THE EXPECTED PERIMETER IN EDEN AND RELATED GROWTH PROCESSES

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ABSTRACT. Following Richardson and using results of Kesten on First-passage percolation, we obtain an upper bound on the expected perimeter in an Eden Growth Process. Using results of the author from a problem in Statistical Mechanics, we show that the average perimeter of the lattice animals resulting from a very natural family of "growth histories" does not obey a similar bound.

1. INTRODUCTION

An Eden growth process on $\mathbb{Z}^d$ is a discrete-time Markov process. The state space at step n is the collection of all connected subsets of $\mathbb{Z}^d$ (or *lattice animals*) of size $n + 1$ containing the origin. Given a lattice animal $L(n - 1)$ containing n lattice points, the possible lattice animals at step n are those which can be realized by adding a lattice point from the perimeter of $L(n - 1)$. (Here a lattice point $y \notin L(n - 1)$ is on the perimeter of $L(n - 1)$ if there exists $x \in L(n - 1)$ such that $\{x, y\}$ is a nearest-neighbor pair. We will also call such a nearest-neighbor pair a *perimeter edge*.) In Eden's original formulation (which is the only one we will consider here), the probability of choosing any particular lattice point y on the perimeter is

$$(1.1) \qquad \frac{\text{number of perimeter edges containing } y}{\text{total number of perimeter edges for } L(n - 1)}.$$

Computer simulations of a two-dimensional Eden growth process demonstrate that the typical lattice animal containing a large number of lattice points grown by such a method is very nearly a ball [5]. In addition, if the radius of this "ball" is t, then simulations also suggest that nearly all of the perimeter sites are contained in a surface layer of thickness $t^{\frac{1}{3}}$. It is straightforward to turn this estimate of the surface layer thickness into an upper bound on the expected perimeter. Results of Kesten [3] on first-passage percolation and a method of Richardson [4] for associating an Eden growth process with a continuous-time process imply an upper bound on the thickness of the surface layer, and so we are able to demonstrate the following.

Theorem 1. *The expected perimeter in a d-dimensional Eden growth process is bounded above by $Kn^{1 - \frac{1}{d(2d+5)+1}}$ for some constant K.*

2. First-passage Percolation and the Eden Growth Process

We follow [3] for the setup of first-passage percolation, and then mention the key results we will need. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\{\tau(\{x, y\}) \mid x, y \in \mathbb{Z}^d, \operatorname{dist}(x, y) = 1\}$ a collection of independent random variables on Ω each having the exponential distribution with rate 1. A *path* r from $x \in \mathbb{Z}^d$ to $y \in \mathbb{Z}^d$ is a finite sequence of ordered pairs $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ such that $x_1 = x$ and $x_n = y$. The passage time of such a path r is

$$(2.1) \qquad T(r)(\omega) := \sum_{i=1}^{n-1} \tau(\{x_i, x_{i+1}\})(\omega).$$

The passage time from x to y is

$$(2.2) \qquad T(x, y)(\omega) := \inf\{T(r)(\omega) \mid r \text{ is a path from } x \text{ to } y\},$$

and we define

$$(2.3) \qquad \tilde{B}(t) = \{v \mid T(0, v) \leq t\}$$

$$(2.4) \qquad B(t) = \left\{v + x \mid v \in \tilde{B}(t), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d\right\}.$$

In (Kesten reference) Kesten shows that there exists a compact convex set $B_0 \subset \mathbb{R}^d$ with nonempty interior such that for $t \geq 1$:

$$(2.5) \qquad \mathcal{P}\left\{\frac{B(t)}{t} \subset \left(1 + \frac{x}{\sqrt{t}}\right)B_0\right\} \geq 1 - C_1 t^{2d} e^{-C_2 x} \quad \text{if} \quad x \leq \sqrt{t} \quad \text{and}$$

$$(2.6) \qquad \mathcal{P}\left\{\left(1 - C_3 t^{-\frac{1}{2d+4}} (\log t)^{\frac{1}{d+2}}\right)B_0 \subset \frac{B(t)}{t}\right\}$$
$$\geq 1 - C_4 t^d \exp\left(-C_5 t^{\frac{d+1}{2d+4}} (\log t)^{\frac{1}{d+2}}\right).$$

Moreover,

$$(2.7) \qquad \mathcal{P}\left\{\left(1 - 2C_3 t^{-\frac{1}{2d+4}} (\log t)^{\frac{1}{d+2}}\right)B_0 \subset \frac{B(t)}{t}\right.$$
$$\left. \subset \left(1 + C_6 \frac{\log t}{\sqrt{t}}\right)B_0 \text{ for all large } t\right\} = 1.$$

We would like to relate these results on first-passage percolation to the Eden growth process.

