LAGRANGIAN AVERAGED NAVIER-STOKES EQUATIONS WITH ROUGH DATA IN SOBOLEV SPACE

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ABSTRACT. We prove the existence of short time, low regularity solutions to the incompressible, isotropic Lagrangian Averaged Navier-Stokes equations with initial data in Sobolev spaces. In the special case of initial datum in the Sobolev space $H^{3/2,2}(\mathbb{R}^3)$, we obtain a global solution, improving on previous results, which required data in $H^{3,2}(\mathbb{R}^3)$.

1. Introduction

The incompressible Navier-Stokes equations govern the motion of incompressible fluids and are given by

$$\partial_t u + (u \cdot \nabla)u - \nu A u = 0, \tag{1.1}$$

where A is (essentially) the Laplacian, ν is a constant greater than zero due to the viscosity of the fluid, and u is a velocity field, which means $u(t, x)$ is the velocity of the particle of fluid located at position x a time of t units after the fluid is put in motion. These equations are derived from the Euler Equations, and setting $\nu = 0$ recovers the Euler Equations.

The Navier-Stokes equations govern the behavior of many physical phenomena, including ocean currents, the weather and water flowing through a pipe. The question of global existence for the Navier-Stokes equations is one of the most significant remaining open problems in mathematics, and because of the intractability of the Navier-Stokes equations, several different equations that approximate the Navier-Stokes equations have been studied. A recently derived approximating equation is the Lagrangian Averaged Navier-Stokes equations (LANS). The LANS equations come from the Lagrangian Averaged Euler (LAE) equations in the same way that the Navier-Stokes equations come from the Euler equations. Like the Euler equations, the LAE equations are the geodesics of a specific functional. For the Euler equation, this is the Energy functional. For the LAE equations, the functional is derived via an averaging process, with the averaging occurring at the level of the initial data. For an exhaustive treatment of this process, see [10], [11], [8] and [5]. In [9] and [14], the authors discuss the numerical improvements that use of the LANS equation provides over more common approximation techniques of the Navier-Stokes equations.

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Like the Navier-Stokes equations, the LANS equations have both a compressible and an incompressible formulation. The compressible LANS equations are derived and studied in [6]. The incompressible LANS equations exist most generally in the anisotropic form, and are derived and studied in [5]. In this paper, we will consider a special case of these anisotropic equations called the isotropic incompressible LANS equations. One form of the incompressible, isotropic LANS equations on a region without boundary is

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \text{div } \tau^\alpha u &= -(1-\alpha^2\triangle)^{-1}\text{grad } p + \nu\triangle u \\ \text{div } u = 0,\ u(t,x)|_{t=0} &= \varphi(x), \end{aligned} \tag{1.2}$$

where $\alpha > 0$ and φ is the initial data. The Reynolds stress τ^α is given by

$$\tau^\alpha u = \alpha^2(1-\alpha^2\triangle)^{-1}[Def(u) \cdot Rot(u)] \tag{1.3}$$

where $Rot(u) = (\nabla u - \nabla u^T)/2$ is the antisymmetric part of the velocity gradient and $Def(u) = (\nabla u + \nabla u^T)/2$. Lastly, $(1-\alpha^2\triangle)$ is the Helmholtz operator.

Setting $\nu = 1$, we write (1.2) as

$$\begin{aligned} \partial_t u - Au + P^\alpha(\text{div } \cdot (u\otimes u) + \text{div } \tau^\alpha u) = 0, \\ x \in \mathbb{R}^n,\ n \geq 2,\ t \geq 0,\ u(0) = \varphi = P^\alpha \varphi \end{aligned} \tag{1.4}$$

where $u = u(t) : \mathbb{R}^n \to \mathbb{R}^n$, $A = P^\alpha\triangle$, $u \otimes u$ is the tensor with jk-components $u_j u_k$ and $\text{div} \cdot (u \otimes u)$ is the vector with j-component $\sum_k \partial_k(u_j u_k)$. P^α is the Stokes Projector defined as

$$P^\alpha(w) = w - (1-\alpha^2\triangle)^{-1}\text{grad } f \tag{1.5}$$

where f is a solution of the Stokes problem: Given w, there is a unique v and a unique (up to additive constants) function f such that

$$(1-\alpha^2\triangle)v + \text{grad } f = (1-\alpha^2\triangle)w \tag{1.6}$$

with $\text{div } v = 0$. For a more explicit treatment of the Stokes Projector, see Theorem 4 of [11].

The averaging process has a smoothing effect on the resulting PDE. In [4], this smoothing is exploited to show the existence of a global solution to (1.2) for initial data of any size in $H^{3,2}(\mathbb{R}^3)$. This is in stark contrast to the case for the Navier-Stokes equations, where such a global existence result is one of the great remaining open problems in mathematics.

Inspired by the approach used for the Navier-Stokes equations in [2], we prove local existence of solutionsto the LANS equations in the class of weighted continuous functions in time and in the class of integral norms in time. When $n = 3$ and $p = 2$, we extend these local solutions to global solutions with initial data in $H^{s,2}(\mathbb{R}^3)$, where $s \geq 3/2$, improving on the result from [4].

The paper is organized as follows. The main results in this article are Theorems A, B, and C, and their full statements can be found in Section 3.1, Section 5.1, and Section 7, respectively. In Section 2, we state the most important special cases and consequences of these three main theorems. In Section 3, we define the weighted continuous in time spaces, state Theorem A, and prove some preliminary results. The proof of Theorem A is in Section 4. In Sections 5 and 6, we repeat this process for the integral norms in time and Theorem B. In Section 7, we state and prove Theorem C, which is our global existence result.

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2. Special Cases of the main theorems

Our first result is a special case of Theorem A, and comes from setting $b' = 1$ in (4.18).

Theorem 1. *For any $\varphi = P^\alpha \varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in \overline{C}_{r,p} \cap \dot{C}_{a;k,c} \tag{2.1}$$

provided the parameters (with $r = n/p + b$) satisfy (4.19)*. If $\|\varphi\|_{r,p}$ is sufficiently small, $T = \infty$.*

The function spaces $\overline{C}$ and $\cdot C$ are defined in Section 3. We record two additional special cases that illustrate our "best" result, emphasizing that we can achieve a local existence result for regularity arbitrarily close to zero if we allow sufficiently large p.

Theorem 2. *Let $r = n/p$, $n < p$, and $n \geq 2$. Then for any $\varphi = P^\alpha \varphi \in H^{r,p}(\mathbb{R}^n)$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in \overline{C}_{r,p} \cap \dot{C}_{(1-n/p)/2;1,p}. \tag{2.2}$$

We note that the result from [2] gives local existence for the standard Navier-Stokes equations with initial data in $H^{n/p-1,p}(\mathbb{R}^n)$.

We also record the result in the special case $n = 3$ and $p = 2$, which requires a different choice of parameters.

