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A law of motion for spherical shells in special relativity

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Abstract

Self-similar solutions to the problem of a special relativistic law of motion for thin shells of matter are calculated. These solutions represent the special relativistic generalization of momentum conservation for the thin layer approximation in classical physics. The analytical and numerical results are applied to Supernova Remnant 1987 A.

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1 Introduction

The study of the relativistic dynamics of thin shells of matter is a current subject of investigation when the framework of general relativity is adopted [1, 2, 3, 4]. Here, we will explore how the framework of special relativity can produce a law of motion which can be compared with expansion data of a supernova remnant (SNR). From a classical point of view, the temporal evolution of the SNR is modelled by adopting different physical approaches that are not invariant under relativistic transformations. As an example, a classical analytical solution of the Sedov type [5, 6, 7] is equation (10.27) in [8]

$$R(t) = (\frac{25Et^2}{4\pi\rho})^{1/5} \quad , \tag{1}$$

where ρ is the density of the surrounding medium which is supposed to be constant, E is the energy of the explosion and t is the age of the SNR. In this paper, we show that it is possible to deduce the relativistic law of motion starting from the conservation of relativistic momentum. The relativistic equation of a SNR is deduced as a non-linear relationship between radius and time which can be solved numerically. The self-similar relativistic solutions are accurate for sufficiently large time after the formation of the SNR. The theory is applied to the SNR connected with Supernova (SN) 1987A.

2 Classical and relativistic laws of motion

The thin layer approximation in classical physics assumes that all the swept-up gas accumulates with infinite density in a thin shell just after the shock front. The conservation of radial momentum requires that, after the initial radius R_0,

$$\frac{4}{3}\pi R^3\rho V = \frac{4}{3}\pi R_0^3\rho V_0 \quad , \tag{2}$$

where R and V are the radius and velocity of the advancing shock wave, ρ is the density of the ambient medium and V_0 is the initial velocity, see [9, 10]. In classical physics, the velocity as a function of radius is:

$$V = V_0(\frac{R_0}{R})^3 \quad , \tag{3}$$

the law of motion is:

$$R = R_0\left(1 + 4\frac{V_0}{R_0}(t - t_0)\right)^{\frac{1}{4}} \quad , \tag{4}$$

where t is time and t_0 is the initial time. In classical physics, the velocity as a function of time is:

$$V = V_0\left(1 + 4\frac{V_0}{R_0}(t - t_0)\right)^{-\frac{3}{4}} \quad . \tag{5}$$

Equation (3) can also be solved with a similar solution of type $R = K(t-t_0)^\alpha$, k being a constant, and the classical result is:

$$R = \sqrt[4]{4}\sqrt[4]{\beta_0 {R_0}^3 c}(t - t_0)^{\frac{1}{4}} \quad , \tag{6}$$

where $\beta_0 = \frac{V_0}{c}$ has been introduced in order to make a comparison with the relativistic case.

Newton's law in special relativity is:

$$F = \frac{dp}{dt} = \frac{d}{dt}(mV) \quad , \tag{7}$$

where F is the force, p the relativistic momentum, m the relativistic mass, m_0 the mass at rest and V the velocity, see equation (7.16) in [11]. In the case of the relativistic expansion of a shell in which all the swept material resides at two different points, denoted by radius R and R_0, we have:

$$\frac{\rho\frac{4}{3}\pi R^3\beta}{\sqrt{1-\beta^2}} = \frac{\rho\frac{4}{3}\pi R_0^3\beta_0}{\sqrt{1-\beta_0^2}} \quad , \tag{8}$$

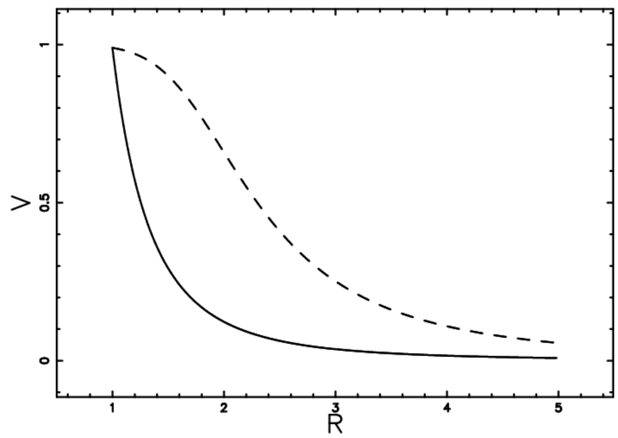

Figure 1: Velocity as a function of radius when $R_0 = 1$, $c = 1$, $v_0/c = 0.99$. (relativistic case, equation (9)) (dashed-line) and (classical case, equation (3)) (full-line)

where β_0=V_0/c, β=V/c and c is the velocity of light. The velocity of the relativistic expanding shell is:

$$\beta = \frac{{R_0}^3 \beta_0}{\sqrt{{R_0}^6 {\beta_0}^2 + R^6 - R^6 {\beta_0}^2}} \quad . \tag{9}$$

Figure 1 shows the classical and relativistic behaviors of the velocity as a function of radius R.

