ON THE STABILITY OF WEIGHT SPACES OF ENVELOPING ALGEBRA IN PRIME CHARACTERISTIC

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Abstract. By the result of Dixmier, any weight space of enveloping algebra of Lie algebra L over a field of characteristic 0 is adL stable. In this paper we will show that this result need not be true, if F is replaced by a field of prime characteristic. A condition will be given, so a weight space will be adL stable.

1. Introduction

Let L be a finite dimensional Lie algebra over a field F, $U(L)$ its enveloping algebra with center $Z(U(L))$. For each linear form $\lambda : L \longrightarrow F$ we denote

$$U(L)_\lambda = \{u \in U(L) \mid [x, u] = \lambda(x)u \ for \ all \ x \in L\}.$$

Clearly $U(L)_\lambda$ is a submodule of $U(L)$ and $U(L)_\lambda U(L)_\mu \subseteq U(L)_{\lambda+\mu}$. We call $U(L)_\lambda$ a weight space of $U(L)$ and the sum of all $U(L)_\lambda$ is direct and is denoted by $Sz(U(L))$, the semi-center of $U(L)$. Semi-center was introduced by Dixmier [4, 4.3] and its crucial properties in characteristic 0 is that any two sided ideal of $U(L)$ has non trivial intersection with semi-center ([4, 4.4.1]). If L is nilpotent or $[L, L] = L$ then $Z(U(L)) = Sz(U(L))$. In addition if L is nilpotent and $charF = 0$, then by [4] $Sz(U(L))$ is factorial domain and if $charF = p > 0$ then $Sz(U(L))$ is again factorial by [1]. If $charF = 0$ and L is solvable, then $Sz(U(L))$ is factorial due to Moeglin [5]. However, if char$F = p > 0$, then by recent result [2], it is shown that $Sz(U(L))$ need not be factorial.

Consider $L = Fx + H$ a Lie algebra over an algebraically closed field F where H is an ideal in L. A weight space $U(H)_\lambda$ is called adL stable if $[L, U(H)_\lambda] \subseteq U(H)_\lambda$. By Dixmier [4, 1.3.11], when $charF = 0$, $U(H)_\lambda$ is always adL stable which implies that $Sz(U(H))$ is also adL stable. The techniques used by Dixmier are characteristic zero dependent and so we cannot adapt them for the case of $charF = p > 0$. In this paper we focus on the problem related stability of $U(H)_\lambda$ in case L is a finite dimensional Lie algebra over a field F of char$F = p > 0$. Unlike in the $charF = 0$ case, it appears that $U(H)_\lambda$ is not necessarily adL stable. We will provide conditions, so that $U(H)_\lambda$ will be adL stable, as well we will show a surprising result that $Sz(U(H))$ is adL stable if and only if $U(H)_\lambda$ is adL stable for every weight λ.

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2. Main result

Let $L = Fx + H$ be a finite dimensional Lie algebra over a field F and H an ideal in L. As was mentioned, by [4, 1.3.11] if $charF = 0$ then $U(H)_\lambda$ is adL stable. The following discussion will show that this result need not be true if $charF = p > 0$. For the simplicity we assume $p = 3$, although the techniques we use, can be adapted for any $p > 0$.

Result 2.1. Let L be 5-dimensional Lie algebra over a field F of char$F = 3$, with the following multiplication table

$$[x, e_1] = 0,\ [x, e_2] = e_2,\ [x, e_3] = 2e_3, [y, e_1] = e_3,\ [y, e_2] = e_1,\ [y, e_3] = e_2,$$
$$[x, y] = 2y,\ the\ rest\ of\ products\ 0.$$

The subspace $K \equiv span\{e_1, e_2, e_3\}$ is an ideal of L and consider $L = Fx + H$, $H = Fy + K$. Clearly H is a codimension one ideal in L. Then by [2, Section 9]

$$(1) \qquad Sz(U(L)) = U(K)^{ady}[x^p - x, y^{p^2} - y^p],\ Z(U(H)) = U(K)^{ady}[y^{p^2} - y^p].$$

Therefore $Sz(U(L)) = Z(U(H))[x^p - x]$ and using direct calculations we can show that

$$Z(U(H))) = F[y^{p^2} - y^p, e_1^p, e_2^p, e_3^p, e_1e_2^2 + e_1^2e_3 + e_2e_3^2].$$

Consider $u \equiv e_1 + e_2 + e_3$ and $v \equiv e_2^2 + e_1e_3 + 2e_2e_3 + 2e_3^2$. Now $[y, u] = u$ and $[y, v] = 2e_1e_2 + e_3^2 + e_1e_2 + 2e_1e_3 + 2e_2^2 + e_2e_3 = 2e_2^2 + 2e_1e_3 + e_2e_3 + e_3^2 = 2v$. Hence $u \in U(H)_1$ (the weight space of $U(H)$ related to eigen value 1) and $v \in U(H)_2$. It can be seen that no more semi-invariant for $U(H)$ exists and $Sz(U(H)) = Z(U(H))[u, v]$. But then $[x, u] = e_2 + 2e_3$ and this implies that u is not semi-invariant of $U(L)$ and consequently $U(H)_1$ is not adL stable. In a similar way, we see that $U(H)_2$ is not adL stable, since $[x, v] \notin U(H)_2$.