Theorem 3. *For any $\varphi = P^\alpha \varphi \in H^{3/2,2}(\mathbb{R}^3)$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in \overline{C}_{3/2,2} \cap \dot{C}_{1/4;2,2}. \tag{2.3}$$

In Theorem C, we extend this special case to a global existence result. For the details, see section 7. We compare this with the result in [4], where the authors achieve global existence for solutions to 1.2 for initial data in $H^{3,2}(\mathbb{R}^3)$.

Our next series of results generate solutions to (1.2) in a slightly different functional setting. We begin with a special case of Theorem B obtained by setting $b' = 1$ in (6.4).

Theorem 4. *For any $\varphi = P^\alpha \varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in BC([0,T) : H^{r,p}) \cap L^a((0,T) : H^{k,c}) \tag{2.4}$$

provided the parameters (with $r = n/p+b$) satisfy (6.5)*. If $\|\varphi\|_{r,p}$ is sufficiently small, then $T = \infty$. Lastly, we have that solutions depend continuously on the initial data.*

We also state a further special case.

Theorem 5. *Let $r = n/p$ and assume $p > 5/4$ and $2 \leq n \leq 5/4 + p$. Then for any $\varphi = P^\alpha \varphi \in H^{r,p}(\mathbb{R}^n)$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in BC([0,T) : H^{r,p}) \cap L^a((0,T) : H^{k,n}) \tag{2.5}$$

where $k = 5p/4 + 1$ and $a = 8p/5$.

We note that the case $n = 3$ and $p = 2$ satisfies these conditions, and that Theorem C extends the result in this case to a global solution.

We conclude this introduction with a comment about the regularity of our solutions. For example, for $n \geq p$, choosing $k = r + 1/4$, $c = p$, $a = 1/8$, $b' = 1$ and $s' = n/p - 3/4$ satisfies (4.18). Using these choices for the parameters, let $\varphi \in H^{r,p}(\mathbb{R}^n)$ be our chosen initial data and let

$$u \in BC([0,T) : H^{r,p}) \cap \dot{C}_{1/8;r+1/4,p} \tag{2.6}$$

be the solution to (1.2) given by Theorem A. Then, for any $0 < t' < T$, define $\varphi' = u(t')$. Viewing φ' as "new" initial data, applying Theorem A gives the existence of a solution v to (1.2) such that

$$v \in BC([0,T) : H^{r+1/4,p}) \cap \dot{C}_{1/8;r+1/2,p} \tag{2.7}$$

where $v(0) = \varphi' = u(t')$. Because

$$BC([0,T) : H^{r+1/4,p}) \cap \dot{C}_{1/8;r+1/2,p} \subset BC([0,T) : H^{r,p}) \cap \dot{C}_{1/8;r+1/4,p}, \tag{2.8}$$

uniqueness of our solution gives that u and v are the same solution. Each iteration of this process results in a "gain" of one-quarter of a derivative, and thus for any $0 < t' < T$, $u(t') \in H^{s,p}$ for any $s \in \mathbb{R}$. We record this as a corollary of Theorem A, but first we remark that for $n < p$, choosing $k = b+1$, $c = p$, $a = (1-n/p)/2$, $b' = 1$ and $s' = 0$ also satisfies (4.18), so we have the same result for the $n < p$ case.

Corollary 1. *Let $\varphi \in H^{r,p}(\mathbb{R}^n)$, with $n \geq p$. If $n \geq p$, let*

$$u \in BC([0,T) : H^{r,p}) \cap \dot{C}_{1/8;r+1/4,p} \tag{2.9}$$

be the solution to (1.2) *given by Theorem A. Then for any $0 < t < T$, $u(t) \in H^{s,p}(\mathbb{R}^n)$ for any real s. If $n < p$, let*

$$v \in BC([0,T) : H^{r,p}) \cap \dot{C}_{(1-n/p)/2;b+1,p} \tag{2.10}$$

be the solution to (1.2) *given by Theorem A. Then for any $0 < t < T$, $v(t) \in H^{s,p}(\mathbb{R}^n)$ for any real s.*

We also remark that a similar corollary applies to Theorem B.

3. Weighted continuous functions and preliminary results

3.1. **Definition of weighted continuous functions in time and statement of Theorem A.** Fixing $0 < T \leq \infty$, for any $k \geq 0$, we define the space

$$C^T_{k;s,q} = \{f \in C((0,T) : H^{s,q}) : \|f\|_{k;s,q} < \infty\} \tag{3.1}$$

where

$$\|f\|_{k;s,q} = \sup\{t^k \|f(t)\|_{s,q} : t \in (0,T)\}. \tag{3.2}$$

$\dot{C}^T_{k;s,q}$ denotes the subspace of $C^T_{k;s,q}$ consisting of f such that

$$\lim_{t \to 0^+} t^k f(t) = 0 \text{ (in } H^{s,q}). \tag{3.3}$$

If $k = 0$, we write $\overline{C}^T_{s,q}$ for $BC([0,T) : H^{s,q})$, the space of bounded, continuous functions from $[0,T)$ to $H^{s,q}$. We will typically write $C^T_{k;s,q}$ and $\overline{C}^T_{s,q}$ as $C_{k;s,q}$ and $\overline{C}_{s,q}$, respectively, suppressing the T dependence.

We now state our first theorem in its full generality.

Theorem A. *For any $\varphi = P^\alpha \varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in \overline{C}_{r,p} \cap \dot{C}_{a;k,c} \tag{3.4}$$

provided there exists a real number b' such that the list of conditions (4.18) *is satisfied (where $r = n/p + b$). If $\|\varphi\|_{r,p}$ is sufficiently small, $T = \infty$.*

This result is in the spirit of Theorem 2.1 in [2], which gave local existence results for the Navier-Stokes equations.

3.2. **Preliminary work.** Before proving the theorem, we do some preliminary work. We begin by using Duhamel's principle to write (1.4) as the integral equation

$$u = \Gamma\varphi - G \cdot P^\alpha(\operatorname{div}(u \otimes u + \tau^\alpha(u))) \tag{3.5}$$

with

$$(\Gamma\varphi)(t) = e^{tA}\varphi, \tag{3.6}$$

where A agrees with $\triangle$ when restricted to $P^\alpha H^{s,p}$, and

$$G \cdot g(t) = \int_0^t e^{(t-s)A} \cdot g(s) ds. \tag{3.7}$$

Our plan is to construct a contraction mapping based on (3.5). This requires appropriate estimates for Γ, G, and the Reynolds stress term τ^α. We begin by examining the Reynolds stress term.