The previous formula can be expressed in differential form as:

$$\sqrt{{R_0}^6 {\beta_0}^2 + R^6 - R^6 {\beta_0}^2}\, dR = {R_0}^3 \beta_0 c dt \quad . \tag{10}$$

The integral on the lhs of the previous equation can be evaluated with a first transformation $\mu = \frac{1}{{R_0}^6 {\beta_0}^2} - \frac{1}{R_0^6}$ and $x = R$,

$$\int \sqrt{1 + \mu\, x^6} dx \quad . \tag{11}$$

A second transformation $y = x^6$ changes the integral into:

$$\int \frac{1}{6} \frac{\sqrt{1 + \mu\, y}}{y^{5/6}} dy \quad . \tag{12}$$

This integral is of the same type as formula 3.194.1 in [12]

$$\int_0^u \frac{x^{\mu-1}}{(1 + \beta\, x)^\nu} dx = \frac{u^\mu \ {}_2F_1(\mu, \nu; 1 + \mu; -\beta\, u)}{\mu} \quad , \tag{13}$$

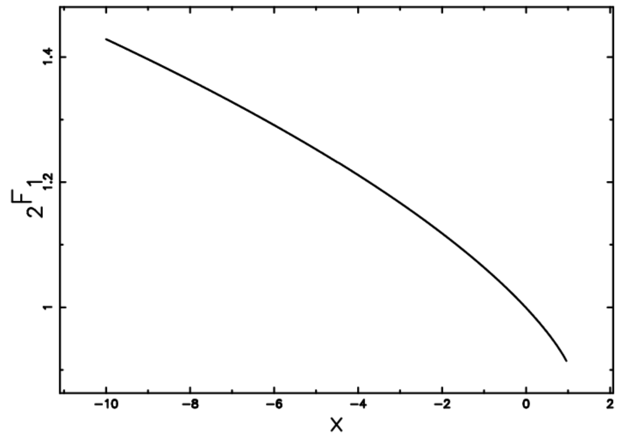

Figure 2: Plot of the hypergeometric function $_2F_1(-1/2, 1/6;\, 7/6;\, x)$ as a function of x (full line) for $-10 \leq x < +1$.

where $_2F_1(a, b;\, c;\, z)$ is a regularized hypergeometric function [13, 14, 15, 12]. In our case, $\nu = \frac{1}{2}$ and $\mu = \frac{1}{6}$. The hypergeometric function is defined by the following power series expansion:

$$_2F_1(a, b;\, c\,; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad , \tag{14}$$

where $(w)_n$ is the Pochhammer symbol

$$(w)_n = w(w+1)\ldots(w+n-1) \quad , \tag{15}$$

(a, b, c) is a triplet of real numbers with c not belonging to the set of negative integers and z is a real number < 1.

We are now ready to integrate formula (10) and the result is a non-linear equation, $\mathcal{F}_{NL}$, in R:

$$\begin{aligned} \mathcal{F}_{NL} = \\ R\ {_2F_1}(-1/2, 1/6;\, 7/6;\, \frac{R^6 \left(-1 + {\beta_0}^2\right)}{{R_0}^6 {\beta_0}^2}) \\ -R_0\ {_2F_1}(-1/2, 1/6;\, 7/6;\, \frac{-1 + {\beta_0}^2}{{\beta_0}^2}) \\ -c(t - t_0){R_0}^3 \beta_0 = 0 \end{aligned} \tag{16}$$

From a numerical point of view, the hypergeometric function can be evaluated with the FORTRAN subroutine HYGFX extracted from [16] and a typical plot of $_2F_1(1/6, 1/2;\, 7/6;\, x)$ is reported in Figure 2.