We would like to notice that although both u and v are not L-semi-invariant, $uv = w$ where $w \equiv e_1e_2^2 + e_1^2e_3 + e_2e_3^2 + e_2^3 + e_3^3$ and it can be seen that $w \in Z(U(H))$ or by (1), $w \in Sz(U(L))$. We recall the following result of Moeglin [5, Lemma 2]. Let L be solvable Lie algebra over a field of characteristic 0 and let u, v be non zero elements of $U(L)$, then if uv is semi-invariant, then so are u and v. In addition if F is algebraically closed, then $uv \in Sz(U(L))$ implies that $u, v \in Sz(U(L))$. This result was also extended in [3, Proposition 1.3] and it seems to be crucial in showing that $Sz(U(L))$ is factorial in characteristic 0. Similar result appears to be wrong if $charF = p > 0$. Clearly if L is Lie algebra over a field F of $charF = p > 0$ defined by $[x, y] = y$, $[x, z] = -z$, $[y, z] = t$ and $[t, L] = 0$, then $yy^{p-1} = y^p \in U(L)_0$ but both $y, y^{p-1} \notin U(L)_0$. However, we are interested in the following condition

Condition 2.2. *Let L be solvable finite dimensional Lie algebra over an algebraically closed field F of $charF = p > 0$. Let $u, v \in U(L)$ be non zero such that u and v are linearly independent over $U(L)$. If uv is semi-invariant, then so are u and v.*

Result (2.1) is the evidence where this condition fails and by [2] $Sz(U(L))$ in this case is not factorial. Based on many observations, we have a strong assumption to believe there is linkage between Condition (2.2) and factoriality of $Sz(U(L))$ where L is a finite dimensional Lie algebra over algebraically closed field F of $charF = p > 0$, however we are not able not supply any general proof here.

We would like to mention that being adL stable, does not imply that $U(H)_\lambda$ is a weight space of $U(L)$, as can be seen in the following example

Example 2.3. Let L be 4 dimensional Lie algebra spanned by $\{x, y, u_1, u_2\}$ with a multiplication table $[x, u_2] = u_1$, $[y, u_1] = u_1$, $[y, u_2] = u_2$ and the rest of the products are 0. Then $L = Fx + H$ where $H = span\{y, u_1, u_2\}$. Then $U(H)_\lambda = span\{u_1, u_2\}$. Clearly $\lambda = 1$ and $U(H)_1$ is adx stable. Notice that $adx(u_2) = u_1$ but $U(H)_1 \not\subset Sz(U(L))$.

The following result will establishes a condition on L for $U(H)_\lambda$ to be adL stable.

Lemma 2.4. *Let $L = Fx + H$ be a finite dimensional Lie algebra over an algebraically closed field F, of characteristic $p > 0$ and H an ideal in L. If $[L, L]$ is nilpotent , then $U(H)_\lambda$ is adx stable and consequently is adL stable.*

Proof. Let $0 \neq u \in U(H)_\lambda$. We need to show that for every $y \in H$, $[y, [x, u]] = \lambda(y)[x, u]$. Indeed

$$[y, [x, u]] = [[y, x], u] + [x, [y, u]] = [[y, x], u] + \lambda(y)[x, u].$$

Consider $[[y, x], u]$. Since $y \in H$ then $[y, x] \in H$. Hence

$$[[y, x], u] = \lambda([y, x])u$$

and we need to show that $\lambda([y, x]) = 0$. Now $[y, x] \in [L, L]$ and so $[y, x]$ is nilpotent, therefore $ad[y, x]$ is nilpotent. Let k be its nilpotency index. Therefore

$$0 = (ad[y, x])^k(u) = \lambda([y, x])^k u$$

hence $\lambda([y, x])^k = 0$ which implies that $\lambda([y, x]) = 0$. □

The previous lemma is somewhat misleading. Unlike in the char$F = 0$, case, $U(H)_\lambda$ is not necessarily an L-module and consequently $\lambda([x, y])$, $x, y \in L$, may be non-zero. The assumption $[L, L]$ being nilpotent is therefore essential here. The next result links between Lemma (2.4) and completely solvable Lie algebra. Recall that a Lie algebra L is called a completely solvable if there is a finite family $\{I_i\}$ of ideals in L such that

$$L = I_1 \supset I_2 \supset \cdots \supset I_k \supset I_{k+1} = (0) \text{ and } I_i = Fe_i + I_{i+1}.$$

Corollary 2.5. *Let L be a solvable finite dimensional Lie algebra over an algebraically closed field F, with char$F = p$. Then $[L, L]$ is nilpotent if and only if L is completely solvable.*

Proof. The proof is similar to one of Dixmier [4, 1.3.12] where [4, 1.3.11] which is used in its proof is replaced by Lemma (2.4). □

The following result extends Lemma (2.4).