Lemma 1. *Let $r \in [1, \infty)$ and $1 < q, p < \infty$, with $2/p - 1/q < 1$ and $0 \leq n(2q - p)/pq \leq r - 1$. Then $div\ \tau^\alpha : H^{r,p} \to H^{r,q}$, where we recall $\tau^\alpha(u, v) = \alpha^2(1 - \alpha^2\triangle)^{-1}(Def(u)) \cdot (Rot(v))$. Specifically, we have the estimate*

$$\|div\ \tau^\alpha(u)\|_{r,q} \leq C\|u\|^2_{r,p} \tag{3.8}$$

Proof. Recalling the definitions of $Def(u)$ and $Rot(u)$ and applying Proposition 1.1 from [13] (which has its origins in [3], [1], and [7]) we get

$$\begin{aligned}\|\tau^{\alpha}(u)\|_{r+1,q} &\leq C\|[Def(u)\cdot Rot(u)]\|_{r-1,q}\\ &\leq C\|\nabla u\|_{r-1,p}^{2}\\ &\leq C\|u\|_{r,p}^{2}.\end{aligned} \tag{3.9}$$

Since the divergence is a degree one differential operator, we get

$$\|\text{div }\tau^{\alpha}(u)\|_{r,q}\leq\|\tau^{\alpha}(u)\|_{r+1,q}\leq C\|u\|_{r,p}^{2} \tag{3.10}$$

which proves the lemma. □

This immediately gives

$$t^{2a}\|\text{div }\tau^{\alpha}(u)\|_{r,q}\leq C(t^{a}\|u\|_{r,p})^{2} \tag{3.11}$$

which proves the following corollary.

Corollary 2. *div* $\tau^{\alpha}:\dot{C}_{a;r,p}\to\dot{C}_{2a;r,q}$*, with the estimate* $\|div\,\tau^{\alpha}(u)\|_{2a;r,q}\leq C\|u\|_{a;r,p}^{2}$.

We remark here that $r\geq 1$ is forced by the need to estimate the τ^{α} term. For the Navier-Stokes equation, we would only need $r\geq 0$.

Our next task is to establish some properties of the operator V^{α} defined by

$$V^{\alpha}(u,v)=\text{div }u\otimes v+\text{div }\tau^{\alpha}(u,v). \tag{3.12}$$

Abusing notation, we will write $V^{\alpha}(u,u)=V^{\alpha}(u)$. We also observe that V^{α} is linear in each of its arguments.

Proposition 1. *Let* $a\geq 0$*,* $b\geq 1$*,* $1<q,p<\infty$*, and* $q=\frac{np}{2n-s'p}$ *where* $0\leq s'\leq b-1$ *and* $s'p<n$*. Then*

$$V^{\alpha}:\dot{C}_{a;b,p}\times\dot{C}_{a;b,p}\to\dot{C}_{2a;b-1,q} \tag{3.13}$$

with the estimate

$$\|V^{\alpha}(u,v)\|_{2a;b-1,q}\leq\|u\|_{a;b,p}\|v\|_{a;b,p}. \tag{3.14}$$

This follows directly from Corollary 2.

Next, we observe that

$$V^{\alpha}(u)-V^{\alpha}(v)=-(V^{\alpha}(u,u-v)+V^{\alpha}(u-v,v)). \tag{3.15}$$

Using Proposition 1, we have

$$\|V^{\alpha}(u,u-v)\|_{b-1,q}\leq\|u\|_{b,p}\|u-v\|_{b,p} \tag{3.16}$$

and

$$\|V^{\alpha}(u-v,v)\|_{b-1,q}\leq\|v\|_{b,p}\|u-v\|_{b,p}. \tag{3.17}$$

These estimates give that

$$\|V^{\alpha}(u(s))-V^{\alpha}(v(s))\|_{b-1,q}\leq C\left(\|u(s)\|_{b,p}+\|v(s)\|_{b,p}\right)\|u(s)-v(s)\|_{b,p}. \tag{3.18}$$

Multiplying both sides by t^{a} and distributing through the right hand side, we get the following corollary to Proposition 1.

Corollary 3. *With the same assumptions on the parameters as in Proposition* 1*, we have that if* $u, v \in \dot{C}_{a/2;b,q}$ *then*

$$\|V^\alpha(u(s)) - V^\alpha(v(s))\|_{a;b-1,q} \le C(\|u\|_{a/2;b,p} + \|v\|_{a/2;b,p})\|u-v\|_{a/2;b,p}. \tag{3.19}$$

Our next topic is the operator Γ.

Proposition 2. *Let* $s' \le s''$, $1 < q' \le q'' < \infty$, *and define* $k'' = (n/q' - n/q'' + s'' - s')/2$. *Then*

$$\|\Gamma f\|_{k'';s'',q''} \le C\|f\|_{s',q'} \tag{3.20}$$

and for any $\varepsilon > 0$, *there exists sufficiently small* T *such that*

$$\|\Gamma f\|_{k'';s'',q''} \le \varepsilon, \tag{3.21}$$

provided $k'' > 0$.

This is an immediate consequence of equation 1.15 in Chapter 15 of [12].

We now turn our attention to the operator G. Assuming $s' \le s''$, $q' \le q''$, and $u \in \dot{C}_{k';s',q'}$, we formally calculate

$$\begin{aligned} \|G \cdot u\|_{s'',q''} &= \|\int_0^t e^{(t-s)A} u(s) ds\|_{s'',q''} \\ &\le C \int_0^t \|e^{(t-s)A} u(s)\|_{s'',q''} ds \\ &\le C \int_0^t (t-s)^{-(s''-s'+n/q'-n/q'')/2} \|u(s)\|_{s',q'} ds \\ &\le C \int_0^t (t-s)^z s^{-k'} s^{k'} \|u(s)\|_{s',q'} ds \\ &\le C t^{z-k'+1} \|u\|_{k';s',q'} \end{aligned} \tag{3.22}$$

where $z = -(s'' - s' + n/q' - n/q'')/2$. This result will hold provided $0 \le (s'' - s' + n/q' - n/q'')/2 < 1$ and $k' < 1$, and this leads to our first result involving G.

Proposition 3. *With* $s' \le s''$, $q' \le q''$ *and setting* $k'' = k' - 1 + (s'' - s' + n/q' - n/q'')/2$, G *continuously maps* $\dot{C}^T_{k';s',q'}$ *into* $\dot{C}^T_{k'';s'',q''}$ *with* $0 \le (s'' - s' + m/q' - m/q'')/2 < 1$ *and* $k' < 1$ *with the estimate*

$$\|G \cdot u\|_{k'';s'',q''} \le C\|u\|_{k';s',q'}. \tag{3.23}$$

4. Proof of Theorem A

To prove Theorem A, we begin by constructing the nonlinear map

$$\Phi u = \Gamma\varphi - G \cdot P^\alpha(\text{div } (u \otimes u) + \text{div } \tau^\alpha u). \tag{4.1}$$

Our goal is to show that this map is a contraction on an appropriate function space. Using (3.12), Φ can be re-written as

$$\Phi u = \Gamma\varphi - G \cdot P^\alpha(V^\alpha(u)). \tag{4.2}$$

Beginning with initial data $\varphi \in H^{r,p}(\mathbb{R}^n)$ where $r = \frac{n}{p} + b$, we construct the space

$$E_{T,M} = \{v \in \bar{C}_{r,p} \cap \dot{C}_{a;k,c} : \|v - \Gamma\varphi\|_{0;r,p} + \|v\|_{a;k,c} \leq M\}, \tag{4.3}$$

recalling that the definition of $\bar{C}_{r,p}$ and $\dot{C}_{a;k,c}$ requires a choice of T. Our goal will be to show that Φ is a contraction on this space for appropriate choices of parameters.