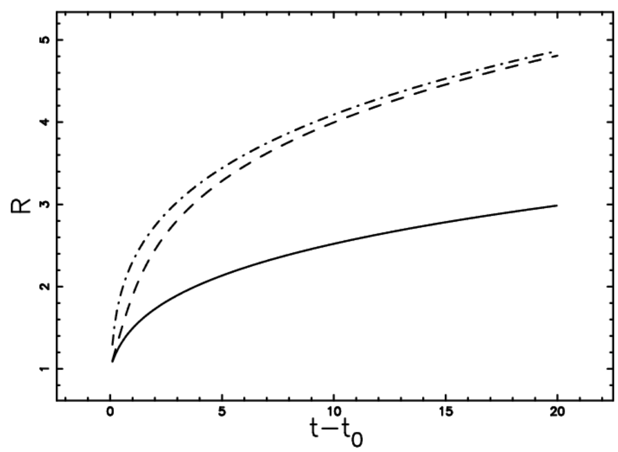

Figure 3: Radius as a function of time when $R_0 = 1$, $\beta_0 = 0.99$, $c = 1$, (asymptotic relativistic case, equation (17)) (dot-dash-dot-dash line), (relativistic case, equation (16)) (dashed-line) and (classical case, equation (4))(full-line)

Once the hypergeometric function is implemented, we can solve the non-linear equation (16) with the FORTRAN subroutine ZRIDDR from [17] and a typical example is shown in Figure (3). Once a numerical expression for the relativistic radius is obtained, we can easily obtain a velocity-time relationship from equation (9), see Figure 4.

An approximate solution of equation 9 can be found by imposing $R(t) = k(t-t_0)^\alpha$. The self-similar solutions of the relativistic case under the assumption $R^6(1-\beta_0^2) \gg R_0^6\beta_0^2$ are:

$$R(t) = \sqrt{2}\,\sqrt[4]{\frac{\beta_0 {R_0}^3 c}{\sqrt{1-{\beta_0}^2}}}\,\sqrt[4]{(t-t_0)} \quad , \tag{17}$$

and

$$\beta(t) = 1/4\,\sqrt{2}\,\sqrt[4]{\frac{\beta_0 {R_0}^3 c}{\sqrt{1-{\beta_0}^2}}}(t-t_0)^{-3/4}c^{-1} \quad . \tag{18}$$

3 Application to SN 1987A

The SN 1987A exploded in the Large Magellanic Cloud in 1987. The distance of this SN is $\approx$ 50 *kpc* (163050 *ly*) and a detailed analysis of the distance, D, gives $D = 51.4$ *kpc* [18] and $D = 50.18$ *kpc* [19]. After 7987 *days*, the diameter of the SNR was $0.77''$ and it's velocity $\approx$ 1412 km/s [20][21]. In this Section

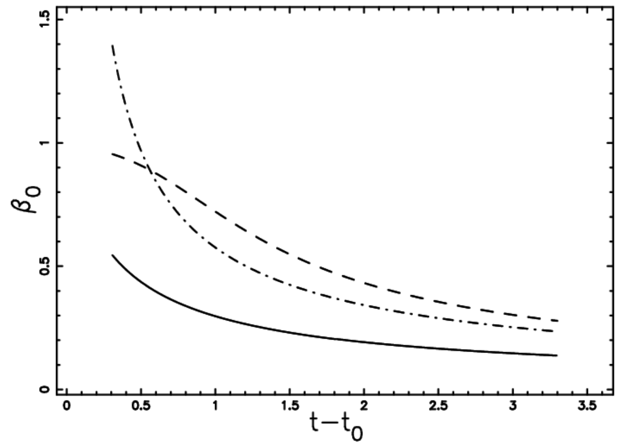

Figure 4: Velocity as a function of time when $R_0 = 1$, $\beta_0 = 0.99$ and $c = 1$. The dashed-line represents the relativistic case, the dot-dash-dot-dash represents the asymptotic relativistic case and the full line the classical case.

we will adopt year (yr) as a time unit and light year (ly) as a length unit; with these natural units $c = 1$. The radius of the SNR after 21.86 yr is

$$R = r_{77} \times D_{50} 0.61 \; ly \quad , \tag{19}$$

where r_{77} is the radius in arcsec divided by 0.77 and D_{50} the distance in pc divided by 50000. Next, we attempt to evaluate the initial conditions R_0 and β_0 after $t = 21.86 \; yr$ and $R = 0.61 \; ly$ are given. The approximate self-similar relativistic solution for the radius as represented by equation (17) allows a relationship to be determined between these two unknown variables,

$$\beta_0 = \frac{R^4}{\sqrt{R^8 + 16 \, {R_0}^6 c^2 t^2}} \quad , \tag{20}$$

and Figure 5 reports such correlated initial conditions. Figure 6 shows the behavior of the radius as a function of time after setting initial conditions given by equation (20) and extracting the data of SNR 1987A from Figure 2 in [21].

