [BuKa] and from Section 4 of Ruh's improvement of Gromov's theorem in [Ru]. The estimates which we assert below hold if $\varepsilon(n)$ is chosen sufficiently small. The arguments in the proofs of these assertions are elementary albeit intricate.

For any curve $c : [a, b] \to N$, denote by $L(c)$ the length of c and by $h(c)$ parallel translation along c. For orthogonal transformations A and B between equi-dimensional Euclidean spaces V and W, we follow [Ru] and let $d(A, B)$ be the maximal angle $\angle(Av, Bv)$, where v runs over non-zero vectors in V. This is a non-smooth Finsler metric on the space of all orthogonal transformations from V to W, invariant under precomposition and postcomposition by orthogonal transformations of V and W, respectively, with injectivity radius and diameter π.

We begin with results from Chapters 2 and 3 in [BuKa]. Normalize the Riemannian metric of N so that $\operatorname{diam} N = 1$, and assume, correspondingly, that $\sqrt{K} \le \varepsilon(n)$. As in [Ru], let

$$w = 2 \cdot 14^{\dim \mathrm{SO}(n)} \quad \text{and} \quad \rho \ge 10^4 w. \tag{6.2}$$

Let x and y be points in N. Then, if c_0 and c_1 are geodesics segments from x to y of length $< \rho$ such that $h(c_0)$ and $h(c_1)$ are 10^{-1}-close, then $h(c_0)$ and $h(c_1)$ are actually 10^{-5}-close. The relation $h(c_0) \sim h(c_1)$ iff $h(c_0)$ and $h(c_1)$ are 10^{-1}-close is an equivalence relation among the holonomies of geodesic segments from x to y of length $< \rho$. For each such equivalence class of holonomies, there is a geodesic segment from x to y of length $< 2 \cdot 10^{-4}\rho$ such that its holonomy belongs to the given equivalence class.

Let c_0 and c_1 be geodesic loops at x such that $L(c_0)+L(c_1) < \rho$. Then there is a unique geodesic loop $c_0 * c_1$ at x of length $< \rho$ homotopic to the concatenation of c_0 and c_1, and $h(c_0 * c_1)$ is 10^{-5}-close to $h(c_1) \circ h(c_0)$. This turns the set H of equivalence classes of holonomies along geodesic loops at x of length $< \rho$ into a group, and the order of H is at most w.

Next we explain Ruh's construction of a flat metric connection on N from [Ru]. Fix an orthonormal frame $F_0 : \mathbb{R}^n \to T_xN$ to identify T_xN with $\mathbb{R}^n$. For each equivalence class $h \in H$ of holonomies along geodesic loops at x of length $< \rho$, let $b_0(h) \in \mathrm{O}(T_xN) \simeq \mathrm{O}(n)$ be its barycenter. This defines an almost homomorphism $b_0 : H \to \mathrm{O}(n)$ in the sense of [GKR] and b_0 is 10^{-4}-close to a homomorphism $b : H \to \mathrm{O}(n)$, by Theorem 3.8 of [GKR]. It follows that b is injective, and we use b to identify H with its image in $\mathrm{O}(n)$.

Let c_0 be a geodesic segment from x to y of length $< \rho$. For each geodesic segment c of length $< \rho$ from x to y, there is precisely one $h \in H$ such that $h(c) \circ h$ is 10^{-4}-close to $h(c_0)$. Enrich the equivalence class of $h(c_0)$ as above by all such $h(c) \circ h$.

Choose a smooth monotone function $\chi : \mathbb{R} \to \mathbb{R}$ with $\chi(r) = 1$ for $r \le \rho/3$, $\chi(r) = 0$ for $r \ge 2\rho/3$, and $|\chi'| \le 10/\rho$. For any enriched equivalence class $[h(c) \circ h]$ of holonomies as above, let $b([h(c) \circ h])$ be its barycenter with respect to the weights $\chi(L(c))/\nu$, where ν is the order of H times the sum of the $\chi(L(c))$, over all geodesic segments c from x to y of length $< \rho$. By the equivariance of barycenters with respect to orthogonal transformations, the set of the barycenters $b([h(c) \circ h])$ is invariant under right multiplication by elements from H, and hence the frames $b \circ F_0$, where b runs over the above barycenters, define a reduction of the principal bundle of orthonormal frames of N to a principal subbundle with structure group H. In other words, we get a flat metric connection $\bar\nabla$ on N with holonomy in H.

To estimate the norm of the difference between $\bar\nabla$ and the Levi-Civita connection ∇ of N, we go back one step and consider the situation before taking barycenters. Let $v \in T_yN$ and $\sigma = \sigma(s)$ be a curve through y with s-derivative $\dot\sigma(0) = v$. Let $c_0 : [0,1] \to N$ be a geodesic segment from x to y with $L(c_0) < \rho$. There is a unique geodesic variation $c = c_s(t)$ of c_0 with $c_s(0) = x$ and $c_s(1) = \sigma(s)$, and then $L(c_s) < \rho$ for all (sufficiently small) s. Let $u \in T_xN$ and $X = X(s,t)$ be the vector field along c such that $X(s,0) = u$ and such that X is parallel along the segments c_s. Note that parallel translation along σ with respect to $\bar\nabla$ corresponds to taking barycenters of such $X(s,1)$ along σ, arising from geodesic segments from x to y of length $< \rho$.

We have $\nabla_t\nabla_s X = R(c', J)X$, where the s-derivative $J := \dot c$ of c is a Jacobi field along each of the c_s which vanishes at $t = 0$ and is equal to $\dot\sigma(s)$ at $t = 1$. It follows that

$$|(\nabla_t\nabla_s X)(0,t)| \le C_0 K\rho t|v||X|, \tag{6.3}$$

where C_0 is a universal constant. Since taking barycenters depends smoothly on points and weights, we conclude that

$$|\bar\nabla_v X - \nabla_v X| \le C_1\big(K\rho + \frac{1}{\rho}\big)|v||X|. \tag{6.4}$$

Now, for any given $\delta > 0$, we may choose ρ so large and, accordingly, $\varepsilon = \varepsilon(n,\delta)$ so small, that the right hand side of (6.4) is $< \delta|v||X|$. Hence, reversing the normalization of the diameter, we get that

$$|\bar\nabla - \nabla| \le \delta \operatorname{diam} N, \tag{6.5}$$

where we recall that scaling does not change the Levi-Civita connection. This finishes the exposition of results from [BuKa] and [Ru].

Proof of Theorem 1.13. In the above constructions, it is understood, in the literature, that the Riemannian manifold N is smooth. We want to apply it in our situation of straight ends, where the Riemannian metric of the cross sections $N_t \subseteq U \simeq [0,\infty) \times N$ is, in general, only C^1. To overcome this technical difficulty, we note that f can be approximated, locally uniformly in the C^2 topology, by a sequence of smooth functions $f_k : U \to \mathbb{R}$. Then, for any given cross section N_t, the level sets $L_k = f_k^{-1}(t)$ approximate N_t in the sense that there is a C^1 diffeomorphism between them such that Riemannian metric, Levi-Civita connection, and Weingarten map on N_t are approximated by the corresponding objects on L_k. In particular, diameter and modulus of the sectional curvature of the connected components of the levels L_k are bounded from above by

$$d_t + \alpha \quad \text{and} \quad K = C_R + 2C_W^2 + \alpha,$$

for any given $\alpha > 0$ and all sufficiently large k, where d_t is an upper bound for the diameter of the connected components of N_t and where we use the Gauss equation for the second estimate. Thus the above constructions apply to N_t if

$$\sqrt{K} d_t \leq \varepsilon < \varepsilon(m-1, 1),$$

where $K = C_R + 2C_W^2 + 1$ and $\varepsilon(m-1,1) = \varepsilon(n, \delta)$ is as in the discussion of (6.5) above, and they guarantee a flat connection $\bar{\nabla}^t$ on N_t such that

$$|\bar{\nabla}^t - \nabla^t| \leq d_t,$$

where ∇^t denotes the Levi-Civita connection of N_t (in difference to our convention as in Lemma 3.42).

Suppose now that $E \to M$ is a Dirac bundle of the type required in Theorem 1.13. Then the restrictions of E to any given cross section N_t is of the corresponding type. Let $\bar{\nabla}^t$ be the flat metric connection on N_t as above. By the assumption on the type of the bundle, $\bar{\nabla}^t$ induces a flat Hermitian connection $\bar{\nabla}^{t,E}$ on the restriction $E_t = E|_{N_t}$ with holonomy of order at most w over each connected component of N_t.

For convenience, assume now that N is connected. Decompose E_t into holonomy irreducible components, and let $F \to N_t$ be any such component. Then F has a twisted parallel orthonormal frame

$$\Phi = (\sigma_1, \dots, \sigma_k), \tag{6.6}$$

that is, the sections σ_i of F are well defined and parallel on the induced bundle with induced flat connection over the universal covering of N_t. We think of them as sections of E over N_t which transform according to the holonomy of F. Approximating the Riemannian metric on N_t by a smooth ϵ-flat Riemannian metric as above, we see that we can apply the usual estimates for the Rayleigh quotient of sections of F, that is, the estimate of Li and Yau [LiYa] in the case where F is the trivial complex line bundle and the corresponding estimate in [BBC1] in the other cases. The outcome is an estimate as follows: If σ is a section of E over N_t and σ is orthogonal to the globally $\bar{\nabla}$-parallel sections of E over N_t, then

$$\|\bar{\nabla}^{t,E}\sigma\|_{N_t}^2 \geq \frac{C(C_R, C_W, m)}{\varepsilon^2} \|\sigma\|_{N_t}^2. \tag{6.7}$$

Here we use, in the twisted case, that the holonomy of F is non-trivial in the sense that, for each unit vector v in F, there is a loop c in N_t (of length at most ρ) such that the angle between v and hv is at least $\pi/2$, since otherwise the holonomy orbit of v would be contained in an open spherical ball of radius π and would have a fixed point. Hence, for each unit vector v in F, there is a loop c in N_t of length at most $2d_t$ such that the angle between v and $h(v)$ is at least $\pi/2w$.

Now the estimate $|\bar{\nabla}^t - \nabla^t| \le d_t$ implies that

$$|\bar{\nabla}^t - \nabla|_{N_t}| \le d_t + C_W,$$

where ∇ denotes the Levi-Civita connections of M. Hence

$$|\bar{\nabla}^{t,E} - \nabla^E|_{N_t}| \le C(d_t + C_W),$$

where C is a constant which depends only on the type of E. It follows that the difference between the Rayleigh quotients for $\nabla^E|_{N_t}$ and $\bar{\nabla}^{t,E}$ is uniformly bounded. We conclude that the assumptions of Proposition 4.45 are satisfied. □

7. Cuspidal Ends

Assume from now that the ends of M are cuspidal. In the setup of Definition 1.8, denote by $\mathcal{D}$ the Dirac system associated to E over U as in Section 3.3. Clearly, for any $\epsilon > 0$, the cross sections N_t are ϵ-flat for all sufficiently large t so that Theorem 1.13 applies. On the other hand, in this chapter, we aim at more specific results. In addition, we do not need to rely on the proof of Gromov's theorem on almost flat manifolds.

7.1. **The Flat Connection.** Over U, define a tensor field $\bar{S}$ of bilinear maps on $TM \oplus TM$ with values in TM by

$$\langle \bar{S}(u,v), w\rangle = -\int_s^\infty \langle R(J,T)X,Y\rangle(t,x)\,dt, \tag{7.1}$$

where $u, v, w \in T_{(s,x)}M$, J is the T-Jacobi field along $\gamma_{(s,x)}$ with $J(s) = u$, and X, Y are the parallel vector fields along $\gamma_{(s,x)}$ with $X(s) = v$, $Y(s) = w$. The integral converges uniformly, by (1.1) and since the ends are cuspidal. Hence $\bar{S}$ is continuous and uniformly bounded. We let C_S be an upper bound for the operator norm of $\bar{S}$.

In the analogous way, define a field $\bar{S}^E$ of bilinear maps on $TM \oplus E$ with values in E,

$$\langle \bar{S}^E(u,v), w\rangle = -\int_s^\infty \langle R^E(J,T)\sigma_1,\sigma_2\rangle(t,x)\,dt, \tag{7.2}$$

where now $v, w \in E_{(s,x)}$ and σ_1, σ_2 are the parallel sections along $\gamma_{(s,x)}$ with $\sigma(s) = v$, $\tau(s) = w$. Again, the integral converges uniformly, by (1.1) and since the ends are cuspidal. Hence $\bar{S}^E$ is also continuous and uniformly bounded. We let C_S^E be an upper bound for the operator norm of $\bar{S}^E$.

The arguments in Section 3 of [BB2] carry over word by word and show that the continuous metric connections

$$\bar\nabla := \nabla - \bar S \quad \text{and} \quad \bar\nabla^E := \nabla^E - \bar S^E \tag{7.3}$$

on TM and E over U are flat in the sense of the existence of parallel C^1 frames over simply connected domains in U. The difference to the situation in Section 6 is that we do not assume that E is geometric and that we have to pay for it by making stronger assumptions on the smallness of the Riemannian metrics g_t and by loosing control on the holonomy of $\bar\nabla$ and $\bar\nabla^E$.

It is easy to see that

$$\bar S^E(X, Y\sigma) = \bar S(X,Y)\sigma + Y\bar S^E(X,\sigma), \tag{7.4}$$

hence the new connections are compatible with Clifford multiplication as well, that is,

$$\bar\nabla^E_X(Y\sigma) = (\bar\nabla_X Y)\sigma + Y\bar\nabla^E_X\sigma \tag{7.5}$$

By definition,

$$\bar\nabla_T = \nabla_T, \quad \bar\nabla^E_T = \nabla^E_T, \quad \text{and} \quad \bar\nabla T = 0. \tag{7.6}$$

For each $t \in \mathbb{R}_+$, the restriction of $\bar\nabla$ and $\bar S$ to N_t will be denoted by $\bar\nabla_t$ and $\bar S_t$, and similarly for $\bar\nabla^E$ and $\bar S^E$. We also consider $\bar\nabla^E_t$ as a first order differential operator on $H^1(N_t, E)$ with values in $L^2(T^*N_t \otimes E)$. The formal adjoint of $\bar\nabla^E_t$ is denoted $(\bar\nabla^E_t)^*$.

Remark 7.7. The above construction of a flat connection is taken from [BB2] (where it is considered for a narrower class of bundles E). In Appendix C of [BeKa], Igor Belegradek and Vitali Kapovitch remark that this connection coincides with the flat connection introduced by Brian Bowditch in [Bow] (in the case of the tangent bundle), who uses a kind of parallel translations through infinity (which, in turn, coincides with the *horospherical translations* in Section 2 of [BrKa]).

7.2. **The Splitting.** To keep the notation simple, it will be convenient to assume in this section that N is connected. It will be obvious that, mutatis mutandis, the results also apply in the case where N is not connected.

For each $t \in \mathbb{R}_+$, we let H^c_t be the space of $\bar\nabla^E$-parallel sections of E over N_t, that is, H^c_t is the kernel of $\bar\nabla^E_t$. Here the superscript c stands for *constant*. We note that the spaces H^c_t are invariant under Clifford multiplication by T, by (7.5) and (7.6). It is also clear that parallel translation in the T-direction identifies the different spaces H^c_t, $t \in \mathbb{R}_+$. In particular, we may and will fix a family of $\bar\nabla^E$-parallel sections $(\sigma_1, \dots, \sigma_k)$ of E over U which are pointwise orthonormal and whose restriction to N_t forms an orthogonal basis of H_t^{c}, for all $t \in \mathbb{R}_+$ simultaneously.

We let H_t^{h} be the orthogonal complement of H_t^{c} in $L^2(N_t, E)$. Thus we obtain two families $\mathcal{H}^{\mathrm{c}} = (H_t^{\mathrm{c}})$ and $\mathcal{H}^{\mathrm{h}} = (H_t^{\mathrm{h}})$ of Hilbert spaces, both of them invariant under Clifford multiplication by T. Note, however, that $\mathcal{H}^{\mathrm{h}}$ is not parallel in the T direction if $\mathcal{H}^{\mathrm{c}}$ is non-trivial and the volume density $j = j(t, x)$ as in Section 3.2 does not only depend on t, but also on x, compare (7.9).

As before, we use parallel translation to identify the spaces H_t^{c} with H_0^{c}, endowed with the inner products $(.,.)_t = (j_t.,.)_0$. Since T is parallel in the T direction, Clifford multiplication by T does not depend on t after this identification.

Let $\bar{P}_t$ and $\bar{Q}_t := I - \bar{P}_t$ be the orthogonal projections in H_t onto H_t^{c} and H_t^{h}, respectively. By definition,

$$\bar{P}_t\sigma = \frac{1}{\operatorname{vol} N_t} \sum_{1\le i\le k} (\sigma_i, \sigma)_t \sigma_i. \tag{7.8}$$

For any function $\psi = \psi(t, x)$ on U we denote by $\bar{\psi} = \bar{\psi}(t)$ the function which associates to $t \in \mathbb{R}_+$ the mean of ψ over the cross section N_t. By (3.32) and (7.8), we have

$$(\nabla_T \bar{P})\sigma = \bar{P}(\kappa\sigma) - \bar{\kappa}\bar{P}\sigma. \tag{7.9}$$

Associated to the projections $\bar{P}$ and $\bar{Q}$, we consider the operators

$$D^{\mathrm{c}} := \bar{P}D\bar{P}, \quad D^{\mathrm{h}} := \bar{Q}D\bar{Q}, \quad D^{\mathrm{ch}} := \bar{P}D\bar{Q}, \quad D^{\mathrm{hc}} := \bar{Q}D\bar{P}. \tag{7.10}$$

We use corresponding notations and conventions in other cases.