For almost every ω it is the case that for every $z \in \mathbb{Z}^d$, $\infty > T(0, z) = T(r)$ for some path r. (Otherwise, the infimum in the definition of $T(0, z)$ (see (2.2)) is not

achieved for some z and, therefore, there are infinitely many paths from 0 to z with passage time less that $T(0, z) + 1$. Hence, $B(T(0, z) + 1)$ is not contained in any ball. By (2.7) this can happen only for ω in a set of probability 0. Since $\mathbb{Z}^d$ is countable, the result follows.) Also, it is easy to see that for almost every ω, $T(r_1) \neq T(r_2)$ for all pairs of paths such that $r_1 \neq r_2$. Thus, to almost every ω there exists a unique increasing sequence of times $0 < t_1 < t_2 < \ldots$ such that:

$$(2.8) \qquad \lim_{n \to \infty} t_n = \infty$$

$$(2.9) \qquad |\tilde{B}(t_1)| = 2$$

$$(2.10) \qquad |\tilde{B}(t_n) - \tilde{B}(t_{n-1})| = 1$$

$$(2.11) \qquad |\tilde{B}(t_n) - \lim_{t \uparrow t_n} \tilde{B}(t)| = 1 \,.$$

It is clear that $\tilde{B}(t_n)$ is a lattice animal $L(n)$ of size $n + 1$ and that t_n is the smallest t for which $\tilde{B}(t)$ contains $n + 1$ points. So, to almost every ω we can associate a unique sequence of lattice animals $L(0)(\omega), L(1)(\omega), L(2)(\omega), L(3)(\omega), \ldots$ such that:

   (1) $L(0)(\omega)$ is the origin;
   (2) $|L(n + 1)(\omega) - L(n)(\omega)| = 1$;
   (3) $L(n + 1)(\omega) - L(n)(\omega)$ is a perimeter vertex of $L(n)(\omega)$.

Without loss of generality, we redefine Ω to be the set containing only those ω which satisfy the desired properties outlined in the previous two paragraphs.

Lemma 2.1. *The sequence of random variables $\{L(n)\}_{n=0}^{\infty}$ is an Eden growth process.*

Proof. We will demonstrate that $L(n+1)$ depends only on $L(n)$ and not on $L(0), \ldots, L(n-1)$ and that $\mathcal{P}(\{L(n + 1) = L(n) \cup \{v\}\})$ for some v on the perimeter of $L(n)$ is given by

$$(2.12) \qquad \frac{\text{number of perimeter edges containing } v}{|p(L(n))|}$$

where $p(L(n))$ is the collection of perimeter edges of $L(n)$.

Fix a valid lattice animal evolution $l_0, l_1, l_2, \ldots, l_n$ where l_0 is the origin. (That is, l_j is a lattice animal consisting of l_{j-1} plus a vertex on the perimeter of l_{j-1}.) For any lattice animal L, we make the following definition.

$$(2.13) \qquad E^{\text{int}}(L) := \{e = \{v_1, v_2\} \text{ an edge} \mid v_1, v_2 \in L\}$$

If $|E^{\text{int}}(l_n)| = N$, then define $X_i := \tau(e_i)$, the passage time of the ith interior edge, $i = 1, \ldots, N$ for some enumeration of the edges in $E^{\text{int}}(l_n)$. Similarly, if $|p(l_n)| = M$, then define $Y_j := \tau(g_j), j = 1, \ldots, M$ for some enumeration of the edges in $p(l_n)$. Define $v_s := l_s - l_{s-1}$ for $s = 1, \ldots, n$ and $v_0 := 0$ (the origin). Then, the conditions

$L(1)(\omega) = l_1, \ldots, L(n)(\omega) = l_n$ can be written as conditions on $\mathbf{X} := (X_1, \ldots, X_N)$ and $\mathbf{Y} := (Y_1, \ldots, Y_M)$.