Proposition 2.6. *Let $L = Fx + H$ be a finite dimensional Lie algebra over an algebraically closed field F of char$F = p > 0$ and H a codimension one ideal in L. Then $\lambda([L, L]) = 0$ if and only if $U(H)_\lambda$ is adx stable.*

Proof. Assume first that $U(H)_\lambda$ is adx stable for every $\lambda \in H^*$. Hence for every $w \in U(H)_\lambda$,$[x, w] \in U(H)_\lambda$. Therefore for every $y \in H$,

$$[y, [x, w]] = \lambda([y, x])w + \lambda(y)[x, w]. \tag{2}$$

If $[x, w] = 0$, then (2) implies that

$$0 = \lambda([y, x])w + 0$$

and so $\lambda([y, x]) = 0$. Assume next that $[x, w] \neq 0$. If $[y, [x, w]] = 0$ then since $[x, w] \in U(H)_\lambda$ we get $\lambda(y) = 0$. By (2), $0 = \lambda([y, x])w + 0$ and so $\lambda[y, x] = 0$. If $[y, [x, w]] \neq 0$, then the stability of $U(H)_\lambda$ implies that $\lambda([y, x]) = 0$. . Finally if $[y, x] = 0$ then $\lambda([y, x]) = 0$. Therefore $\lambda([y, x]) = 0$ for every $y \in H$ and so $\lambda([L, L]) = 0$. By Lemma (2.4), the reverse direction is trivial. □

Theorem 2.7. *Let $L = Fx + H$ be a finite dimensional Lie algebra over a field F and H an ideal in L. Then $Sz(U(H))$ is adx stable if and only if $U(H)_\lambda$ is adx stable for every λ.*

Proof. Clearly if $Sz(U(H)) = Z(U(H))$, then $Sz(U(H))$ is adL stable. So we assume that $Z(U(H)) \subset Sz(U(H))$. Let $\Delta = \{\mu | \mu \text{ is weight on } U(H)\}$. Now since $charF = p > 0$, then $Sz(U(L))$ is finitely generated as a $Z(U(L))$ module, which implies that Δ is finite and we assume $dim_F \Delta = r$. Assume that $Sz(U(H))$ is adx stable and assume by negation that there exists some $1 \leq k \leq r$ such such that $U(H)_{\mu_k}$ is not adx stable. Let $0 \neq u \in U(H)_{\mu_k}$, hence $[x, u] \in \Sigma_{i \leq r} U(H)_{\mu_i}$, in particular

$$[x, u] = \Sigma_{i \leq r} u_i, \ u_i \in U(H)_{\mu_i}. \tag{3}$$

Then for every $y \in H$,

$$[y, [x, u]] = \Sigma_{\mu_i \in \Delta} \mu_i(y) u_i. \tag{4}$$

In addition

$$[y, [x, u]] = [[y, x], u]] + [x, [y, u]] = \mu_k([y, x])u + \mu_k(y)[x, u]. \tag{5}$$

Therefore combining (3), (4) and (5) we get

$$\mu_k([y, x])u + \mu_k(y)\Sigma_{i \leq r} u_i = \Sigma_{\mu_i \in \Delta} \mu_i(y) u_i. \tag{6}$$

Since u_i belongs to different weight spaces, they are linearly independent, hence the only solution to (6) is $\mu_k([x, y]) = 0$ and $\mu_k(y) = \mu_i(y)$ for every $i \leq r$, which is impossible by the assumption, that $U(H)_{\mu_k}$ is not adx stable. Hence $[x, u] \in U(H)_{\mu_k}$ and we are done. The second direction is trivial. □

Corollary 2.8. *Let L be a finite dimensional Lie algebra over an algebraically closed field F, of $charF = p > 0$ and H any ideal of L such that $[L, L] \subseteq H$. Then $Sz(U(H))$ is adL stable if and only if $\lambda([L, L]) = 0$ for every $\lambda \in H^*$.*

As was seen in [2, Theorem 4.7], if $[L, L]$ is nilpotent then $Sz(U(L))$ is factorial. By Lemma (2.4) $[L, L]$ being nilpotent, implies that $U(H)_\lambda$ is adL stable and consequently by Lemma (2.6) $\lambda([L, L]) = 0$. Therefore $Sz(U(L))$ is factorial if $Sz(U(H))$ is adL stable. This make certain relation between factoriality of $Sz(U(L))$ and stability of semi-center of certain subalgebras. Although the reverse direction is wrong, as $Sz(U(L))$ is factorial does not implies that $Sz(U(H))$ is adL stable. We hope this relation will help on further research related factoriality of semi-center.

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