Proposition 7.11. *The family*

$$\mathcal{D}^{\mathrm{c}} := (\mathcal{H}^{\mathrm{c}}, \mathcal{A}^{\mathrm{c}}, T)$$

is a Dirac system in the sense of Section 3.1 with

$$\partial^{\mathrm{c}} = \frac{d}{dt} + \frac{\bar{\kappa}}{2} \quad \textit{and} \quad D^{\mathrm{c}} = T(\partial^{\mathrm{c}} + A^{\mathrm{c}}).$$

Proof. The sections $\sigma_1, \ldots, \sigma_k$ as above are C^1, so that the image H_t^{c} of $\bar{P}_t$ consists of C^1 sections of E over N_t. Hence H_t^{c} is contained in H_A, for all $t \in \mathbb{R}_+$. Furthermore, $A_t^{\mathrm{c}} = \bar{P}_t A_t \bar{P}_t$ is a bounded and symmetric operator on H_t^{c}. Clearly, for $\sigma_1, \sigma_2 \in H_0^{\mathrm{c}}$,

$$|(\bar{P}_t A_t \bar{P}_t \sigma_1, \sigma_2)_t - (\bar{P}_s A_s \bar{P}_s \sigma_1, \sigma_2)_s| = |(A_t\sigma_1, \sigma_2)_t - (A_s\sigma_1, \sigma_2)_s|. \qquad \square$$

Associated to the decomposition into constant sections and sections perpendicular to them, we get an orthogonal splitting

$$L^2(\mathcal{D}) = L^2(\mathcal{H}) = L^{2,\mathrm{c}}(\mathcal{H}) \oplus L^{2,\mathrm{h}}(\mathcal{H}). \tag{7.12}$$

where

$$L^{2,\mathrm{c}}(\mathcal{H}) := L^2(\mathcal{H}^{\mathrm{c}}) \quad \text{and} \qquad (7.13)$$
$$L^{2,\mathrm{h}}(\mathcal{H}) := \{\sigma \in L^2(\mathcal{H}) : \bar{P}\sigma = 0\}.$$

We use corresponding notations for other spaces of sections.

LEMMA 7.14. *The projections $\bar{P}$ and $\bar{Q}$ are continuous on $H^1(\mathcal{D})$. In particular, as topological vector spaces,*

$$H^1(\mathcal{D}) = H^{1,\mathrm{c}}(\mathcal{D}) \oplus H^{1,\mathrm{h}}(\mathcal{D}),$$
$$H^1_{\mathrm{loc}}(\mathcal{D}) = H^{1,\mathrm{c}}_{\mathrm{loc}}(\mathcal{D}) \oplus H^{1,\mathrm{c}}_{\mathrm{loc}}(\mathcal{D}).$$

Proof. Since $\sigma_1, \dots, \sigma_k$ and $\operatorname{vol} N_t$ are C^1, we conclude that

$$\bar{P}(H^1(\mathcal{D})) \subseteq H^1(\mathcal{D}) \quad \text{and} \quad \bar{Q}(H^1(\mathcal{D})) \subseteq H^1(\mathcal{D}),$$

by (7.8). Hence $\bar{P}$ and $\bar{Q} = I - \bar{P}$ are continuous with respect to the H^1-norm, by the closed graph theorem. □

LEMMA 7.15. *The Rayleigh quotients*

$$\bar{\rho}_t := \inf\{\|\bar{\nabla}^E_t\sigma\|^2_t/\|\sigma\|^2_t : \sigma \in H^{\mathrm{h}}_t \cap H_A, \sigma \neq 0\}, \qquad (1)$$

$$\rho_t := \inf\{\|\nabla^E_t\sigma\|^2_t/\|\sigma\|^2_t : \sigma \in H^{\mathrm{h}}_t \cap H_A, \sigma \neq 0\} \qquad (2)$$

tend to infinity as t tends to infinity. Here ∇^E_t and $\bar{\nabla}^E_t$ denote the restrictions of ∇^E and $\bar{\nabla}^E$ to N_t.

Proof. We discuss the Rayleigh quotients associated to $\bar{\nabla}^E$ first. Split $H^{\mathrm{h}}_t \cap H_A = U_t \oplus V_t$, where U_t consists of sections in $H^1(N_t, E)$ which are linear combinations $\sum \varphi_i\sigma_i$ of the basis $(\sigma_1, \dots, \sigma_k)$ as above and where V_t consists of sections in $H^1(N_t, E)$ which are pointwise perpendicular to $\sigma_1, \dots, \sigma_k$. Note that U_t and V_t are invariant under $\bar{\nabla}^E_t$ and perpendicular to each other, and thus it suffices to consider them separately.

Let $\sigma = \sum \varphi_i\sigma_i \in U_t$, $\sigma \neq 0$. To be perpendicular to H^{c}_t in $L^2(N_t, E)$ means that the coefficient functions φ_i integrate to 0. Moreover, the Rayleigh quotient of σ is given by the sum of the Rayleigh quotients corresponding to the Laplace operator on functions on N_t. Hence

$$\frac{\|\bar{\nabla}^E_t\sigma\|^2_t}{\|\sigma\|^2_t} = \frac{\sum \|\operatorname{grad}\varphi_i\|^2_t}{\sum \|\varphi_i\|^2_t} \geq ce^{ct},$$

for some constant $c > 0$, by Theorem 7 in [LiYa].

Now we consider V_t. Perpendicular to $(\sigma_1, \dots, \sigma_k)$, the holonomy of $\bar{\nabla}$ does not have non-trivial invariant vectors. Since loops in $N = N_0$ of length at most $2 \operatorname{diam} N$ generate the fundamental group of N, there is a constant $\alpha >$ such that, for each vector u in some fiber of E over

N, there is a loop c in N of length at most $2 \operatorname{diam} N$ such that the holonomy h_c of $\bar\nabla$ along c satisfies $|h_c u - u| \geq \alpha |u|$. For each $t \geq 0$, the $\bar\nabla$-holonomy about the curve c shifted to N_t is the same. We conclude that, for each $t \in \mathbb{R}_+$ and vector u in some fiber of E over N_t, there is a loop c in N_t of length at most $2\varphi(t) \operatorname{diam} N$ such that the holonomy h_c of $\bar\nabla$ along c satisfies the same inequality,

$$|h_c u - u| \geq \alpha |u|.$$

Hence Theorem 5 in [BBC1] applies and shows that the Rayleigh quotient of $\bar\nabla^E_t$ on V_t tends to infinity as t tends to infinity. This shows the first claim. As for the Rayleigh quotients associated to ∇^E_t, we recall that the difference $\|\bar\nabla^E_t - \nabla^E_t\| \leq C^E_S$. □

Theorem 7.16. *There are constants* $\lambda_0, \Lambda_t \geq 0$ *with* $\lim_{t\to\infty} \Lambda_t = \infty$ *such that* $\operatorname{spec} A_t \cap (\lambda_0, \Lambda_t) = \emptyset$ *or, more precisely, such that*

$$\|D_t \sigma\|_t \leq \lambda_0 \|\sigma\|_t \quad \textit{for all } \sigma \in H^{\mathrm{c}}_t, \tag{1}$$

$$\|D_t \sigma\|_t \geq \Lambda_t \|\sigma\|_t \quad \textit{for all } \sigma \in H^{\mathrm{h}}_t. \tag{2}$$

In particular, for all sufficiently large t,

(1) $\mathcal{D}_t$ *satisfies the hypothesis of Proposition 4.45,*
(2) D *is non-parabolic with respect to* M_t.

Proof. By (3.43) and (2.16),

$$\big| \|D_t \sigma\|^2_t - \|\nabla^t \sigma\|^2_t \big| \leq C_K \|\sigma\|^2_t.$$ □

7.3. **Explicit Index Formulas.** Assume from now on that the ends of M are smooth, that is, the associated distance function f on U is smooth. Since the ends of M are cuspidal and the curvatures of M and E and the second fundamental forms of the cross sections are uniformly bounded,

$$\lim_{t\to\infty} \int_{M_t} \omega_{D^+} = \int_M \omega_{D^+} \quad \text{and} \quad \lim_{t\to\infty} \int_{N_t} \tau_{D^+} = 0, \tag{7.17}$$

compare (5.18). By Theorem 7.16, we may fix the starting time $t = 0$ such that the condition

$$(\Lambda_t - \lambda_0)^2 > 4c_0(c_0 + 2 + \lambda_0 + \Lambda_t) \tag{7.18}$$

of Proposition 4.45 is satisfied for all $t \in \mathbb{R}_+$, where λ and Λ there correspond to λ_0 and Λ_t here.

Proposition 7.19. *If $w > 0$ satisfies $(w-\lambda_0)^2 > c_0(c_0+2+2w)$, then*

$$\begin{aligned}\operatorname{ind} D^+_{-w} &= \int_M \omega_{D^+} + \frac{1}{2}\big(\dim H^+_{[-\lambda_0,\lambda_0]}(A^+_0) + \lim_{t\to\infty} \eta^{\rm he}(A^+_t)\big),\\ \operatorname{ind} D^+_{\rm ext} &= \operatorname{ind} D^+_{-w} \ - \dim\ker D^-_{U_0,\le\lambda_0,\max}.\end{aligned}$$

Proof. This is immediate from Corollary 5.23 and (7.17), where we observe that $\dim H^+_{[-\lambda_0,\lambda_0]}(A_t)$ is independent of $t \in \mathbb{R}_+$. □

To get an explicit formula for the extended index of D, we assume from now on in addition that

$$\kappa = \bar\kappa \quad \text{and} \quad A\sigma_i = \sum\nolimits_j \bar a^j_i \sigma_j, \tag{7.20}$$

for some (constant) Hermitian matrix $\bar A = (\bar a^j_i) \in \mathrm{Gl}(k,\mathbb{C})$. These conditions hold for homogeneous cusps as discussed further on.

The second condition of (7.20) requires that the space H^c_t of constant sections in H_t is invariant under A_t. By Theorem 7.16, we get that

$$H^{\rm le}_t = H^{\rm c}_t = H_{[-\lambda_0,\lambda_0]}(A_t) \quad \text{and} \quad H^{\rm he}_t = H^{\rm h} = H_{\mathbb{R}\setminus[-\lambda_0,\lambda_0]}(A_t), \tag{7.21}$$

compare (5.19) and (5.20). The additional assumption $\kappa = \bar\kappa$ implies that the high energy family $\mathcal{H}^{\rm he} = (H^{\rm he}_t)$ is invariant under parallel translation so that it defines a Dirac subsystem $\mathcal{D}^{\rm he}$ of $\mathcal{D}$, as in the case of the low energy system $\mathcal{D}^{\rm le} := \mathcal{D}^{\rm c}$; compare (7.9) and Proposition 7.11. We obtain corresponding low and high energy Dirac operators $D^{\rm le}$ and $D^{\rm he}$, decomposing the original Dirac operator D.

Lemma 7.22. *Under the above assumptions,*

$$D^{\rm he}_{U_t,<\lambda,\rm ext} = D^{\rm he}_{U_t,\le\lambda,\rm ext} = D^{\rm he}_{U_t,<\Lambda_t,\rm ext}.$$

and $D^{\rm he}_{U_t,<\lambda,\rm ext}$ and $D^{\rm he,\pm}_{U_t,<\lambda,\rm ext}$ are isomorphisms, for all $t \ge 0$ and $-\Lambda_t < \lambda < \Lambda_t$. In particular, for all such t and λ,

$$\operatorname{ind} D^+_{U_t,<\lambda,\rm ext} = \operatorname{ind} D^{\rm le,+}_{U_t,<\lambda,\rm ext}.$$

Proof. The fist assertion is clear since the spectrum of $A^{\rm he}_t$ does not intersect the interval $(-\Lambda_t,\Lambda_t)$. Furthermore, $D^{\rm he}_{U_t,<0,\rm ext}$ is injective, by Corollary 4.43. Now $D^{\rm he}_{U_t,<\lambda,\rm ext}$ and $D^{\rm he}_{U_t,\le-\lambda,\rm ext}$ are adjoints of each other, hence $D^{\rm he}_{U_t,\le 0,\rm ext}$ is surjective. □

Since κ depends (at most) on t and j solves the initial value problem $j' = \kappa j$ with $j_0 = 1$, we conclude that $j = j(t,x)$ depends only on t as well. Then the linear map

$$\Phi : L^2(\mathbb{R}_+,\mathbb{R}^k) \to L^2(\mathcal{H}^{\rm le}), \quad \Phi(\varphi) = j^{-1/2}\sum\nolimits_i \varphi^i\sigma_i, \tag{7.23}$$

is a unitary isomorphism such that

$$\Phi^{-1} D^{\mathrm{le}} \Phi = \bar{T}\left(\frac{d}{dt} + \bar{A}\right), \tag{7.24}$$

where $\bar{T} = \Phi^{-1} T \Phi$. This is a finite dimensional constant coefficient Dirac system. In the super-symmetric case, we get a system of the form

$$\Phi^{-1} D^{\mathrm{le}} \Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{d}{dt} + \begin{pmatrix} \bar{A}^+ & 0 \\ 0 & \bar{A}^- \end{pmatrix}\right), \tag{7.25}$$

where $\bar{A}^- = -\bar{A}^+$.

Proposition 7.26. *Under the above assumptions, $D^{\mathrm{le}}_{<0,\mathrm{ext}}$ and $D^{\mathrm{le},+}_{<0,\mathrm{ext}}$ are isomorphisms.*

Proof. The Dirac system 7.24 does not have extended or L^2-solutions σ with $\sigma(0)$ in $H^{\mathrm{le}}_{<0}$ or $H^{\mathrm{le}}_{\leq 0}$, respectively. □

In what follows, we use that $\bar{A}$ is the matrix of A^{le}_t associated to the basis (σ_j) of H^{le}_t, for all $t \in \mathbb{R}_+$. In particular, the quantities $\eta^{\mathrm{le}}(A^+_t)$ and $\dim\ker A^{\mathrm{le},+}_t$ do not depend on $t \in \mathbb{R}_+$.

Theorem 7.27. *If all ends of M are smooth and* (7.20) *holds, then*

$$\operatorname{ind} D^+_{\mathrm{ext}} = \int_M \omega_{D^+} + \frac{1}{2}\left(\lim_{t\to\infty} \eta^{\mathrm{he}}(A^+_t) + \eta^{\mathrm{le}}(A^+_0) + \dim\ker A^{\mathrm{le},+}_0\right).$$

Proof. Immediate from (5.18), (7.17), Theorem 5.13, Lemma 7.22, and Proposition 7.26. □

The quantities $h^{\pm}_\infty := \dim\ker D^{\pm}_{\mathrm{ext}} - \dim\ker D^{\pm}_{\max}$ determine the difference between the extended and L^2-indices of $D^{\pm}$,

$$\operatorname{ind} D^{\pm}_{\mathrm{ext}} = \operatorname{ind}_{L^2} D^{\pm} + h^{\pm}_\infty, \tag{7.28}$$

where $\operatorname{ind}_{L^2} D^{\pm} := \dim\ker D^{\pm} - \dim\ker D^{\mp}$.

Theorem 7.29. *If all ends of M are smooth and* (7.20) *holds, then*

$$\operatorname{ind}_{L^2} D^+ = \int_M \omega_{D^+} + \frac{1}{2}\left(\lim_{t\to\infty} \eta^{\mathrm{he}}(A^+_t) + \eta^{\mathrm{le}}(A^+_0) - h^+_\infty + h^-_\infty\right).$$

Proof. Since D is formally self-adjoint, the L^2-index of D vanishes and therefore

$$\operatorname{ind} D_{\mathrm{ext}} = \operatorname{ind} D^+_{\mathrm{ext}} + \operatorname{ind} D^-_{\mathrm{ext}} = h^+_\infty + h^-_\infty.$$

On the other hand, we have

$$\omega_{D^-} = -\omega_{D^+} \quad \text{and} \quad A^-_t = -A^+_t,$$

for all $t \in \mathbb{R}_+$. Therefore, applying Theorem 7.27 to D^+ and D^-,

$$\begin{aligned}\operatorname{ind} D_{\mathrm{ext}} &= \operatorname{ind} D^+_{\mathrm{ext}} + \operatorname{ind} D^-_{\mathrm{ext}} \\ &= \frac{1}{2}\big(\dim\ker A_0^{\mathrm{le},+} + \dim\ker A_0^{\mathrm{le},-} \big) = \dim\ker A_0^{\mathrm{le},+}\end{aligned}$$

since the integral and η terms for D^+ and D^- cancel each other. We conclude that

$$h^+_\infty + h^-_\infty = \dim\ker A_0^{\mathrm{le},+} \tag{7.30}$$

and hence that

$$\operatorname{ind} D^+_{\mathrm{ext}} - \operatorname{ind}_{L^2} D^+ = h^+_\infty = \frac{1}{2}\big(h^+_\infty - h^-_\infty + \dim\ker A_0^{\mathrm{le},+}\big). \qquad \square \tag{7.31}$$

Remarks 7.32. 1) In examples, the non-local term $h^+_\infty - h^-_\infty$ is a contribution of zero energy resonances and can be computed from the scattering matrix at zero energy, see [Mü1],[Mü2].
2) D is of Fredholm type if and only if the kernel of $\bar{A}^+$ vanishes or, equivalently, if and only if $h^+_\infty = h^-_\infty = 0$.