Define

$$(2.14) \quad f_s(\mathbf{X}) := \min \left\{ X_{i_1} + \cdots + X_{i_k} \,|\, (e_{i_1}, \ldots, e_{i_k}) \text{ is a path in } l_s \text{ joining } 0 \text{ to } v_s \right\}.$$

The first set of conditions is

$$(2.15) \qquad\qquad\qquad f_1(\mathbf{X}) < f_2(\mathbf{X}) < \ldots < f_n(\mathbf{X}).$$

For each $g_j \in p(l_n)$, we have the additional conditions

$$(2.16) \qquad Y_j + f_{r_j}(\mathbf{X}) > f_n(\mathbf{X}), \text{ where } g_j = \{v_{r_j}, w_j\}, w_j \in p(l_n).$$

Define $Z_j := Y_j + f_{r_j}(\mathbf{X}) - f_n(\mathbf{X})$. Let $\tilde{\Omega} \subset \Omega$ be the set where (2.15) and (2.16) are satisfied. We want to calculate

$$\mathcal{P}(Z_j \le t \,|\, \tilde{\Omega}) \quad \text{and} \quad \mathcal{P}(Z_1 \le t_1, \ldots, Z_M \le t_M \,|\, \tilde{\Omega}).$$

By definition

$$(2.17) \qquad \mathcal{P}(Z_j \le t \,|\, \tilde{\Omega}) = \frac{\mathcal{P}((Z_j \le t) \cap \tilde{\Omega})}{\mathcal{P}(\tilde{\Omega})} = \frac{\mathbb{E}(\chi_{[Z_j \le t] \cap \tilde{\Omega}})}{\mathbb{E}(\chi_{\tilde{\Omega}})}.$$

Let A be the (Borel) subset of $\mathbb{R}_+^N \times \mathbb{R}_+^M$ where the following inequalities are satisfied:

$$(2.18) \qquad\qquad\qquad 0 < f_1(x) < \ldots < f_n(x)$$
$$(2.19) \qquad\qquad\qquad 0 < y_j + f_{r_j}(x) - f_n(x) \le t$$
$$(2.20) \qquad\qquad 0 < y_k + f_{r_k}(x) - f_n(x) \text{ for } k \in \{1, \ldots, M\} - \{j\}$$

and let $B = \{x \in \mathbb{R}_+^N \,|\, 0 < f_1(x) < \ldots < f_n(x)\}$. Then,

(2.21)

$$\mathbb{E}(\chi_{[Z_j \le t] \cap \tilde{\Omega}}) = \int_{\mathbb{R}^N_+ \times \mathbb{R}^M_+} \chi_A e^{-x_1 - \dots - x_N - y_1 - \dots - y_M} \, dx \, dy$$

$$(2.22) \qquad = \int_{\mathbb{R}^N_+} e^{-x_1 - \dots - x_N} \left(\int_{\mathbb{R}^M_+} \chi_A(x, y) e^{-y_1 - \dots - y_M} \, dy \right) dx$$

$$(2.23) \qquad = \int_B e^{-x_1 - \dots - x_N} \left[\left(\int_{f_n(x) - f_{r_j}(x)}^{t + f_n(x) - f_{r_j}(x)} e^{-y_j} \, dy_j \right) \left(\prod_{k \ne j} \int_{f_n(x) - f_{r_k}(x)}^{\infty} e^{-y_k} \, dy_k \right) \right] dx$$

$$(2.24) \qquad = \int_B e^{-x_1 - \dots - x_N} e^{-\sum_{k \ne j} \left[f_n(x) - f_{r_k}(x) \right]} \left[e^{-\left[f_n(x) - f_{r_j}(x) \right]} - e^{-\left[t + f_n(x) - f_{r_j}(x) \right]} \right] dx$$

$$(2.25) \qquad = \left(1 - e^{-t} \right) \int_B e^{-x_1 - \dots - x_N} e^{-\sum_{k=1}^M \left[f_n(x) - f_{r_k}(x) \right]} \, dx$$