A further test can be done by inserting in formula (18) for the self-similar relativistic velocity the boundary conditions used to deduce the trajectory, i.e. $R_0 = 0.079$, $\beta_0 = 0.95$ and $t = 21.86 \; yr$; the theoretical velocity turns out to be 2672 $\frac{km}{s}$ against the observed 1412 $\frac{km}{s}$ [21]. The proper time of the world line is

$$\tau - \tau_0 = \int_{\tau_0}^{\tau} \sqrt{1 - \beta(t)^2} dt \quad . \tag{21}$$

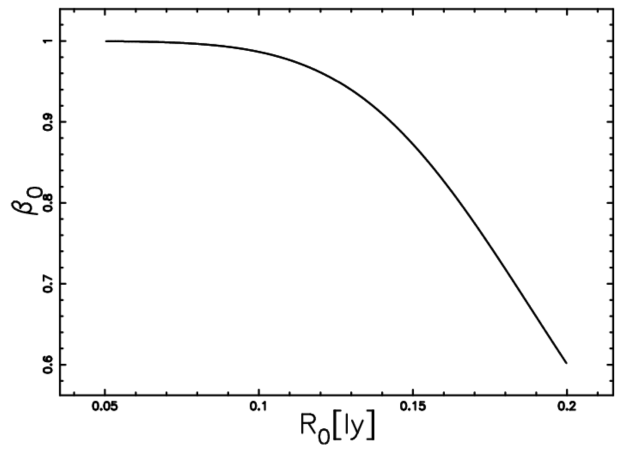

Figure 5: Initial velocity β_0 as a function of initial radius when $R = 0.61$ ly, $t = 21.86$ yr and $c = 1$.

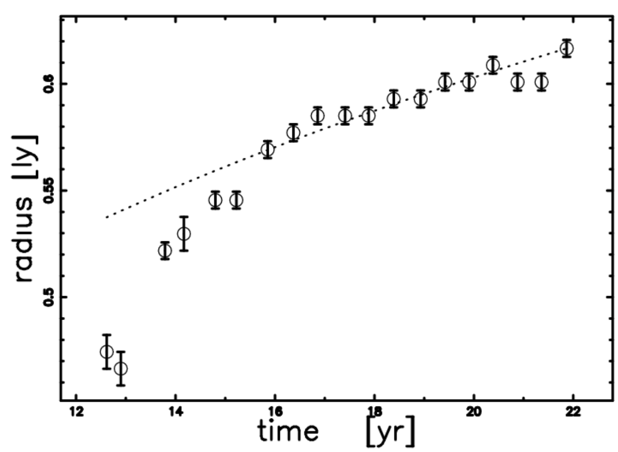

Figure 6: Radius as a function of time when $R_0 = 0.079$, $\beta_0 = 0.95$, $c = 1$ (asymptotic relativistic case, equation (17)) (dotted line) with the addition of the observed radius with relative error extracted from [21]

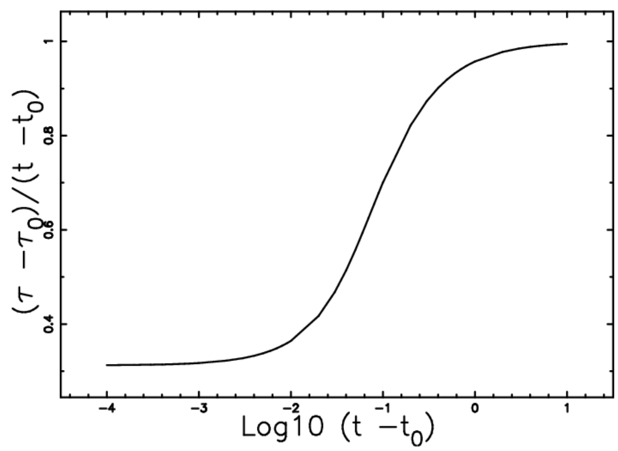

Figure 7: The time contraction r of the clock on the expanding surface as a function of the decimal logarithm in laboratory time when $R_0 = 0.079$, $\beta_0 = 0.95$ and $c = 1$.

This means a relativistic time contraction, r, for the clock which follows the expansion

$$r = \frac{\tau - \tau_0}{t - t_0} \quad , \tag{22}$$

and Figure 7 shows such a contraction of the framework which sees the expansion when the velocity is evaluated as a function of time from equation (9).

4 Conclusions

The introduction of a relativistic framework into the equation of a SNR under the hypothesis of the thin layer approximation avoids the paradox of an initial velocity greater than the velocity of light. The self-similar solutions for radius and velocity, respectively eqns. (17) and (18), are found under the approximation $R^6(1-\beta_0^2) \gg R_0^6\beta_0^2$. The application of these new formulae to SNR1987A produces acceptable results. The conservation of classical and relativistic momentum adopted here does not take into account the momentum carried away by photons.

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