8. Homogeneous Cusps

Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Fix a left-invariant Riemannian metric g on N, and let W be a negative definite and symmetric derivation of $\mathfrak{n}$. Then $(\exp(-tW))_{t\in\mathbb{R}}$ is a one-parameter group of automorphisms of $\mathfrak{n}$ which induces a one-parameter group $(\Phi_t)_{t\in\mathbb{R}}$ of automorphisms of N. The associated semi-direct product $S := \mathbb{R} \ltimes N$, where

$$(s,x)(t,y) := (s+t, x\Phi_s(y)), \tag{8.1}$$

is a simply connected solvable Lie group containing $N \cong \{0\} \times N$ as a subgroup of codimension one. The vector field $T := \partial/\partial t$ on S is left-invariant, and the Lie algebra $\mathfrak{s}$ of S extends $\mathfrak{n}$ by

$$[T, X] = -WX, \tag{8.2}$$

where $X \in \mathfrak{n}$. For later use, we note that left translation, right translation, and conjugation with $(t,e) \in S$ are given by

$$\begin{aligned} L_{(t,e)}(s,x) &= (s+t, \Phi_t(x)), \\ R_{(t,e)}(s,x) &= (s+t, x), \\ (t,e)(s,x)(-t,e) &= (s, \Phi_t(x)), \end{aligned} \tag{8.3}$$

respectively. In particular, the shift by t along the T-lines is obtained by right translation with (t,e). Moreover, for $X \in \mathfrak{n} \subseteq \mathfrak{s}$,

$$\begin{aligned} R_{(t,e)*}X_{(s,x)} &= L_{(s,x)*}L_{(t,e)*}L_{(-t,e)*}R_{(t,e)*}X \\ &= L_{(s+t,x)*}(\mathrm{Ad}^{-1}_{(t,e)}X) = L_{(s+t,x)*}(\exp(tW)X), \end{aligned} \tag{8.4}$$

where we recall that (Φ_t) is the one-parameter group of automorphism of N associated to $-W$ (and where we identify $\mathfrak{s} \ni X = X_e \in T_eS$).

Endow S with the left-invariant Riemannian metric which agrees with g along N and such that $\mathbb{R}$ and $\mathfrak{n}$ are pairwise perpendicular with $|T| = 1$. Note that T is a unit normal field along the cross sections $N_t := \{t\} \times N$ and that the T-lines are unit speed geodesics. In particular,

$$f : S \to \mathbb{R}, \quad f(t,x) := t, \tag{8.5}$$

is a smooth distance function on S such that $\operatorname{grad} f = T$ and such that the associated diffeomorphism F is the identity on $S = \mathbb{R} \times N$. By the Koszul formula and the symmetry of W,

$$\nabla_T X = 0, \tag{8.6}$$

for any $X \in \mathfrak{s}$. For any $X \in \mathfrak{n} \subseteq \mathfrak{s}$,

$$\nabla_X T = WX, \tag{8.7}$$

by (8.2) and (8.6); that is, except for the compactness of the cross sections, we are in the situation of Section 3.2. By (8.6) and (8.7),

$$R(T,X) = \nabla_{[X,T]} = \nabla_{WX}. \tag{8.8}$$

In particular,

$$R(X,T)T = -W^2X, \tag{8.9}$$

and hence the sectional curvature of tangential 2-planes of S containing T is strictly negative.

Let $\Gamma \subseteq N$ be a discrete subgroup such that the quotient $\Gamma\backslash N$ is compact. Since $\Gamma \subseteq N$, the distance function f as in (8.5) is well defined on $\Gamma\backslash S$. We keep the notation f and $T = \operatorname{grad} f$ on the quotient. The cross sections of f are given by $\{t\} \times \Gamma\backslash N$, and right translation by (t,e) induces the shift F_t from $\Gamma\backslash N$ to $\{t\} \times \Gamma\backslash N$, see (8.3). By (8.4), F_t has derivative $F_{t*} = \exp(tW)$. The Jacobian of F_t is given by $j(t) = \exp(\kappa t)$, where $\kappa = \operatorname{tr} W$ as in Section 3.2. It only depends on t and not on $x \in \Gamma\backslash N$. Moreover, since W is negative definite, F_t is contracting for $t > 0$: If we order the eigenvalues of W,

$$\kappa_2 \le \ldots \le \kappa_m < 0, \tag{8.10}$$

then any part $[t_0,\infty)\times N$ of $\mathbb{R}\times N$ models cuspidal ends as in Definition 1.12 with $c=-2\kappa_m$ and $C=1$. We call such ends *homogeneous cusps*.

If $X_i\in\mathfrak{n}$ is a unit eigenvector of W for the eigenvalue κ_i, then $\exp(\kappa_i t)X_i$ is a Jacobi field along each T-line and

$$\langle\nabla_X Y,Z\rangle=-\int_0^\infty\langle R(T,e^{\kappa_i t}X_i)Y,Z\rangle, \tag{8.11}$$

for all $Y,Z\in\mathfrak{s}$. It follows that the flat connection $\bar\nabla$ associated to the cusp as in Section 7.1 defines left-invariant vector fields on S or, rather, their image in $\Gamma\backslash S$ to be $\bar\nabla$-parallel.

Let K_0 be a connected Lie subgroup of the orthogonal group $\mathrm{SO}(\mathfrak{s})$ which contains the holonomy group of S at e. Denote the Lie algebra of K_0 by $\mathfrak{k}$. Consider the principal bundle $\mathcal{P}_0:=S\times K_0$ over S, with structure group K_0, where we view $p=(s,k)\in\mathcal{P}_0$ as representing the frame $L_s\circ k:T_eS\to T_sS$ of S, where L_s denotes left-translation by s (and its derivative). This interpretation corresponds to an embedding of $\mathcal{P}_0$ into the principal bundle of orthonormal frames of S. The group S acts on $\mathcal{P}_0$ by left translation, $s(s',k):=(ss',k)$, and the orbits of this action are the left-invariant frames $F_k:=\{(s,k)\mid s\in S\}$ over S.

Lemma 8.12. *The Levi-Civita connection ∇ and flat connection $\bar\nabla$ of S reduce to $\mathcal{P}_0$. That is, if $c:I\to S$ is a smooth curve and F is a parallel frame along c with respect to ∇ or $\bar\nabla$ such that $F(t_0)\in\mathcal{P}_0$ for some $t_0\in I$, then $F(t)\in\mathcal{P}_0$ for all $t\in I$.*

Proof. Let F be an orthonormal frame along c, and write $F(t)=L_{c(t)}f(t)$, where $f:I\to\mathrm{O}(\mathfrak{s})$. Then the covariant derivative of F along c with respect to ∇ is given by

$$F'(t)=L_{c(t)}(f'(t)+A_{c'(t)}f(t)), \tag{8.13}$$

where

$$A_X=\begin{cases}R(T,W^{-1}X) & \text{for } X\in\mathfrak{n},\\ 0 & \text{for } X=T,\end{cases} \tag{8.14}$$

by (8.6) and (8.8). By (8.13), F is ∇-parallel if $f'+A_cf=0$.

Now $R(Y,Z)$ is in the Lie algebra of the holonomy group of S at e, for all $Y,Z\in\mathfrak{s}$, hence also $A_{c'(t)}$, for all $t\in I$. Since K_0 contains the holonomy group of S at e, we get that $A_{c'(t)}\in\mathfrak{k}$, for all $t\in I$. It follows that a solution of $f'+A_cf=0$ is contained in K_0 if $f(t_0)$ is in K_0, for some $t_0\in I$. This proves the assertion for ∇.

By what we said above, a frame is $\bar\nabla$-parallel if and only if it is left-invariant under S. Hence the $\bar\nabla$-parallel frames along c are of the form

$F(t) = L_{c(t)}k$, $t \in I$, where $k \in \mathrm{O}(\mathfrak{s})$. Hence, if $F(t_0) \in \mathcal{P}_0$ for some $t_0 \in I$, then $k \in K_0$, and then $F = F_k$ is contained in $\mathcal{P}_0$. □

Let $K \to K_0$ be a covering homomorphism, where K is a connected Lie group, and let $\mathcal{P} := S \times K$ be the corresponding covering space of $\mathcal{P}_0$, a principal bundle over S with structure group K. Via the projection $K \to K_0$, identify the Lie algebra of K with the Lie algebra $\mathfrak{k}$ of K_0. As in the case of $\mathcal{P}_0$, S acts by left translations on $\mathcal{P}$, and we have the corresponding orbits F_k, $k \in K$. Moreover, since $\mathcal{P} \to \mathcal{P}_0$ is a covering projection, Levi-Civita and flat connection lift from $\mathcal{P}_0$ to $\mathcal{P}$.

Denote by $\hat{\alpha}_* : \mathfrak{k} \to \mathfrak{u}(\Sigma_{\mathfrak{s}})$ the composition of the differential of $\alpha : K \to K_0 \subseteq \mathrm{SO}(\mathfrak{s})$ with the differential of the spinor representation $\Sigma_{\mathfrak{s}}$ of $\mathfrak{so}(\mathfrak{s}) \simeq \mathfrak{spin}(\mathfrak{s})$. Let V be a finite dimensional Hermitian vector space and $\pi_* : \mathfrak{k} \to \mathfrak{u}(V)$ be a unitary representation. Suppose that there is a unitary representation $\beta : K \to \Sigma_{\mathfrak{s}} \otimes V$ with

$$\hat{\alpha}_* \otimes \mathrm{id} + \mathrm{id} \otimes \pi_* = \beta_*, \tag{8.15}$$

and let $E = \mathcal{P} \times_\beta (\Sigma_{\mathfrak{s}} \otimes V)$ be the associated Hermitian vector bundle over S. Levi-Civita and flat connection on $\mathcal{P}$ induce Hermitian connections ∇^E and $\bar{\nabla}^E$ on E, respectively. We extend Clifford multiplication to $\Sigma_{\mathfrak{s}} \otimes V$ by

$$X \cdot (u \otimes v) := (X \cdot u) \otimes v, \tag{8.16}$$

where $X \in \mathfrak{s}$, $u \in \Sigma_{\mathfrak{s}}$, and $v \in V$. By (8.15) and since K is connected, Clifford multiplication commutes with β, that is

$$\beta(k)(Xw) = X(\beta(k)w), \tag{8.17}$$

for all $k \in K$, $X \in \mathfrak{s}$, and $w \in \Sigma_{\mathfrak{s}} \otimes V$. Hence (8.16) induces a Clifford multiplication on E which turns E into a Dirac bundle over S. The canonical action of S on E preserves the Dirac data of E; we say that E is a *homogeneous* Dirac bundle over S.

Using the left-invariant orbit F_e in $\mathcal{P}$, we view sections of E as smooth maps $\sigma : S \to \Sigma_m \otimes V$. In this interpretation, covariant derivatives and Dirac operator are given by

$$\nabla^E_X \sigma = X(\sigma) + \beta_*(A_X)\sigma, \quad \bar{\nabla}^E_X \sigma = X(\sigma), \tag{8.18}$$

and

$$D\sigma = \sum\nolimits_j X_j \cdot (X_j(\sigma) + \beta_*(A_{X_j})\sigma), \tag{8.19}$$

where X is a vector field on S, $(X_1, \dots, X_m)$ is an orthonormal frame of S, and A_X is as in (8.14). In particular, σ is $\bar{\nabla}^E$-parallel if and only if σ is constant.

Let τ be a unitary representation of Γ on V, the *twist*, and assume that τ and π_* commute, that is,

$$\tau(\gamma)\pi_*(Y) = \pi_*(Y)\tau(\gamma), \tag{8.20}$$

for all $\gamma \in \Gamma$ and $Y \in \mathfrak{k}$.

LEMMA 8.21. *Extend τ by the trivial representation on $\Sigma_{\mathfrak{s}}$ to $\Sigma_{\mathfrak{s}} \otimes V$. Then τ commutes with β and Clifford multiplication,*

$$\tau(\gamma)(\beta(k)w) = \beta(k)(\tau(\gamma)w),$$
$$\tau(\gamma)(Xw) = X(\tau(\gamma)w),$$

for all $\gamma \in \Gamma$, $k \in K$, $X \in \mathfrak{s}$, and $w \in \Sigma_{\mathfrak{s}} \otimes V$.

Proof. Since K is connected, the first assertion follows from the corresponding infinitesimal properties in (8.15) and (8.20). As for the second assertion, we note that τ acts trivially on the first and Clifford multiplication trivially on the second factor of $\Sigma_{\mathfrak{s}} \otimes V$. □

By Lemma 8.21, τ induces a Hermitian bundle E_τ over $\Gamma\backslash S$ such that sections of E_τ correspond to maps $\sigma : S \to \Sigma_{\mathfrak{s}} \otimes V$ which satisfy

$$\sigma(\gamma s) = \tau(\gamma)\sigma(s), \tag{8.22}$$

for all $s \in S$. The connections ∇^E and $\bar{\nabla}^E$ on E descend to Hermitian connections on E_τ, also denoted by ∇^E and $\bar{\nabla}^E$, respectively. Moreover, E_τ inherits Clifford multiplication from E and thus turns into a Dirac bundle over $\Gamma\backslash S$.

EXAMPLES 8.23. 1) (Spinor bundles) Since S is contractible, spin structures over $\Gamma\backslash S$ are determined by homomorphisms $\tau : \Gamma \to \{+1, -1\}$. In our setup, the corresponding spinor bundles over $\Gamma\backslash S$ can be given by the data: $K_0 = \mathrm{SO}(\mathfrak{s})$ and $K = \mathrm{Spin}(\mathfrak{s})$, $\alpha : \mathrm{Spin}(\mathfrak{s}) \to \mathrm{SO}(\mathfrak{s})$ the canonical covering map, $V = \mathbb{C}$, $\pi_* = 0$, β the spinor representation, extended trivially to the factor $\mathbb{C}$ of $\Sigma_{\mathfrak{s}} \otimes \mathbb{C}$, and finally the twist defined by τ, where γ acts by multiplication with $\tau(\gamma) = \pm 1$ on $\mathbb{C}$.

2) (Clifford bundle) If m is even, then $\mathbb{C}\mathrm{l}(\mathfrak{s}) = \Sigma_{\mathfrak{s}} \otimes \Sigma_{\mathfrak{s}}$. Thus, to obtain the Clifford bundle over $\Gamma\backslash S$, we may take $K_0 = K = \mathrm{SO}(\mathfrak{s})$, $\alpha = \mathrm{id}$, $V = \Sigma_{\mathfrak{s}}$, β_* the differential of the spinor representation, and τ the trivial representation of Γ on $\Sigma_{\mathfrak{s}}$.

If the dimension m of S is even, then the ± 1-eigenspaces $\Sigma_{\mathfrak{s}}^{\pm} \otimes V$ of multiplication by the complex volume form (compare Section 2.2) are invariant under β, by (8.17). By Lemma 8.21, they are also invariant under τ. Thus the complex volume form yields the super-symmetry $E = E^+ \oplus E^-$ with

$$E^{\pm} = \mathcal{P} \times_\beta (\Sigma_{\mathfrak{s}}^{\pm} \otimes V). \tag{8.24}$$

In the case of the Clifford bundle, there is another natural supersymmetry, namely the even-odd decomposition. Our methods also allow for a discussion of the latter, but here and below we concentrate on the decomposition given by the complex volume form.

We now pass to the Dirac system associated to the distance function f and the Dirac bundle E_τ over $\Gamma\backslash S$. We identify sections of E_τ over $\{t\} \times \Gamma\backslash N$ with maps $\sigma : N \to \Sigma_{\mathfrak{s}} \otimes V$ satisfying (8.22). Under this identification, parallel translation along the T-lines is the identity, and the Hilbert space $L^2(\{t\} \times \Gamma\backslash N, E_\tau)$ corresponds to the Hilbert space of measurable maps $N \to \Sigma_{\mathfrak{s}} \otimes V$ satisfying (8.22) which are square integrable over a fundamental domain of Γ. In the notation of (3.46),

$$A_t\sigma = -\sum_{2\le j\le m} e^{-\kappa_j t} TX_j \cdot X_j(\sigma) - \sum_{2\le j\le m} TX_j \cdot \beta_*(A_{X_j})\sigma - \frac{\kappa}{2}\sigma, \tag{8.25}$$

where $(X_2, \dots, X_m)$ is an orthonormal basis of $\mathfrak{n}$ consisting of eigenvectors of W, $WX_i = \kappa_i X_i$.

We may also have a different view on E_τ over $\{t\} \times \Gamma\backslash N$: $L_{(t,e)}$ is an isometry of S which maps N to $\{t\} \times N$ and which leaves the normal field T to the cross sections $\{t\} \times N$ invariant. Suppressing the coordinate t in $\{t\} \times N$, $L_{(t,e)}$ corresponds to Φ_t, by (8.3). That is, E_τ over $\{t\} \times \Gamma\backslash N$ corresponds to $E_{\Phi_t\tau\Phi_t^{-1}}$ over $\Phi_t(\Gamma)\backslash N$, where N is endowed with the fixed left-invariant metric g. Under this correspondence, the exponential factors in the expression for A_t in (8.25) disappear. More precisely, $-A_t$ corresponds to the Dirac operator

$$D_t\sigma = \sum_{2\le j\le m} TX_j \cdot X_j(\sigma) + \sum_{2\le j\le m} TX_j \cdot \beta_*(A_{X_j})\sigma + \frac{\kappa}{2}\sigma, \tag{8.26}$$

where σ satisfies the twist data with respect to $\Phi_t\tau\Phi_t^{-1}$. In particular, the local data for the different operators D_t coincide under the correspondence.

8.1. **Asymptotic η-Invariants.**

Let $L^{2,\pm}(t)$ be the Hilbert space of measurable maps $N \to \Sigma_{\mathfrak{s}}^\pm \otimes V$ satisfying (8.22) with respect to $\Phi_t\tau\Phi_t^{-1}$ which are square integrable over a fundamental domain of $\Phi_t(\Gamma)$. Then $D_t^\pm = -A_t^\pm$ is an unbounded self-adjoint operator on $L^{2,\pm}(t)$.

For the computation of the asymptotic high energy η-invariant of D_t^+, it will be useful to consider the *flat Dirac operator* $\bar{D}_t^+$, defined by

$$\bar{D}_t^+\sigma = \sum\nolimits_{2\le j\le m} TX_j \cdot X_j(\sigma). \tag{8.27}$$

We note that $\bar{D}_t^+$ is a formally self-adjoint operator and that $D_t^+ - \bar{D}_t^+$ is left-invariant of order zero. In particular, the principal symbols of

D_t^+ and $\bar{D}_t^+$ are the same. We have

$$(\bar{D}_t^+)^2\sigma = \Delta\sigma + \sum\nolimits_{2\le j<k} X_jX_k \cdot [X_j, X_k](\sigma). \tag{8.28}$$

If $\mathfrak{n}$ is nilpotent of rank at most two, then the Lie brackets $[X_j, X_k]$ in the second term on the right are in the center of $\mathfrak{n}$, and then the operator defined by the second term commutes with Δ.