Now let C be the (Borel) subset of $\mathbb{R}^N_+ \times \mathbb{R}^M_+$ where the following inequalities are satisfied:

$$(2.26) \qquad\qquad 0 < f_1(x) < \dots < f_n(x)$$
$$(2.27) \qquad\qquad 0 < y_k + f_{r_k}(x) - f_n(x) \quad \text{for} \quad k \in \{1, \dots, M\}.$$

Then,

$$(2.28) \qquad \mathbb{E}(\chi_{\tilde{\Omega}}) = \int_{\mathbb{R}^N_+ \times \mathbb{R}^M_+} \chi_C(x, y) e^{-x_1 - \dots - x_N - y_1 - \dots - y_M} \, dx \, dy$$

$$(2.29) \qquad = \int_B e^{-x_1 - \dots - x_N} \left(\prod_{k=1}^M \int_{f_n(x) - f_{r_k}(x)}^{\infty} e^{-y_k} \, dy_k \right) dx$$

$$(2.30) \qquad = \int_B e^{-x_1 - \dots - x_N} e^{-\sum_{k=1}^M \left[f_n(x) - f_{r_k}(x) \right]} \, dx$$

Thus,

$$(2.31) \qquad\qquad \mathcal{P}(Z_j \le t \,|\, \tilde{\Omega}) = \left(1 - e^{-t} \right).$$

A completely analogous calculation gives

$$(2.32) \qquad\qquad \mathcal{P}(Z_1 \le t_1, \dots, Z_M \le t_M \,|\, \tilde{\Omega}) = \prod_{k=1}^M \left(1 - e^{-t_k} \right).$$

Now note that the random variable Z_k is the expected additional waiting time (beyond the time the $(n + 1)^{\text{st}}$ site was added to the lattice animal evolution) for a path containing the perimeter edge g_k to be traversed. Equations (2.31) and (2.32) show that these waiting times are i.i.d. and depend only on l_n. Thus, if v is a perimeter

vertex for $L(n)$, the probability that $L(n + 1) = L(n) \cup \{v\}$ is given by (2.12). This proves the lemma.

$\square$

3. An Upper Bound on the Expected Perimeter

Now that we have this link between first-passage percolation and Eden growth processes, we will follow ideas of Davidson and use Kesten's results to say something about the expected perimeter of the lattice animals in an Eden growth process. The strategy is the following. For large n, we will find times t_1 and t_2 between which $\tilde{B}(t)$ is overwhelmingly likely to contain n lattice points. We will then show that between these two times, the boundary sites of $\tilde{B}(t)$ are, with probability nearly 1, contained in a region of very small volume. The desired result will follow.

Lemma 3.1. *Let* $s_1(n) = \inf \{s \mid sB_0 \text{ intersects } n \text{ unit cubes}\}$. *(Here the unit cubes are centered at points of* $\mathbb{Z}^d$.*) Then there exists a constant* K_1 *depending only on* B_0 *and the dimension* d *of the space such that, for large* n, $(s_1(n) + K_1) B_0$ *contains at least* n *cells.*

Proof. Let B_1 be a closed ball centered at the origin and contained in the interior of B_0. Let ρ_1 be the radius of B_1. It is well-known that the map $\Phi : \partial B_1 \to \partial B_0$ that takes $x \in \partial B_1$ to a positive scalar multiple of itself is a bijective Lipschitz continuous map. Let C be the Lipschitz constant. Tessellate $\mathbb{R}^d$ with cubes having edges of length $\frac{1}{t}$, where $\frac{1}{t}$ is less than the distance from B_1 to ∂B_0, the edges are parallel to the coordinate axes and such that the origin of $\mathbb{R}^d$ is the center of some cube. (In other words, just rescale the tiling of $\mathbb{R}^d$ referred to in the statement of the lemma by a factor of $\frac{1}{t}$.) We would like to know the smallest α such that αB_0 contains every cube that intersects B_0.

