A TOPOLOGICAL CENTRAL POINT THEOREM

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Abstract. In this paper a generalized topological central point theorem is proved for maps of a simplex to finite-dimensional metric spaces. Similar generalizations of the Tverberg theorem are considered.

1. Introduction

Let us state the the discrete version of the Neumann–Rado theorem [7, 8, 4] (see also the reviews [3] and [2]).

Theorem (The discrete central point theorem). *Suppose $X \subset \mathbb{R}^d$ is a finite set with $|X| = (d+1)(r-1)+1$. Then there exists $x \in \mathbb{R}^d$ such that for any halfspace $H \ni x$*

$$|H \cap X| \geq r.$$

An important and nontrivial generalization of this theorem is proved in [9].

Theorem (Tverberg's theorem). *Consider a finite set $X \in \mathbb{R}^d$ with $|X| = (d+1)(r-1)+1$. Then X can be partitioned into r subsets $X_1, \dots, X_r$ so that*

$$\bigcap_{i=1}^{r} \operatorname{conv} X_i \neq \emptyset.$$

In [1, 10] the following topological generalization of the Tverberg theorem was established.

Theorem (The topological Tverberg theorem). *Let $n = (d+1)(r-1)+1$, r be a prime power, and let Δ^{n-1} be the $(n-1)$-dimensional simplex. Suppose $f : \Delta^{n-1} \to \mathbb{R}^d$ is a continuous map. Then there exist r disjoint faces $F_1, \dots, F_r \subset \Delta^{n-1}$ such that*

$$\bigcap_{i=1}^{r} f(F_i) \neq \emptyset.$$

It is still unknown whether such a theorem holds for r not equal to a prime power. But if we return to the central point theorem, we see that the following topological version holds without restrictions on r. Moreover, the target space can be any d-dimensional metric space, not necessarily $\mathbb{R}^d$.

Theorem 1.1. *Let $n = (d+1)(r-1)+1$, and let Δ^{n-1} be the $(n-1)$-dimensional simplex, let W be a d-dimensional metric space. Suppose $f : \Delta^{n-1} \to W$ is a continuous map. Then*

$$\bigcap_{\substack{F \subset \Delta^{n-1} \\ \dim F = d(r-1)}} f(F) \neq \emptyset,$$

where the intersection is taken over all faces of dimension $d(r-1)$.

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Note that for $W = \mathbb{R}^d$ this theorem can also be deduced from the topological Tverberg theorem (see Section 4 for details). The goal of this paper is another proof of Theorem 1.1, valid for any d-dimensional W. In Section 5 we show that a similar generalization of the Tverberg theorem for maps into finite-dimensional spaces essentially needs larger values of n.

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2. The index of $\mathbb{Z}_2$-spaces

Let us remind some basic facts on the homological index of $\mathbb{Z}_2$-action ($\mathbb{Z}_2$ is a group with two elements), the reader may consult the book [6] for more details. Denote $G = \mathbb{Z}_2$, if we consider $\mathbb{Z}_2$ as a transformation group. The algebra $H^*(BG, \mathbb{F}_2)$ is a polynomial ring $\mathbb{F}_2[c]$ with the one-dimensional generator c.

In this section we consider the cohomology with $\mathbb{F}_2$ coefficients, the coefficients are omitted in the notation. Define the equivariant cohomology for a space X with continuous action of G (a G-space) by

$$H_G^*(X) = H^*(X \times_G EG) = H^*((X \times EG)/G),$$

thus we have $H_G^*(\mathrm{pt}) = H^*(BG)$ for a one-point space with trivial action of G and $H_G^*(X) = H^*(X/G)$ for a free G-space. For $G = \mathbb{Z}_2$ we may take EG to be the infinite-dimensional sphere S^∞ with the antipodal action of G, and $BG = \mathbb{R}P^\infty$. For any G-space X the natural map $X \to \mathrm{pt}$ induces the natural cohomology map

$$\pi_X^* : H_G^*(\mathrm{pt}) = H^*(BG) \to H_G^*(X).$$

Definition 2.1. For a G-space X define $\operatorname{ind}_G X$ to be the maximal n such that $\pi_X^*(c^n) \neq 0 \in H_G^*(X)$.

We need the following two properties of index. The first is the generalized Borsuk-Ulam theorem.

Lemma 2.2. *Let $\operatorname{ind}_G X = n$, let V be an n-dimensional vector space with antipodal G-action. Then for every continuous G-equivariant map $f : X \to V$*

$$f^{-1}(0) \neq \emptyset.$$

The following lemma is proved in [14], see also [5].