The idea to consider $\bar{D}_t^+$ is taken from [DeSi]. The proof of our main result in this direction, Theorem 8.29 below, is a variation of arguments in §5 of [DeSi]. This line of reasoning was also used by Cheeger and Gromov in order to show that their ρ-invariant is the limit of the (signature) η-invariant under a collapse of the corresponding manifold with bounded covering geometry [ChGr].

THEOREM 8.29. *For D_t^+ and $\bar{D}_t^+$ as above, we have*

$$\lim_{t\to\infty} \eta^{\mathrm{he}}(D_t^+) = \lim_{t\to\infty} \eta(\bar{D}_t^+),$$

Proof. For all sufficiently large t, the kernel of the operator $\bar{D}_t^+$ consists precisely of the left-invariant sections in $L^{2,+}(t)$, by Theorem 7.16. Let $P_t : L^{2,+}(t) \to L^{2,+}(t)$ be the orthogonal projection onto this space. Then P_t commutes with $\bar{D}_t^+$ and $D_{t,c}^+$, where we write $D_{t,c}^+ = D_t^+ - \bar{D}_t^+$. For fixed t, consider the family of operators

$$D_{t,u}^+ := \bar{D}_t^+ + u(I-P_t)D_{t,c}^+(I-P_t) + P_t, \quad 0 \le u \le 1.$$

By definition,

$$\eta(D_{t,1}^+) = \eta^{\mathrm{he}}(D_t^+) + \dim\operatorname{im} P_t,$$
$$\eta(D_{t,0}^+) = \eta(\bar{D}_t^+) + \dim\operatorname{im} P_t.$$

The non-zero eigenvalues of $\bar{D}_t^+$ tend to infinity as t tends to ∞, whereas $D_{t,c}^+$ is uniformly bounded independently of t. It follows that $D_{t,u}^+$ is invertible, for all sufficiently large t. Now by Proposition 2.12 in [APS3] and the invertibility of $D_{t,u}^+$,

$$\frac{d}{du}\eta(D_{t,u}^+)$$

is a local invariant[10], given by an explicit integral formula constructed out of the complete symbols of $D_{t,u}^+$ and $(I-P_t)D_{t,c}^+(I-P_t)$. On the other hand, P_t is (infinitely) smoothing, and hence the complete symbol of $D_{t,u}^+$ and $(I-P_t)D_{t,c}^+(I-P_t)$ are the same as those of

$$L_{t,u} := \bar{D}_t^+ + uD_{t,c}^+ \quad \text{and} \quad D_{t,c}^+.$$

[10]In [APS3] this assertion is only stated for the η-invariant modulo $\mathbb{Z}$. However, as is clear from the remarks preceding Proposition 2.12 in [APS3], this is only because of the possibility of eigenvalues crossing 0, which is excluded by invertibility.

Now the symbols of $L_{t,u}$ and $D^+_{t,c}$ do not depend on t, by (8.26) and (8.27). It follows that the local invariant for $d\eta(D^+_{t,u})/du$ is bounded in modulus by a continuous function $b = b(u)$ which does not depend on t. Therefore we have

$$\begin{aligned}|\eta^{\mathrm{he}}(D^+_t) - \eta(\bar{D}^+_t)| &= |\eta(D^+_{t,1}) - \eta(D^+_{t,0})| \\ &\leq \mathrm{const} \cdot \mathrm{vol}(\Phi_t(\Gamma)\backslash N) \to 0. \qquad \square\end{aligned}$$

The fact that the high energy η-invariant has no spectral flow is perhaps an indication that its limit deserves to be investigated along the lines of the discussion of the ρ-invariant in [ChGr].

8.2. **Vanishing of η-Invariants.** Let Z belong to the center of $\mathfrak{n}$.

LEMMA 8.30. *Clifford multiplication with Z commutes with $(\bar{D}^+_t)^2$.*

Proof. We can assume that Z has norm one. Choosing $X_2 = Z$, then, in the second sum on the right in (8.28) above, the terms with $i = 2$ vanish since Z commutes with all the X_i, $i > 2$. $\square$

THEOREM 8.31. *If the center of N has dimension at least two, then the spectrum of $\bar{D}^+_t$, including multiplicities, is symmetric about zero. In other words, the eta function of $\bar{D}^+_t$ vanishes identically.*

Proof. Choose orthonormal vector fields Z and Z' in the center of $\mathfrak{n}$ and let $W_\pm$ be the eigenspaces of the involution iZ in $\Sigma^+_{\mathfrak{s}} \otimes V$ for the eigenvalues ± 1. Since $(\bar{D}^+_t)^2$ commutes with iZ, see Lemma 8.30, it leaves the spaces of sections with values in W_+ and W_- invariant. In particular, if $\lambda > 0$ is an eigenvalue of $(\bar{D}^+_t)^2$ and $\mathcal{S}(\lambda)$ denotes the corresponding eigenspace of sections, then

$$\mathcal{S}(\lambda) = \mathcal{S}_+(\lambda) \oplus \mathcal{S}_-(\lambda),$$

where $\mathcal{S}_+(\lambda)$ and $\mathcal{S}_-(\lambda)$ consist of eigensections in $\mathcal{S}(\lambda)$ with values in W_+ and W_-, respectively.

We note that $\mathcal{S}(\lambda)$ is invariant under $\bar{D}^+_t$ and that $\bar{D}^+_t$ has eigenvalues $\pm\sqrt{\lambda}$ on $\mathcal{S}(\lambda)$. Furthermore, the multiplicities of $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ as eigenvalue of $\bar{D}^+_t$ coincide if and only if the trace of $\bar{D}^+_t$ on $\mathcal{S}(\lambda)$ vanishes.

We let $X_2 = Z$. Then $X_i W_+ = W_-$ and $X_i W_- = W_+$ for $3 \leq i \leq m$, and hence the corresponding terms of $\bar{D}^+_t$ do not contribute to the trace of $\bar{D}^+_t$ on $\mathcal{S}(\lambda)$. Now the remaining term $X_2 \cdot X_2(\sigma) = Z \cdot Z(\sigma)$ of $\bar{D}^+_t \sigma$ leaves $\mathcal{S}(\lambda)$ invariant, and its trace on $\mathcal{S}(\lambda)$ is equal to the trace of $\bar{D}^+_t$ on $\mathcal{S}(\lambda)$, by what we just said.

Clifford multiplication with Z' leaves $\mathcal{S}(\lambda)$ invariant, by Lemma 8.30. On the other hand,

$$Z \cdot Z(Z' \cdot \sigma) = Z \cdot (Z' \cdot Z(\sigma)) = -Z' \cdot (Z \cdot Z(\sigma)),$$

that is, the involution iZ' anticommutes with the operator which sends σ to $Z \cdot Z(\sigma)$. It follows that the trace of $\bar{D}_t^+$ on $\mathcal{S}(\lambda)$ vanishes. □

COROLLARY 8.32. *If the center of N has dimension at least two, then the asymptotic high energy η-invariant* $\lim_{t\to\infty} \eta^{\mathrm{he}}(A_t^+) = 0$.

Proof. Recall that $A_t^+ = -D_t^+$ and apply Theorems 8.29 and 8.31. □

9. η-Invariants for Heisenberg Manifolds

The only simply connected nilpotent Lie groups of rank two not covered by Theorem 8.31 are the standard Heisenberg groups $N = G_n$, where here $m - 1 = \dim N = 2n + 1$; see Appendix A for notation and definitions. In this chapter, we study the η-invariant of the operator $\bar{D}_t^+$ as in (8.27). The solvable extension S of $N = G_n$ as in Chapter 8 and the connection ∇^E do not enter in this discussion. We recall though that $\Sigma_{\mathfrak{s}}^+ \simeq \Sigma_{\mathfrak{n}}$, where $\mathfrak{n}$ denotes the Lie algebra of G_n and where Clifford multiplication with X in $\Sigma_{\mathfrak{n}}$ corresponds to Clifford multiplication with TX in $\Sigma_{\mathfrak{s}}^+$, for all $X \in \mathfrak{n}$. This should be kept in mind, see e.g. (9.4).

Let Γ be a lattice in G_n of type d and set

$$|\Gamma| := d_1 \cdots d_n, \tag{9.1}$$

following the notation in [GoWi]. It is clear from (A.3) that there is a smallest $s > 0$ such that $\zeta := \exp(s^2 Z)$ is contained in Γ and that ζ is a generator of the center of Γ. The automorphism $\Phi(x, y, z) = (sx, sy, s^2 z)$ of G_n maps $\exp Z$ to ζ, and, therefore, we may assume that

$$\zeta = \exp Z \tag{9.2}$$

generates the center of Γ. For any left-invariant Riemannian metric on G_n, $N = \Gamma \backslash G_n$ is a Riemannian submersion over a flat torus with closed geodesics as fibers, given as orbits of the one-parameter group generated by Z. By our normalization (9.2), the length of the fibers is given by $|Z|$.

Let τ be an irreducible unitary representation of Γ_d on a finite dimensional Hermitian vector space V as in Appendix A and extend τ by the trivial representation on $\Sigma_{\mathfrak{n}}$ to $\Sigma_{\mathfrak{n}} \otimes V$ as in Chapter 8. Recall from Appendix A that ζ acts by multiplication with $\exp(2\pi i c)$ for some $c = c(\tau) \in [0, 1) \cap \mathbb{Q}$ and that

$$\dim V = \delta(c, d) := m_1 \cdots m_n, \tag{9.3}$$

where $d = d(\Gamma)$ and m_j is the denominator of cd_j. In the notation of this chapter, and in terms of an orthonormal frame (E_j) of G_n, we study the unbounded operator

$$\bar{D}\sigma = \sum E_j \cdot E_j(\sigma), \tag{9.4}$$

in the Hilbert space $L^2(\tau)$ of measurable maps $G_n \to \Sigma_{\mathfrak{n}} \otimes V$ satisfying (8.22) which are square integrable over a fundamental domain of $\Gamma = \Gamma_d$ in the Heisenberg group G_n.

Before stating the next result, we recall the definition of the *Hurwitz zeta function*, for $c > 0$ and $\operatorname{Re} s > 1$ given by the infinite sum

$$\zeta_c(s) = \zeta(s, c) := \sum\nolimits_{k \geq 0} (k + c)^{-s}. \tag{9.5}$$

We have $\zeta_1 = \zeta$, the *Riemann zeta function*. We also set $\zeta_0 := \zeta$. For each $c \geq 0$, ζ_c can be extended to a meromorphic function on the complex plane, defined for all $s \neq 1$, and with a simple pole at $s = 1$, where the residue is equal to 1.

It is maybe interesting to note that, for $0 < c < 1$,

$$\zeta_c(s) - \zeta_{1-c}(s) \quad \text{and} \quad \zeta_c(2s) + \zeta_{1-c}(2s) \tag{9.6}$$

are the eta and zeta function of the operator id/dt and $-d^2/dt^2$, respectively, on the Hermitian line bundle over $\mathbb{R}/2\pi\mathbb{Z}$ with twist $e^{-2\pi i c}$.

THEOREM 9.7. *Endow G_n with a left-invariant Riemannian metric, let Γ be a lattice in G_n such that $\zeta = \exp Z$ generates the center of Γ, and set $r := 1/|Z|$. Consider a Clifford module $\Sigma_{\mathfrak{n}} \otimes V$ as above and let $c = c(\tau)$. Then we have, for all $s \in \mathbb{C}$ with sufficiently large real part,*

(1) $$\eta(\bar{D}, s) = |\Gamma| \dim V (2\pi r)^{-s} (\zeta_c(s - n) - \zeta_{1-c}(s - n))$$

if n is even

(2) $$\eta(\bar{D}, s) = -|\Gamma| \dim V (2\pi r)^{-s} (\zeta_c(s - n) + \zeta_{1-c}(s - n))$$

if n is odd.

We conclude that, under the assumptions of the above theorem, the eta function of $\bar{D}$ is holomorphic if n is even and is meromorphic with a simple pole at $s = n + 1$ if n is odd. We also see that the η-invariant $\eta(\bar{D}) = \eta(\bar{D}, 0)$ of $\bar{D}$ only depends on n, the type of Γ, and c.

Proof of Theorem 9.7. The main argument in the proof is modeled along the lines of the proof of Proposition 4.1 of [DeSi]. We rely on the discussion in Appendix A. For $w \equiv c$ modulo integers, we let

$$L^2(\tau, w) := \{\sigma \in L^2(\tau) : \sigma(x, y, z + t) = e^{2\pi i w t} \sigma(x, y, z)\} \tag{9.8}$$

and get an orthogonal decomposition

$$L^2(\tau) = \oplus_{w \equiv c} L^2(\tau, w), \tag{9.9}$$

where $L^2(\tau)$ is the Hilbert space of measurable maps $G_n \to \Sigma_{\mathfrak{n}} \otimes V$ satisfying (8.22) which are square integrable over a fundamental domain of $\Gamma = \Gamma_d$ in G_n as above. Since the spaces $L^2(\tau, w)$ are invariant under $\bar{D}$, the eta function of $\bar{D}$ is the sum of the eta functions of the restrictions of $\bar{D}$ to the different $L^2(\tau, w)$. Thus we can consider the latter separately.

There are two cases, $w = 0$ and $w \neq 0$. As for $w = 0$, we note that $Z(\sigma) = 0$ for any $\sigma \in L^2(\tau, 0)$. Hence the unitary involution ω_0 of $L^2(\tau, 0)$ given by Clifford multiplication with irZ anti-commutes with $\bar{D}$. Hence the spectrum of $\bar{D}$ is symmetric about 0, and, therefore, the eta function of $\bar{D}$ on $L^2(\tau, 0)$ vanishes identically.

Suppose now that $w \neq 0$. We want to apply the results from Appendix A and note, to that end, that the spaces $L^2(\tau)$ and $L^2(\tau, w)$ here are isomorphic to the corresponding spaces there, tensored with $\Sigma_{\mathfrak{n}}$.

It follows from the discussion in Appendix A that, except for the determination of multiplicities, the particular lattice does not enter into the discussion. By what we explain in Subsection A.2, we can assume that

$$r_1 X_1, r_1 Y_1, \dots, r_n X_n, r_n Y_n, rZ \tag{9.10}$$

is an orthonormal basis of the given left-invariant metric on G_n. Then (9.4) turns into

$$\bar{D}\sigma = \sum_{1 \le j \le n} r_j \big(X_j(\sigma) + Y_j(\sigma) \big) + rZ(\sigma), \tag{9.11}$$

and (8.28) turns into

$$\bar{D}^2 \sigma = \Delta\sigma + \sum_{1 \le j \le n} r_j^4 X_j Y_j \cdot Z(\sigma), \tag{9.12}$$

where $\sigma \in L^2(\tau, w)$ is smooth.

We let ω_j, $1 \le j \le n$, be the unitary involutions on $\Sigma_{\mathfrak{n}} \otimes V$ and $L^2(\tau, w)$ given by Clifford multiplication with $ir_j^2 X_j Y_j$, respectively. Then

$$\Sigma_{\mathfrak{n}} = \oplus_{\varepsilon \in \{1,-1\}^n} \Sigma_\varepsilon, \tag{9.13}$$

where

$$\Sigma_\varepsilon = \{ \sigma \in \Sigma_{\mathfrak{n}} : \omega_j \sigma = \varepsilon_j \sigma \text{ for all } 1 \le j \le n \}. \tag{9.14}$$

Now the unitary involutions ω_j commute with Δ. Thus on

$$L^2(\tau, w, \varepsilon) := \{\sigma \in L(\tau, w) : \sigma \text{ has values in } \Sigma_\varepsilon \otimes V\}, \tag{9.15}$$

$\bar{D}^2$ has eigenvalues

$$\begin{aligned} \lambda(w, p, \varepsilon) &= \lambda(w, p) + 2\pi w(r_1^2\varepsilon_1 + \cdots + r_n^2\varepsilon_n) \\ &= 4\pi^2 w^2 r^2 + 2\pi|w| \sum_{1\le j\le n} (2p_j + 1 + \varepsilon_j \operatorname{sign} w) r_j^2 \end{aligned} \tag{9.16}$$

with multiplicity $2^n m_1 d_1 \cdots m_n d_n |w|^n$, where p runs over all n-tuples of non-negative integers, by (A.38) and (A.39). For all p, we have

$$\lambda(w, p, \varepsilon) \ge 4\pi^2 w^2 r^2 > 0. \tag{9.17}$$

Let W be an eigenspace of $\bar{D}^2$ in $L^2(\tau, w)$ for the eigenvalue λ, and recall from Subsection A.2 that W is independent of the parameter r of the metric. Since $\bar{D}^2$ commutes with the involutions ω_j, W has an orthonormal basis consisting of eigensections of $\bar{D}^2$ such that each of them belongs to some $L^2(\tau, w, \varepsilon)$, where p and ε satisfy

$$S := 2\pi|w| \sum_{1\le j\le n} (2p_j + 1 + \varepsilon_j \operatorname{sign} w) r_j^2 = \lambda - 4\pi^2 w^2 r^2, \tag{9.18}$$

by (9.16). Now Clifford multiplication by the unit vector rZ commutes with $\bar{D}^2$ and leaves the subspaces $L^2(\tau, w, \varepsilon)$ invariant, whereas Clifford multiplication by the unit vectors $r_j X_j$ and $r_j Y_j$ maps $L^2(\tau, w, \varepsilon)$ to $L^2(\tau, w, \delta)$ for $\delta \ne \varepsilon$. Hence using an orthonormal basis of eigensections of W as above, we see that the trace of $\bar{D}$ on W is equal to an integral multiple $k2\pi wr$ of $2\pi wr$. On the other hand, the trace of $\bar{D}$ on W is also equal to $l\sqrt{\lambda}$ for some integer l. Now 0 is not an eigenvalue of $\bar{D}^2$ on W, independently of $r > 0$. Hence k and l do not depend on r, and we get an equality of functions of $r \in (0, \infty)$,

$$k^2 4\pi^2 w^2 r^2 = l^2(4\pi^2 w^2 r^2 + S)^2. \tag{9.19}$$

If $l = 0$, then the eigenvalues $\pm\sqrt{\lambda}$ of $\bar{D}$ occur with equal multiplicity in W and, therefore, their contributions to the eta function of $\bar{D}$ on $L^2(\tau, w)$ cancel. If $l \ne 0$, then $S = 0$, since S does not depend on r. But then, since $w \ne 0$, $p_j \ge 0$, and $\varepsilon_j = \pm 1$ for all j, we conclude that $\lambda(w, p, \varepsilon) = 4\pi^2 w^2 r^2$ and that

$$p_1 = \cdots = p_n = 0 \quad \text{and} \quad \varepsilon_1 = \cdots = \varepsilon_n = -\operatorname{sign} w \tag{9.20}$$

for $1 \le j \le n$. This will be denoted by $p = 0$ and $\varepsilon = -\operatorname{sign} w$.