Let M_1 be a cube that intersects the boundary of B_0, but is not contained in B_0. Suppose $x \in M_1 \cap B_0^c$, and let $y \in M_1 \cap \partial B_0$. Then $\|x - y\| < \frac{\sqrt{d}}{t}$ and

$$
(3.1) \qquad \left\| \frac{\rho_1}{\|x\|} x - \frac{\rho_1}{\|y\|} y \right\| < \frac{\sqrt{d}}{t} .
$$

Further,

$$
(3.2) \qquad C \frac{\sqrt{d}}{t} \geq \left\| \Phi\left(\frac{\rho_1}{\|x\|} x \right) - \Phi\left(\frac{\rho_1}{\|y\|} y \right) \right\| = \left\| \Phi\left(\frac{\rho_1}{\|x\|} x \right) - y \right\| ,
$$

which implies

$$
(3.3) \qquad \left\| x - \Phi\left(\frac{\rho_1}{\|x\|} x \right) \right\| \leq \frac{\sqrt{d}}{t} + C \frac{\sqrt{d}}{t} = (C + 1) \frac{\sqrt{d}}{t} .
$$

Also $\left\| \Phi\left(\frac{\rho_1}{\|x\|} x \right) \right\| > \rho_1$. Thus, if $\alpha \Phi\left(\frac{\rho_1}{\|x\|} x \right) = x$, then

$$\text{(3.4)} \qquad \alpha = \frac{\left\|\Phi\left(\frac{\rho_1}{\|x\|}x\right)\right\| + \left\|x - \Phi\left(\frac{\rho_1}{\|x\|}x\right)\right\|}{\left\|\Phi\left(\frac{\rho_1}{\|x\|}x\right)\right\|}$$

$$\text{(3.5)} \qquad \leq 1 + \frac{(C+1)}{\rho_1}\frac{\sqrt{d}}{t}.$$

If we now stretch by a factor of t, then $t\left(1 + \frac{(C+1)}{\rho_1}\frac{\sqrt{d}}{t}\right)B_0$ will contain every unit cube that intersects tB_0. Letting $K_1 := \frac{(C+1)\sqrt{d}}{\rho_1}$, the lemma is proved. $\qquad\square$

Theorem 1. *In an Eden growth process, the expected perimeter of the lattice animal containing n lattice points is bounded above by $Kn^{1-\frac{1}{d(2d+5)+1}}$ for some constant K.*

Proof. Let s_1 be the smallest s such that sB_0 intersects n cells. Then s_1B_0 contains less than n lattice points and, by the previous lemma, $(s_1 + K_1)B_0$ contains at least n cells, hence, at least n lattice points. So, if s_2 is such that $\text{vol}(s_2B_0) = n$, then

$$\text{(3.6)} \qquad s_1 \leq s_2 \leq s_1 + K_1 =: s_3.$$

To simplify the notation we make the following definitions:

$$\text{(3.7)} \qquad f(t) := C_3 t^{-\frac{1}{2d+4}}(\log t)^{\frac{1}{d+2}}$$

$$\text{(3.8)} \qquad g(t) := C_1 t^{2d} e^{-C_2 t^{\frac{1}{4}}}$$

$$\text{(3.9)} \qquad h(t) := C_4 t^d \exp\left(-C_5 t^{\frac{d+1}{2d+4}}(\log t)^{\frac{1}{d+2}}\right).$$

We also set x in (2.5) equal to $t^{\frac{1}{4}}$. With this inequality in mind, we want to find t_1 such that

$$\text{(3.10)} \qquad t_1\left(1 + t_1^{-\frac{1}{4}}\right) = s_1.$$

It is straightforward to check that, for large enough s_1,

$$\text{(3.11)} \qquad s_1 - s_1^{\frac{3}{4}} < t_1 < s_1 - s_1^{\frac{3}{4}} + \frac{3}{4}s_1^{\frac{1}{2}}.$$

We would like to adjust (2.6) slightly to have something that will be easier to calculate with. Choose a constant $\tilde{C}_3$ such that

$$\text{(3.12)} \qquad C_3 t^{-\frac{1}{2d+4}}(\log t)^{\frac{1}{d+2}} \leq \tilde{C}_3 t^{-\frac{1}{2d+5}} \quad \text{for all } t \geq 1.$$

