Orders induced by segments in floorplan partitions and (2-14-3, 3-41-2)-avoiding permutations

Andrei Asinowski * Gill Barequet † Mireille Bousquet-Mélou ‡ Toufik Mansour §
Ron Y. Pinter ¶

November 9, 2010

Abstract

Floorplan partitions are certain tilings of a rectangle by other rectangles. There are natural ways to order their elements (rectangles and segments). In particular, Ackerman, Barequet, and Pinter studied a pair of orders induced by neighborhood relations between *rectangles* of a floorplan partition, and obtained a natural bijection between these pairs and (2-41-3, 3-14-2)-avoiding permutations (also known as Baxter permutations).

In the present paper, we study a pair of orders induced by neighborhood relations between *segments* of a floorplan partition. We obtain a natural bijection between these pairs and another family of permutations, namely (2-14-3, 3-41-2)-avoiding permutations. We also enumerate these permutations, investigate relations between the two kinds of pairs of orders — and correspondingly, between (2-14-3, 3-41-2)-avoiding permutations and Baxter permutations — and study the special case of "guillotine" partitions.

Keywords: Floorplan partitions, Permutation patterns, Baxter permutations, Generating functions.

1 Introduction

A *floorplan partition* is a partition of a rectangle into interior-disjoint rectangles (Fig. 1). It is stipulated that a point may belong to the boundary of at most three rectangles in the partition. In other words, the segments forming a floorplan partition do not cross, and a meeting of segments can have one of the following forms: $\dashv$, $\perp$, $\vdash$, $\top$ (but not $+$). In particular, this implies that the number of (internal) segments in a floorplan partition is less than the number of rectangles by 1. *Throughout the paper, for a given floorplan partition P, the number of segments in P is denoted by n, and accordingly, $n+1$ is the number of rectangles in P.* We say that P has *size* $n+1$. For instance, the trivial partition formed of a single rectangle and no (internal) segment has size 1.

Recently, Ackerman, Barequet, and Pinter studied a representation of *neighborhood relations* between rectangles in floorplan partitions by means of permutations [1]. These neighborhood relations are defined as follows. A rectangle A is a *left-neighbor* of B (equivalently, B is a *right-neighbor* of A) if there is a vertical segment in the partition that contains the right side of A and the left side of B (note that the right side of A and the left side of B may be disjoint). The relation *"A is to the left of B"* (equivalently, B *is to the right of* A), denoted by $A \leftarrow B$, is defined as the transitive closure of the relation "A is a left-neighbor of B."

*Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel. E-mail andrei@tx.technion.ac.il.
†Department of Computer Science, Technion - Israel Institute of Technology, Haifa 32000, Israel, *and* Department of Computer Science, Tufts University, Medford, MA 02155. E-mail barequet@cs.technion.ac.il.
‡CNRS, LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France. E-mail bousquet@labri.fr.
§Department of Mathematics, University of Haifa, Mount Carmel, Haifa 31905, Israel. E-mail toufik@math.haifa.ac.il.
¶Department of Computer Science, Technion - Israel Institute of Technology, Haifa 32000, Israel. E-mail pinter@cs.technion.ac.il.

Finally, the relation $\twoheadleftarrow$ is defined as follows: $A \twoheadleftarrow B$ if $A = B$ or $A \leftarrow B$. The terms A *is a below-neighbor of* B (equivalently, B *is an above-neighbor of* A) and A *is below* B (equivalently, B *is above* A) are defined similarly, as well as the notation $A \downarrow B$ for "A is below B," and $A \twoheaddownarrow B$ for "$A = B$ or $A \downarrow B$." It is not hard to see that the relations $\twoheadleftarrow$ and $\twoheaddownarrow$ are partial orders. In the partitions of Fig. 1, there holds $A \downarrow D$ and $B \leftarrow C$.

Two floorplan partitions P_1 and P_2 of size $n+1$ are said to be *R-equivalent* if there exists a labeling of the rectangles of P_1 by $A_1, A_2, \dots, A_{n+1}$ and a labeling of the rectangles of P_2 by $B_1, B_2, \dots, B_{n+1}$ such that for all $k, m \in [n+1] := \{1, 2, \dots, n+1\}$, the rectangles A_k and A_m stand in the same neighborhood relations as B_k and B_m. See Fig. 1 for an example[1].

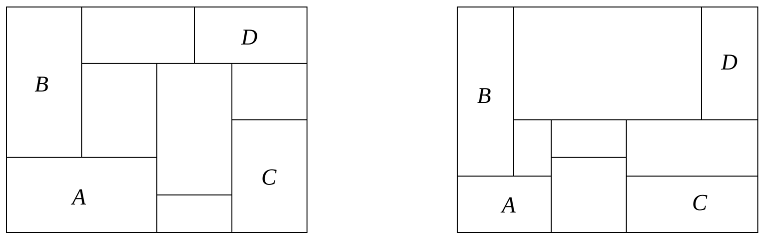


Figure 1: Two R-equivalent floorplan partitions.

The following results are proved (in slightly different terms) in [1]. Let P be a floorplan partition of size $n+1$. Two distinct rectangles A and B of P are in exactly one of the relations $A \leftarrow B$, $B \leftarrow A$, $A \downarrow B$, or $B \downarrow A$. It follows that the relations $\swarrow$ and $\nwarrow$ between rectangles of P defined by

$$\begin{array}{lll} A \swarrow B & \textit{if} & A = B, \textit{ or } A \textit{ is to the left of } B, \textit{ or } A \textit{ is below } B \quad (A = B, \text{ or } A \leftarrow B, \text{ or } A \downarrow B), \\ A \nwarrow B & \textit{if} & A = B, \textit{ or } A \textit{ is to the left of } B, \textit{ or } A \textit{ is above } B \quad (A = B, \text{ or } A \leftarrow B, \text{ or } B \downarrow A), \end{array}$$

are linear orders. Each of these orders can be used for labeling the rectangles of P by $1, 2, \dots, n+1$. In the $\swarrow$ order, the rectangle in the lower left corner will be labeled 1, and the rectangle in the upper right corner $n+1$. In the $\nwarrow$ order, the rectangle in the upper left corner will be labeled 1, and the rectangle in the lower right corner $n+1$. Let $R(P)$ be the sequence $a_1, a_2, \dots, a_{n+1}$, where, for all $1 \le i \le n+1$, a_i is the label in the $\swarrow$ order of the rectangle which is labeled i in the $\nwarrow$ order. It is clear that $R(P)$ is a permutation of $[n+1]$; we call it the *R-permutation* of P. Loosely speaking, $R(P)$ is obtained by labeling the rectangles according to the $\swarrow$ order, and then reading these labels while passing the rectangles according to the $\nwarrow$ order. Fig. 2 shows a floorplan partition and the corresponding R-permutation. Note that by construction, R-equivalent partitions have the same R-permutation.

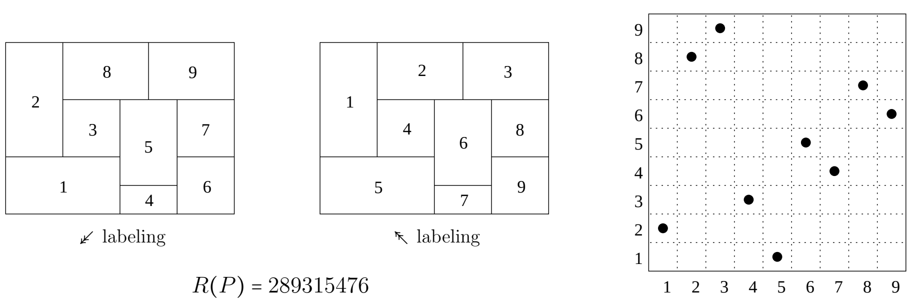


Figure 2: Constructing the R-permutation of a floorplan partition P.

The main result of [1] states that R, regarded as a function from R-equivalence classes of partitions to

[1] In [1], two R-equivalent partitions are considered to be the same.

(2-41-3, 3-14-2)-avoiding permutations[2] (originally known as *Baxter permutations*), is a bijection. Through this correspondence, the number of rectangles becomes the size of the permutation, and the neighborhood relations between rectangles in P can be easily read from $R(P)$. The papers [8, 16, 19] suggest that *bipolar orientations of planar maps*, which are also in bijection with Baxter permutations, provide a convenient geometric description of R-equivalence classes of floorplan partitions. Besides, several other classes of objects in bijection with Baxter permutations are mentioned in [16].

There are many works about floorplan partitions to be found in the literature. Their study in computational geometry [11, 24, 26] is motivated, in particular, by the fact that their generation is a critical stage in integrated circuit layout [22, 23, 27, 34, 35], in architectural designs [7, 14, 18, 30, 31], etc.

In the present work we study neighborhood relations between *segments* forming a floorplan partition. We define these relations in Section 2. This leads to the notion of S-equivalent partitions[3]. Then we construct a permutation from these relations in a way similar to that described above; we call it the *S-permutation* of P, and denote it by $S(P)$. In Section 3 we prove the main results: all S-permutations are (2-14-3, 3-41-2)-avoiding permutations, and S, regarded as a function from S-equivalence classes of floorplan partitions to (2-14-3, 3-41-2)-avoiding permutations, is a bijection. In Section 4 we show that R-equivalence of partitions implies their S-equivalence (this means that S-equivalence classes are unions of R-equivalence classes), and explain how $S(P)$ can be constructed directly from $R(P)$. We also describe in terms of R when two floorplan partitions give rise to the same S-permutation. In Section 5 we construct the generating tree of (2-14-3, 3-41-2)-avoiding permutations and enumerate them. Their generating function turns out to be simply related to that of Baxter permutations, but we have not found any direct combinatorial explanation of this fact. In Section 6 we study in more details S-permutations corresponding to the so-called guillotine partitions, and obtain several results for their multidimensional generalization.

2 Orders between segments of a floorplan partition

In this section we define neighborhood relations between segments in a floorplan partition, and construct from them four order relations:

- two partial orders: ↞ (West – East) and ↡ (South – North); and
- two linear orders: ↙ (South-West – North-East) and ↖ (North-West – South-East);

and prove several facts about these relations (most of which are analogous to the ones for the relations between rectangles mentioned in the introduction, and proved in [1]).

Definition 2.1. Let P be a floorplan partition. Let I and J be two segments in P. We say that I is a *left-neighbor* of J (equivalently, J is a *right-neighbor* of I) if one of the following holds:

- I and J are vertical, there is a rectangle A in P such that the left side of A is contained in I and the right side of A is contained in J; moreover, the vertical projection of A coincides with the intersection of the vertical projections of I and J;
- I is vertical, J is horizontal, and the left endpoint of J lies in I; or
- I is horizontal, J is vertical, and the right endpoint of I lies in J.

The terms *I is a below-neighbor of J* (equivalently, *J is an above-neighbor of I*) and *I is below J* (equivalently, *J is above I*) are defined similarly.

Typical examples are shown in Fig. 3.

Note that a horizontal segment I has at most one left-neighbor and at most one right-neighbor (exactly one when the corresponding endpoint(s) of I do not lie on the boundary), which are both vertical segments.

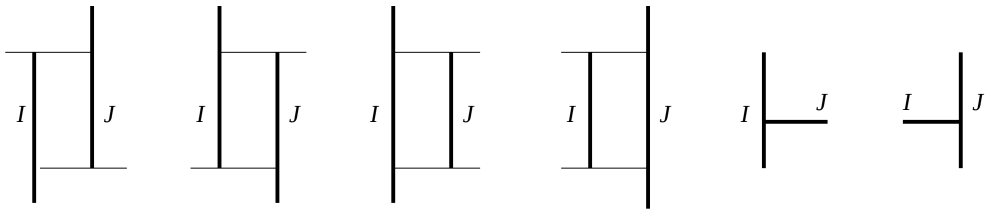


Figure 3: The segment I is a left-neighbor of the segment J.

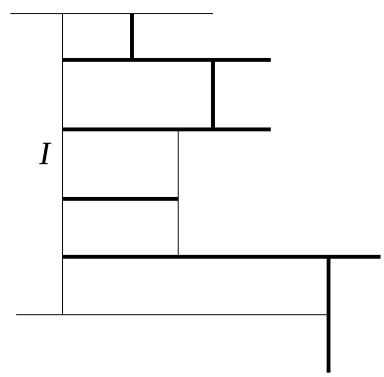


Figure 4: The right-neighbors of a vertical segment I (thick segments).