To show Φ is a contraction, we use the mapping properties of G to send each component space of $E_{T,M}$ into an intermediate space, and then use Proposition 1. Our intermediate space will be of the form $\dot{C}_{2a;k-b',\bar{c}}$.

Our first task is to show that Φ maps $E_{T,M}$ into $E_{T,M}$. To do this, we need to estimate

$$\|\Phi(u) - \Gamma\varphi\|_{0;r,p} = \|G \cdot P^\alpha V^\alpha(u)\|_{0;r,p} \tag{4.4}$$

and

$$\|\Phi(u)\|_{a;k,c} = \|\Gamma\varphi - G \cdot P^\alpha V^\alpha(u)\|_{a;k,c}. \tag{4.5}$$

To estimate (4.4), we note that by Proposition 3 and that P^α is a projection, we have that

$$\|GV^\alpha u\|_{0;\frac{n}{p}+b,p} \leq C\|V^\alpha u\|_{2a;k-b',\bar{c}} \tag{4.6}$$

will hold provided

$$\begin{aligned} 0 &= 2a - 1 + \left(\frac{n}{p} + b - (k - b') + \frac{n}{\bar{c}} - \frac{n}{p}\right)/2 \\ 2a &< 1 \\ 0 &\leq (n/p + b - (k - b') + n/\bar{c} - n/p)/2 < 1 \\ k - b' &\leq n/p + b \\ \bar{c} &\leq p. \end{aligned} \tag{4.7}$$

Proposition 1 gives

$$\|V^\alpha u\|_{2a;k-b',\bar{c}} \leq C\|u\|^2_{a;k,c} \tag{4.8}$$

provided

$$\begin{aligned} k, b' &\geq 1, \\ c &> 1, \\ \bar{c} &= \frac{nc}{2n - s'c} \\ 0 \leq s' &\leq k - 1 \\ s'c &< n. \end{aligned} \tag{4.9}$$

These combine to give our estimate on (4.4). To estimate $\|G \cdot P^\alpha V^\alpha(u)\|_{a;k,c}$, we have

$$\|GV^\alpha u\|_{a;k,c} \leq C\|V^\alpha u\|_{2a;k-b',\bar{c}} \leq C\|u\|^2_{a;k,c} \tag{4.10}$$

will hold provided

$$
\begin{aligned}
a &= 2a - 1 + \left(k - (k - b') + \frac{n}{\bar{c}} - \frac{n}{c}\right)/2 \\
2a &< 1 \\
\bar{c} &\leq c \\
0 &\leq (k - (k - b') + n/\bar{c} - n/c)/2 < 1.
\end{aligned} \tag{4.11}
$$

Using (4.6) and (4.10), we have

$$
\|\Phi(u) - \Gamma\varphi\|_{0;r,p} + \|\Phi(u)\|_{a;k,c} \leq C\|u\|^2_{a;k,c} + \|\Gamma\varphi\|_{a;k,c}. \tag{4.12}
$$

By assumption, $u \in E_{T,M}$, so $\|u\|^2_{a;k,c} \leq M^2$. So our last task is to estimate $\|\Gamma\varphi\|_{a;k,c}$. From Proposition 2, we have that

$$
\Gamma : \dot{C}_{0;\frac{n}{p}+b,p} \to \dot{C}_{a;k,c} \tag{4.13}
$$

if $a > 0$, $k \geq \frac{n}{p} + b$, $c \leq p$, and

$$
a = \left(\frac{n}{p} - \frac{n}{c} + k - \left(\frac{n}{p} + b\right)\right)/2 \tag{4.14}
$$

which simplifies to

$$
2a = k - \frac{n}{c} - b. \tag{4.15}
$$

Because $\Gamma\varphi \in \dot{C}_{a;k,c}$, there exists a T, depending only on M and the norm of the initial data φ, such that $\|\Gamma\varphi\|_{a;k,c} \leq M/2$. So by choosing a sufficiently small M and an appropriate T, we have that $\Phi : E_{T,M} \to E_{T,M}$.

Now we seek to show that Φ is a contraction map. Let $u, v \in E^{T,M}$. Then by Corollary 3 we have

$$
\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_{r,p} &= \|G(P^\alpha V^\alpha u - P^\alpha V^\alpha(v))\|_{r,p} \\
&\leq C\|V^\alpha u - V^\alpha(v)\|_{2a;k-b',\bar{c}} \\
&\leq C(\|u\|_{a;k,c} + \|v\|_{a;k,c})\|u - v\|_{a;k,c} \\
&\leq CM\|u - v\|_{a;k,c},
\end{aligned} \tag{4.16}
$$

and similarly we have

$$
\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_{a;k,c} &= \|G(V^\alpha u - V^\alpha(v))\|_{a;k,c} \\
&\leq C\|V^\alpha u - V^\alpha(v)\|_{2a;k-b',\bar{c}} \\
&\leq C(\|u\|_{a;k,c} + \|v\|_{a;k,c})\|u - v\|_{a;k,c} \\
&\leq CM\|u - v\|_{a;k,c}.
\end{aligned} \tag{4.17}
$$

So for a sufficiently small choice of M, we can choose a T such that Φ sends $E_{T,M}$ into itself and is a contraction on $E_{T,M}$. So by the contraction mapping principle, we have a unique fixed point $u \in E_{T,M}$ provided our parameters satisfy all the requisite inequalities. Combining and simplifying these inequalities, and allowing

$s' = k - 2 - b + b'$ to define s', we get the following list of restrictions on the parameters:

$$
\begin{aligned}
&1 < p \le c < \infty \\
&s' := k - 2 - b + b' \\
&k \ge 1,\ b' \ge 1,\ s'c < n \\
&0 < 2a = k - n/c - b < 1 \\
&0 \le s' \le k - 1 \\
&1 < \frac{nc}{2n - s'c} \le p \\
&1 \ge b' - b \\
&1 \le b' + \frac{n}{c} - s' < 2 \\
&2 - 2b' + s' \le \frac{n}{p} \le 2 - b' + s'.
\end{aligned} \tag{4.18}
$$

This is not optimal, because of the presence of the "extra" parameter b'. However, this version does make it easy to ascertain certain bounds on the original parameters. For example, the second and seventh conditions require that $b \ge 0$, which provides a lower bound of n/p on the regularity of our initial data.