Then, (2.6) can be replaced with

$$(3.13) \qquad \mathcal{P}\left\{\left(1 - \tilde{C}_3 t^{-\frac{1}{2d+5}}\right) B_0 \subset \frac{B(t)}{t}\right\} \geq 1 - C_4 t^d \exp\left(-C_5 t^{\frac{d+1}{2d+4}} (\log t)^{\frac{1}{d+2}}\right).$$

In light of (2.6), we want to find t_2 such that

$$(3.14) \qquad t_2\left(1 - \tilde{C}_3 t_2^{-\frac{1}{2d+5}}\right) = s_3.$$

Again it is straightforward to check that, for large enough s_3,

$$(3.15) \qquad t_2 < s_3 + \tilde{C}_3 s_3^{\frac{2d+4}{2d+5}} + 2\left(\tilde{C}_3\right)^2 \left(\frac{2d+4}{2d+5}\right) s_3^{\frac{2d+3}{2d+5}}.$$

So, with probability at least $1 - g(t_1) - h(t_2)$, our continuously evolving lattice animal will have exactly n lattice points at some time between t_1 and t_2. Further,

$$(3.16) \qquad t_1\left(1 - \tilde{C}_3 t_1^{-\frac{1}{2d+5}}\right) B_0 \subset B(t_1) \subset t_1\left(1 + t_1^{-\frac{1}{4}}\right) B_0$$

with probability at least $1 - g(t_1) - h(t_1)$, and

$$(3.17) \qquad t_2\left(1 - \tilde{C}_3 t_2^{-\frac{1}{2d+5}}\right) B_0 \subset B(t_2) \subset t_2\left(1 + t_2^{-\frac{1}{4}}\right) B_0$$

with probability at least $1 - g(t_2) - h(t_2)$. So, both (3.16) and (3.17) are both satisfied with probability at least $1 - g(t_1) - g(t_2) - h(t_1) - h(t_2)$. For the Eden growth process defined on the same probability space, we can conclude that the lattice animal $L(n-1)$ will contain all the lattice points contained in $t_1\left(1 - \tilde{C}_3 t_1^{-\frac{1}{2d+5}}\right) B_0$ and will be contained in $t_2\left(1 + t_2^{-\frac{1}{4}}\right) B_0$ with probability at least $1 - g(t_1) - g(t_2) - h(t_1) - h(t_2)$.

Therefore, for large n, with probability at least $1 - g(t_1) - g(t_2) - h(t_1) - h(t_2)$, all of the boundary lattice points of $L(n-1)$ are contained in

$$(3.18) \qquad \left[t_2\left(1 + t_2^{-\frac{1}{4}}\right) + K_1\right] B_0 \setminus \left[t_1\left(1 - \tilde{C}_3 t_1^{-\frac{1}{2d+5}}\right) - K_1\right] B_0.$$

So, the number of boundary lattice points in $L(n-1)$ is bounded above by

$$(3.19) \qquad \text{vol}\left\{\left[t_2\left(1 + t_2^{-\frac{1}{4}}\right) + 2K_1\right] B_0\right\} - \text{vol}\left\{\left[t_1\left(1 - \tilde{C}_3 t_1^{-\frac{1}{2d+5}}\right) - 2K_1\right] B_0\right\}.$$

We note that (3.6) implies that

$$(3.20) \qquad s_3 \leq s_2 + K_1 = \left[\frac{n}{\text{vol}(B_0)}\right]^{\frac{1}{d}} + K_1$$

and

$$(3.21) \qquad s_1 \geq s_2 - K_1 = \left[\frac{n}{\text{vol}(B_0)}\right]^{\frac{1}{d}} - K_1 .$$

Using also (3.11) and (3.15) and performing straightforward but somewhat tedious calculations, we find that, for large enough n, with probability at least $1 - g(t_1) - g(t_2) - h(t_1) - h(t_2)$, the number of boundary lattice points in $L(n-1)$ is bounded above by

$$(3.22) \qquad K_2 \left[\frac{n}{\text{vol}(B_0)}\right]^{1-\frac{1}{d(2d+5)}}$$

for some constant K_2. Therefore, the number of perimeter edges is bounded above by

$$(3.23) \qquad 2dK_2 \left[\frac{n}{\text{vol}(B_0)}\right]^{1-\frac{1}{d(2d+5)}} .$$