Lemma 2.3. *Let X be a compact metric G-space, $\operatorname{ind}_G X \geq (d+1)k$, let W be a d-dimensional metric space with trivial G-action. Then for every continuous G-equivariant map $f : X \to W$ there exists $x \in W$ such that*

$$\operatorname{ind}_G f^{-1}(x) \geq k.$$

In this lemma it is important to use the Čech cohomology, which is assumed in the sequel.

3. Proof of Theorem 1.1

Let us map the $(n-1)$-dimensional sphere S^{n-1} to Δ^{n-1} by the following formula

$$g(x_1, \dots, x_n) = (x_1^2, \dots, x_n^2).$$

Apply Lemma 2.3 to the composition $f \circ g$, it is possible because $g(x) = g(-x)$. We obtain a point $x \in W$ such that for $Z = (f \circ g)^{-1}(x)$ we have $\operatorname{ind}_G Z \geq r$.

We are going to show that x is the required intersection point. Assume the contrary: a face $F \subseteq \Delta^{n-1}$ of dimension $d(r-1)$ does not intersect $g(Z)$. Without loss of generality, let $g^{-1}(F)$ be defined by the equations

$$x_1 = \dots = x_{r-1} = 0.$$

Note that the $r-1$ coordinates $x_1, \dots, x_{r-1}$ give a continuous G-equivariant map $h : S^{n-1} \to \mathbb{R}^{r-1}$, where G acts on $\mathbb{R}^{r-1}$ antipodally. By Lemma 2.2 the intersection $g^{-1}(F) \cap Z = h^{-1}(0) \cap Z = h|_Z^{-1}(0)$ should be nonempty. The proof is complete.

4. The case $W = \mathbb{R}^d$ of Theorem 1.1

We are going remind the known fact: the case $W = \mathbb{R}^d$ of Theorem 1.1 follows from the topological Tverberg theorem (only the case of prime r is needed).

The proof goes as follows. Consider a simplicial map $\varphi : \Delta^{N-1} \to \Delta^{n-1}$, where $R = k(r-1)+1$ is a prime (for some k it is so by the Dirichlet theorem on arithmetic progressions), $N = (R-1)(d+1)+k$, and there are k vertices of Δ^{N-1} in the preimage of every vertex of Δ^{n-1}. For Δ^{N-1} the topological Tverberg theorem holds (since $N \geq (R-1)(d+1)+1$), and there exist R disjoint faces $\tilde{F}_1, \dots, \tilde{F}_R$ of Δ^{N-1} such that

$$\bigcap_{i=1}^{R} f(\varphi(\tilde{F}_i)) \ni x.$$

Consider a face $F \subseteq \Delta^{n-1}$ of dimension $d(r-1)$ and assume that $\varphi^{-1}(F)$ does not contain any of $\tilde{F}_i$, then N should be at least the number of vertices in $\varphi^{-1}(F)$ plus R, that is

$$N \geq k(r-1)d + k + R = (R-1)d + k + R = N+1,$$

which is a contradiction. So $\varphi^{-1}(F)$ contains some of $\tilde{F}_i$, and $f(F) \ni x$.

5. Tverberg type theorems for maps to finite-dimensional spaces

It would be natural to ask whether the corresponding version of the Tverberg theorem holds for maps from Δ^{n-1} to a d-dimensional metric space, at least for r being a prime power. In fact, the number $n = (d+1)(r-1)+1$ should be increased, as claimed by the following:

Theorem 5.1. *Let $n = (d+1)r - 1$. Then there exists a d-dimensional polyhedron W and a continuous map $f : \Delta^{n-1} \to W$ with the following properties. For any pairwise disjoint faces $F_1, \dots, F_r \subseteq \Delta^{n-1}$ there exists i such that*

$$f(F_i) \cap f(F_j) = \emptyset$$

for all $j \neq i$.

This theorem also shows that our way to prove Theorem 1.1 cannot be applied to the topological Tverberg theorem. Indeed, this proof does not distinguish between $\mathbb{R}^d$ and any metric d-dimensional space, but the topological Tverberg theorem does not hold for maps to d-dimensional metric spaces.

Proof of Theorem 5.1. The construction in the proof is taken from [13]. Let Δ^{n-1} be a regular simplex in $\mathbb{R}^{n-1}$, centered at the origin. Denote by Δ^{n-1}_{d-1} its $(d-1)$-skeleton, and $W = C\Delta^{n-1}_{d-1}$ the cone (centered at the origin) of this skeleton. Define the PL-map (of the barycentric subdivision to the barycentric subdivision) $f : \Delta^{n-1}_{d-1} \to W$ as follows. For every face $F \subseteq \Delta^{n-1}$ of dimension $\leq d-1$ its center is mapped to itself, for every face $F \subseteq \Delta^{n-1}$ of dimension $\geq d$ its center is mapped to the origin.