To determine the contribution of the corresponding eigenspaces, we note that, by our identification $\Sigma_{\mathfrak{n}} = \Sigma_{\mathfrak{s}}^+$, Clifford multiplication by $irZ\omega_1 \cdots \omega_n$ is equal to the identity on $\Sigma_{\mathfrak{n}}$. Since Clifford multiplication with irZ commutes with Clifford multiplication with the ω_j, it leaves the subspaces Σ_ε invariant and acts by multiplication with $\varepsilon_1 \cdots \varepsilon_n$ on them. Now $Z(\sigma) = 2\pi i w \sigma$ for any σ in $L^2(\tau, w)$. Hence the eigenspace for $\bar{D}^2$ in $L^2(\tau, w)$ with eigenvalue $\lambda(w, 0, -\operatorname{sign} w) = 4\pi^2 w^2 r^2$ is an eigenspace of $\bar{D}$ with eigenvalue

$$\begin{aligned} 2\pi w r \quad & \text{if } n \text{ is even,} \\ -2\pi |w| r \quad & \text{if } n \text{ is odd,} \end{aligned} \tag{9.21}$$

and dimension $m_1 \dots m_n d_1 \dots d_n |w|^n = |\Gamma| \dim V$. Thus, for all $s \in \mathbb{C}$ with sufficiently large real part,

$$\eta(\bar{D}, s) = |\Gamma| \dim V (2\pi r)^{-s} \sum_{w \equiv c,\, w \neq 0} \operatorname{sign}(w) |w|^{n-s} \tag{9.22}$$

if n is even and

$$\eta(\bar{D}, s) = -|\Gamma| \dim V (2\pi r)^{-s} \sum_{w \equiv c,\, w \neq 0} |w|^{n-s} \tag{9.23}$$

if n is odd. □

We apply the results of this chapter to Dirac operators on homogeneous vector bundles over complex hyperbolic cusps of complex dimension n. Such cusps are homogeneous in the sense of Chapter 8, where the nilpotent Lie group is given by the Heisenberg group $N = G_{n-1}$ of dimension $2n-1$ and $\Gamma \subseteq G_{n-1}$ is a lattice. In our formulas above we therefore need to substitute n by $n-1$.

Corollary 9.24. *In the sense of Chapter 8, suppose that a complex hyperbolic cusp is determined by a lattice $\Gamma \subseteq G_{n-1}$ and that the homogeneous Dirac bundle over the cusp is given by unitary representations π_* of $\mathfrak{u}(n)$ and τ of Γ on a Hermitian vector space V. Assume that V is irreducible as a joint $\mathfrak{u}(n)$ and Γ module. Then the twist parameter c of τ is well defined and*

$$\lim_{t\to\infty} \eta^{\mathrm{he}}(A_t^+) = (-1)^n |\Gamma| \dim V \big(\zeta_c(1-n) + (-1)^n \zeta_{1-c}(1-n) \big).$$

Proof. We recall that $A_t = -D_t$, see (3.45). By Theorem 8.29, we have $\lim_{t\to\infty} \eta^{\mathrm{he}}(D_t^+) = \lim_{t\to\infty} \eta(\bar{D}_t^+)$. Now the operator $\bar{D}_t^+$ corresponds to the operator $\bar{D}$ considered above, where the left-invariant metric on G_n comes from the cross section $\{t\} \times N$ in S. Since V is irreducible as a joint $\mathfrak{u}(n)$ and Γ module, it is a direct sum of isotypical irreducible representations of Γ as used in Theorem 9.7 so that the number c is the same for each summand. Hence Theorem 9.7 applies and shows that $\eta(\bar{D}_t^+)$ does not depend on t and that it is given by the formula in Corollary 9.24 □

Example 9.25. Spinor bundles as in Example 8.23 are given by the trivial representation of $\mathfrak{u}(n)$ and classified by twists $\tau : \Gamma_d \to \{+1, -1\}$. Since $\tau(\zeta) = \pm 1$, we have $c = 0$ or $c = 1/2$. Hence the asymptotic high energy η-invariant of A_t^+ vanishes identically if n is odd. If n is even and $c = 0$, then

$$\lim_{t\to\infty} \eta^{\mathrm{he}}(A_t^+) = 2|\Gamma|\zeta(1-n), \tag{9.26}$$

which agrees with Proposition 4.1 in [DeSi] in the case $\Gamma = \Gamma_{(1,\ldots,1)}$ considered there (with a different choice of orientation). If n is even and $c = 1/2$, then

$$\lim_{t\to\infty} \eta^{\mathrm{he}}(A_t^+) = 2(2^{1-n} - 1)|\Gamma|\zeta(1-n), \tag{9.27}$$

where we use that $\zeta_{1/2}(s) = (2^s - 1)\zeta(s)$. Recall also that

$$\zeta(1-n) = -B_n/n, \tag{9.28}$$

where B_n denotes the n-th Bernoulli number.

10. Low Energy η-Invariants

10.1. **General Remarks and Computations.** We return to the situation and notation considered in Chapter 8 and let $E = \mathcal{P} \times_\beta (\Sigma_{\mathfrak{s}} \otimes V)$ be a homogeneous Dirac bundle over S. As in Chapter 9, we view sections of E^+ as smooth maps $\sigma : S \to \Sigma_{\mathfrak{n}} \otimes V$.

The vector field T is left-invariant and a global unit normal field along the hypersurfaces $N_t := \{t\} \times N$ of S. In accordance with this, we choose frames $(X_1, \ldots, X_m)$ of S to be left-invariant and orthonormal with $X_1 = T$. Then $X_2, \ldots, X_m$ are tangent to the hypersurfaces N_t.

Let $\Gamma \subseteq N$ be a lattice, $\tau : \Gamma \to V$ be a unitary representation, and E_τ be the induced Dirac bundle over $\Gamma\backslash S = \mathbb{R} \times (\Gamma\backslash N)$. Then we have, for any $t \in \mathbb{R}$, the orthogonal decomposition

$$L^2(N_t, E_\tau^+) = H^{\mathrm{le},+}(A_t) \oplus H^{\mathrm{he},+}(A_t), \tag{10.1}$$

where $H^{\mathrm{le},+}(A_t)$ is the space of constant maps $N_t \to \Sigma_{\mathfrak{n}}^+ \otimes V$, compare Chapter 7 and, in particular, (7.21).

Proposition 10.2. *If V is irreducible as a joint $\mathfrak{k}$ and Γ module and τ is non-trivial, then the low energy spaces $H_t^{\mathrm{le},+}(A_t)$ are trivial and, therefore, $\eta^{\mathrm{le}}(A_t^+) = 0$, for all $t \in \mathbb{R}$.* □

Thus the low energy η-invariant can only be non-trivial when τ is trivial. We refer to this as the *untwisted case* and assume for the rest of this section that we are in this case, whether V is irreducible as a $\mathfrak{k}$ module or not. Then the space $H^{\mathrm{le},+}(A_t)$ is isomorphic to $\Sigma_{\mathfrak{n}}^+ \otimes V$, by identifying constant maps with their respective values.

For $\sigma \in H^{\mathrm{le},+}(A_t)$ and with A_X as in (8.14), we have

$$D_t^+\sigma = \sum_{2\le j\le m} TX_j \cdot \beta_*(A_{X_j})\sigma + \frac{\kappa}{2}\sigma, \tag{10.3}$$

by (8.25), where we recall our convention $X_1 = T$. Our objective in this chapter is the η-invariant of D_t^+ on $H^{\mathrm{le}}(A_t^+)$. We view elements of $H^{\mathrm{le},+}(A_t)$ as constant maps on S. Then $H^{\mathrm{le},+}(A_t)$ becomes independent of Γ and t. By (10.3), D_t^+ does not depend on t either. As a shorthand, we will write

$$H_N^{\mathrm{le}} \text{ for } H^{\mathrm{le},+}(A_t) \quad \text{and} \quad D_N^{\mathrm{le}} \text{ for } D_t^+. \tag{10.4}$$

Recall that $\beta_* = \hat\alpha_* \otimes \mathrm{id} + \mathrm{id} \otimes \pi_*$, by (8.15), and that

$$\hat\alpha_*(A_X) = \frac{1}{2}\sum_{1\le j<k\le m} \langle \nabla_X X_j, X_k\rangle X_j X_k. \tag{10.5}$$

where X_jX_k stands for Clifford multiplication by X_jX_k. With our convention $X_1 = T$, (10.5) turns into

$$\hat\alpha_*(A_X) = \frac{1}{2}T\nabla_X T + \frac{1}{2}\sum\nolimits_{2\le j<k\le m} \langle \nabla_X X_j, X_k\rangle X_j X_k. \tag{10.6}$$

It follows that (2.8) and (2.9) define the Dirac structure on E associated to the Riemannian metric of N.

Choose an orthonormal frame $(X_2, \dots, X_m)$ of $\mathfrak{n}$ such that $[X_j, X_k]$ is contained in the linear hull of the X_l with $l < \min\{j,k\}$. On H_N^{le}, we then obtain

$$\begin{aligned}
8D_N^{\mathrm{le}} - 8\sum_{j\ge 2} & TX_j \otimes \pi_*(A_{X_j}) \\
&= \;\; 4\sum_{j\ge 2\le k<l} TX_j \langle \nabla_{X_j} X_k, X_l\rangle X_k X_l \\
&= -2\sum_{j\ge 2\le k<l} TX_j\big(\langle X_j, [X_k,X_l]\rangle + \langle X_k, [X_j,X_l]\rangle\big) X_k X_l \\
&= -2\sum_{2\le k<l} T[X_k,X_l]X_kX_l - 2\sum_{2\le j,l} TX_j[X_j,X_l]X_l \\
&= \;\; 2\sum\nolimits_{2<j<k} T[X_j,X_k]X_jX_k \\
&= \;\; \sum\nolimits_{j,k>2} T[X_j,X_k]X_jX_k,
\end{aligned} \tag{10.7}$$

where we use the Koszul formula and where we note that X_2 is central.

We now come to our main example, the case where N is of Heisenberg type. That is, we are given an orthogonal decomposition

$$\mathfrak{n} = \mathfrak{z} + \mathfrak{x}, \tag{10.8}$$

where $\mathfrak{z}$ is contained in the center of $\mathfrak{n}$, and a linear map J from $\mathfrak{z}$ into the space of skew-symmetric endomorphisms of $\mathfrak{x}$ such that the Clifford relations hold,

$$J_{Z_1}J_{Z_2} + J_{Z_2}J_{Z_1} + 2\langle Z_1, Z_2\rangle = 0, \tag{10.9}$$

for all $Z_1, Z_2 \in \mathfrak{z}$. Moreover, the Lie brackets of vectors in $\mathfrak{x}$ are contained in $\mathfrak{z}$ and satisfy, by definition,

$$\langle [X_1, X_2], Z\rangle = 2c\langle J_Z X_1, X_2\rangle, \tag{10.10}$$

for all $X_1, X_2 \in \mathfrak{x}$ and $Z \in \mathfrak{z}$. Here $c > 0$ is some chosen constant and the derivation W is defined to have $\mathfrak{x}$ and $\mathfrak{z}$ as eigenspaces with $-c$ and $-2c$ as respective eigenvalues. This normalization has the following amazing formula as a consequence.

Lemma 10.11. *For all $Z \in \mathfrak{z}$ and $X \in \mathfrak{x}$, we have*

$$R(Z, X) = R(J_Z X, T).$$

Remark 10.12. If N is the standard Heisenberg group of dimension $2n+1$, then S is isometric to the complex hyperbolic space $\mathbb{C}H^{n+1}$ of dimension $2n+2$ with sectional curvature in $[-4c^2, -c^2]$ and complex structure J with $JT = Z$ and such that J coincides with J_Z on N. In this case, the equation in Lemma 10.11 is a special case of the more general $R(JU, V) = -R(U, JV)$ which says that the curvature tensor of $\mathbb{C}H^{n+1}$ is a differential form of type $(1, 1)$.

Proof of Lemma 10.11. By straightforward computations, using (8.7), (8.8), (10.9), and (10.10), ☐

Let $Z \in \mathfrak{z}$ with $|Z| = 1$. Then J_Z is an orthogonal complex structure on $\mathfrak{x}$. In particular, the dimension of $\mathfrak{x}$ is even, and we denote it by $2n$. Moreover, there is an orthonormal basis $(X_1, \ldots, X_{2n})$ of $\mathfrak{x}$ such that $J_Z X_{2j-1} = X_{2j}$, for $1 \le j \le n$. Given any such basis, set

$$D_Z := \frac{c}{2}\sum\nolimits_{1\le j\le n} TZX_{2j-1}X_{2j} + TZ \otimes \pi_*(A_Z). \tag{10.13}$$

Observe that, for any orthonormal basis $(Y_1, \ldots, Y_{2n})$ of $\mathfrak{x}$,

$$\begin{aligned} D_Z &= \frac{1}{8}\sum\nolimits_{j,k}\langle [Y_j, Y_k], Z\rangle TZY_jY_k + TZ \otimes \pi_*(A_Z) \\ &= \frac{c}{4}\sum\nolimits_{j,k}\langle J_Z Y_j, Y_k\rangle TZY_jY_k + TZ \otimes \pi_*(A_Z) \\ &= \frac{c}{2}\sum\nolimits_{j<k}\langle J_Z Y_j, Y_k\rangle TZY_jY_k + TZ \otimes \pi_*(A_Z) \end{aligned} \tag{10.14}$$

In what follows, let $\{A, B\} := AB + BA$.

LEMMA 10.15. *For any $X \in \mathfrak{x}$, we have*

$$\{D_Z, TX \otimes \pi_*(A_X) + TJ_ZX \otimes \pi_*(A_{J_ZX})\} = 0.$$

Proof. We have $cA_X = R(X,T)$, by (8.8), and hence

$$c\{TZXJ_ZX, TX \otimes \pi_*(A_X)\} = 2ZJ_ZX \otimes \pi_*(R(X,T)). \tag{10.16}$$

By substituting J_ZX for X in (10.16), we obtain

$$c\{TZXJ_ZX, TJ_ZX \otimes \pi_*(A_{J_ZX})\} = -2ZX \otimes \pi_*(R(J_ZX,T)). \tag{10.17}$$

We also have $[Z,X] = 0$, hence $[A_Z, A_X] = R(Z,X)$. Furthermore, $R(Z,X) = R(J_ZX,T)$, by Lemma 10.11, hence

$$\{TZ \otimes \pi_*(A_Z), TX \otimes \pi_*(A_X)\} = ZX \otimes \pi_*(R(J_ZX,T)). \tag{10.18}$$

By substituting J_ZX for X in (10.18), we obtain

$$\{TZ \otimes \pi_*(A_Z), TJ_ZX \otimes \pi_*(A_{J_ZX})\} = -ZJ_ZX \otimes \pi_*(R(X,T)). \tag{10.19}$$

Moreover, we have

$$\begin{aligned} &\{TZYJ_ZY, TX \otimes \pi_*(A_X)\} \\ &\qquad = \{TZYJ_ZY, TJ_ZX \otimes \pi_*(A_{J_ZX})\} = 0, \end{aligned} \tag{10.20}$$

for all $Y \in \mathfrak{x}$ perpendicular to X and J_ZX. Now we may assume that X is of norm 1. Then there is an orthonormal basis $(X_1, \dots, X_{2n})$ of $\mathfrak{x}$ such that $J_ZX_{2j-1} = X_{2j}$, for $1 \le j \le n$, and such that $X = X_1$. By (10.20), the terms of D_Z involving $TZX_{2j-1}X_{2j}$, $j \ge 2$, do not contribute to the anti-commutator $\{D_Z, TX \otimes \pi_*(A_X) + TJ_ZX \otimes \pi_*(A_{J_ZX})\}$. The four remaining terms cancel pairwise, by (10.16)–(10.19). □

For an orthonormal basis $(X_1, \dots, X_{2n})$ of $\mathfrak{x}$, set

$$D_{\mathfrak{x}} := TX_1 \otimes \pi_*(A_{X_1}) + \cdots + TX_{2n} \otimes \pi_*(A_{X_{2n}}), \tag{10.21}$$

and note that $D_{\mathfrak{x}}$ does not depend on the choice of $(X_1, \dots, X_{2n})$.

REMARK 10.22. If $\mathfrak{z} = 0$, then $\mathfrak{n}$ is Abelian and we are in the case of real hyperbolic spaces or cusps, respectively, and we get $D_N^{\mathrm{le}} = D_{\mathfrak{x}}$ on H_N^{le}. The contribution of cusps in the case $\dim N = 1$ follows easily from the more general discussion in [BB1]. If $\dim N \ge 2$, then the arguments in the proof of Theorem 8.31 apply and show that the low energy η-invariant vanishes.