In contrast, a vertical segment may have several left- and right-neighbors, which may be horizontal or vertical. This is illustrated in Fig. 4.

Definition 2.2. The relation *"I is to the left of J"* (equivalently, *J is to the right of I*), denoted by $I \leftarrow J$, is the transitive closure of the relation "I is a left-neighbor of J." The relation $\twoheadleftarrow$ between segments of P is defined as follows: $I \twoheadleftarrow J$ if $I = J$ or $I \leftarrow J$.

The relations $I \downarrow J$ (for "I is below J") and $I \downarrow\!\!\downarrow J$ (for "$I = J$ or $I \downarrow J$") are defined similarly.

Observation 2.3. *The relations $\twoheadleftarrow$ and $\downarrow\!\!\downarrow$ are partial orders.*

Proof. We prove the claim for the relation $\twoheadleftarrow$.

Reflexivity and transitivity are clear from the definition.

For antisymmetry, note that $I \leftarrow J$ and $J \leftarrow I$ cannot hold simultaneously because if $I \leftarrow J$, then any interior point of I has a smaller abscissa than any interior point of J. □

Definition 2.4. Let P_1 and P_2 be two floorplan partitions of size $n+1$. We say that P_1 and P_2 are *S-equivalent* if it is possible to label the segments of P_1 by $I_1, I_2, \ldots, I_n$ and the segments of P_2 by $J_1, J_2, \ldots, J_n$ so that for all $k, m \in [n]$, the segments I_k and I_m stand in exactly the same neighborhood relations as J_k and J_m.

Fig. 5 shows two floorplan partitions which are S-equivalent: in both cases, the left-right *neighborhood* relations are $1 \leftarrow 4,\ 2 \leftarrow 4,\ 3 \leftarrow 4,\ 4 \leftarrow 5,\ 4 \leftarrow 6$, and the below-above *neighborhood* relations are $2 \downarrow 1,\ 3 \downarrow 2,\ 6 \downarrow 5$. We will see in Section 4 that two R-equivalent partitions are always S-equivalent; but the partitions of the figure are *not* R-equivalent.

[2]This notation is defined in Section 3.2.
[3]Of course, R stands for *rectangles* and S for *segments*.

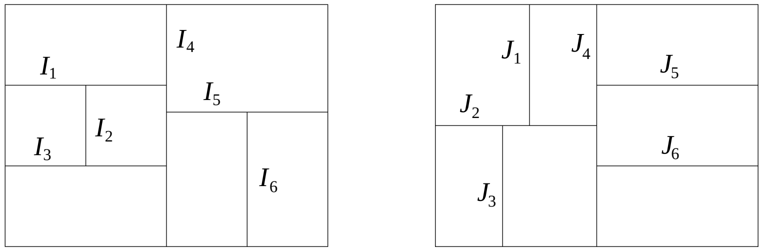


Figure 5: Two S-equivalent (but not R-equivalent) floorplan partitions.

The following observation may help to understand the $\twoheadleftarrow$ order. If I and J are vertical segments and right-left neighbors, let us create a horizontal edge, called *traversing edge*, in the rectangle A that lies between them. Fig. 6 shows a chain of neighbors in the $\twoheadleftarrow$ order, starting from a vertical segment I, and the corresponding traversing edges (dashed lines).

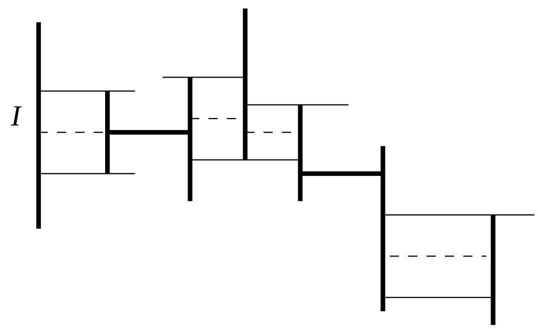


Figure 6: A chain in the $\twoheadleftarrow$ order (thick segments), and the corresponding traversing edges (dashed lines).

Observation 2.5. *Assume $I \twoheadleftarrow J$. Then any point of J lies weakly to the right of any point of I (that is, its abscissa is at least as large).*

Let x (respectively, y) be a point of minimal (respectively, maximal) abscissa on I (respectively, J). Then there exists a path from x to y formed of vertical and horizontal sections, such that

- *each vertical section is part of a vertical segment of P,*
- *each horizontal section is an (entire) horizontal segment of P, or a traversing edge of P, visited from left to right,*
- *if I (respectively, J) is horizontal, it is entirely included in the path.*

It suffices to prove these properties when J is a right-neighbor of I, and they are obvious in this case (see Fig. 3).

Lemma 2.6. *In the $\twoheadleftarrow$ order, J covers I if and only if J is a right-neighbor of I. A similar statement holds for the $\twoheaddownarrow$ order.*

Proof. Since $\twoheadleftarrow$ is constructed as the transitive closure of the right-left neighborhood relation, every covering relation is a neighborhood relation.

Conversely, let us prove that any neighborhood relation is a covering relation. Equivalently, this means that the right-neighbors of any segment I form an antichain. If I is horizontal, it has at most one right-neighbor, and there is nothing to prove. Assume I is vertical (as in Fig. 4), and that two of its neighbors, J_1

and J_2, are comparable (but distinct): $J_1 \leftarrow J_2$. By the first part of Observation 2.5, J_2 cannot be horizontal (because its leftmost point would then lie on I). Hence J_2 is vertical. Let y be a point of J_2, and let x be a point of J_1 of minimal abscissa. Consider the path from x to y described in Observation 2.5. The last horizontal section of this path cannot be a horizontal segment (all segments that end on J_2 start to the left of I). Hence the last horizontal section is a traversing edge. If the vertical projection of J_2 is included in the vertical projection of I (as is the case for the highest two vertical neighbors of I in Fig. 4), any traversing edge ending on J_2 starts on I, so that the path cannot have started from J_1. Thus the vertical projection of J_2 is not included in the vertical projection of I, and there exist points of J_2 that have (for instance) a smaller ordinate than all points of I (as is the case for the lowest vertical neighbor of I in Fig. 4). Then the path from x to y must cross the above-neighbor of J_2. But segments do not cross in a floorplan partition. Hence the neighbors of I form an antichain, and the covering relation coincides with the neighborhood relation. □

Lemma 2.7. *Let I and J be two different segments in a floorplan partition P. Then exactly one of the relations: $I \leftarrow J$, $J \leftarrow I$, $I \downarrow J$, or $J \downarrow I$, holds.*

Proof. We first observe that, as $\leftarrow$ and $\downarrow$ are antisymmetric, there can not hold simultaneously $I \leftarrow J$ and $J \leftarrow I$, nor $I \downarrow J$ and $J \downarrow I$. Assume without loss of generality that I is a horizontal segment. Construct the *NE-sequence* $K_1, K_2, \ldots$ of I as follows (see Fig. 7 for an illustration): K_1 is the right-neighbor of I, K_2 the above-neighbor of K_1, K_3 the right-neighbor of K_2, and so on, until the boundary is reached. Construct similarly the SE-, NW-, and SW-sequences of I. These sequences partition the rectangle into four (or less, if some endpoints of I lie on the boundary) regions; each segment of P (except I and those belonging to either of the sequences) lies in precisely one of them. Also, if J is in the interior of a region, then its neighbors are either in the same region, or in one of the sequences that bound the region.

It is not hard to see that the vertical segments of the NE-sequence are to the right of I, while horizontal segments are above I. A horizontal segment K_{2i} cannot be to the left of I, since it ends to the right of I. Let us prove that K_{2i} cannot be the right of I either. Assume this is the case, and consider the path going from the leftmost point of I to the rightmost point of K_{2i}, as described in Observation 2.5. The last section of this path is K_{2i}. Hence the path has points in the interior of the region comprised between the NW- and NE-sequences. But since the path follows entirely every horizontal segment it visits, it can never enter the interior of this region. Thus K_{2i} cannot be to the right of I. Thus, its only relation to I is $I \downarrow K_{2i}$. Similar arguments apply for vertical segments of the NE-sequence, and for the other three sequences.

Consider now a segment J that lies in the region bounded by the NE-sequence, the NW-sequence, and the boundary. Then I is below J: if we consider the below-neighbors of J, then their below-neighbors, and so on, then we necessarily reach one of the horizontal segments of the NW- or NE-sequence, which, as we have seen, are above I (we cannot reach a vertical segment of the sequences without having reached an horizontal segment first).

To prove that J cannot be to the right of I, we argue as we did for K_{2i}: the path from I to J starting from the leftmost point of I cannot enter the North region. Similarly, J cannot be to the left of I. This completes the proof. □

Definition 2.8. The relations $\Swarrow$ and $\Nwarrow$ between segments of a floorplan partition P are defined as follows:

$$
\begin{array}{lll}
I \Swarrow J & \textit{if} & I = J, \textit{ or } I \textit{ is to the left of } J, \textit{ or } I \textit{ is below } J \quad (I = J, \text{ or } I \leftarrow J, \text{ or } I \downarrow J), \\
I \Nwarrow J & \textit{if} & I = J, \textit{ or } I \textit{ is to the left of } J, \textit{ or } I \textit{ is above } J \quad (I = J, \text{ or } I \leftarrow J, \text{ or } J \downarrow I).
\end{array}
$$

We also write $I \swarrow J$ when $I \Swarrow J$ and $I \neq J$; and $I \nwarrow J$ when $I \Nwarrow J$ and $I \neq J$.

Observation 2.9. *The relations $\Swarrow$ and $\Nwarrow$ are linear orders.*

Proof. We prove the claim for the relation $\Swarrow$.

Reflexivity follows directly from the definition.

Antisymmetry follows from the fact that $\twoheadleftarrow$ and $\twoheaddownarrow$ are order relations, and from Lemma 2.7.

For transitivity, assume that $I \swarrow J$ and $J \swarrow K$. If $I \leftarrow J$ and $J \leftarrow K$ (respectively, $I \downarrow J$ and $J \downarrow K$), then we have $I \leftarrow K$ (respectively, $I \downarrow K$) by the transitivity of $\leftarrow$ (respectively, $\downarrow$). Assume now that $I \leftarrow J$

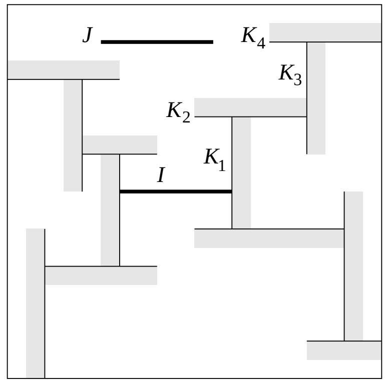


Figure 7: Four regions determining in which relation I stands with other segments of the partition.

and $J \downarrow K$ (the case $I \downarrow J$ and $J \leftarrow K$ is proven similarly). By Lemma 2.7, $I = K$ is impossible, and we have either $I \leftarrow K$, $K \leftarrow I$, $I \downarrow K$, or $K \downarrow I$. However, the combination of $K \leftarrow I$ and $I \leftarrow J$ yields $K \leftarrow J$, contradicting the assumption that $J \downarrow K$. Similarly, combining $K \downarrow I$ with $J \downarrow K$ yields $J \downarrow I$, contradicting the assumption that $I \leftarrow J$. Therefore, we have either $I \leftarrow K$ or $I \downarrow K$; that is, $I \swarrow K$.

Linearity follows from Lemma 2.7. □

Observation 2.10. *The orders $\twoheadleftarrow$ and $\twoheaddownarrow$ can be recovered from $\swarrow$ and $\nwarrow$. Indeed, $I \twoheadleftarrow J$ if and only if $I \swarrow J$ and $I \nwarrow J$; moreover, $I \twoheaddownarrow J$ if and only if $I \swarrow J$ and $J \nwarrow I$.*

The signs $\nwarrow$ and $\swarrow$ are intended to resemble the inequality sign $\leq$. *Throughout the paper, the ith segment in the $\nwarrow$ order ($1 \leq i \leq n$) will be denoted by I_i.* See Fig. 5 for examples.