To eliminate the extra parameter b', we remark that the conditions force $1 \le b' < 2$, and our optimal case ($b = 0$) requires $b' = 1$. So setting $b' = 1$, we let $k = 1 + b + s'$ define s', and our list of conditions becomes

$$
\begin{aligned}
&1 < p \le c < \infty \\
&b \ge 0 \\
&s' := k - 1 - b \\
&k \ge 1,\ s'c < n \\
&0 < 2a = k - n/c - b < 1 \\
&1 < \frac{nc}{2n - s'c} \le p \\
&0 \le \frac{n}{c} - s' < 1 \\
&s' \le \frac{n}{p} \le 1 + s'.
\end{aligned} \tag{4.19}
$$

To get Theorem 2, we choose $p > n$, $b = 0$, $k = 1$, $c = p$ and $a = 1 - n/p$. To get Theorem 3, we choose $p = c = k = 2$, $n = 3$, $b = 0$, and $a = 1/4$.

5. Integral norms in time and preliminary results

5.1. Definition of integral norms in time and statement of Theorem B. We now seek to solve (1.2) in a different function space. We fix $T > 0$ and let $\mathbb{M}((0,T) : \mathbb{E})$ be the set of measurable functions defined on $(0,T)$ with values in the space $\mathbb{E}$. Then we define

$$L^\sigma((0,T):H^{s,q}) = \{f \in \mathbb{M}((0,T):H^{s,q}) : (\int_0^T \|f(t)\|_{s,q}^\sigma dt)^{1/\sigma} < \infty\}. \tag{5.1}$$

We state our second theorem.

Theorem B. *For any $\varphi = P^\alpha \varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.2) *such that*

$$u \in BC([0,T):H^{r,p}) \cap L^a((0,T):H^{k,c}) \tag{5.2}$$

provided the parameters (with $r = n/p+b$) satisfy (6.4)*. If $\|\varphi\|_{r,p}$ is sufficiently small, then $T = \infty$. Lastly, we have that solutions depend continuously on the initial data.*

This is analogous to Theorem 3.1 in [2], where the authors prove a similar result for Navier-Stokes equations.

5.2. **Preliminary results.** Our first supporting result is Lemma 3.2 in [2] and involves the operator Γ.

Proposition 4. *Let $1 < q_0 \le q_1 < \infty$, $s_0 \le s_1$, and assume $0 < (s_1 - s_0 + n/q_0 - n/q_1)/2 = 1/\sigma \le 1/q_0$. Then Γ maps H^{s_0,q_0} continuously into $L^\sigma((0,\infty):H^{s_1,q_1})$, with the estimate*

$$\left(\int_0^\infty \|\Gamma u\|_{H^{s_1,q_1}}^\sigma\right)^{1/\sigma} \le C\|u\|_{H^{s_0,q_0}}. \tag{5.3}$$

Proof. We first observe that $(s_1 - s_0 + n/q_0 - n/q_1)/2 = 1/\sigma < 1$ implies $s_1 - s_0 < 2$. So without loss of generality, we assume $s_0 = 0$ and $s_1 \in [0,2)$. Next, we define the quasi-linear operator K by

$$(Kf)(t) = \|e^{t\triangle} f\|_{H^{s_1,q_1}}, \tag{5.4}$$

where s_1 and q_1 have been fixed. Using the heat kernel estimate, we have that

$$(Kf)(t) = \|e^{t\triangle} f\|_{H^{s_1,q_1}} \le Ct^{-1/\sigma}\|f\|_{L^{q_0}} = C(Tf)(t), \tag{5.5}$$

so K is a quasi-linear map that satisfies

$$K : L^{q_0}(\mathbb{R}^n) \to L_{1/\sigma,\infty}(I). \tag{5.6}$$

The Proposition then follows from an application of the Marcinkiewicz interpolation theorem. □

The following corollary follows from an application of the dominated convergence theorem.

Corollary 4. *For any $\varepsilon > 0$, there exists a T which depends only on ε and $\|u\|_{H^{s_0,q_0}}$ such that*

$$\left(\int_0^T \|e^{t\triangle} u\|_{H^{s_1,q_1}}^\sigma dt\right)^{1/\sigma} \le \varepsilon, \tag{5.7}$$

for all $0 < t < T$.

Next, we consider the operator V^α on our integral norm space.

Proposition 5. *Let $b \geq 1$, $1 < q, p < \infty$, with $2/p - 1/q < 1$ and $0 \leq n(2q-p)/pq \leq b-1$. Then*

$$V^\alpha : L^\sigma((0,T) : H^{b,p}) \to L^{\sigma/2}((0,T) : H^{b-1,q}) \tag{5.8}$$

with the estimate

$$\left(\int_0^T \|V^\alpha(u(s))\|_{b-1,q}^{\sigma/2} ds\right)^{2/\sigma} \leq \left(\int_0^T \|u(s)\|_{b,p}^{\sigma} ds\right)^{2/\sigma}. \tag{5.9}$$

This follows directly from Proposition 1. We also have that

$$\begin{aligned} &\left(\int_0^T \|V^\alpha(u(s)) - V^\alpha(v(s))\|_{b-1,q}^{\sigma/2} ds\right)^{2/\sigma} \\ &\leq \left(\int_0^T (\|v(s)\|_{b,p} + \|u(s)\|_{b,p})^{\sigma/2} (\|v(s) - u(s)\|_{b,p})^{\sigma/2} ds\right)^{2/\sigma} \\ &\leq \left(\int_0^T (\|v(s)\|_{b,p} + \|u(s)\|_{b,p})^{\sigma} ds\right)^{2/\sigma} \left(\int_0^T \|v(s) - u(s)\|_{b,p}^{\sigma} ds\right)^{2/\sigma} \end{aligned} \tag{5.10}$$

where we used Holder's inequality and Minkowski's inequality. This gives an analog to Corollary 3.

Corollary 5. *With the same assumptions on the parameters as in Proposition 5, we have that if $u, v \in L^\sigma((0,T) : H^{b,p})$ then*

$$\begin{aligned} &\left(\int_0^T \|V^\alpha(u(s)) - V^\alpha(v(s))\|_{b-1,q}^{\sigma/2} ds\right)^{2/\sigma} \\ &\leq \left(\int_0^T (\|v(s)\|_{b,p} + \|u(s)\|_{b,p})^{\sigma} ds\right)^{2/\sigma} \left(\int_0^T \|v(s) - u(s)\|_{b,p}^{\sigma} ds\right)^{2/\sigma}. \end{aligned} \tag{5.11}$$

Our next set of results involve the operator G.