Let $F_1, \dots, F_r \subseteq \Delta^{n-1}$ be a set of r pairwise disjoint faces. For some i the dimension $\dim F_i$ is less than $d-1$ by the pigeonhole principle. For such face we have $f(F_i) = F_i$, and

$$f(F_i) \cap f(F_j) \subseteq F_i \cap f(F_j) \subseteq \partial\Delta^{n-1}.$$

Since $f(F_j) \cap \partial\Delta^{n-1} \subseteq F_j$ we obtain

$$f(F_i) \cap f(F_j) \subseteq F_i \cap F_j = \emptyset.$$

□

The following positive result for larger n is a direct consequence of the reasonings in [12].

Theorem 5.2. *Let $n = (d+1)r$, let r be a prime power. Suppose $f : \Delta^{n-1} \to W$ is a continuous map to d-dimensional metric space W. Then there exist r disjoint faces $F_1, \dots, F_r \subset \Delta^{n-1}$ such that*

$$\bigcap_{i=1}^{r} f(F_i) \neq \emptyset.$$

Proof. Without loss of generality we may assume W to be a finite d-dimensional polyhedron. Assume the contrary and denote $\Delta^{n-1} = K$ for brevity. Then there exists a map

$$\tilde{f} : K^{*r}_{\Delta(2)} \to W^{*r}_{\Delta(r)}$$

from the r-fold pairwise deleted join $K^{*r}_{\Delta(2)}$ in the simplicial sense to the r-fold r-wise deleted join $W^{*r}_{\Delta(r)}$ in the topological sense (the notation is from [6]). Following mostly [10], put $r = p^\alpha$ and consider the group $G = (\mathbb{Z}_p)^\alpha$ and let G act on the factors of the deleted join transitively. The rest of the reasoning is based on the following facts from [11, 12].

Definition 5.3. Let X be a connected G-space. Consider the Leray-Serre spectral sequence with

$$E_2^{*,*} = H^*(BG, H^*(X, \mathbb{F}_p))$$

converging to $H^*_G(X, \mathbb{F}_p)$. Denote by $i_G(X)$ the minimum r such that the differential d_r of this spectral sequence has nontrivial image in the bottom row.

The index i_G has the following properties, if G is a p-torus $G = (\mathbb{Z}_p)^\alpha$.

(1) (Monotonicity) If there is a G-map $f : X \to Y$, then $i_G(X) \le i_G(Y)$. If in addition $i_G(X) = i_G(Y) = n+1$ then the map $f^* : H^n(Y, \mathbb{F}_p) \to H^n(X, \mathbb{F}_p)$ is nontrivial.
(2) (Dimension upper bound) $i_G(X) \le \operatorname{hdim}_{\mathbb{F}_p} X + 1$.
(3) (Cohomology lower bound) If X is connected and acyclic over $\mathbb{F}_p$ in degrees $\le N-1$, then $i_G(X) \ge N+1$.

Now note that from the cohomology lower bound it follows that $i_G\left(K^{*r}_{\Delta(2)}\right) \ge n$, from the dimension upped bound it follows that $i_G\left(W^{*r}_{\Delta(r)}\right) \le n$, and from (1) the map

$$\tilde{f}^* : H^{n-1}\left(W^{*r}_{\Delta(r)}, \mathbb{F}_p\right) \to H^{n-1}\left(K^{*r}_{\Delta(2)}, \mathbb{F}_p\right)$$

should be nontrivial. From the cohomology exact sequence of a pair it follows that the natural map

$$g^* : H^{n-1}\left(W^{*r}, \mathbb{F}_p\right) \to H^{n-1}\left(W^{*r}_{\Delta(r)}, \mathbb{F}_p\right)$$

is surjective because $H^n(W^{*r}, W^{*r}_{\Delta(r)}, \mathbb{F}_p) = 0$ by the dimension considerations. Now it follows that the map

$$(g \circ \tilde{f})^* : H^{n-1}\left(W^{*r}, \mathbb{F}_p\right) \to H^{n-1}\left(K^{*r}_{\Delta(2)}, \mathbb{F}_p\right)$$

should be nontrivial. But the map $g \circ \tilde{f}$ is a composition of the natural inclusion

$$h : K^{*r}_{\Delta(2)} \to K^{*r}$$

with the map

$$f^{*r} : K^{*r} \to W^{*r}.$$

The latter map has contractible domain, and therefore induces a zero map on the cohomology $H^{n-1}(\cdot, \mathbb{F}_p)$. We obtain a contradiction. $\square$

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