Consider the segment I_i in a floorplan partition P. The following observation describes the segment I_{i+1} (that is, the immediate successor of I_i in the $\nwarrow$ order). By Lemma 2.6, the segment I_{i+1} is either a right- or below-neighbor of I_i. There are several cases depending on the existence of such neighbors and the relations between them. We use the following notation. For a horizontal segment I, we denote by $\mathrm{R}(I)$ the right-neighbor of I (when it exists). By Lemma 2.6, the below-neighbors of I form an antichain of the $\twoheaddownarrow$ order. Since $\nwarrow$ is a linear order, they are totally ordered for the $\twoheadleftarrow$ order. By the first part of Observation 2.5, the smallest of them is also the leftmost one, denoted $\mathrm{LB}(I)$. Thus $\mathrm{LB}(I)$ is either $\mathrm{LVB}(I)$ (the leftmost *vertical* below-neighbor of I) or $\mathrm{LHB}(I)$ (the leftmost *horizontal* below-neighbor of I). Similarly, for a vertical segment I, we denote by $\mathrm{B}(I)$ the below-neighbor of I; by $\mathrm{UR}(I)$ the highest right-neighbor of I, and by $\mathrm{UHR}(I)$ (respectively, $\mathrm{UVR}(I)$) the highest horizontal (respectively, vertical) right-neighbor of I.

Observation 2.11. *Let I_i be a segment in a floorplan partition P of size $n+1$. If I_i is horizontal, then I_{i+1} is either $\mathrm{R}(I_i)$ or $\mathrm{LB}(I_i)$. More precisely,*

- *If none of $\mathrm{LB}(I_i)$ and $\mathrm{R}(I_i)$ exists, then I_i is the last segment in the $\nwarrow$ order (that is, $i = n$).*
- *If exactly one of $\mathrm{LB}(I_i)$ and $\mathrm{R}(I_i)$ exists, then I_{i+1} is this segment.*
- *As soon as I_i has at least one vertical below-neighbor, $I_{i+1} = \mathrm{LB}(I_i)$. This segment is $\mathrm{LHB}(I_i)$ if it exists, and otherwise $\mathrm{LVB}(I_i)$.*
- *If $\mathrm{LVB}(I_i)$ does not exist but $\mathrm{LHB}(I_i)$ and $\mathrm{R}(I_i)$ exist, then*
  - *If the join of $\mathrm{LHB}(I_i)$ and $\mathrm{R}(I_i)$ is of type $\dashv$, then $I_{i+1} = \mathrm{LHB}(I_i)$.*
  - *If the join of $\mathrm{LHB}(I_i)$ and $\mathrm{R}(I_i)$ is of type $\perp$, then $I_{i+1} = \mathrm{R}(I_i)$.*

If I_i is vertical, then I_{i+1} is either $\mathrm{B}(I_i)$, $\mathrm{UHR}(I_i)$, or $\mathrm{UVR}(I_i)$ (the details are similar to those in the case of a horizontal segment).

See Fig. 8 for an illustration. In all configurations it is assumed that there are no other candidates to be I_{i+1} except those depicted. The dashed lines belong to the boundary.

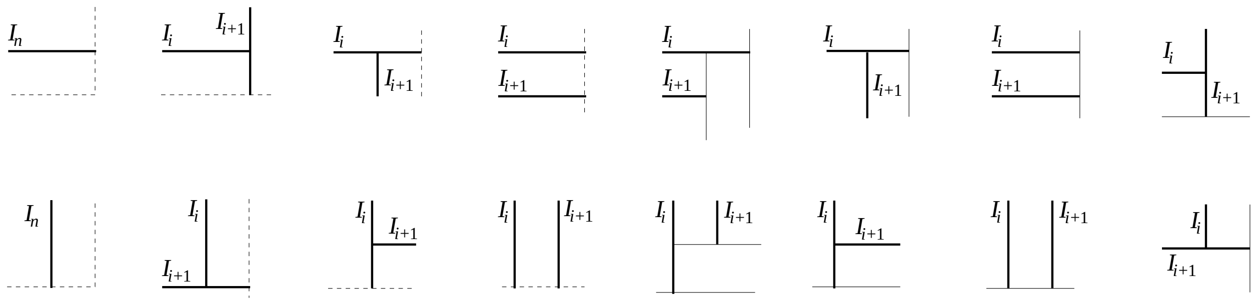


Figure 8: The segment I_{i+1} is the immediate successor of I_i in the $\nwarrow$ order.

The group of symmetries of the square acts on floorplan partitions (one should draw these partitions in a square rather than a rectangle before talking about this action). It is thus worth examining how the orders are transformed when applying such symmetries. As this symmetry group is generated by two generators, for instance the reflections across the first diagonal and across a horizontal line, it suffices to study these two transformations. The following observation easily follows from the neighborhood relations of Fig. 3.

Observation 2.12. *Let P be a floorplan partition, and P' be obtained by reflecting P across the first diagonal. If I is a segment of P, let I' denote the corresponding segment of P'. Then*

$$\begin{array}{lcl} I \twoheadleftarrow J & \Leftrightarrow & I' \twoheaddownarrow J', \\ I \twoheaddownarrow J & \Leftrightarrow & I' \twoheadleftarrow J', \\ I \swarrow J & \Leftrightarrow & I' \swarrow J', \\ I \nwarrow J & \Leftrightarrow & J' \nwarrow I'. \end{array}$$

If instead P' is obtained by reflecting P across a horizontal line,

$$\begin{array}{lcl} I \twoheadleftarrow J & \Leftrightarrow & I' \twoheadleftarrow J', \\ I \twoheaddownarrow J & \Leftrightarrow & J' \twoheaddownarrow I', \\ I \swarrow J & \Leftrightarrow & I' \nwarrow J', \\ I \nwarrow J & \Leftrightarrow & I' \swarrow J'. \end{array}$$

One consequence of this observation is that if P' is obtained by applying a half-turn rotation to P, then $I \twoheadleftarrow J \Leftrightarrow J' \twoheadleftarrow I'$, and similarly for the other three orders. Also, if P' is obtained by applying a clockwise quarter-turn rotation to P, then $I \nwarrow J \Leftrightarrow J' \swarrow I'$.

3 A bijection between S-equivalence classes of floorplan partitions and (2-14-3, 3-41-2)-avoiding permutations

In this section we define a map S from floorplan partitions to permutations. We show that $S(P)$ encodes all neighborhood relations — and hence the four order relations defined in Section 2 — between segments of P. In fact, S can be seen as an injection from S-equivalence classes of floorplan partitions to permutations. We then prove that the class of permutations obtained from floorplan partitions can be described in terms of (generalized) patterns, in a way that is reminiscent of Baxter permutations.

3.1 S-permutations

Let P be a floorplan partition of size $n+1$. There are n segments in P. Let $S(P)$ be the sequence $b_1, b_2, \ldots, b_n$, where b_i is the label in the $\swarrow$ order of the segment labeled i in the $\nwarrow$ order, for all $1 \le i \le n$. Then $S(P)$ is a permutation of $[n] = \{1, 2, \ldots, n\}$; we call it the *S-permutation* of P and denote it by $S(P)$. Equivalently, if $I_1, \ldots, I_n$ is the list of segments in the $\nwarrow$ order, then $I_{\sigma^{-1}(1)}, \ldots, I_{\sigma^{-1}(n)}$ is the list of segments in the $\swarrow$ order. An example is shown in Fig. 9.

Thus, we assign a permutation to a floorplan partition in a way similar to that used in [1], but this time we use order relations between segments rather than rectangles. Note that $S(P)$ is a permutation of $[n]$, while $R(P)$ is a permutation of $[n+1]$.

For σ, a permutation of $[n]$, the *graph of* σ is the point set $\{(i, \sigma(i)) : i \in [n]\}$. By definition of $S(P)$, if a segment of P is labeled i in the $\nwarrow$ order and j in the $\swarrow$ order, then $S(P)(i) = j$. In other words, the graph of $S(P)$ contains the point (i, j), which will be denoted by N_i.

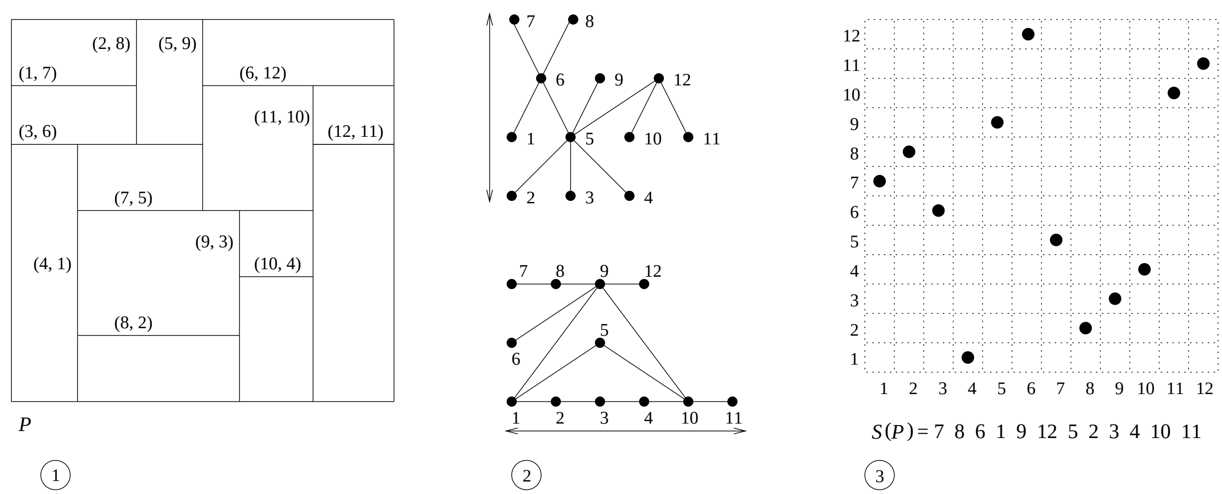


Figure 9: (1) A floorplan partition P, with segments labeled (i, j), where i (respectively, j) is the label according to the $\nwarrow$ (respectively, $\swarrow$) order; (2) Hasse diagrams of the $\downarrow$ and $\twoheadleftarrow$ orders (the minimal elements are at the bottom in the first diagram, but on the left in the second diagram); (3) The corresponding S-permutation.

It follows from Observation 2.12 that the map S has a simple behavior with respect to symmetries.

Observation 3.1. *Let P be a floorplan partition, and P' be obtained by reflecting P across the first diagonal. Let $\sigma = S(P)$ and $\sigma' = S(P')$. Then σ' is obtained by reading σ from right to left or equivalently, by reflecting the graph of σ across a vertical line.*

If instead P' is obtained by reflecting P across a horizontal line, then $\sigma' = \sigma^{-1}$. Equivalently, σ' is obtained by reflecting the graph of σ across the first diagonal.

Since these two reflections generate the group of symmetries of the square, we can describe what happens for the other elements of this group: applying a rotation to P boils down to applying the same rotation to $S(P)$, and reflecting P across Δ, a symmetry axis of the bounding square, boils down to reflecting $S(P)$ across Δ', a line obtained by rotating Δ of 45° in counterclockwise direction. These properties will be extremely useful to decrease the number of cases we have to study in certain proofs.

We will now prove that $S(P)$ characterizes the S-equivalence class of P. Clearly, two S-equivalent partitions give rise to the same orders, and thus to the same S-permutation. Conversely, let us define neighborhood relations between points in the graph of a permutation as follows. Let σ be a permutation.

Let $N_i = (i, \sigma(i))$, $N_j = (j, \sigma(j))$ be two points in the graph of σ. If $i < j$ and $\sigma(i) < \sigma(j)$, then *the point N_j is to the NE of the point N_i*. If, in addition, there is no i' such that $i < i' < j$ and $\sigma(i) < \sigma(i') < \sigma(j)$, then *the point N_j is a NE-neighbor of the point N_i*. In a similar way we define when *the point N_j is to the SE / SW / NW of the point N_i*, and when *the point N_j is a SE- / SW- / NW-neighbor of the point N_i*. For example, in the graph in Fig. 9(3), the points $(1,7)$, $(2,8)$, $(3,6)$, $(5,9)$ and $(6,12)$ are to the NW of $N_7 = (7,5)$; among them, $(3,6)$, $(5,9)$ and $(6,12)$ are NW-neighbors of N_7.