LEMMA 10.23. *For any unit vectors $Z \in \mathfrak{z}$,*

$$\{D_Z, D_{\mathfrak{x}}\} = 0.$$

Proof. Apply Lemma 10.15, using an orthonormal basis $(X_1, \dots, X_{2n})$ of $\mathfrak{x}$ with $J_ZX_{2j-1} = X_{2j}$, for $1 \le j \le n$. □

Assume from now on that $\mathfrak{z} \neq 0$, compare Remark 10.22. For an orthonormal basis $(Z_1, \ldots, Z_\ell)$ of $\mathfrak{z}$, set

$$D_{\mathfrak{z}} := D_{Z_1} + \cdots + D_{Z_\ell}, \tag{10.24}$$

and note that $D_{\mathfrak{z}}$ does not depend on the choice of $(Z_1, \ldots, Z_\ell)$.

Corollary 10.25. *On H_N^{le}, we have*

$$D_N^{\mathrm{le}} = D_{\mathfrak{z}} + D_{\mathfrak{x}} \quad \text{and} \quad \{D_{\mathfrak{z}}, D_{\mathfrak{x}}\} = 0. \quad \square$$

Proposition 10.26. *On H_N^{le}, we have*

(1) $$\ker(D_N^{\mathrm{le}}) = \ker D_{\mathfrak{z}} \cap \ker D_{\mathfrak{x}},$$

(2) $$\eta(D_N^{\mathrm{le}}) = \eta(D_{\mathfrak{z}}) = \eta(D_{\mathfrak{z}}|_{\ker D_{\mathfrak{x}}}).$$

Proof. By Corollary 10.25, (1) is clear and

$$\begin{aligned} \eta(D_N^{\mathrm{le}}) &= \eta(D_{\mathfrak{z}}|_{\ker D_{\mathfrak{x}}}) + \eta(D_{\mathfrak{x}}|_{\ker D_{\mathfrak{z}}}), \\ \eta(D_{\mathfrak{z}}|_{\ker D_{\mathfrak{x}}}) &= \eta(D_{\mathfrak{z}}), \\ \eta(D_{\mathfrak{x}}|_{\ker D_{\mathfrak{z}}}) &= \eta(D_{\mathfrak{x}}). \end{aligned}$$

Now $D_{\mathfrak{x}}$ anticommutes with the involution TZ_1 of $\Sigma^+ \otimes V$, hence $\eta(D_{\mathfrak{x}}) = 0$, hence (2). $\square$

10.2. **Contribution of Complex Hyperbolic Cusps.** We represent complex hyperbolic space $\mathbb{C}H^n$ as in Section 2.3. For any $X \in \mathfrak{su}(1,n)$, we write $X = X^{\mathfrak{p}} + X^{\mathfrak{k}}$ with $X^{\mathfrak{p}} \in \mathfrak{p}$ and $X^{\mathfrak{k}} \in \mathfrak{k} = \mathfrak{u}(n)$. We recall that, after identification of $\mathfrak{p}$ with the tangent space of $\mathbb{C}H^n$ at the point fixed by $\mathrm{U}(n)$ as usual, we have

$$R(X,Y)Z = -[[X,Y],Z], \tag{10.27}$$

for all $X, Y, Z \in \mathfrak{p}$.

Let $X \in \mathfrak{n}$. By (2.33) and (2.35), we have $[T, X] = -WX$ and hence

$$[T, X^{\mathfrak{p}}] = -(WX)^{\mathfrak{k}}. \tag{10.28}$$

Using (8.8), the identification $S \simeq \mathbb{C}H^n$ as in (2.37), and (10.27), we obtain therefore that

$$A_{WX}Y = R(T, X^{\mathfrak{p}})Y^{\mathfrak{p}} = -[[T, X^{\mathfrak{p}}], Y^{\mathfrak{p}}] = [(WX)^{\mathfrak{k}}, Y^{\mathfrak{p}}]. \tag{10.29}$$

Since W is invertible, we conclude that, for any $X \in \mathfrak{n}$,

$$A_X = X^{\mathfrak{k}}. \tag{10.30}$$

With α as in (2.30), we let $\hat{\alpha}_* : \mathfrak{u}(n) \to \mathfrak{u}(\Sigma)$ be the composition of the differential α_* of α with the differential of the spinor representation of $\mathfrak{so}(\mathfrak{p}) \simeq \mathfrak{spin}(\mathfrak{p})$ on $\Sigma := \Sigma_{\mathfrak{p}}$. Following Chapter 8, we choose $K = \mathrm{U}(n)$ and let π_* be a unitary representation of $\mathfrak{u}(n)$ on a Hermitian vector space V. We assume that there exists a unitary representation β of $K = \mathrm{U}(n)$ on $\Sigma \otimes V$ satisfying (8.15) and get the associated Dirac bundle E over $\mathbb{C}H^n$. Clifford multiplication by the complex volume form $\omega_{\mathbb{C}}$ determines a super-symmetry $E = E^+ \oplus E^-$, and this super-symmetry is induced by the corresponding decomposition $\Sigma = \Sigma^+ \oplus \Sigma^-$.

To distinguish it from multiplication with i in $\mathbb{C}^n \simeq \mathfrak{p}$, we denote the complex structure in $\mathbb{C}\mathrm{l}(\mathfrak{p})$ by $\sqrt{-1}$. With the corresponding changes in notation, we follow Section 2.2 and set

$$\omega_j := \sqrt{-1} X_j^{\mathfrak{p}} Y_j^{\mathfrak{p}}, \quad 1 \le j \le n, \tag{10.31}$$

where $X_1 = T, Y_1 = Z, X_2, Y_2, \dots, X_n, Y_n$ are as in (2.41). By the discussion in Section 2.2, we have

$$\Sigma^+ = \oplus_{\epsilon \in \{-1,1\}^{n-1}} \Sigma^+_\epsilon, \tag{10.32}$$

where

$$\Sigma^+_\epsilon := \{\sigma \in \Sigma^+ : \omega_j \sigma = \epsilon_j \sigma \text{ for } 2 \le j \le n\}. \tag{10.33}$$

Since X_j commutes with ω_k for $k \neq j$ and anti-commutes with ω_j, all the subspaces Σ_ϵ are isomorphic. In particular, for all $\epsilon \in \{-1,1\}^{n-1}$,

$$\dim \Sigma^+_\epsilon = \dim \Sigma^+ / \operatorname{card}\{-1,1\}^{n-1} = 1. \tag{10.34}$$

For any $\epsilon \in \{-1,1\}^{n-1}$, let $\nu(\epsilon) \in \{0, \dots, n-1\}$ be the number of j with $\epsilon_j = -1$, for $2 \le j \le n$. Then

$$\Sigma^+ = \oplus_k \Sigma^+_k, \quad \text{where} \quad \Sigma^+_k := \oplus_{\nu(\epsilon)=k} \Sigma^+_\epsilon. \tag{10.35}$$

By definition, $\omega_{\mathbb{C}}$ acts as identity on Σ^+, hence $\omega_1 = \omega_2 \cdots \omega_n$ on Σ^+. Therefore

$$\Sigma^+_{\mathrm{even}} = \oplus_{k \text{ even}} \Sigma^+_k \quad \text{and} \quad \Sigma^+_{\mathrm{odd}} = \oplus_{k \text{ odd}} \Sigma^+_k \tag{10.36}$$

are the eigenspaces of ω_1 for the eigenvalues 1 and -1, respectively. In passing we note that the left side of (10.35) gives the decomposition of Σ^+ into irreducible representations of the stabilizer of T in $\mathrm{U}(n)$, by work of Camporesi and Pedon, see [CamP, Lemma 3.1].

We recall that the complexification of $\mathfrak{u}(n)$ is $\mathfrak{gl}(n,\mathbb{C})$, where the complex structure of $\mathfrak{gl}(n,\mathbb{C})$ is given by multiplication of matrix coefficients with i. The space $\mathfrak{h} \subseteq \mathfrak{gl}(n,\mathbb{C})$ of diagonal matrices is a Cartan subalgebra of $\mathfrak{gl}(n,\mathbb{C})$, and the roots

$$\rho_j(\operatorname{diag}(h_1, ..., h_n)) := h_j \tag{10.37}$$

constitute a basis of $\mathfrak{h}^*$. The associated Weyl group $\mathcal{W}$ of automorphisms of $\mathfrak{h}$ leaves the set $\{\rho_1, \dots, \rho_n\}$ invariant and acts on it as the (complete) group of permutations. As usual, we choose

$$\{h = \operatorname{diag}(h_1, ..., h_n) : h_j \in \mathbb{R}, h_1 > h_2 > \cdots > h_n\} \tag{10.38}$$

as positive Weyl chamber. The corresponding set of positive roots of $\mathfrak{gl}(n,\mathbb{C})$ is given by

$$\Delta^+ = \{\rho_j - \rho_k : 1 \leq j < k \leq n\}. \tag{10.39}$$

Irreducible complex representations of $\mathfrak{u}(n)$ are classified by their *highest weight* $\lambda = \sum \lambda_j \rho_j$, where λ is *dominant*, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and algebraically integral, that is, $\lambda_i - \lambda_j \in \mathbb{Z}$ for all i, j. The dimension of the corresponding representation space V_λ is

$$\dim V_\lambda = \prod_{j<k} \frac{k - j - \lambda_k + \lambda_j}{k - j}, \tag{10.40}$$

by the Weyl character formula. The irreducible representation with highest weight λ is induced by a representation of $\mathrm{U}(n)$ if all the λ_j are integral. The representation α as above is the irreducible representation of $\mathrm{U}(n)$ with highest weight $(2, 1, \ldots, 1)$ (and complex dimension n).

For the discussion of $\hat{\alpha}_*$, we identify $\mathfrak{p} = \mathbb{R}^{2n}$ and $\Sigma = \Sigma_{2n}$. We let $(e_1, \ldots, e_{2n})$ be the standard basis of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ with $e_{2j} = ie_{2j-1}$, $1 \leq j \leq n$, and denote the complex structure of Σ_{2n} by $\sqrt{-1}$ as above. For $h = (it_1, \ldots, it_n)$ in $\mathfrak{h} \cap \mathfrak{u}(n)$, we get

$$\begin{aligned} \hat{\alpha}_*(h) &= \frac{1}{4} \sum_{1 \leq j \leq 2n} e_j \big(it_j e_j + (it_1 + \cdots + it_n) e_j \big) \\ &= -\frac{1}{2} \sqrt{-1} \sum_{1 \leq j \leq n} (t_1 + \cdots + 2t_j + \cdots + t_n) \omega_j. \end{aligned} \tag{10.41}$$

by the Parthasarathy formula [Pa, Lemma 2.1], where e_j and ω_j stand for Clifford multiplication by e_j and ω_j, respectively. Hence the subspaces Σ_ε of Σ_m as in Section 2.2 are weight spaces. For $0 \leq l \leq n$, we let V_l be the sum over all Σ_ε such that l is the number of j with $\varepsilon_j = -1$, that is, $\varepsilon_1 + \cdots + \varepsilon_n = n - 2l$. Then V_l is the irreducible representation of $\mathfrak{u}(n)$ with heighest weight

$$\lambda_1 = \cdots = \lambda_l = l - \frac{n-1}{2} > \lambda_{l+1} = \cdots = \lambda_n = l - \frac{n+1}{2}, \tag{10.42}$$

and is of dimension $\binom{n}{l}$, in agreement with Weyl's character formula.

As an example, we discuss differential forms. Since α is the irreducible representation with maximal weight $(2, 1, \ldots, 1)$, the bundles of differential forms of type $(p, 0)$ and $(0, q)$ are associated to the irreducible representations β of $\mathrm{U}(n)$ with maximal weights

$$\lambda_1 = \cdots = \lambda_{n-p} = -p > \lambda_{n-p+1} = \cdots = \lambda_n = -(p+1) \tag{10.43}$$

and

$$\lambda_1 = \cdots = \lambda_q = q+1 > \lambda_{q+1} = \cdots = \lambda_n = q, \tag{10.44}$$

respectively. We see that the sum of the bundles of differential forms of type $(0,q)$, $0 \le q \le n$, is given by $\Sigma \otimes V_n$, where V_n is as above. That is, π_* is the one-dimensional irreducible representation of $\mathfrak{u}(n)$ with highest weight $\lambda_j = (n+1)/2$, $1 \le j \le n$.

REMARK 10.45. From (10.42), we see that $\hat{\alpha}_*$ comes from a representation of $\mathrm{U}(n)$ if n is odd, and then the spinor bundle of $\mathbb{C}H^n$ descends to quotients of $\mathbb{C}H^n$ by discrete subgroups of $\mathrm{SU}(1,n)$. On the other hand, if $\widetilde{\mathrm{SU}}(1,n)$ denotes the non-trivial twofold cover of $\mathrm{SU}(1,n)$, then $\hat{\alpha}_*$ comes from a representation of the corresponding twofold cover $\tilde{\mathrm{U}}(n)$ of $\mathrm{U}(n)$, for all n. Hence, if the discrete subgroup of $\mathrm{SU}(n)$ under consideration admits a lift into $\tilde{\mathrm{SU}}(1,n)$, then the spinor bundle also decends to the corresponding quotient of $\mathbb{C}H^n$. A similar remark applies to β_*.

We note that $D_{\mathfrak{r}}$ is an odd operator with respect to the grading

$$\Sigma^+ \otimes V_\pi = (\Sigma^+_{\mathrm{even}} \otimes V_\pi) \oplus (\Sigma^+_{\mathrm{odd}} \otimes V_\pi), \tag{10.46}$$

whereas $D_{\mathfrak{z}} = D_Z$ is an even operator.

THEOREM 10.47. *With* $H^k := \ker D_{\mathfrak{r}} \cap (\Sigma^+_k \otimes V_\pi)$ *and* $b_k := \dim H^k(\pi)$, *for* $0 \le k \le n-1$, *we have*

(1) $\ker D_{\mathfrak{r}} = \oplus H^k$,

(2) $$b_k = (n-1)!\dim V_\pi \prod_{\substack{1\le j\le n\\ j\ne k+1}} |\lambda_j - \lambda_{k+1} + k + 1 - j|^{-1},$$

(3) $D_Z|_{H^k} = (-1)^k(2k - 2\lambda_{k+1} - n + 1)/2.$

Proof. Our proof relies on Kostant's theorem, see [Ko] or Theorem 4.139 in [KnVo]. We start by describing an explicit model of Σ, compare Chapter 5 of [Wu]. For $1 \le j \le n$, let

$$F_j := \frac{1}{2}(X^{\mathfrak{p}}_j - \sqrt{-1}Y^{\mathfrak{p}}_j) \quad \text{and} \quad \bar{F}_j := \frac{1}{2}(X^{\mathfrak{p}}_j + \sqrt{-1}Y^{\mathfrak{p}}_j). \tag{10.48}$$

As elements of $\mathbb{C}\mathrm{l}(\mathfrak{p})$, they satisfy

$$F_jF_j = \bar{F}_j\bar{F}_j = 0, \quad \bar{F}_jF_j = -F_j\bar{F}_j - 1, \tag{10.49}$$

for $1 \le j \le n$, and

$$F_jF_k = -F_kF_j, \quad \bar{F}_j\bar{F}_k = -\bar{F}_k\bar{F}_j, \quad F_j\bar{F}_k = -\bar{F}_kF_j, \tag{10.50}$$

for $1 \le j \ne k \le n$. We identify Σ with the left ideal in the Clifford algebra generated by $\bar{F} = \bar{F}_1 \cdots \bar{F}_n$. Then the monomials $F_I\bar{F}$ over all $0 \leq k \leq n$ and multi-indices $I = \{i_1, \cdots, i_k\}$ with $i_1 < \cdots < i_k$ constitute a basis of Σ. The relations (10.49) and (10.50) determine an isomorphism $\Sigma \simeq \Lambda(\mathbb{C}^n)$, where $\mathbb{C}^n$ is spanned by $F_1, \dots, F_n$. We have

$$\omega_j \cdot F_I \bar{F} = \begin{cases} F_I \bar{F} & \text{if } j \notin I, \\ -F_I \bar{F} & \text{if } j \in I. \end{cases} \tag{10.51}$$

so that, under the identification $\Sigma \simeq \Lambda(\mathbb{C}^n)$,

$$\Sigma_k^+ \simeq \begin{cases} \Lambda^k(\mathbb{C}^{n-1}) & \text{if } k \text{ is even}, \\ F_1 \wedge \Lambda^k(\mathbb{C}^{n-1}) \simeq \Lambda^k(\mathbb{C}^{n-1}) & \text{if } k \text{ is odd}, \end{cases} \tag{10.52}$$

where $\mathbb{C}^{n-1}$ is spanned by $F_2, \dots, F_n$.