For any lattice animal in $\mathbb{Z}^d$ with n lattice points it is always the case that the number of perimeter edges is bounded above by $2dn$. For large t, $g(t) > h(t)$. So, for large n,

$$(3.24) \quad 1 - g(t_1) - g(t_2) - h(t_1) - h(t_2) > 1 - 4g(t_1) > 1 - 4e^{-t_1^{\frac{1}{5}}} > 1 - 4e^{-\left[\frac{1}{2}\left(\frac{n}{\text{vol}(B_0)}\right)^{\frac{1}{d}}\right]^{\frac{1}{5}}} .$$

Thus, for large n, the expected perimeter of $L(n-1)$ is bounded above by

$$(3.25) \quad 1 - 4e^{-\left[\frac{1}{2}\left(\frac{n}{\text{vol}(B_0)}\right)^{\frac{1}{d}}\right]^{\frac{1}{5}}} \cdot 2dK_2 \left[\frac{n}{\text{vol}(B_0)}\right]^{1-\frac{1}{d(2d+5)}} + 4e^{-\left[\frac{1}{2}\left(\frac{n}{\text{vol}(B_0)}\right)^{\frac{1}{d}}\right]^{\frac{1}{5}}} \cdot 2dn$$

$$< K_3 n^{1-\frac{1}{d(2d+5)+1}}$$

for some constant K_3. The result follows. $\qquad \square$

4. Average Perimeter Over Lattice Animal Histories

In the context of a problem in Statistical Mechanics [1], the author found it natural to consider an average perimeter that is closely related to the expected perimeter in an Eden growth process. Let $e_1, e_2, \ldots, e_n$, where $e_j = \{x_j, y_j\}$, be a sequence of edges satisfying the following two conditions:

(1) Either x_1 is the origin or y_1 is the origin.
(2) For $2 \leq j \leq n$, exactly one of $\{x_j, y_j\}$ is in $\{x_1, y_1, \ldots, x_{j-1}, y_{j-1}\}$.
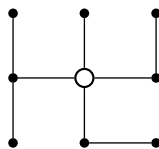
FIGURE 1. A Lattice Animal with "Small" Perimeter and Underlying Spanning Tree
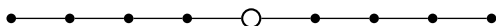

FIGURE 2. A Lattice Animal with "Large" Perimeter and Underlying Spanning Tree

We call such a sequence of edges a *lattice animal history of length n*. We also define $L(e_1, \ldots, e_n) := \{x_1, y_1, \ldots, x_n, y_n\}$, the lattice animal associated with this sequence of edges. What can we say about the average perimeter of $L(e_1, \ldots, e_n)$, where the average perimeter is taken over all lattice animal histories of length n? In an Eden growth process, all lattice animal histories of length n are not equally likely. Therefore we cannot expect the average perimeter in an Eden growth process to be the same as the average perimeter over all lattice animal histories. But should we expect this latter average perimeter to obey a bound similar to that found for the average perimeter in an Eden growth process?

It seems to the author that there is good reason to think so. First, it is plausible that lattice animals with small perimeters will have many more lattice trees spanning their collection of vertices than those with large perimeters. Consider, for example, the two lattice animals shown below. The first has many spanning trees while the second has only one.

In addition, many of the lattice trees that span lattice animals of small perimeter will "branch" often. Trees that branch often will have many different orderings of their edges that form valid lattice animal histories. Consider the spanning trees of the lattice animals from the previous figure shown in figure blank. It can be shown (see [1]) that 1680 orderings of the edges in the first tree form lattice tree histories, while 70 orderings of the edges in the second tree form lattice tree histories.

With these considerations in mind, it is perhaps surprising to find the following.

Theorem 2. *No bound of the form $\bar{p}_n \leq C_1 \cdot n^\alpha$ with $\alpha < 1$ and C_1 independent of n exists for the average perimeter taken over all lattice animal histories of length $n - 1$.*

The proof of this theorem can be found in the author's companion work [1].

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