The neighborhood relations between segments of P correspond to the neighborhood relations in the graph of $S(P)$ in the following way.

Observation 3.2. *Let P be a floorplan partition, and let I_i and I_j be two segments in P.*

The segment I_j is to the right of I_i if and only if the point N_j lies to the NE of N_i. Consequently, I_j is a right-neighbor of I_i if and only if N_j is a NE-neighbor of N_i.

Similar statements hold for the other directions: left (respectively, above, below) neighbors in segments correspond to SW- (respectively, NW-, SE-) neighbors in points.

Proof. The segment I_j is to the right of I_i if and only if $I_i \swarrow I_j$ and $I_i \nwarrow I_j$. By construction of $S(P)$, this means that $i < j$ and $S(P)(i) < S(P)(j)$. Equivalently, N_j lies to the NE of N_i. ☐

Remark. An analogous fact holds for *rectangles* of a floorplan partition and points in the graph of the corresponding *R-permutation*. It is not stated explicitly in [1], but follows directly from the definitions in the same way as our Observation 3.2 does.

Since the neighborhood relations characterize the S-equivalence class, we have proved the following result.

Corollary 3.3. *Two floorplan partitions are S-equivalent if and only if they have the same S-permutation.*

3.2 (2-14-3, 3-41-2)-Avoiding permutations

In this section we first discuss the *dash notation* and *bar notation* for pattern avoidance in permutations, and then prove several facts about (2-14-3, 3-41-2)-avoiding permutations. As we will see in Section 3.3, these are precisely the permutations obtained from floorplan partitions.

In the classical notation, a permutation $\pi = a_1 a_2 \dots a_n$ *avoids* a permutation (*a pattern*) $\tau = b_1 b_2 \dots b_k$ if there are no $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $a_{i_1} a_{i_2} \dots a_{i_k}$ (a subpermutation of π) is order isomorphic to τ ($b_x < b_y$ if and only if $a_{i_x} < a_{i_y}$).

The dash notation and the bar notation generalize the classical notation and provide a convenient way to define more classes of restricted permutations. For a recent survey, see [32].

In the dash notation, some letters corresponding to those from the pattern τ may be required to be adjacent in the permutation π, in the following way. If there is a dash between two letters in τ, the corresponding letters in π may occur at any distance from each other; if there is no dash, they must be adjacent in π. For example, $\pi = a_1 a_2 \dots a_n$ avoids 2-14-3 if there are no $1 \le i < j < \ell < m$ such that $\ell = j + 1$ and $a_j < a_i < a_m < a_\ell$.

In the bar notation, some letters of τ may have bars. A permutation π avoids a barred pattern τ if every occurrence of the unbarred part of τ is a sub-occurrence of τ (with bars removed). For example, $\pi = a_1 a_2 \dots a_n$ avoids $21\bar{3}54$ if for any $1 \le i < j < \ell < m$ such that $a_j < a_i < a_m < a_\ell$, there exists k such that $j < k < \ell$ and $a_i < a_k < a_m$ (any occurrence of the pattern 2154 is a sub-occurrence of the pattern 21354).

A *Baxter permutation*, first defined in [6], is a permutation of $[n]$ satisfying the following condition:

There are no $i, j, \ell, m \in [n]$ satisfying $i < j < \ell < m$, $\ell = j + 1$, such that
either $\pi(j) < \pi(m) < \pi(i) < \pi(\ell)$ and $\pi(i) = \pi(m) + 1$,
or $\pi(\ell) < \pi(i) < \pi(m) < \pi(j)$ and $\pi(m) = \pi(i) + 1$.

In the dash notation, Baxter permutations are those avoiding (2-41-3, 3-14-2), and in the bar notation, Baxter permutations are those avoiding $(41\bar{3}52, 25\bar{3}14)$ (see [20] or [32, Sec. 7]). As proved in [1], the permutations that are obtained as R-permutations are precisely the Baxter permutations. It turns out that the permutations that are obtained as S-permutations may be defined by similar conditions, given below in Proposition 3.5. As in the Baxter case, these conditions can be defined in three different ways.

Lemma 3.4. *Let π be a permutation of $[n]$. The following conditions are equivalent:*

1. *There are no $i,j,\ell,m \in [n]$ such that $i < j < \ell < m$, $\ell = j+1$, $\pi(j) < \pi(i) < \pi(m) < \pi(\ell)$, $\pi(m) = \pi(i)+1$.*
2. *In the dash notation, π avoids* 2-14-3.
3. *In the bar notation, π avoids* $21\bar{3}54$.

Fig. 10 illustrates these three conditions. The rows (respectively, columns) marked by dots in parts (1) and (2) denote adjacent rows (respectively, columns). The shaded area in part (3) does not contain points of the graph.

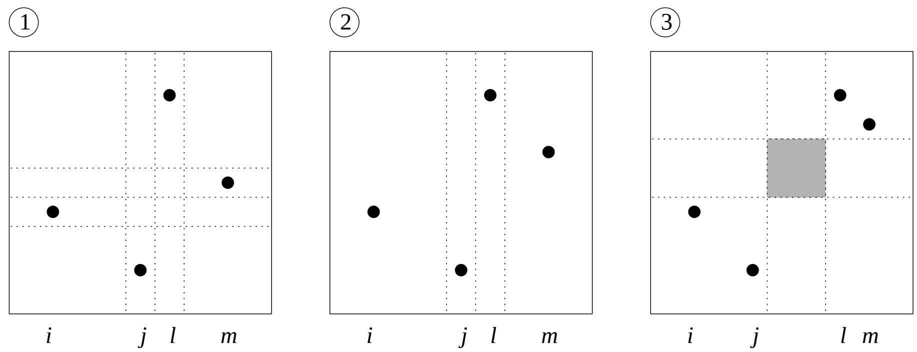


Figure 10: Three ways to define permutations avoiding 2-14-3.

Proof. It is clear that $3 \Rightarrow 2 \Rightarrow 1$: the four points displayed in Fig. 10(1) form an occurrence of the pattern of Fig. 10(2), and the four points displayed in Fig. 10(2) form an occurrence of the pattern of Fig. 10(3).

Conversely, let us prove that if a permutation π contains the pattern $21\bar{3}54$, then there exist i',j',ℓ',m' as in the first condition. Assume that there are $1 \le i < j < \ell < m \le n$ such that $\pi(j) < \pi(i) < \pi(m) < \pi(\ell)$, and there is no k such that $j < k < \ell$, $\pi(i) < \pi(k) < \pi(m)$. Let j' be the maximal number for which $j \le j' < \ell$ and $\pi(j') < \pi(i)$. Let $\ell' = j'+1$. Then $\pi(\ell') > \pi(m)$, and we have a pattern 2-14-3 with i,j',ℓ',m.

Furthermore, let i' be the number satisfying $i' < j'$ and $\pi(i) \le \pi(i') < \pi(m)$, for which $\pi(i')$ is the maximal possible. Let $m' = \pi^{-1}(\pi(i')+1)$. Then $m' > \ell'$ and $\pi(m') = \pi(i')+1$, and, thus, the first condition holds with i',j',ℓ',m'. ☐

A similar result holds for permutations that avoid 3-41-2. Therefore, the following proposition holds.

Proposition 3.5. *Let σ be a permutation of $[n]$. The following statements are equivalent:*

1. *There are no $i,j,\ell,m \in [n]$ satisfying $i < j < \ell < m$, $\ell = j+1$, such that*
   either $\sigma(j) < \sigma(i) < \sigma(m) < \sigma(\ell)$ and $\sigma(m) = \sigma(i)+1$,
   or $\sigma(\ell) < \sigma(m) < \sigma(i) < \sigma(j)$ and $\sigma(i) = \sigma(m)+1$.
2. *In the dash notation, σ avoids* 2-14-3 *and* 3-41-2.
3. *In the bar notation, σ avoids* $21\bar{3}54$ *and* $45\bar{3}12$.

Corollary 3.6. *The group of symmetries of the square leaves invariant the set of* (2-14-3, 3-41-2)*-avoiding permutations.*

Proof. The second description in Proposition 3.5 shows that the set of (2-14-3, 3-41-2)-avoiding permutations is closed under reading the permutations from right to left. The first (or third) description shows that it is invariant under taking inverses, and these two transformations generate the symmetries of the square. □

We shall also use the following fact.

Proposition 3.7. *Let σ be a* (2-14-3, 3-41-2)*-avoiding permutation of* $[n]$*. Let* $N_i = (i, \sigma(i))$ *be a point in the graph of* σ*. Then each of the following situations is impossible:*

- *N_i has several NW-neighbors and several NE-neighbors;*
- *N_i has several NE-neighbors and several SE-neighbors;*
- *N_i has several SE-neighbors and several SW-neighbors;*
- *N_i has several SW-neighbors and several NW-neighbors.*

Proof. Due to symmetry, it suffices to prove the impossibility of the first situation. Assume that N_i has several NW-neighbors and several NE-neighbors.

Let i' be the maximal number for which $N_{i'}$ is an NW-neighbor of N_i, and let N_j be another NW-neighbor of N_i. Then we have $j < i'$ and $\sigma(i) < \sigma(j) < \sigma(i')$. We conclude that $i' = i - 1$: otherwise $\sigma(i'+1) < \sigma(i)$ and, therefore, $j, i', i'+1, i$ form the forbidden pattern 3-41-2 (see Fig. 11 (1)), which is a contradiction.

Similarly, if i'' is the minimal number for which $N_{i''}$ is an NE-neighbor of N_i, then $i'' = i+1$. Let N_k be another NE-neighbor of N_i. We have $\sigma(i) < \sigma(k) < \sigma(i+1)$.

Assume without loss of generality that $\sigma(i-1) < \sigma(i+1)$. Now, if $\sigma(j) < \sigma(k)$, then $j, i, i+1, k$ form the forbidden pattern 2-14-3; and if $\sigma(k) < \sigma(j)$, then $j, i-1, i, k$ form the forbidden pattern 3-41-2 (see Fig. 11 (2)), which is, again, a contradiction. □

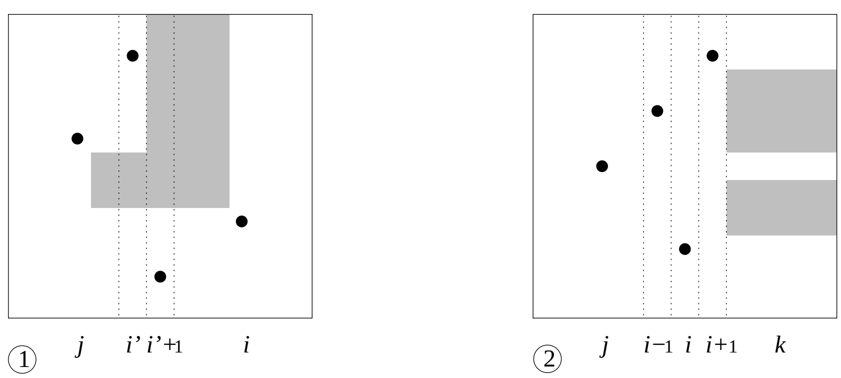


Figure 11: Illustration for the proof of Proposition 3.7.

3.3 S-permutations coincide with (2-14-3, 3-41-2)-avoiding permutations

We have already seen that the map S induces an injection from S-equivalence classes of partitions to permutations (Corollary 3.3). Here, we characterize the image of S.

Theorem 3.8. *The map S induces a bijection between S-equivalence classes of floorplan partitions of size $n+1$ and* (2-14-3, 3-41-2)*-avoiding permutations of size n.*

The proof involves two steps: In Proposition 3.10 we prove that all S-permutations are (2-14-3, 3-41-2)-avoiding. Then, in Proposition 3.11, we show that for any (2-14-3, 3-41-2)-avoiding permutation σ of $[n]$, there exists a floorplan partition P with n segments such that $S(P) = \sigma$.

Recall that a horizontal segment has at most one left-neighbor and at most one right-neighbor, and a vertical segment has at most one below-neighbor and at most one above-neighbor. Therefore, we have the following.

