HEURISTIC FORMULA FOR LOGARITHM OF THE FROBENIUS MORPHISM

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ABSTRACT. We show that the logarithm $\log_q$ of the Frobenius morphism $x \to x^q$ is given by the formula $x \to x \log x$ (the natural logarithm). In particular, it does not depend on q. This is the explicit (although heuristical) formula for the operator conjectured by Hilbert whose eigenvalues coincide with the zeroes of the zeta function.

The aim of this note is to make a remark clarifying the relation between the Riemann hypothesis on the zeroes of the zeta function [1] and the Weil conjectures proved by P. Deligne (see, for example, [2] for a brief exposition). When trying to prove the Riemann hypothesis, it is natural to take into account its generalizations to algebraic varieties, in particular to varieties over finite field $\mathbb{F}_q$. These Weil conjectures state that the analog of the zeta function of a variety X over $\mathbb{F}_q$, denote it by $\zeta_X(u)$, is a rational function of u whose zeroes and poles lie on the circles of radius $|u| = q^{j/2}$, $0 \leq j \leq \dim X$, and coincide with the eigenvalues of the Frobenius morphism $F : x \mapsto x^q$ on the j-th l-adic cohomology of X. The analogy with the Riemann hypothesis on the Riemann zeta function $\zeta(s)$ is achieved by putting $u = q^s$. Further, Hilbert conjectured that there exists a Hilbert space H with a self-adjoint operator A with discrete spectrum in H such that the eigenvalues of A coincide, up to the transformation $s \to 1/2 + is$, with the critical zeroes of $\zeta(s)$.

Comparing these conjectures, one finds it natural to find the operator B on cohomology of X such that $q^B = F$. The aim of this note is to write down a heuristic formula for the operator B. This can be fruitful for the proof of the Riemann hypothesis.

Consider a 1-parametric discrete semigroup generated by F, $F^n : x \mapsto x^{q^n}$. Assume, heuristically, that this 1-parametric family is extended to a continuous semigroup, $F^t : x \mapsto x^{q^t}$, $t \in \mathbb{R}$, $t \geq 0$. Let us find the infinitesimal generator of this semigroup B such that $F^t = \exp(Bt \log q)$. To this end, it suffices to differentiate F^t with respect to t:

$$\frac{1}{\log q}\frac{d}{dt}x^{q^t} = \frac{1}{\log q}x^{q^t}\log x \cdot q^t \log q = x^{q^t}\log x^{q^t}. \tag{1}$$

In particular, for $t = 0$ we obtain

$$B : x \mapsto x \log x. \tag{2}$$

As expected, this formula does not depend on q.

The aim of further work would be to give a (homological) sense to formula (2). The operator B should coincide with the above mentioned operator A (up to the transform $s \mapsto 1/2 + is$).

Note that this operator B is almost a derivation, i. e., it satisfies the property

$$B(xy) = Bx \cdot y + x \cdot By, \tag{3}$$

but it is not linear: $B(x + y) \neq Bx + By$.

References

[1] E. C. Titchmarsh, The theory of the Riemann zeta function, Oxford, 1951.
[2] R. Hartshorne, Algebraic geometry, Springer, 1977.

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