Proposition 6. *Let $1 \leq q' \leq q'' < \infty$, $s' \leq s''$, $1 < \sigma' < \sigma'' < \infty$, and let $1/\sigma' - 1/\sigma'' = 1 - (s'' - s' + n/q' - n/q'')/2$. Then for any $T \in (0, \infty]$, G maps $L^{\sigma'}((0,T) : H^{s',q'})$ continuously into $L^{\sigma''}((0,T) : H^{s'',q''})$.*

Using Proposition 3, we observe that

$$\|Gu(t)\|_{s'',q''} \leq C \int_0^T |t-s|^{(s''-s'+m/q'-m/q'')/2} \|u(s)\|_{s',q'} ds \tag{5.12}$$

where $1/r = (s'' - s' + n/p' - n/p'')/2$. Applying the Hardy-Littlewood-Sobolev Theorem, we have

$$(\int_0^T \|Gu\|_{s'',q''}^{\sigma''} dt)^{1/\sigma''} \leq C\|f\|_{L^{\sigma'}(I)} \tag{5.13}$$

where $f(t) = \|u(t)\|_{s',q'}$. This completes the proof.

Our next result also involves the operator G.

Proposition 7. *Let* $1 < q' \leq q'' < \infty$, $s' \leq s''$ *and assume* $1/q'' \leq 1/\sigma = 1 - (s'' - s' + n/q' - n/q'')/2 \leq 1$. *Then* G *maps* $L^\sigma((0,T) : H^{s',q'})$ *continuously into* $BC([0,T) : H^{s'',q''})$.

The proof is a duality argument, which we include for completeness. We begin by defining H by $(Hf)(s,x) = e^{s\triangle} f(s,x)$. We recall that the dual of the space $L^q((0,T) : L^p)$ is $L^{\bar{q}}((0,T) : L^{\bar{p}})$ (where $\bar{q}$ and $\bar{p}$ denote the conjugate exponents to q and p and $1 \leq p, q < \infty$). So for any $g \in L^\infty((0,T) : H^{-s'',\bar{q}''})$, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} g(s,x) e^{s\triangle}(1-\triangle)^{s''/2} f(s,x)\,dx ds \\
\leq &C \int_0^T \int_{\mathbb{R}^n} e^{s\triangle}(1-\triangle)^{(s''-s')/2} g(s,x)(1-\triangle)^{s'/2} f(s,x)\,dx ds \\
\leq &C \int_0^T \|e^{s\triangle} g(s)\|_{H^{s''-s',\bar{q}'}} \|f(s)\|_{H^{s',q'}}\,ds \\
\leq &C\Big(\int_0^T \|e^{s\triangle} g(s)\|^{\bar{\sigma}}_{H^{s''-s',\bar{q}'}}\,ds\Big)^{1/\bar{\sigma}} \Big(\int_0^T \|f(s)\|^{\sigma}_{H^{s',q'}}\,ds\Big)^{1/\sigma} \\
\leq &C \sup_s \|g(s)\|_{\bar{q}''} \Big(\int_0^T \|f(s)\|^{\sigma}_{H^{s',q'}}\,ds\Big)^{1/\sigma},
\end{aligned} \tag{5.14}
$$

where the last line is a slight generalization of Proposition 4. Since g is an arbitrary element of the dual space of $L^1((0,T) : H^{s'',q''})$, we have

$$
\int_0^T \|e^{s\triangle} f(s)\|_{H^{s'',q''}}\,ds \leq C\Big(\int_0^T \|f(s)\|^{\sigma}_{H^{s',q'}}\,ds\Big)^{1/\sigma}. \tag{5.15}
$$

To finish the proposition, making liberal use of the change of variables formula and using (5.15), we have

$$
\begin{aligned}
\|G \cdot f(t)\|_{s'',q''} &= \Big\|\int_0^t e^{(t-s)\triangle} f(s)\,ds\Big\|_{s'',q''} \\
&\leq \int_0^t \|e^{(t-s)\triangle} f(s)\|_{s'',q''}\,ds \\
&= \int_0^t \|e^{s\triangle} f(t-s)\|_{s'',q''}\,ds \\
&\leq C\Big(\int_0^t \|f(t-s)\|^{\sigma}_{s',q'}\,ds\Big)^{1/\sigma} \\
&\leq C\Big(\int_0^T \|f(s)\|^{\sigma}_{s',q'}\,ds\Big)^{1/\sigma},
\end{aligned} \tag{5.16}
$$

which proves the proposition.

6. Proof of Theorem *B*

As in Section 4, we begin with the nonlinear map

$$
\Phi u = \Gamma\varphi - G \cdot P^\alpha(V^\alpha(u)) \tag{6.1}
$$

and the space $F_{T,M}$ defined to be the space of all

$$v \in BC([0,T) : H^{r,p}) \cap L^a((0,T) : H^{k,c}) \tag{6.2}$$

such that

$$\sup_{0\le t\le T} \|v(t) - \Gamma\varphi\|_{r,p} + \left(\int_0^T \|v(s)\|_{k,c}^a ds\right)^{1/a} \le M. \tag{6.3}$$

Using the same argument used in Section 4, we get that Φ will be a contraction mapping provided the following list of conditions is satisfied

$$\begin{aligned}
&1 < \bar{c} \le p \le c < \infty\\
&s' := k - 2 + b' - b\\
&k \ge 1,\ b' \ge 1,\ s'c < n\\
&0 < 2/a = k - n/c - b < 1\\
&0 \le s' \le k-1\\
&\bar{c} = \frac{nc}{2n - s'c}\\
&1 \ge b' - b\\
&k - b' \le \frac{n}{p} + b \le k\\
&a/2 \le p \le a.
\end{aligned} \tag{6.4}$$

We observe that as in the previous case, these conditions require that $b \ge 0$. We also record the simplified list that arises from setting $b' = 1$:

$$\begin{aligned}
&1 < \bar{c} \le p \le c < \infty\\
&s' := k - 1 - b\\
&k \ge 1,\ s'c < n\\
&0 < 2/a = k - n/c - b < 1\\
&0 \le k - 1 - b\\
&k = 1 + b + \frac{2n}{c} - n\bar{c}\\
&k - 1 \le \frac{n}{p} + b \le k\\
&a/2 \le p \le a.
\end{aligned} \tag{6.5}$$

We record the result for the special case $p = 2$, $n = 3$.

Theorem 6. *For any $\varphi = P^\alpha\varphi \in H^{3/2,2}$ there is a unique global solution to* (1.2) *such that*

$$u \in BC([0,T) : H^{3/2,2}) \cap L^{5/2}((0,T) : H^{2,5/2}). \tag{6.6}$$

The local solution follows by choosing $p = k = c = 2$, $n = 3$, $b = 0$, $b' = s' = 1$ and $a = 4$.

The local result extends to a global result via an argument similar to the one used in Section 7, and the continuous dependence on the initial data follows from the argument in 4.