Observation 3.9. *Let I_i be a segment in a floorplan partition P, and let N_i be the corresponding point in the graph of $S(P)$. If I_i is a horizontal segment, then the point N_i has at most one NE-neighbor and at most one SW-neighbor. Similarly, if I_i is a vertical segment, then N_i has at most one SE-neighbor and at most one NW-neighbor.*

Proposition 3.10. *Let P be a floorplan partition. Then $S(P)$ avoids* 2-14-3 *and* 3-41-2*.*

Proof. By Observation 3.1, the image of S is invariant by all symmetries of the square. Hence it suffices to prove that $\sigma = S(P)$ avoids 2-14-3.

Assume that σ contains 2-14-3. By Lemma 3.4, there exist $i < j < \ell < m$, $\ell = j + 1$ such that $\sigma(j) < \sigma(i) < \sigma(m) < \sigma(\ell)$ and $\sigma(m) = \sigma(i) + 1$ (see Fig. 12(1)). We claim that the four segments I_i, I_j, I_ℓ, I_m are vertical.

Consider I_j. The point N_ℓ is an NE-neighbor of N_j. Consider the set $\{x : x > \ell, \sigma(j) < \sigma(x) < \sigma(\ell)\}$. This set is not empty since it contains m. Let p be the smallest element in this set. Then N_p is an NE-neighbor of N_j. Thus, N_j has at least two NE-neighbors, N_ℓ and N_p. Therefore, I_j is vertical by Observation 3.9. In a similar way one can show that I_i, I_ℓ, I_m are also vertical.

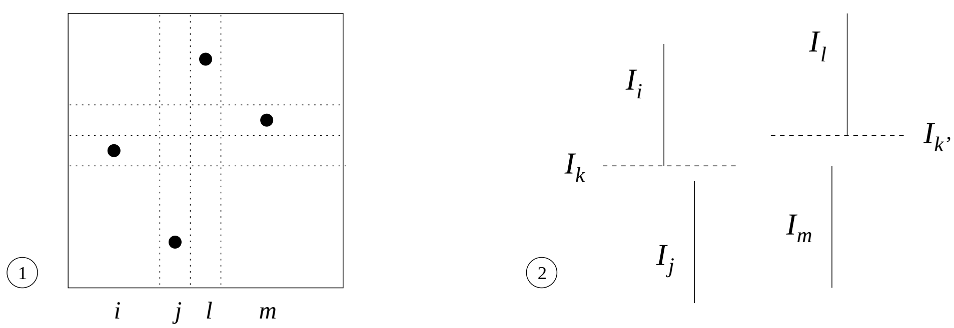


Figure 12: The pattern 2-14-3 never occurs in an S-permutation.

By Observation 3.2 we have that: $I_j \downarrow I_i$, $I_m \downarrow I_\ell$; $I_i \leftarrow I_\ell$, $I_j \leftarrow I_m$, $I_i \leftarrow I_m$, $I_j \leftarrow I_\ell$. Moreover, the last two relations are neighborhood relations. Let I_k be the below-neighbor of I_i, and let $I_{k'}$ be the below-neighbor of I_ℓ (see Fig. 12 (2)). The segments I_k and $I_{k'}$ are horizontal. If the line supporting I_k is (weakly) lower than the line supporting $I_{k'}$, then I_j cannot be a left-neighbor of I_ℓ since the interiors of their vertical projections do not intersect. Similarly, if the line supporting I_k is (weakly) higher than the line supporting $I_{k'}$, then I_i cannot be a left-neighbor of I_m. We have thus reached a contradiction, and σ cannot contain 2-14-3. □

Proposition 3.11. *For each* (2-14-3, 3-41-2)*-avoiding permutation σ of $[n]$, there exists a floorplan partition P with n segments such that $S(P) = \sigma$.*

Proof. Consider the graph of σ. We construct P on this graph. The boundary of the graph is also the boundary of P. For each point $N_i = (i, \sigma(i))$ of the graph, we draw a segment K_i passing through N_i according to certain rules. We first determine the direction of the segments K_i (Point 1 below), and then the coordinates of their endpoints (Point 2). We prove that we indeed obtain a floorplan partition (Point 3), and that its S-permutation is σ (Point 4).

1. The directions of the segments K_i
Let $N_i = (i, \sigma(i))$ be a point in the graph of σ. Our first two rules are forced by Observation 3.9. They are illustrated in Fig. 13 (no point of the graph lies in the shaded areas):

- If N_i has several NW-neighbors or several SE-neighbors, then K_i is horizontal.
- If N_i has several SW-neighbors or several NE-neighbors, then K_i is vertical.

By Proposition 3.7, these two rules never apply simultaneously to the same point. If one of them applies to N_i, we say that N_i is a *strong point*. Otherwise, we say that N_i is a *weak point*. This means that N_i has at most one neighbor in each direction.

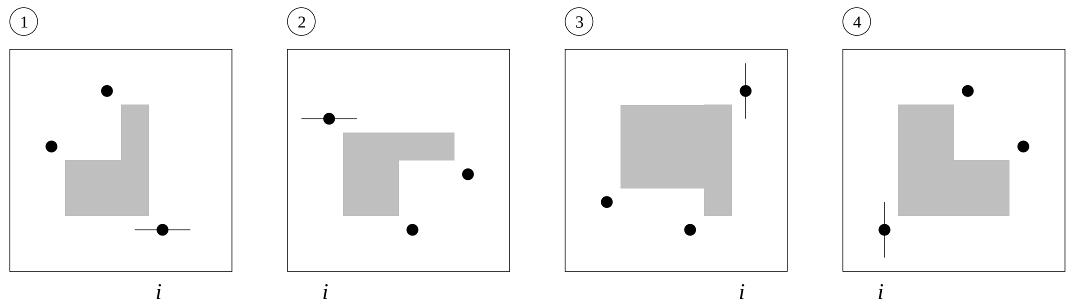


Figure 13: Rules for determining the direction of the segment K_i passing through a strong point.

We claim that if N_i and N_j are weak points, then they are in adjacent rows if and only if they are in adjacent columns. Due to symmetry, it suffices to show the *if* direction. Let N_i and N_{i+1} be weak points, and assume without loss of generality that $\sigma(i) < \sigma(i+1)$. If $\sigma(i+1) - \sigma(i) > 1$, then there are points of the graph of σ in the rows between the rows that contain N_i and N_{i+1}; thus, either N_i has at least two NE-neighbors or N_{i+1} has at least two SW-neighbors. Therefore, it is impossible to have both N_i and N_{i+1} weak.

Thus, weak points appear as ascending or descending sequences of adjacent neighbors: $N_i, N_{i+1}, \ldots, N_{i+\ell}$ with $\sigma(i) = \sigma(i+1) - 1 = \cdots = \sigma(i+\ell) - \ell$ or $\sigma(i) = \sigma(i+1) + 1 = \cdots = \sigma(i+\ell) + \ell$. Note that a weak point N_i can be isolated.

For weak points, the direction of the corresponding segments is determined as follows:

- If $N_i, N_{i+1}, \ldots, N_{i+\ell}$ is a maximal *ascending* sequence of weak points, then the directions of $K_i, K_{i+1}, \ldots, K_{i+\ell}$ are chosen in such a way that K_j and K_{j+1} are never both horizontal, for $i \le j < i+\ell$. Hence several choices are possible.
- If $N_i, N_{i+1}, \ldots, N_{i+\ell}$ is a maximal *descending* sequence of weak points, then the directions of $K_i, K_{i+1}, \ldots, K_{i+\ell}$ are chosen in such a way that K_j and K_{j+1} are never both vertical, for $i \le j < i+\ell$.

In particular, for an isolated weak point N_i, the direction of K_i can be chosen arbitrarily.

2. The endpoints of the segments K_i
Once the directions of all K_i's are chosen, their endpoints are set as follows (see Fig. 14 for an illustration):

- If K_i is vertical (which implies that it has at most one NW-neighbor and at most one SE-neighbor):

  If N_i has an NW-neighbor N_j, then the upper endpoint of K_i is set to be the point $(i, \sigma(j))$. We say that N_j *bounds K_i from above*.

  Otherwise (if N_i has no NW-neighbor), K_i reaches the upper side of the boundary.

If N_i has an SE-neighbor N_k, then the lower endpoint of K_i is set to be the point $(i, \sigma(k))$. We say that N_k *bounds* K_i *from below*.

Otherwise (if N_i has no SW-neighbor), K_i reaches the lower side of the boundary.

- If K_i is horizontal (which implies that it has at most one SW-neighbor and at most one NE-neighbor):

  If N_i has an SW-neighbor N_j, then the left endpoint of K_i is set to be the point $(j, \sigma(i))$. We say that N_j *bounds* K_i *from the left*.

  Otherwise (if N_i has no SW-neighbor), K_i reaches the left side of the boundary.

  If N_i has an NE-neighbor N_k, then the right endpoint of K_i is set to be the point $(k, \sigma(i))$. We say that N_k *bounds* K_i *from the right*.

  Otherwise (if N_i has no NE-neighbor), K_i reaches the right side of the boundary.

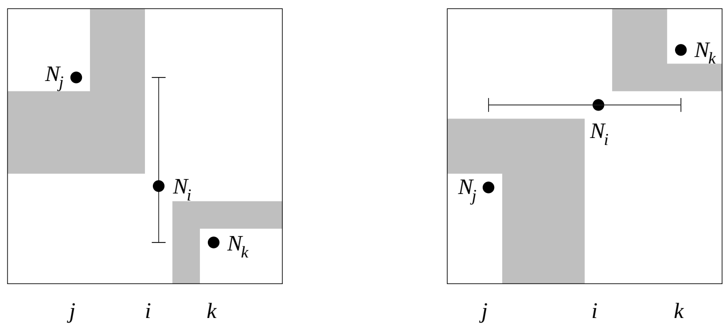


Figure 14: Determining the endpoints of the segment K_i: the points N_j and N_k bound the segment K_i.

Fig. 15 presents an example of the whole construction: in Part 1, the directions are determined for strong (black) points, and chosen for weak (gray) points; in Part 2, the endpoints are determined and a floorplan partition is obtained. Notice that σ is the S-permutation associated with the floorplan partition P in Fig. 9, but here we have obtained a different floorplan partition, P'. However, we could have also obtained P had we chosen the appropriate directions of segments passing through weak points. The question of when $S(P) = S(P')$ will be studied in Section 4.2.

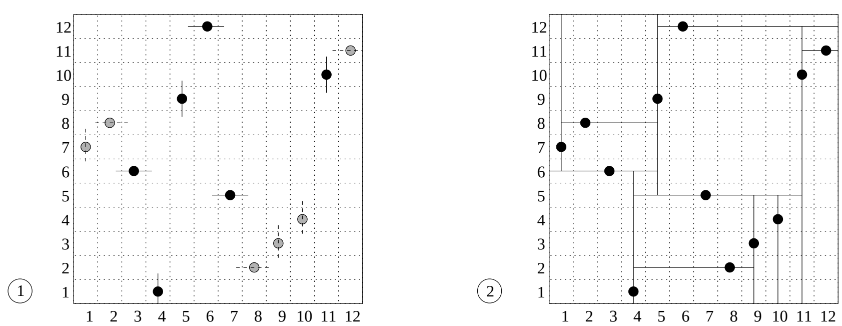


Figure 15: Constructing a floorplan partition from a (2-14-3, 3-41-2)-avoiding permutation.

3. The construction above indeed determines a floorplan partition

In order to prove this, we need to show that two segments never cross, and that the endpoints of any segment K_i are contained in segments perpendicular to K_i (unless they lie on the boundary). The following observation will simplify some of our proofs.

Observation 3.12. *Let σ be a $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$-avoiding permutation, and let σ' be obtained by applying a rotation ρ (a quarter turn or a half-turn) to (the graph of) σ. If P is a configuration of segments obtained from σ by applying the rules of Points 1 and 2 above, then $\rho(P)$ can be obtained from σ' using those rules.*

To prove this, it suffices to check that the rules are invariant by a 90° rotation, which is immediate. That the construction has the other symmetries of Observation 3.1 is also true, but less obvious.

3.1 Let K_i be a vertical (respectively, horizontal) segment, and let N_j and N_k be the points that bound it. Then the segments K_j and K_k are horizontal (respectively, vertical).
Due to symmetry, it suffices to prove this claim for a *vertical* segment K_i and for the point N_j that bounds it *from above*. We need to prove that K_j is a horizontal segment.