Recall that, by complexification, π_* induces a representation of $\mathfrak{gl}(n, \mathbb{C})$. Following the exposition in [LaMi, §IV.8], we set

$$\mathcal{D}_{\mathfrak{r}} := \frac{1}{2} \sum_{2 \leq j \leq n} T(X_j^{\mathfrak{p}} - \sqrt{-1} Y_j^{\mathfrak{p}}) \otimes \pi_*(X_j^{\mathfrak{k}} + i Y_j^{\mathfrak{k}}) = 2 \sum_{2 \leq j \leq n} T F_j \otimes \pi_*(E_{1j}), \tag{10.53}$$

$$\bar{\mathcal{D}}_{\mathfrak{r}} := \frac{1}{2} \sum_{2 \leq j \leq n} T(X_j^{\mathfrak{p}} + \sqrt{-1} Y_j^{\mathfrak{p}}) \otimes \pi_*(X_j^{\mathfrak{k}} - i Y_j^{\mathfrak{k}}) = -2 \sum_{2 \leq j \leq n} T \bar{F}_j \otimes \pi_*(E_{j1}), \tag{10.54}$$

where we note that factors $\sqrt{-1}$ on the left and i on the right of $\otimes$ multiply to -1 in the tensor product. Using (10.30), we have

$$D_{\mathfrak{r}} = \mathcal{D}_{\mathfrak{r}} + \bar{\mathcal{D}}_{\mathfrak{r}}, \quad \bar{\mathcal{D}}_{\mathfrak{r}} = \mathcal{D}_{\mathfrak{r}}^*, \quad \text{and} \quad \mathcal{D}_{\mathfrak{r}} \mathcal{D}_{\mathfrak{r}} = \bar{\mathcal{D}}_{\mathfrak{r}} \bar{\mathcal{D}}_{\mathfrak{r}} = 0. \tag{10.55}$$

Moreover,

$$\mathcal{D}_{\mathfrak{r}}(\Sigma_k^+ \otimes V_\pi) \subset \Sigma_{k+1}^+ \otimes V_\pi \quad \text{and} \quad \bar{\mathcal{D}}_{\mathfrak{r}}(\Sigma_k^+ \otimes V_\pi) \subset \Sigma_{k-1}^+ \otimes V_\pi. \tag{10.56}$$

Hence $\ker D_{\mathfrak{r}}$ is equal to the space of $\mathcal{D}_{\mathfrak{r}}$-harmonic cocycles of the cochain complex

$$\cdots \xrightarrow{\mathcal{D}_{\mathfrak{r}}} \Sigma_{k-1}^+ \otimes V_\pi \xrightarrow{\mathcal{D}_{\mathfrak{r}}} \Sigma_k^+ \otimes V_\pi \xrightarrow{\mathcal{D}_{\mathfrak{r}}} \Sigma_{k+1}^+ \otimes V_\pi \xrightarrow{\mathcal{D}_{\mathfrak{r}}} \cdots \tag{10.57}$$

This shows the first assertion of the theorem and that $\ker D_{\mathfrak{r}}$ is isomorphic to the cohomology of the complex. Moreover, under the above identification $\Sigma^+ = \Lambda(\mathbb{C}^{n-1})$, we have

$$\mathcal{D}_{\mathfrak{r}}(\omega \otimes v) = 2 \sum_{2 \leq j \leq n} (F_j \wedge \omega) \otimes \pi_*(E_{1j}) v. \tag{10.58}$$

Following the notation in [KnVo], we consider the subalgebras $\mathfrak{u}$ and $\mathfrak{l}$ of $\mathfrak{gl}(n, \mathbb{C})$, where $\mathfrak{u}$ is spanned by the E_{1j}, $2 \le j \le n$, and

$$\mathfrak{l} := \left\{ \begin{pmatrix} x & 0 \\ 0 & B \end{pmatrix} : x \in \mathbb{C} \text{ and } B \in \mathfrak{gl}(n-1, \mathbb{C}) \right\}. \tag{10.59}$$

Then $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. By (10.58), the kernel of the restriction of $D_{\mathfrak{r}}$ is isomorphic to $H^k(\mathfrak{u}, \pi)$, the Lie algebra cohomology of $\mathfrak{u}$ with respect to π. Now Kostant's theorem determines the latter as an $\mathfrak{l}$-module, where $l \in \mathfrak{l}$ acts on $\Lambda^k(\mathfrak{u}) \otimes V_\pi$ by

$$-\operatorname{ad}(l)^* \otimes \mathrm{id} + \mathrm{id} \otimes \pi_*(l), \tag{10.60}$$

see (4.138b) in [KnVo]. To apply Kostant's theorem, we introduce

$$\Delta^+(\mathfrak{u}) = \{\rho_1 - \rho_j : 2 \le j \le n\}, \tag{10.61}$$

$$\Delta^+(\mathfrak{l}) = \{\rho_i - \rho_j : 2 \le i < j \le n\}. \tag{10.62}$$

For $w \in \mathcal{W}$, we also introduce

$$\Delta^+(w) := \{\lambda \in \Delta^+ : w^{-1}\lambda < 0\}, \quad \ell(w) := |\Delta^+(w)|, \tag{10.63}$$

and

$$\mathcal{W}^1 := \{w \in \mathcal{W} : \Delta^+(w) \subseteq \Delta^+(\mathfrak{u})\}. \tag{10.64}$$

Then $\mathcal{W}^1 = \{w_0, \dots, w_{n-1}\}$, where $w_0 = \mathrm{id}$ and

$$w_k^{-1} = \begin{pmatrix} 1 & 2 & \cdots & k+1 \\ k+1 & 1 & \cdots & k \end{pmatrix}, \tag{10.65}$$

for $1 \le k \le n-1$. We note that $\ell(w_k) = k$, for $0 \le k \le n-1$.

Kostant's theorem implies that, as an $\mathfrak{l}$-module, $H^k(\mathfrak{u}, \pi)$ is the irreducible representation of $\mathfrak{l}$ with highest weight

$$w_k(\lambda + \delta) - \delta = (\lambda_{k+1} - k)\rho_1 + \sum_{2 \le j \le k+1} (\lambda_{j-1} + 1)\rho_j + \sum_{j > k+1} \lambda_j \rho_j, \tag{10.66}$$

where δ is the half sum of the positive roots of $\mathfrak{gl}(n, \mathbb{C})$,

$$\delta := \frac{1}{2} \sum_{j=1}^{n} (n + 1 - 2j)\rho_j. \tag{10.67}$$

Moreover, the action of the $\mathfrak{k}$-component $Z^{\mathfrak{k}} \simeq -iE_{1,1}$ of Z on $H^k(\mathfrak{u}, \pi)$ is given by multiplication with

$$ik - i\lambda_{k+1} = ik + \mathrm{id} \otimes \pi(Z^{\mathfrak{k}}). \tag{10.68}$$

In particular, $\mathrm{id} \otimes \pi(Z^{\mathfrak{k}}) = -i\lambda_{k+1}$. It follows that, on $H^k(\mathfrak{u}, \pi)$,

$$D_{\mathfrak{z}} = D_Z = (-1)^k \frac{1}{2}(2k - 2\lambda_{k+1} - n + 1), \tag{10.69}$$

which is the third assertion of the theorem. We have

$$b_k = \dim H^k(\mathfrak{u}, \pi) = \prod_{\alpha \in \Delta^+(\mathfrak{l})} \frac{(\alpha, w_k(\lambda + \delta) - \delta + \delta_{\mathfrak{l}})}{(\alpha, \delta_{\mathfrak{l}})}, \tag{10.70}$$

by Weyl's dimension formula, where $\delta_{\mathfrak{l}}$ is the half sum of the positive roots of $\mathfrak{l}$,

$$\delta_{\mathfrak{l}} = \frac{1}{2} \sum_{j=2}^{n} (n + 2 - 2j)\rho_j. \tag{10.71}$$

The second assertion of the theorem is an easy consequence. □

We recall that $A_t^{\mathrm{le},+}$ corresponds to the operator $-D_N^{\mathrm{le}}$ considered above, see (3.45) and (10.4).

Theorem 10.72. *D is a Fredholm operator if and only if*

$$2\lambda_{k+1} \neq 2k + 1 - n, \quad \textit{for all } 0 \leq k \leq n - 1.$$

Furthermore,

$$\dim \ker A_t^{\mathrm{le},+} = \sum b_k,$$
$$\eta^{\mathrm{le}}(A_t^+) = \eta(A_t^{\mathrm{le},+}) = \sum (-1)^k b_k \,\mathrm{sign}(n - 1 - 2k + 2\lambda_{k+1}),$$

where the first sum is over all $0 \leq k \leq n-1$ with $2\lambda_{k+1} = 2k + 1 - n$ and the second sum is over the remaining k. □

10.3. **Examples.** Before going into examples, we note that

$$\sum_k (-1)^k \dim H^k = \sum_k (-1)^k \dim \left(\Lambda^k(\mathfrak{u}^*) \otimes V_\pi\right) = 0, \tag{10.73}$$

a formula which is not a priori evident from the explicit formula for the dimensions of the H^k.

1) Dirac operator on spinors: In this case, π is the irreducible representation with highest weight $\lambda = 0$ (where the spin structure along the cusps is trivial). If n is even, D is a Fredholm operator, that is, $\ker A_t^{\mathrm{le},+} = 0$. Moreover, each cusp contributes a low energy

η-invariant,

$$\begin{aligned} \eta^{\mathrm{le}}(A_t^+) &= \sum_{0\le k\le n-1} (-1)^k \tbinom{n-1}{k} \operatorname{sign}(n-1-2k) \\ &= 2\sum_{0\le 2k\le n-2} (-1)^k \tbinom{n-1}{k} \\ &= 2\sum_{0\le 2k\le n-2} (-1)^k \left(\tbinom{n-2}{k} + \tbinom{n-2}{k-1}\right) \\ &= 2(-1)^{\frac{n-2}{2}} \tbinom{n-2}{\frac{n-2}{2}}. \end{aligned} \tag{10.74}$$

If n is odd, the low energy eta invariant of A_t^+ vanishes. Furthermore, D is not a Fredholm operator and each cusp contributes to the kernel,

$$\dim\ker A_t^{\mathrm{le},+} = \tbinom{n-1}{\frac{n-1}{2}}. \tag{10.75}$$

2) Dolbeault operator on forms of bi-degree $(0,q)$: In this case, π is the irreducible representation with highest weight $\lambda_j = (n+1)/2$, $1\le j\le n$. We compute

$$b_k = \dim H^k(\pi) = \tbinom{n-1}{k} \tag{10.76}$$

and

$$D_Z|_{H^k(\pi)} = (-1)^k(k-n). \tag{10.77}$$

In particular, D is a Fredholm operator and $\eta^{\mathrm{le}}(A_t^+) = 0$.

3) Signature operator: In this case, π is the spin representation $\Sigma = \Sigma_{\mathfrak{p}}$, which is the sum of the irreducible representations V_l with highest weight as in (10.42), where $0\le l\le n$. As for the dimension $b_{k,l}$ of $H^k(\mathfrak{u}, V_l)$, there are two cases:

$$b_{k,l} = \begin{cases} \tbinom{n}{l}\tbinom{n}{k}\frac{l-k}{n} & \text{if } k<l, \\ \tbinom{n}{l}\tbinom{n}{k+1}\frac{k+1-l}{n} & \text{if } k\ge l. \end{cases} \tag{10.78}$$

Furthermore, we have

$$D_Z|_{H^k(V_l)} = \begin{cases} (-1)^k(k-l) & \text{if } k<l, \\ (-1)^k(k+1-l) & \text{if } k\ge l. \end{cases} \tag{10.79}$$

Hence

$$\eta^{\mathrm{le}}(A_t^+) = \sum_{k<l} (-1)^k \tbinom{n}{l}\tbinom{n}{k}\frac{l-k}{n} + \sum_{k\ge l} (-1)^{k+1} \tbinom{n}{l}\tbinom{n}{k+1}\frac{k+1-l}{n}. \tag{10.80}$$

If we change l in $n-l$ and k in $n-1-k$ in the second sum, we obtain

$$\eta^{\mathrm{le}}(A_t^+) = 0 \quad \text{if } n \text{ is odd}. \tag{10.81}$$

For even n, we get

$$\eta^{\mathrm{le}}(A_t^+) = 2\sum_{k<l}(-1)^k \tbinom{n}{l}\tbinom{n}{k}\tfrac{l-k}{n}. \tag{10.82}$$

For $1 \leq l \leq n$, we have

$$\sum_{0\leq k<l}(-1)^k\tbinom{n}{k} = (-1)^{l-1}\tbinom{n-1}{l-1}$$

and

$$\sum_{0\leq k<l}(-1)^k\tbinom{n}{k}\tfrac{k}{n} = \sum_{0\leq k<l-1}(-1)^{k+1}\tbinom{n-1}{k} = (-1)^{l-1}\tbinom{n-2}{l-2}.$$

Hence

$$\begin{aligned}\eta^{\mathrm{le}}(A_t^+) &= 2\sum_{1\leq l\leq n}(-1)^{l-1}\tbinom{n}{l}\left\{\tbinom{n-1}{l-1}\tfrac{l}{n} - \tbinom{n-2}{l-2}\right\}\\ &= 2\sum_{1\leq l\leq n}(-1)^{l-1}\tbinom{n-1}{l-1}^2 + 2\textstyle\sum_{1\leq l\leq n}(-1)^l\tbinom{n}{l}\tbinom{n-2}{l-2}\end{aligned}$$

The first sum is zero since n is even. The second sum is the coefficient of x^n in $(1-x)^n(1+x)^{n-2} = (1-x)^2(1-x^2)^{n-2}$ and hence

$$\eta^{\mathrm{le}}(A_t^+) = 2(-1)^{n/2}\left(\tbinom{n-2}{n/2} - \tbinom{n-2}{n/2-1}\right). \tag{10.83}$$

Appendix A. Lattices in Heisenberg groups

In this appendix we discuss lattices in the standard Heisenberg groups G_n of $g = (x, y, z)$ with $x, y \in \mathbb{R}^n$, $z \in \mathbb{R}$, and multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy'). \tag{A.1}$$

The left-invariant vector fields

$$X_j := \frac{\partial}{\partial x_j}, \quad Y_j := \frac{\partial}{\partial y_j} + x_j \frac{\partial}{\partial z}, \quad \text{and} \quad Z := \frac{\partial}{\partial z} \tag{A.2}$$

form a basis of the Lie algebra of G_n. They commute pairwise, except for the n Lie brackets $[X_j, Y_j] = Z$.

Lattices in G_n are classified in [GoWi, Section 2]: Let D_n be the set of n-tupels $d = (d_1, \dots, d_n)$ of natural numbers such that d_i divides d_{i+1}, $1 \le i < n$. Then, for any $d \in D_n$,

$$\Gamma_d := \{(x, y, z) \mid x, y \in \mathbb{Z}^n, z \in \mathbb{Z}, d_i \text{ divides } x_i\} \tag{A.3}$$

is a lattice in G_n. The isomorphism type of Γ_d is determined by d and, up automorphism of G_n, any lattice in G_n is equal to some Γ_d, $d \in D_n$.

Fix $d \in D_n$. The $2n + 1$ elements

$$\phi_j := (d_j e_j, 0, 0), \quad \psi_j := (0, e_j, 0), \quad \zeta := (0, 0, 1) \tag{A.4}$$

generate Γ_d. They commute pairwise, except for the n relations

$$\phi_j \psi_j \phi_j^{-1} \psi_j^{-1} = \zeta^{d_j} = (0, 0, d_j). \tag{A.5}$$

Let τ be an irreducible unitary representation of Γ_d on a finite dimensional Hermitian vector space V. Since τ is irreducible and ζ is central, there is a number $c \in [0, 1)$ with

$$\tau(\zeta) = e^{2\pi i c} I. \tag{A.6}$$

Let $A_j := \tau(\phi_j)$ and $B_j := \tau(\psi_j)$, for $1 \le j \le n$. Then, if λ is an eigenvalue of B_j, for some j and some eigenvector $v \in V$, then

$$B_j(A_j v) = e^{-2\pi i c d_j}(A_j B_j A_j^{-1})(A_j v) = e^{-2\pi i c d_j} \lambda A_j v, \tag{A.7}$$

and hence $e^{-2\pi i c d_j} \lambda$ is an eigenvalue of B_j as well. It follows that c is rational, by the finite dimensionality of V.

Let m_j be the denominator of $c d_j$. Consider the sublattice $\Gamma_{md} \subseteq \Gamma_d$, where $md := (m_1 d_1, ..., m_n d_n)$. Then

$$|\Gamma_d / \Gamma_{md}| = m_1 \cdots m_n$$

and τ restricts to an Abelian representation on Γ_{md}. By irreducibility, τ is induced from a one-dimensional representation of Γ_{md}. That is, there are real numbers $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ such that $\phi_j^{m_j}$ and ψ_j act on $\mathbb{C}$

by multiplication with $e^{2\pi i\alpha_j}$ and $e^{2\pi i\beta_j}$, respectively, and τ is induced from this representation of Γ_{md}. In particular,

$$\dim V = m_1 \cdots m_n. \tag{A.8}$$

For any n-tuple

$$b = (b_1, \ldots, b_n) = (\beta_1 + l_1 cd_1, \ldots, \beta_n + l_n cd_n) \in \mathbb{R}^n/\mathbb{Z}^n, \tag{A.9}$$

where $(l_1, \ldots, l_n) \in \mathbb{Z}^n$, we let V_b be the subspace of V on which ψ_j acts by $e^{2\pi i b_j}$. We note that these subspaces V_b are one-dimensional and pairwise orthogonal and that they span V.

A.1. **Twisted Right Regular Representation.** The set

$$F := \{(x, y, z) \in G \mid x \in P, (y, z) \in Q\}, \tag{A.10}$$

where

$$\begin{aligned} P &:= \{x \in \mathbb{R}^n \mid 0 \le x_j \le d_j\}, \\ Q &:= \{(y, z) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \le y_j, z \le 1\}, \end{aligned} \tag{A.11}$$

is a fundamental domain of the action of Γ_d on G_n by left translations. Observe that, by (A.2), the standard Lebesgue measure with respect to the (x, y, z)-coordinates is left-invariant, hence bi-invariant, on G_n.

Fix an irreducible unitary representation τ of Γ_d on a finite dimensional Hermitian vector space V as above. and consider the Hilbert space $L^2(\tau)$ of maps $\sigma : G_n \to V$ such that

$$\sigma(\gamma g) = \tau(\gamma)\sigma(g) \tag{A.12}$$

for all $\gamma \in \Gamma_d$ and $g \in G_n$ which are square integrable over F. The *right regular representation* ρ of G_n acts unitarily on $L^2(\tau)$ by

$$(\rho(g)\sigma)(x, y, z) = \sigma((x, y, z)g), \tag{A.13}$$

and our next aim is to determine the multiplicities of the irreducible unitary representations of G_n in $L^2(\tau)$. Here we recall that irreducible unitary representations of the Heisenberg group G_n correspond to coadjoint orbits of G_n, by the classical theorem of Stone and von Neumann (or by the more general Kirillov theory, respectively). This correspondence will show up in the following discussion.