7. Global Existence in Sobolev space

In this section we extend the result from Theorem 3 to a global existence result. We recall that Theorem 3 says that, given $\varphi \in H^{3/2,2}(\mathbb{R}^3)$, there exists a T and a unique solution u to (1.2) such that

$$u \in \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2}, \tag{7.1}$$

where we again recall that the definition of C implies a choice of T. Following the work in [4], we will extend this to a global result.

Theorem C. *Let $\phi \in H^{3/2,2}(\mathbb{R}^3)$ and let*

$$u \in \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2} \tag{7.2}$$

be the unique solution to (1.2) *with initial data φ on the time strip* $[0, T)$. *Then* $u \in L^\infty([0,T], \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2})$.

Proof. We start by recalling (1.2)

$$\partial_t u + (u \cdot \nabla)u + \text{div } \tau^\alpha u = -(1 - \alpha^2 \triangle)^{-1} \nabla p + \nu \triangle \tag{7.3}$$

and stating an equivalent form (see Section 3 of [4])

$$\begin{aligned} &\partial_t (1 - \alpha^2 \triangle)u + \nabla_u [(1 - \alpha^2 \triangle)u] - \alpha^2 (\nabla u)^T \cdot \triangle u \\ = &- (1 - \alpha^2 \triangle)Au - \nabla p. \end{aligned} \tag{7.4}$$

To start, we take the L^2 product of (7.4) with u. We get

$$I_1 + I_2 + I_3 = J_1 + J_2 \tag{7.5}$$

where

$$\begin{aligned} I_1 &= (\partial_t (1 - \alpha^2 \triangle)u, u) \\ I_2 &= (\nabla_u u, u) \\ I_3 &= -\alpha^2 \left((\nabla_u \triangle u, u) + ((\nabla u)^T \cdot \triangle u, u) \right) \\ J_1 &= -((1 - \alpha^2 \triangle)(Au), u) \\ J_2 &= (\nabla p, u). \end{aligned} \tag{7.6}$$

We start with I_1, which becomes

$$\begin{aligned} I_1 &= (\partial_t u, u) - \alpha^2 (\triangle \partial_t u, u) \\ &= \frac{1}{2} \partial_t (\|u\|_{L^2}^2 + \alpha^2 \|A^{1/2} u\|_{L^2}^2). \end{aligned} \tag{7.7}$$

Applying integration by parts to I_2, I_3, and J_2 and recalling that div $u = 0$, we get that all three terms vanish. For J_1 we have

$$J_1 = -((1 - \alpha^2 \triangle)(Au), u) = -(A^{1/2} u, A^{1/2} u) - \alpha^2 (Au, Au). \tag{7.8}$$

Applying this to (7.4), we get

$$\frac{1}{2}\partial_t(\|u(t)\|_{L^2}^2 + \alpha^2\|u(t)\|_{\dot{H}^{1,2}}^2) \leq -(\|A^{1/2}u(t)\|_{L^2}^2 + \alpha^2\|Au(t)\|_{L^2}^2), \tag{7.9}$$

where $\dot{H}$ denotes the homogeneous Sobolev norm. This proves that $\|u(t)\|_{H^{1,2}}$ is decreasing in time.

For our next estimate, we will apply A to (1.2) and take the L^2 product with Au to get

$$(\partial_t Au, Au) + (A^2u, Au) + (AP^\alpha(\nabla_u u + \text{div } \tau^\alpha u, Au) = 0. \tag{7.10}$$

The first piece satisfies

$$(\partial_t Au, Au) = \frac{1}{2}\partial_t\|Au\|_{L^2}^2 \tag{7.11}$$

and the second satisfies

$$(A^2u, Au) = (A^{3/2}u, A^{3/2}) = \|A^{3/2}\|_{L^2}^2. \tag{7.12}$$

To handle the last term of (7.10), we write it as

$$(AP^\alpha(\nabla_u u), Au) + (AP^\alpha \text{div } \tau^\alpha u, Au) = K_1 + K_2. \tag{7.13}$$

To proceed, we will need two inequalities. The first is the well known Sobolev embedding:

$$\|u\|_{L^\infty} \leq C\|u\|_{H^{k,2}} \tag{7.14}$$

provided $2k > 3$. The second is called a Ladyzhenskaya inequality, and is (5.3) in [4]:

$$\|u\|_{\dot{H}^i} \leq C\|u\|_{L^2}^{1-i/m}\|u\|_{\dot{H}^m}^{i/m}, \tag{7.15}$$

where $\dot{H}$ is the homogeneous Sobolev space.

Starting with K_1, we have

$$\begin{aligned}(AP^\alpha(\nabla_u u), Au) &= (A^{1/2}(\nabla u \cdot u), A^{3/2}u)\\ &\leq C\|A^{3/2}u\|_{L^2}(\|(A^{1/2}\nabla u)u\|_{L^2} + \|(A^{1/2}u)\nabla u\|_{L^2})\\ &\leq C\|u\|_{\dot{H}^3}(\|u\|_{L^\infty}\|A^{1/2}\nabla u\|_{L^2} + \|A^{1/2}u\|_{L^\infty}\|\nabla u\|_{L^2})\\ &\leq C\|u\|_{\dot{H}^3}\left(\|u\|_{L^\infty}\|u\|_{\dot{H}^2} + \|A^{1/2}u\|_{L^\infty}\|u\|_{\dot{H}^1}\right).\end{aligned} \tag{7.16}$$

By Sobolev embedding and the observation that $\|u\|_{H^{s,p}} \leq \|u\|_{L^p} + \|u\|_{\dot{H}^{s,p}}$, we have

$$\begin{aligned}\|u\|_{L^\infty} &\leq C\|u\|_{H^{k_1}} \leq C(\|u\|_{L^2} + \|u\|_{\dot{H}^{k_1}}) \leq C(\|u\|_{H^1} + \|u\|_{\dot{H}^{k_1}})\\ \|A^{1/2}u\|_{L^\infty} &\leq C\|u\|_{H^{k_2}} \leq C(\|\nabla u\|_{L^2} + \|u\|_{\dot{H}^{k_2}}) \leq C(\|u\|_{H^1} + \|u\|_{\dot{H}^{k_2}})\end{aligned} \tag{7.17}$$

where $k_1 = 3/2 + \varepsilon$ and $k_2 = 5/2 + \delta$ for positive numbers ε and δ. So (7.16) becomes

$$\begin{aligned}(AP^\alpha(\nabla_u u), Au) \leq{}& C\|u\|_{\dot{H}^3}\|u\|_{\dot{H}^2}\|u\|_{H^1} + C\|u\|_{\dot{H}^3}\|u\|_{\dot{H}^2}\|u\|_{\dot{H}^{k_1}}\\ &+ C\|u\|_{\dot{H}^3}\|u\|_{H^1}\|u\|_{\dot{H}^{k_2}} + C\|u\|_{\dot{H}^3}\|u\|_{H^1}^2.\end{aligned} \tag{7.18}$$