We have $j < i$ and $\sigma(i) < \sigma(j)$, and, since N_j is an NW-neighbor of N_i, there is no ℓ such that $j < \ell < i$ and $\sigma(i) < \sigma(\ell) < \sigma(j)$. Furthermore, there is no ℓ such that $j < \ell < i$, $\sigma(j) < \sigma(\ell)$, or such that $\ell < j$, $\sigma(i) < \sigma(\ell) < \sigma(j)$: otherwise N_i would have several NW-neighbors and, therefore, K_i would be horizontal. Now, if $i - j > 1$, then there exists ℓ such that $j < \ell < i$, $\sigma(\ell) < \sigma(i)$; and if $\sigma(j) - \sigma(i) > 1$, then there exists m such that $i < m$, $\sigma(i) < \sigma(m) < \sigma(j)$. In both cases N_j has several SE-neighbors, and, therefore, K_j is horizontal as claimed.

It remains to consider the case $j = i - 1$, $\sigma(j) = \sigma(i) + 1$. If the point N_i is strong, then (since K_i is vertical) it has several NE-neighbors or several SW-neighbors. Assume without loss of generality that N_i has several NE-neighbors. Let ℓ be the minimal number such that N_ℓ is an NE-neighbor of N_i, and let N_m be another NE-neighbor of N_i. Then we have $\sigma(i-1) < \sigma(m) < \sigma(\ell)$ and $\sigma(\ell - 1) \leq \sigma(i)$. However, then $i-1, \ell-1, \ell, m$ form a forbidden pattern 2-14-3. Therefore, N_i is a weak point. Clearly, N_{i-1} as a unique SE neighbor (which is N_i). Its NE and SW neighbors coincide with those of N_i, so that there is at most one of each type. Thus if N_{i-1} is strong, it has several NW-neighbors, and K_{i-1} is horizontal, as claimed. If N_{i-1} is weak, then the rules that determine the direction of the segments passing through weak points implies that K_{i-1} and K_i cannot be both vertical. Therefore, $K_j = K_{i-1}$ is horizontal, as claimed.

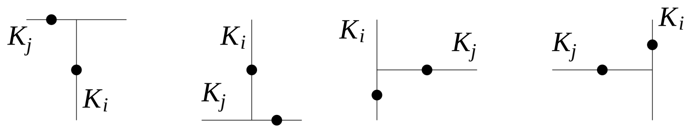
3.2. If N_j and N_k are the points that bound the segment K_i, then the segments K_j and K_k contain the endpoints of K_i
It suffices to show that if K_i is a vertical segment and N_j bounds it from above, then K_j (which is horizontal as shown in Point 3.1 above) contains the point $(i, \sigma(j))$. We saw in Point 3.1 that in this situation there is no ℓ such that $j < \ell < i$, $\sigma(j) < \sigma(\ell)$. This means that there is no point N_ℓ that could bound K_j from the right before it reaches $(i, \sigma(j))$.

3.3. Two segments K_i and K_j cannot cross
Assume that K_i and K_j cross. Assume without loss of generality that K_i is vertical and K_j is horizontal, so that their crossing point is $(i, \sigma(j))$. We have either $i < j$ or $j < i$, and $\sigma(i) < \sigma(j)$ or $\sigma(j) < \sigma(i)$. Assume without loss of generality $j < i$ and $\sigma(i) < \sigma(j)$. Then N_j is to the NW of N_i. The ordinate of the (unique) NE-neighbor of N_i is hence at most $\sigma(j)$. By construction, the upper point of K_i has ordinate at most $\sigma(j)$, while K_j lies at ordinate $\sigma(j)$, and thus K_i and K_j cannot cross.

Let us finish with an observation on joins of segments, which follows from Point 3.2.

Observation 3.13. *Suppose that a vertical segment K_i and a horizontal segment K_j join at the point $(i, \sigma(j))$. Then the rules that determine the endpoints of segments imply the following:*

- *If the join of K_i and K_j is of the type $\top$, then $i > j$.*
- *If the join of K_i and K_j is of the type $\bot$, then $i < j$.*
- *If the join of K_i and K_j is of the type $\vdash$, then $\sigma(i) < \sigma(j)$.*
- *If the join of K_i and K_j is of the type $\dashv$, then $\sigma(i) > \sigma(j)$.*



4. For any floorplan partition P obtained by the construction described above, $S(P) = \sigma$
In order to prove this claim, we will show that for all $1 \leq i < n$, the segment K_{i+1} is the immediate successor of K_i in the $\nwarrow$ order, and that $K_{\sigma^{-1}(i+1)}$ is the immediate successor of $K_{\sigma^{-1}(i)}$ in the $\swarrow$ order. The properties of our construction imply that both statements are equivalent. Assume we have proved the first one, and let us prove the second. Let σ' be obtained by applying a quarter-turn rotation ρ to σ in counterclockwise direction. By Observation 3.12, the partition $P' = \rho(P)$ is associated with σ' by our construction. Let us denote by K_i' the segment of P' containing the point $(i, \sigma'(i))$. Then $\rho(K_{\sigma^{-1}(i)}) = K'_{n+1-i}$. By assumption, K'_{n+1-i} is the successor of K'_{n-i} for the $\nwarrow$ order in P'. By the second remark following Observation 2.12, this means that $K_{\sigma^{-1}(i+1)}$ is the successor of $K_{\sigma^{-1}(i)}$ for the $\swarrow$ order in P.

Thus we only need to prove that K_{i+1} is the immediate successor of K_i in the $\nwarrow$ order. Recall that, by Observation 2.11, the immediate successor of a horizontal (respectively, vertical) segment I in the $\nwarrow$ order is R(I), LVB(I), or LHB(I) (respectively, B(I), UHR(I), or UVR(I))[4], depending on the existence of these segments and the type of joins between them.

There are 8 cases to consider, depending on whether $\sigma(i) < \sigma(i+1)$ or $\sigma(i) > \sigma(i+1)$, and on the directions of K_i and K_{i+i}.

Case 1: $\sigma(i) < \sigma(i+1)$, K_i and K_{i+1} are vertical.

Assume that N_j is the point that bounds K_i from above. Then, as shown in Point 3.1 above, K_j is horizontal; furthermore, K_i and K_j have a $\top$ join at the point $(i, \sigma(j))$. In particular, the rightmost point of K_j has abscissa at least $i+1$.

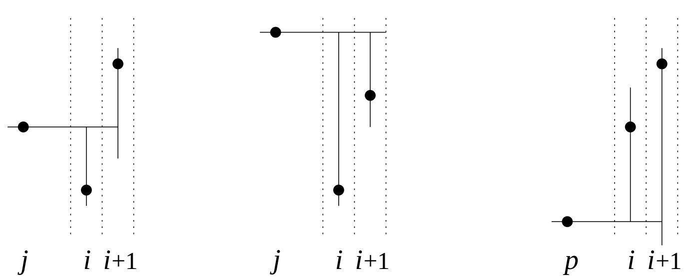
If $\sigma(j) < \sigma(i+1)$, then N_{i+1} bounds K_j from the right. There is a $\dashv$ join of K_j and K_{i+1} at the point $(i+1, \sigma(j))$.

If $\sigma(j) > \sigma(i+1)$, then N_j also bounds K_{i+1} from above.

If N_j does not exist and K_i reaches the upper side of the boundary, then no point can bound K_{i+1} from above, and, thus, K_{i+1} reaches the boundary as well.

In all these cases, it is readily seen that K_{i+1} is UVR(K_i).

By Observation 2.11, UVR(K_i) is the successor of K_i, unless UHR(K_i) does not exist, B(K_i) $:= K_p$ exists and its join with UVR(K_i) is of type $\dashv$. But this would mean that $p < i$, and the positions of N_i and N_p would then contradict Observation 3.13.

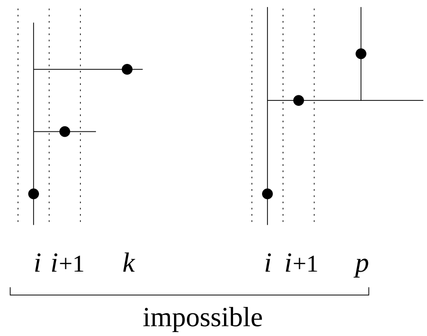


Case 2: $\sigma(i) < \sigma(i+1)$, K_i is vertical and K_{i+1} is horizontal.

The point N_i bounds K_{i+1} from the left. Therefore, there is a $\vdash$ join of K_i and K_{i+1} at the point $(i, \sigma(i+1))$, and K_{i+1} is a horizontal right-neighbor of K_i. Moreover, if K_k is another horizontal right-neighbor of K_i, then $\sigma(k) < \sigma(i+1)$: otherwise N_i cannot be an SW-neighbor of N_k. Therefore, $K_{i+1} = $ UHR(K_i).

[4]This notation is defined before Observation 2.11.

By Observation 2.11, UHR(K_i) is the successor of K_i, unless UVR$(K_i) := K_p$ exists. If this were the case, K_p and K_{i+1} would have a $\perp$ join, and the position of N_{i+1} and N_p would then be incompatible with Observation 3.13.



Case 3: $\sigma(i) < \sigma(i+1)$, K_i is horizontal and K_{i+1} is vertical.

We claim that this case follows from the previous one. Let σ' be obtained by applying a half-turn rotation ρ to (the graph of) σ. By Observation 3.12, the partition $P' = \rho(P)$ is associated with σ'. The points and segments $\rho(N_i)$, $\rho(N_{i+1})$, $\rho(K_i)$, $\rho(K_{i+1})$ in P' are in the configuration described by Case 2, with $\rho(N_{i+1})$ to the left of $\rho(N_i)$. Consequently, $\rho(N_i)$ is the successor of $\rho(N_{i+1})$ in the $\nwarrow$ order in P'. By the remark that follows Observation 2.12, N_{i+1} is the successor of N_i in the $\nwarrow$ order in P.

Case 4: $\sigma(i) < \sigma(i+1)$, K_i and K_{i+1} are horizontal.

If this case, N_i bounds K_{i+1} from the left. Therefore, K_i must be vertical (see Point 3.1 above). So, this case is impossible.

Case 5: $\sigma(i) > \sigma(i+1)$, K_i and K_{i+1} are vertical.

Since K_{i+1} is vertical, N_{i+1} has at most one NW-neighbor, which is then N_i. By Point 3.1 above, K_i is then horizontal. Thus this case is impossible.

Case 6: $\sigma(i) > \sigma(i+1)$, K_i is vertical and K_{i+1} is horizontal

Since the segment K_i is vertical, the point N_i has at most one SE-neighbor, which is then N_{i+1}. Therefore, N_{i+1} bounds K_i from below, and there is a $\perp$ join of K_i and K_{i+1} at the point $(i, \sigma(i+1))$. In particular, $K_{i+1} = \mathrm{B}(K_i)$.

By Observation 2.11, $\mathrm{B}(K_i)$ is the successor of K_i, unless UHR$(K_i) := K_k$ exists, or UHR(K_i) does not exist, but UVR$(K_i) := K_p$ does and forms with $\mathrm{B}(K_i)$ a $\perp$ join. In the former case, N_{i+1} or another point to its right would bound K_k from the left, and, thus, K_k would not reach K_i. In the latter case, K_p and K_{i+1} would form a $\perp$ join, and the positions of N_p and N_{i+1} would contradict Observation 3.13.

Case 7: $\sigma(i) > \sigma(i+1)$, K_i is horizontal and K_{i+1} is vertical.

This case follows from Case 6 by the symmetry argument already used in Case 3.

Case 8: $\sigma(i) > \sigma(i+1)$, K_i and K_{i+1} are horizontal.

The point that bounds K_i from the right, if it exists, lies to the NE of N_{i+1}. Thus the abscissa of the rightmost point of K_i is greater than or equal to the abscissa of the rightmost point of K_{i+1}. Similarly, the abscissa of the leftmost point of K_{i+1} is less than or equal to the abscissa of the leftmost point of K_i.

We will show that $K_{i+1} = \mathrm{LHB}(K_i)$. Once this is proved, Observation 2.11 implies that LHB(K_i) is the successor of K_i, unless LVB(K_i) does not exist, but $R(K_i)$ exists and forms with K_{i+1} a $\perp$ join. But this would mean that K_{i+1} ends further to the right than K_i, which we have just proved to be impossible.