Let $\sigma \in L^2(\tau)$. Then

$$e^{2\pi i c}\sigma(x, y, z) = \tau(\zeta)\sigma(x, y, z) = \sigma(x, y, z + 1). \tag{A.14}$$

Let σ_b be the component of σ in V_b. Then

$$e^{2\pi i b_j}\sigma_b(x, y, z) = B_j\sigma_b(x, y, z) = \sigma_b(x, y + e_j, z). \tag{A.15}$$

The transformation rule with respect to A_j is more complicated,

$$A_j\sigma_b(x, y, z) = \sigma_{b - cd_j e_j}(x + d_j e_j, y, z + d_j y_j). \tag{A.16}$$

By (A.14) and (A.15), we can develop σ_b in a Fourier series,

$$\sigma_b(x,y,z) = \sum_{\substack{v\equiv b\\ w\equiv c}} e^{2\pi i(vy+wz)}\sigma_{v,w}(x) \tag{A.17}$$

where $\equiv$ indicates congruence modulo $\mathbb{Z}^n$. Fix a w congruent to c and consider the space $L^2(\tau, w)$ of $\sigma \in L^2(\tau)$ with

$$\sigma(x,y,z+t) = e^{2\pi i wt}\sigma(x,y,z), \tag{A.18}$$

that is, in the above Fourier development of the components σ_b of σ, only the terms with the given w occur. We obtain an orthogonal decomposition

$$L^2(\tau) = \oplus_{w\equiv c} L^2(\tau, w). \tag{A.19}$$

Now the spaces $L^2(\tau, w)$ are ρ-invariant and, therefore, it remains to investigate ρ on them. For $\sigma \in L^2(\tau, w)$, we have

$$\begin{aligned}\sum_{u\equiv b+cd_je_j} e^{2\pi i(uy+wz)}A_j\sigma_{u,w}(x) &= A_j\sigma_{b+cd_je_j}(x,y,z)\\ &= \sigma_b(x+d_je_j, y, z+d_jy_j)\\ &= e^{2\pi i wd_jy_j}\sigma_b(x+d_je_j,y,z)\\ &= \sum_{v\equiv b} e^{2\pi i((v+wd_je_j)y+wz)}\sigma_{v,w}(x+d_je_j).\end{aligned} \tag{A.20}$$

We conclude that, for any $v \equiv b$ and $x \in \mathbb{R}^n$,

$$\sigma_{v+wd_je_j,w}(x) = A_j^{-1}\sigma_{v,w}(x+d_je_j). \tag{A.21}$$

There are two cases, $w = 0$ and $w \neq 0$, respectively.

If $w = 0$, then $w = c = 0$ and $\dim V = 1$. By (A.21), the Fourier coefficients $\sigma_{v,0}$ are d_j-periodic in x_j up to the twists by the complex numbers A_j of norm one.

Suppose now that $w \neq 0$. Then, by (A.21), the Fourier coefficients $\sigma_{u,w}$ with

$$u = b + k_1e_1 + \cdots + k_ne_n, \quad 0 \le k_j < |w|d_j, \tag{A.22}$$

determine all the Fourier coefficients of σ. We also get

$$||\sigma||^2_{L^2(\tau,w)} = \sum ||\sigma_{u,w}||^2_{L^2(\mathbb{R}^n,V_b)}, \tag{A.23}$$

where the sum is over all u as in (A.22). Here we recall that, on the left hand side, the L^2-norms are given by the corresponding integrals over the fundamental domain F of Γ_d as in (A.10), whereas the integrals on the right hand side are over Euclidean x-space. We obtain

$$L^2(\tau, w) \cong \oplus L^2(\mathbb{R}^n, V_b), \tag{A.24}$$

where we have $m_1 d_1 \cdots m_n d_n |w|^n$ summands $L^2(\mathbb{R}^n, V_b)$ on the right hand side, namely one for each u as in (A.22).

To identify ρ on $\oplus L^2(\mathbb{R}^n, V_b) \cong L^2(\tau, w)$, let $g = (x', y', z') \in G_n$ and recall (A.1) and (A.13). We compute

$$\text{(A.25)} \quad e^{2\pi i(u(y+y')+w(z+z'+xy'))}\sigma_{u,w}(x+x') = e^{2\pi i(uy+wz)}e^{2\pi i(uy'+w(z'+xy'))}\sigma_{u,w}(x+x'),$$

hence g acts on $\sigma_{u,w} \in L^2(\mathbb{R}^n, V_b)$ by

$$\text{(A.26)} \qquad (\rho(g)\sigma_{u,w})(x) = e^{2\pi i(uy'+w(z'+xy'))}\sigma_{u,w}(x+x').$$

Via a unitary identification $V_b \cong \mathbb{C}$ and the substitution $x + u/c$ for x, we see that ρ on $L^2(\mathbb{R}^n, V_b)$ is unitarily equivalent to the irreducible unitary representation ρ_w of G_n on $L^2(\mathbb{R}^n, \mathbb{C})$ with

$$\text{(A.27)} \qquad (\rho_w(g)f)(x) = e^{2\pi i w(z+xy')}f(x+x').$$

This is the standard representation of G_n associated to the coadjoint orbit of linear functionals on the Lie algebra of G_n which send Z to w. Hence $L^2(\tau, w)$ is a corresponding isotypical component of ρ_w in $L^2(\tau)$. By (A.22) and (A.23), the multiplicity of ρ_w in $L^2(\tau)$ and $L^2(\tau, w)$ is

$$\text{(A.28)} \qquad m_1|w|d_1 \cdots m_n|w|d_n = |\Gamma| \dim V |w|^n.$$

A.2. **Spectrum of Twisted Laplacians.** Let $w \neq 0$. To determine the spectrum of the Laplacian Δ_w of a given left-invariant Riemannian metric on $L^2(\mathbb{R}^n, \mathbb{C})$ with respect to the representation ρ_w as in (A.27), we follow the discussion in the proof of Theorem 3.3 of [GoWi]: With respect to the given metric, there is an orthonormal basis

$$\text{(A.29)} \qquad X'_1, \dots, X'_n, Y'_1, \dots Y'_n, Z'$$

of the Lie algebra of G_n with $Z' = rZ$, $r = 1/|Z| > 0$, such that

$$\text{(A.30)} \qquad [X'_j, X'_k] = [X'_j, Y'_k] = [Y'_j, Y'_k] = 0$$

for all $j \neq k$ and such that there are numbers $r_j > 0$ with

$$\text{(A.31)} \qquad [X'_j, Y'_j] = r_j^2 Z.$$

The pull back of the metric under the automorphism Φ of G_n with

$$\text{(A.32)} \qquad \Phi_*(r_j X_j) = X'_j, \quad \Phi_*(r_j Y_j) = Y'_j, \quad \Phi_*(Z) = Z$$

is the left-invariant Riemannian metric on G_n for which the fields

$$\text{(A.33)} \qquad r_1 X_1, \dots, r_n X_n, r_1 Y_1, \dots, r_n Y_n, rZ$$

are orthonormal. Since $\Phi_*(Z) = Z$, $\rho_w \circ \Phi$ is still an irreducible unitary representation of G_n associated to the coadjoint orbit of linear functions on the Lie algebra of G_n which send Z to w, hence $\rho_w \circ \Phi$ is unitarily equivalent to ρ_w. In other words, we can assume without loss of generality that the given left-invariant Riemannian metric on G_n has an orthonormal basis as in (A.33). As for the Laplacian on $L^2(\mathbb{R}^n, \mathbb{C})$ with respect to ρ_w, we obtain

$$\Delta_w = - \sum_{1 \le j \le n} r_j^2 \frac{\partial^2}{\partial x_j^2} + 4\pi^2 w^2 \big(r^2 + \sum_{1 \le j \le n} x_j^2 r_j^2 \big), \tag{A.34}$$

by (A.27). Now the Hermite functions

$$h_p(x) = \exp(x^2/2) \frac{\partial^{p_1 + \cdots + p_n}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}} \exp(-x^2), \tag{A.35}$$

where $p = (p_1, \dots, p_n)$ runs over all n-tuples of non-negative integers, form an orthogonal basis of $L^2(\mathbb{R}^n, \mathbb{C})$ and satisfy

$$x_j^2 h_p - \frac{\partial^2 h_p}{\partial x_j^2} = (2p_j + 1) h_p. \tag{A.36}$$

It follows that the functions $f_p(x) = h_p(\sqrt{2\pi |w|} x)$ are an orthogonal basis of $L^2(\mathbb{R}^n, \mathbb{C})$ and that they satisfy

$$\Delta_w f_p = \lambda(w, p) f_p, \tag{A.37}$$

where

$$\lambda(w, p) := 4\pi^2 w^2 r^2 + 2\pi |w| \sum_{1 \le j \le n} (2p_j + 1) r_j^2. \tag{A.38}$$

Thus, by (A.28), the multiplicity of $\lambda(w, p)$ in $L^2(\tau, w)$ is equal to

$$d_1 \cdots d_n m_1 \cdots m_n |w|^n = |\Gamma| \dim V |w|^n, \tag{A.39}$$

when counted according to the n-tuples p.

In our application of the above in the proof of Theorem 9.7, we will vary the parameter $r = 1/|Z|$ of the metric, keeping

$$r_1 X_1, \dots, r_n X_n, r_1 Y_1, \dots, r_n Y_n$$

orthonormal and perpendicular to Z. Then the above functions f_p remain eigenfunctions of Δ_w in $L^2(\mathbb{R}^n, \mathbb{C})$ and the corresponding eigenvalues vary according to (A.38). Hence the eigensections in $L^2(\tau, w)$ corresponding to the above eigenfunctions f_p remain the same during this variation of the metric and the corresponding eigenvalues vary according to (A.38) as well.

References

[Ag] S. Agmon, *Lectures on elliptic boundary value problems.* Van Nostrand Company, Princeton-Toronto-London, 1965.

[An] N. Anghel, An abstract index theorem on non-compact Riemannian manifolds. *Houston J. Math.* **19** (2) (1993), 223–237.

[APS1] M.F. Atiyah, V.K. Patodi, & I.M. Singer, Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Phil. Soc.* **77** (1975), 43–69.

[APS3] M.F. Atiyah, V.K. Patodi, & I.M. Singer, Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Phil. Soc.* **79** (1976), 71–99.

[Bä] C. Bär, The Dirac operator on hyperbolic manifolds of finite volume. *J. Differential Geom.* **54** (2000), no. 3, 439–488.

[Ba] W. Ballmann, *Lectures on spaces of nonpositive curvature.* With an appendix by Misha Brin. DMV Seminar, 25. Birkhäuser Verlag, Basel, 1995. viii+112 pp.

[BBB] W. Ballmann, M. Brin, & K. Burns, On the differentiability of horocycles and horocycle foliations. *J. Differential Geom.* **26** (1987), no. 2, 337–347.

[BB1] W. Ballmann & J. Brüning, On the spectral theory of surfaces with cusps. In: *Geometric analysis and partial differential equations*, 13–37, Springer, Berlin, 2003.

[BB2] W. Ballmann & J. Brüning, On the spectral theory of manifolds with cusps. *J. Math. Pures Appl.* **80** (2001), 593–625.

[BBC1] W. Ballmann, J. Brüning, & G. Carron, Eigenvalues and holonomy. *Int. Math. Res. Notes* **2003**, 657–665.

[BBC2] W. Ballmann, J. Brüning, & G. Carron, Regularity and index theory for Dirac-Schrödinger systems with Lipschitz coefficients. *J. Math. Pures Appl.* (9) **89** (2008), no. 5, 429–476.

[BGS] W. Ballmann, M. Gromov, & V. Schroeder, *Manifolds of nonpositive curvature.* Progress in Mathematics, 61. Birkhäuser Boston, Inc., Boston, MA, 1985. vi+263 pp.

[BaMo] D. Barbasch & H. Moscovici, L^2-index and the Selberg trace formula. *J. Funct. Anal.* **53** (1983), no. 2, 151–201.

[BeKa] I. Belegradek & V. Kapovitch, Classification of negatively pinched manifolds with amenable fundamental groups. *Acta Math.* **196** (2006), no. 2, 229–260.

[BoCa] A. Borel & W. Casselman, L^2-cohomology of locally symmetric manifolds of finite volume. *Duke Math. J.* **50** (1983), 625–647.

[Bow] B.H. Bowditch, Discrete parabolic groups. *J. Differential Geom.* **38** (1993), no. 3, 559–583.

[BrKa] M. Brin & H. Karcher, Frame flows on manifolds with pinched negative curvature. *Compositio Math.* **52** (1984), no. 3, 275–297.

[BuKa] P. Buser & H. Karcher, *Gromov's almost flat manifolds.* Astérisque, 81. Société Math. de France, Paris, 1981. 148 pp.

[CamP] R. Camporesi & E. Pedon, The continuous spectrum of the Dirac operator on noncompact Riemannian symmetric spaces of rank one. *Proc. Amer. Math. Soc.* **130** (2002), no. 2, 507–516

[Ca1] G. Carron, Un théorème de l'indice relatif. *Pacific J. Math.* **198** (2001), 81–107.

[Ca2] G. Carron, Théorèmes de l'indice sur les variétés non-compactes. *J. Reine Angew. Math.* **541** (2001), 81–115.

[ChGr] J. Cheeger, Jeff & M. Gromov, Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds. *J. Differential Geom.* **21** (1985), no. 1, 1–34.

[DeSi] C. Deninger & W. Singhof, The e-invariant and the spectrum of the Laplacian for compact nilmanifolds covered by Heisenberg groups. *Invent. Math.* **78** (1984), no. 1, 101–112.

[Do] H. Donnelly, Essential spectrum and heat kernel. *J. Funct. Anal.* **75** (1987), no. 2, 362381.

[Eb] P. Eberlein, Lattices in spaces of nonpositive curvature. *Ann. of Math.* (2) **111** (1980), no. 3, 435–476.

[GiTu] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of second order.* Second edition. Grundlehren der Mathematischen Wissenschaften 224. Springer-Verlag, Berlin, 1983. xiii+513 pp.

[Gil1] P.B. Gilkey, On the index of geometrical operators for Riemannian manifolds with boundary. *Adv. Math.* **102** (1993), no. 2, 129–183.

[Gil2] P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995. x+516 pp.

[GoWi] C.S. Gordon & E.N. Wilson, The spectrum of the Laplacian on Riemannian Heisenberg manifolds. *Michigan Math. J.* **33** (1986), no. 2, 253–271.

[Gr] M. Gromov, Almost flat manifolds. *J. Differential Geom.* **13** (1978), no. 2, 231–241.

[GrLa] M. Gromov & H.B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.* **58** (1983), 83–196.

[GKR] K. Grove, H. Karcher, & E.A. Ruh, Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems. *Math. Ann.* **211** (1974), 7–21.

[Ha] G. Harder, A Gauss-Bonnet formula for discrete arithmetically defined groups. *Ann. Sci. École Norm. Sup.* (4) **4** (1971), 409–455.

[HeIH] E. Heintze & H. C. Im Hof, Geometry of horospheres. *J. Differential Geom.* **12** (1977), 481–491.

[Hi] M. Hilsum, L'invariant η pour les variétés lipschitziennes. *J. Differential Geom.* **55** (2000), no. 1, 1–41.

[KnVo] A. Knapp and D. Vogan, *Cohomological induction and unitary representations.* Princeton Mathematical Series, 45. Princeton University Press, Princeton, NJ, 1995.

[Ko] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math.* **74** (1961) 329–387

[LaMi] H.B. Lawson Jr. & M.-L. Michelsohn, *Spin geometry.* Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989. xii+427 pp.

[LiYa] P. Li & S.T. Yau, Eigenvalues of a Compact Riemannian Manifold. *Proc. Symp. Pure Math. 36* (1980), 205–239.

[Lo1] J. Lott, On the spectrum of a finite-volume negatively-curved manifold. *Amer. J. of Math.* **123** (2001) 185–205.

[Lo2] J. Lott, Collapsing and Dirac-type operators. *Geom. Ded.* **91**, (2002), 175-196.

[Mi] M. Mitrea, Generalized Dirac operators on nonsmooth manifolds and Maxwell's equations. *J. Fourier Anal. Appl.* **7** (2001), no. 3, 207–256.

[Mü1] W. Müller, L^2-index and resonances. In *Geometry and Analysis on manifolds*, Katata/Kyoto (1987), Lectures Notes in Math. 1339, 203–211.

[Mü2] W. Müller, *Manifolds with cusps of rank one, spectral theory and L^2-index theorem*, Lecture Notes in Math. 1244, Springer-Verlag, Berlin [e.a.], 1987.

[Pa] R. Parthasarathy, Dirac operator and the discrete series, *Ann. of Math.* **96** (1972), 1–30.

[Ru] E. Ruh, Almost flat manifolds. *J. Differential Geom.* **17** (1982), 1–14.

[St] M. Stern, Eta invariants and Hermitian locally symmetric spaces. *J. Differential Geom.* **31** (1990), no. 3, 771–789.

[Ta] M. Taylor, *Partial Differential Equations I. Basic Theory.* Appl. Math. Sciences 115, Springer, Berlin etc. 1996.

[Wo] J.A. Wolf, Essential self-adjointness for the Dirac operator and its square. *Indiana Univ. Math. J.* **22** (1972/73), 611–640.

[Wu] H.H. Wu, *The Bochner technique in differential geometry.* Math. Rep. 3 (1988), no. 2, i–xii and 289–538.

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn and Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Deutschland,

E-mail address: `hwbllmnn@math.uni-bonn.de`

Institut für Mathematik, Humboldt–Universität, Rudower Chaussee 5, 12489 Berlin, Germany,

E-mail address: `bruening@mathematik.hu-berlin.de`

Département de Mathématiques, Université de Nantes, 2 rue de la Houssiniére, BP 92208, 44322 Nantes Cedex 03, France,

E-mail address: `Gilles.Carron@math.univ-nantes.fr`