By (7.15), we have

$$
\begin{aligned}
\|u\|_{\dot H^2} &= \|\nabla u\|_{\dot H^1} \le C\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{\dot H^2}^{1/2} \le C\|u\|_{\dot H^1}^{1/2}\|u\|_{\dot H^3}^{1/2} \\
\|u\|_{\dot H^{k_1}} &= \|\nabla u\|_{\dot H^{k_1-1}} \le C\|u\|_{\dot H^1}^{1-(k_1-1)/2}\|u\|_{\dot H^3}^{(k_1-1)/2} \\
\|u\|_{\dot H^{k_2}} &= \|\nabla u\|_{\dot H^{k_2-1}} \le C\|u\|_{\dot H^1}^{1-(k_2-1)/2}\|u\|_{\dot H^3}^{(k_2-1)/2}.
\end{aligned} \tag{7.19}
$$

Applying (7.19) to (7.18), we have

$$
\begin{aligned}
(AP^\alpha(\nabla_u u), Au) \le & C\|u\|_{\dot H^3}^{1+k_1/2}\|u\|_{\dot H^1}^{2-k_1/2} + C\|u\|_{\dot H^3}^{(k_2+1)/2}\|u\|_{\dot H^1}^{(k_2+3)/2} \\
& + C\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2.
\end{aligned} \tag{7.20}
$$

Choosing $\varepsilon = \delta = 1/4$, we get

$$
\begin{aligned}
(AP^\alpha(\nabla_u u), Au) \le & C\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \\
& + C\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2,
\end{aligned} \tag{7.21}
$$

which finishes our K_1 estimate. For K_2, we have

$$
(A(\text{div }\tau^\alpha)(u), Au) \le \|u\|_{\dot H^2}\|A(\text{div }\tau^\alpha)(u)\|_{L^2}. \tag{7.22}
$$

To estimate the second term, we remark that it is sufficient to consider $A(1-\alpha^2\triangle)^{-1}\text{div }(\nabla u\cdot\nabla u)$, and we have

$$
\begin{aligned}
\|A(1-\alpha^2\triangle)^{-1}\text{div }(\nabla u\cdot\nabla u)\|_{L^2} &\le \|\text{div }(\nabla u\cdot\nabla u)\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{\dot H^2}.
\end{aligned} \tag{7.23}
$$

Plugging this back into (7.22) and using (7.17) and (7.19) gives

$$
\begin{aligned}
(A(\text{div }\tau^\alpha)(u), Au) &\le C\|u\|_{\dot H^2}^2\|\nabla u\|_{L^\infty} \\
&\le C(\|u\|_{\dot H^2}^2\|u\|_{H^1} + \|u\|_{\dot H^2}^2\|u\|_{\dot H^{k_2}}) \\
&\le C(\|u\|_{\dot H^3}\|u\|_{H^1}^2 + \|u\|_{\dot H^3}^{15/8}\|u\|_{H^1}^{23/8}).
\end{aligned} \tag{7.24}
$$

Combining (7.21) and (7.24) gives

$$
\begin{aligned}
(AP^\alpha(\nabla_u u + \text{div }\tau^\alpha u), Au) \le & C\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \\
& + C\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2.
\end{aligned} \tag{7.25}
$$

Applying Young's multiplicative inequality with $p = 16/15$ and $p' = 16$, we get

$$
\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \le C\varepsilon\|u\|_{\dot H^3}^2 + \frac{C}{\varepsilon}(\|u\|_{H^1}^{18} + \|u\|_{H^1}^{46}). \tag{7.26}
$$

Choosing $\varepsilon = (4C)^{-1}$, (7.26) becomes

$$
\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \le \frac{1}{4}\|u\|_{\dot H^3}^2 + C(\|u\|_{H^1}^{18} + \|u\|_{H^1}^{46}). \tag{7.27}
$$

Similarly, Young's multiplicative inequality applied to the second term on the right hand side of (7.25) gives

$$
\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2 \le \frac{1}{4}\|u\|_{\dot H^3}^2 + C(\|u\|_{H^1}^6 + \|u\|_{H^1}^4). \tag{7.28}
$$

Using (7.27) and (7.28) in (7.25) gives

$$(AP^{\alpha}(\nabla_u u + \operatorname{div} \tau^{\alpha} u), Au) \leq \frac{1}{2}\|u\|_{\dot{H}^3}^2 + C(\|u\|_{H^1}^{18} + \|u\|_{H^1}^{46} + \|u\|_{H^1}^{6} + \|u\|_{H^1}^{4}). \tag{7.29}$$

Finally, using (7.11), (7.12) and (7.29) in (7.10) gives

$$\begin{aligned} \frac{1}{2}\partial_t \|u(t)\|_{\dot{H}^2}^2 &\leq \frac{-1}{2}\|u(t)\|_{\dot{H}^3}^2 + C(\|u(t)\|_{H^1}^{18} + \|u(t)\|_{H^1}^{46} + \|u\|_{H^1}^{6} + \|u\|_{H^1}^{4}) \\ &\leq \frac{-1}{2}\|u(t)\|_{\dot{H}^3}^2 + C(\|\varphi\|_{H^1}^{18} + \|\varphi\|_{H^1}^{46} + \|\varphi\|_{H^1}^{6} + \|\varphi\|_{H^1}^{4}), \end{aligned} \tag{7.30}$$

where the last line used (7.9). So, for any t such that

$$\|u(t)\|_{\dot{H}^3} \geq C(\|\varphi\|_{H^1}^{18} + \|\varphi\|_{H^1}^{46} + \|\varphi\|_{H^1}^{6} + \|\varphi\|_{H^1}^{4})^{1/2}, \tag{7.31}$$

we get that $\|u(t)\|_{\dot{H}^2}$ is decreasing as a function of time at t. So our last task is to show that $\|u\|_{\dot{H}^2}$ is bounded provided

$$\|u(t)\|_{\dot{H}^3} < C(\|\varphi\|_{H^1}^{18} + \|\varphi\|_{H^1}^{46} + \|\varphi\|_{H^1}^{6} + \|\varphi\|_{H^1}^{4})^{1/2}. \tag{7.32}$$

To handle this case, we again use (7.15), and get

$$\|u(t)\|_{\dot{H}^2} \leq C\|u(t)\|_{L^2}^{1/3}\|u(t)\|_{\dot{H}^3}^{2/3} \leq C\|\varphi\|_{L^2}^{1/3}(\|\varphi\|_{H^1}^{10} + \|\varphi\|_{H^1}^{2} + \|\varphi\|_{H^1}^{6} + \|\varphi\|_{H^1}^{4})^{1/3}. \tag{7.33}$$

Since the right hand side has no time dependence, we get that $\|u(t)\|_{\dot{H}^2}$ is bounded independent of time. Combining this with (7.9), we finally get

$$u \in L^{\infty}([0,T], \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2}), \tag{7.34}$$

which proves the Theorem. ∎

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