So let us prove that $K_{i+1} = \mathrm{LHB}(K_i)$. We assume that K_i does not reach the left side of the boundary, and that K_{i+1} does not reach the right side of the boundary (the other cases are proven similarly). Let N_k be the point that bounds K_i from the left, and let N_m be the point that bounds K_{i+1} from the right.

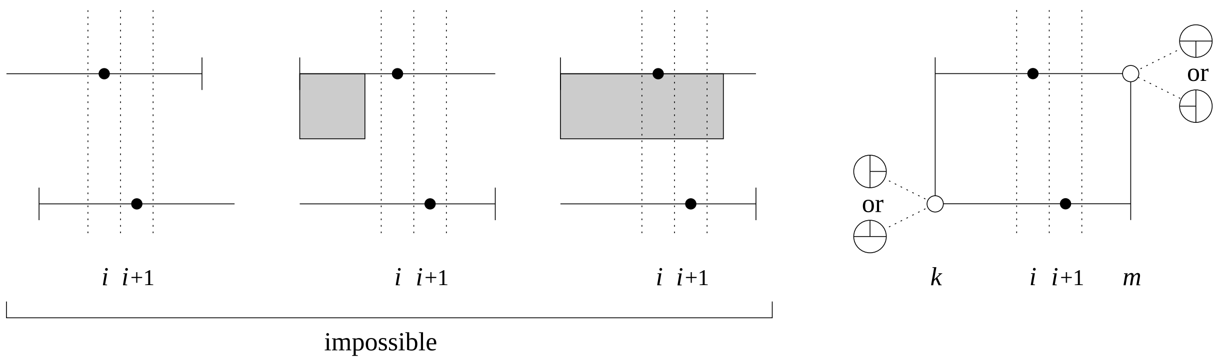
Consider A, the leftmost rectangle whose upper side is contained in K_i. The left side of A is clearly contained in K_k. We claim that the lower side of A is contained in K_{i+1}, and that the right side of A is contained in K_m. Note that this implies $K_{i+1} = \mathrm{LHB}(K_i)$.

Let K_p (respectively, K_q) be the segment that contains the lower (respectively, right) side of A. Clearly, $q > k$. If $q < i$, then K_q is a vertical below-neighbor of K_i, and the positions of N_q and N_i contradict Observation 3.13. Therefore, $q > i + 1$.

Consider now the segment K_p. Clearly, $\sigma(p) \geq \sigma(i+1)$. One cannot have $p > i + 1$: otherwise N_{i+1} (or a point located to the right of N_{i+1}) would bound K_p from the left, and K_p would not reach K_k. One cannot have either $p < i$: otherwise N_i (or a point located to the left of N_i) would bound K_p from the right, and K_p would not reach K_q. Since $p \neq i$, we have proved that $p = i + 1$.

Finally, K_q coincides with K_m: otherwise, $q < m$, and K_q is a vertical above-neighbor of K_{i+1}; however, in this case N_q would bound K_{i+1} from the right, and K_{i+1} would not reach K_m.

We have thus proved that $K_{i+1} = \text{LHB}(K_i)$, and this concludes the study of this final case, and the proof of Proposition 3.11.



□

4 Relations between the R- and S-permutations

In this section we first prove that two R-equivalent partitions have the same S-permutation, and give a simple graphical way to construct $S(P)$ from $R(P)$. Then, we explain how the R-permutations of two S-equivalent partitions are related.

4.1 Constructing $S(P)$ from $R(P)$

Let P be a floorplan partition of size $n + 1$. We draw the graphs of $\rho = R(P)$ and $\sigma = S(P)$ on the same diagram in the following way. For the graph of ρ we use an $(n+1) \times (n+1)$ square whose columns and rows are numbered by $1, 2, \ldots, n + 1$. The points of the graph of ρ are placed at the centers of these squares, and these points are black. The point $(i, \rho(i))$ is denoted by M_i. For the graph of σ we use the grid lines of the same drawing, when the ith vertical (respectively, horizontal) line is the grid line between the ith and the $(i+1)$st columns (respectively, rows). The point $(i, \sigma(i))$, denoted by N_i, is placed at the intersection of the ith vertical grid line and the jth horizontal grid line, where $j = \sigma(i)$. Such points are white. The whole drawing is called *the combined diagram of P*. See Fig. 21 for an example. Note that the extreme (rightmost, leftmost, etc.) grid lines are not used.

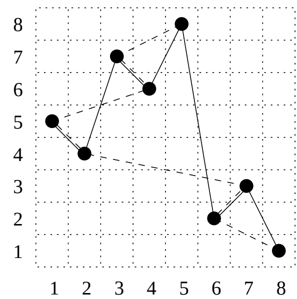
We start with the following fact about Baxter permutations, which is closely related to the three possible definitions of these permutations described in Section 3.2.

Observation 4.1. *Let ρ be a Baxter permutation of $[n+1]$. For each i, $1 \leq i \leq n$ there exists a unique j_i, $1 \leq j_i \leq n$, such that:*

- *if $\rho(i) < \rho(i+1)$, then $\rho(i) \leq j_i < \rho(i+1)$ and $\rho^{-1}(j_i) \leq i < \rho^{-1}(j_i + 1)$;*

- *if* $\rho(i) > \rho(i+1)$, *then* $\rho(i+1) \le j_i < \rho(i)$ *and* $\rho^{-1}(j_i+1) \le i < \rho^{-1}(j_i)$.

In the graph of ρ, Observation 4.1 has the following interpretation. For each segment $M_i M_{i+1}$, there exists a unique segment $M_{\rho^{-1}(j)} M_{\rho^{-1}(j+1)}$ such that the segment $M_i M_{i+1}$ intersects the jth horizontal grid line and the segment $M_{\rho^{-1}(j)} M_{\rho^{-1}(j+1)}$ intersects the ith vertical grid line, and the slopes of these segments have the same sign. See Fig. 16 for an example: the segments $M_i M_{i+1}$ are depicted by solid lines, and the segments $M_{\rho^{-1}(j)} M_{\rho^{-1}(j+1)}$ by dashed lines.



i	1	2	3	4	5	6	7
j_i	4	5	6	7	3	2	1

Figure 16: Illustration for Observation 4.1.

Proof of Observation 4.1. Assume without loss of generality $\rho(i) < \rho(i+1)$.

Let $X_1 = \{x : \ x \le i, \rho(i) \le \rho(x) < \rho(i+1)\}$, $X_2 = \{x : \ x \ge i+1, \rho(i) < \rho(x) \le \rho(i+1)\}$. These sets are nonempty since $i \in X_1$ and $i+1 \in X_2$. For each $x_1 \in X_1$, $x_2 \in X_2$, we have $\rho(x_1) < \rho(x_2)$: otherwise we have $x_1 < i$, $i+1 < x_2$ and $\rho(i) < \rho(x_2) < \rho(x_1) < \rho(i+1)$, and then $x_1, i, i+1, x_2$ form a forbidden pattern 3-14-2. Let k be the element of X_1 with the maximal $\rho(k)$, and let m be the element of X_2 with the minimal $\rho(m)$. Then we have $\rho(m) = \rho(k) + 1$, $k \le i$ and $m \ge i+1$, and the statement holds with $j_i = \rho(k)$. See Fig. 17.

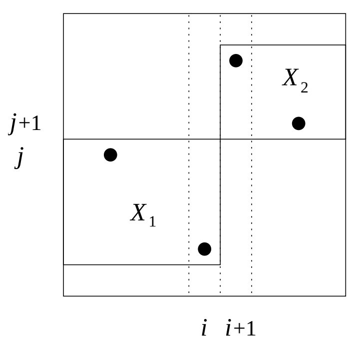


Figure 17: Illustration of the proof of Observation 4.1.

Such j_i is unique since if $\rho(i) \le j' < j_i$, then $\rho^{-1}(j), \rho^{-1}(j+1) \in X_1$, and therefore, we have $\rho^{-1}(j) \le i$ and $\rho^{-1}(j+1) \le i$; similarly, if $j_i < j' < \rho(i+1)$, then $\rho^{-1}(j) \ge i+1$ and $\rho^{-1}(j+1) \ge i+1$. ☐

Theorem 4.2. *Let P be a floorplan partition of size $n+1$, and let $\rho = R(P)$. For $1 \le i \le n$, let j_i be as defined in Observation 4.1. Then $S(P) = (j_1, j_2, \ldots, j_n)$. In particular, R-equivalent partitions are also S-equivalent.*

We refer to Fig. 21 for an example.

In order to prove Theorem 4.2, we shall use two observations, which involve the orders defined on rectangles in the introduction. We use the following notation: A_i is the rectangle labeled i in the ↖ order, and B^j is the rectangle labeled j in the ↙ order.

Observation 4.3. *Let P be a floorplan partition of size $n+1$. For each k, $1 \le k \le n$, the following holds:*

- *If the segments forming the SE-corner of A_k have a $\perp$ join, let J_k be the segment containing the right side of A_k. Then A_{k+1} is the topmost among the rectangles whose left side is contained in J_k.*
- *If the segments forming the SE-corner of A_k have a $\dashv$ join, let J_k be the segment containing the lower side of A_k. Then A_{k+1} is the leftmost among the rectangles whose upper side is contained in J_k.*

Proof. By definition of the $\nwarrow$ order, A_{k+1} is either a right-neighbor or a below-neighbor of A_k. If there is a $\perp$ join in the SE-corner of A_k, then all the right-neighbors of A_k are above all its below-neighbors. Therefore, A_{k+1} is the topmost among them. If there is a $\dashv$ join in the SE-corner of A_k, then all the below-neighbors of A_k are to the left of all its right-neighbors. Therefore, A_{k+1} is the leftmost among them. See Fig. 18. □

Figure 18: The rectangle A_{k+1} is the immediate successor of A_k in the $\nwarrow$ order.

Observation 4.4. *Consider the rectangles A_k, A_{k+1} in a floorplan partition P. The segment denoted by J_k in Observation 4.3 is the* kth *segment in the $\nwarrow$ order of segments, denoted so far by I_k.*

Proof. Observe this directly for $k=1$, and proceed by induction. One has to examine several cases, depending on whether the segments in the SE-corners of A_k and of A_{k+1} have $\perp$ or $\dashv$ joins. In all cases, J_{k+1} is found to be the immediate successor of J_k in the $\nwarrow$ order, as described in Observation 2.11. See Fig. 19 for several typical situations. □

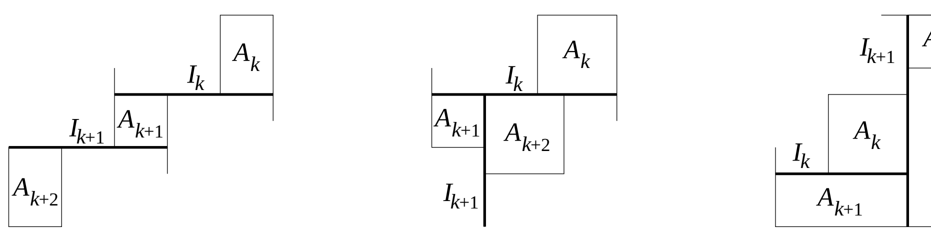


Figure 19: Successors of segments and rectangles for the $\nwarrow$ orders.

Proof of Theorem 4.2. Let $i \in [n]$. Denote $\sigma = S(P)$ and $j = \sigma(i)$. Then the segment I_i labeled i in the $\nwarrow$ order, is labeled j in the $\swarrow$ order. We wish to prove that $j = j_i$.

Assume first that I_i is horizontal. By Observations 4.3 and 4.4, the rightmost among the rectangles whose lower side is contained in I_i is A_i, and the leftmost among the rectangles whose upper side is contained in I_i is A_{i+1}.

By symmetry, since I_i is the jth segment in the $\swarrow$ order, the rightmost among the rectangles whose upper side is contained in I_i is B^j, and the leftmost among the rectangles whose lower side is contained in I_i is B^{j+1}. There holds $A_{i+1} \swarrow B^j \swarrow B^{j+1} \swarrow A_i$ and $B^{j+1} \nwarrow A_i \nwarrow A_{i+1} \nwarrow B^j$ (see Fig. 20). By definition of $R(P)$, this means $\rho(i+1) \le j < j+1 \le \rho(i)$ and $\rho^{-1}(j+1) \le i < i+1 \le \rho^{-1}(j)$. Thus, j satisfies the conditions that define j_i in Observation 4.1 (for the case $\rho(i) > \rho(i+1)$). Since j_i is defined uniquely, we have $j = j_i$, and therefore, $\sigma(i) = j_i$, as claimed.

The case where I_i is vertical is similar, and corresponds to an ascent in ρ. □

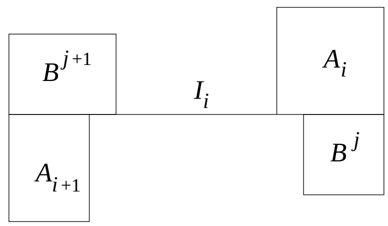


Figure 20: Illustration of the proof of Theorem 4.2.

Theorem 4.2 and Observation 4.1 give a simple graphical way to obtain $S(P)$ from $R(P)$. Draw $R(P)$ as explained in the beginning of Section 4.1. By Observation 4.1, for each $1 \le i \le n$ there exists $1 \le j \le n$ such that the segment M_iM_{i+1} intersects the jth horizontal grid line and the segment $M_{\rho^{-1}(j)}M_{\rho^{-1}(j+1)}$ intersects the ith vertical grid line, and these segments have slopes of the same sign. Put a white point in the intersection of the ith vertical grid line and the jth horizontal grid line. Then the white points form the graph of $\sigma = S(P)$. An example is shown in Fig. 21: in Part 1, *rectangles* are labeled (i, j) where i is the label in the $\nwarrow$ order and j is the label in the $\swarrow$ order; in Part 2, *segments* are labeled in a similar way; in Part 3, the graphs of $R(P)$ and of $S(P)$ are shown together forming the combined diagram: the graph of $R(P)$ with black points in the squares of the grid, the graph of $S(P)$ with white points on the nodes of the grid.

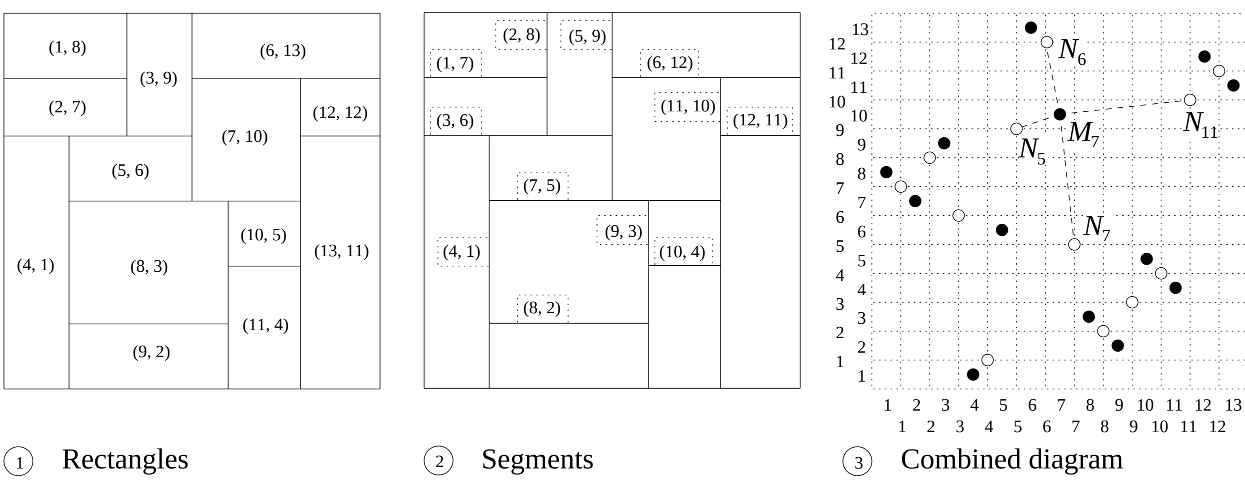


Figure 21: The floorplan partition P from Fig. 9: (1) The labeling of rectangles; (2) The labeling of segments; (3) The combined diagram: $R(P) = 8\ 7\ 9\ 1\ 6\ 13\ 10\ 3\ 2\ 5\ 4\ 12\ 11$ (black points) together with $S(P) = 7\ 8\ 6\ 1\ 9\ 12\ 5\ 2\ 3\ 4\ 10\ 11$ (white points).

Observation 4.5. *Let P be a floorplan partition and let $\rho = R(P)$ be the corresponding Baxter permutation. Let us abuse notation by denoting $S(\rho) := S(P)$. If ρ' is obtained by applying to ρ a symmetry of the square, then the same symmetry, applied to $S(\rho)$, gives $S(\rho')$.*

Remark. The combined diagram is actually the R-permutation of a floorplan partition of size $2n+1$. Indeed, let P be a floorplan partition of size $n+1$. If we inflate segments of P into narrow rectangles, we obtain a new floorplan partition of size $2n+1$, which we denote by $\tilde{P}$ (Fig. 22). Observe that a rectangle of $\tilde{P}$ corresponding to a rectangle A of P has a unique above (respectively, right, below, left) neighbor, which corresponds to the segment of P that contains the above (respectively, right, below, left) side of A.

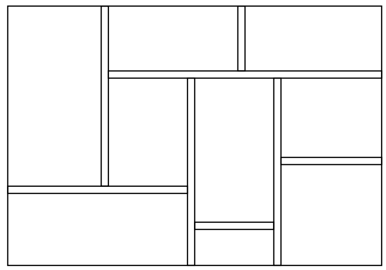


Figure 22: Inflating the segments of a floorplan partition.

It follows from Observation 4.4 and Fig. 18 that the $\nwarrow$ order in $\tilde{P}$ is $A_1 I_1 A_2 I_2 \dots A_n I_n A_{n+1}$. It is thus obtained by shuffling the $\nwarrow$ orders for rectangles and segments of P. Symmetrically, the $\swarrow$ order in $\tilde{P}$ is $A_{\rho^{-1}(1)} I_{\sigma^{-1}(1)} \cdots A_{\rho^{-1}(n)} I_{\sigma^{-1}(n)} A_{\rho^{-1}(n+1)}$. Thus the combined diagram of $R(P)$ and $S(P)$, as in Fig. 21, coincides with the graph of $R(\tilde{P})$.

Now, according to the Remark that follows Observation 3.2, the order/neighborhood relations between rectangles of $\tilde{P}$ can be read from the combined diagram. As observed above, the segments that bound a rectangle A_i in P, become, once inflated, the neighbors of the rectangle corresponding to A_i in $\tilde{P}$. Therefore these segments can be determined from the combined diagram: the segment I_j that bounds A_i from above (respectively, right, below, left), corresponds in the combined diagram to the white point N_j that is the NW- (respectively, NE-, SE-, SW-) neighbor of the black point M_i. Consider, for instance, the floorplan partition P in Fig. 21. In order to read from the combined diagram what segments contain the sides of the rectangle A_7, we look for the white neighbors of the black point $M_7 = (7, 10)$. These neighbors are $N_6 = (6, 12)$, $N_{11} = (11, 10)$, $N_7 = (7, 5)$, $N_5 = (5, 9)$. Therefore the sought-for segments are I_6, I_{11}, I_7, I_5.

4.2 Which partitions produce the same S-permutation

In this section we characterize in terms of their R-permutations the floorplan partitions that have the same S-permutation.

We first describe the floorplan partitions whose S-permutation is $123\dots n$. Such partitions will be called *ascending F-blocks*[5]. It is easy to see that in an ascending F-block, all vertical segments extend from the lower to the upper side of the boundary, and there is at most one horizontal segment between a pair of adjacent vertical segments (this can be shown inductively, by noticing that at most one horizontal segment starts from the left side of the bounding rectangle). Conversely, every floorplan partition of this type has S-permutation $123\dots n$. Therefore, an ascending F-block consists of several rectangles that extend from the lower to the upper side of the boundary, some of them being split into two sub-rectangles by a horizontal segment. The corresponding R-permutations are those that can be obtained from $123\dots(n+1)$ by several disjoint transpositions of adjacent elements; they can also be characterized as the permutations ρ that satisfy $|\rho(i) - i| \le 1$ for all $1 \le i \le n+1$. The number of ascending F-blocks of size $n+1$ (and, therefore, the number of such permutations) is the Fibonacci number F_{n+1} (where $F_0 = F_1 = 1$). Fig. 23 shows all ascending F-blocks and the corresponding R-permutations for $n = 4$.

A similar observation holds for the floorplan partitions whose S-permutation is $n\dots 321$. Such partitions are called *descending F-blocks*. In descending F-blocks, all horizontal segments extend from the left side

[5]The letter F refers to Fibonacci, for reasons that will be explained further down.

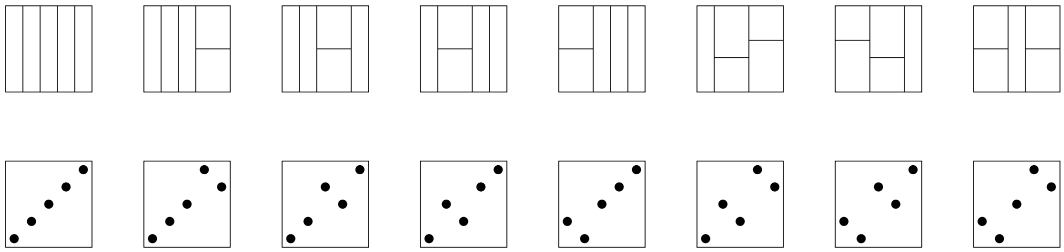


Figure 23: The 8 ascending F-blocks for $n = 4$, and their R-permutations.

to the right side of the boundary, and there is at most one vertical segment between a pair of adjacent horizontal segments. In other words, descending F-blocks consist of several rectangles that extend from the left to the right side of the boundary, some of them being split into two sub-rectangles by a vertical segment. The corresponding R-permutations are those that can be obtained from $(n+1)\dots 321$ by several disjoint transpositions of adjacent elements, and they are characterized by the condition $|\rho(i) - (n+2-i)| \le 1$ for all $1 \le i \le n+1$.

For an F-block F, the size of F (that is, the number of rectangles) will be denoted by $|F|$. If $|F| = 1$, we say that F is a *trivial* F-block. Note that if $|F| \le 2$, then F is both ascending and descending, while if $|F| \ge 3$, then its type (ascending or descending) is uniquely determined.

Let P be a floorplan partition. We define an *F-block in* P as a set of rectangles of P whose union is an F-block, as defined above. In other words, their union is a rectangle, and the S-permutation of the induced subpartition is either $123\dots$ or $\dots 321$. F-blocks in P are partially ordered by inclusion. Since segments of P do not cross, a rectangle in P belongs precisely to one maximal F-block (which may be of size 1). So there is a uniquely determined partition of P into maximal F-blocks (Fig. 24, left).

A *block* in a permutation ρ is an interval $[i,j]$ such that the values $\{\rho(i),\dots,\rho(j)\}$ also form an interval [3]. By extension, we also call a block the corresponding set of points in the graph of ρ. Consider ℓ rectangles in P that form an ascending (respectively, descending) F-block. By Observation 4.3 and the analogous statement for the $\swarrow$ order, these ℓ rectangles form an interval in the $\nwarrow$ and $\swarrow$ orders. Hence the corresponding ℓ points of the graph of $R(P)$ form a block, and their inner order is isomorphic to a permutation τ of $[\ell]$ that satisfies $|\tau(i) - i| \le 1$ (respectively, $|\tau(i) - (\ell+1-i)| \le 1$) for all $1 \le i \le \ell$.

The converse is also true: If ℓ points of the graph of $R(P)$ form an $\ell \times \ell$ block, and their inner order is isomorphic to a permutation τ of $[\ell]$ that satisfies $|\tau(i) - i| \le 1$ (respectively, $|\tau(i) - (\ell+1-i)| \le 1$) for all $1 \le i \le \ell$, then the corresponding rectangles in P form an ascending (respectively, descending) F-block. Indeed, let H be such an ascending block in the graph of $R(P)$. Let us partition the points of H in singletons (formed of points that lie on the diagonal) and pairs (formed of transposed points at adjacent positions). Let $Q_1, Q_2, \dots$ be the list of parts of this partition, read from the SW to the NE corner of H. For each $i = 1, 2, \dots$, the point(s) of Q_{i+1} are the only NE-neighbors of the point(s) of Q_i, and, conversely, the point(s) of Q_i are the only SW-neighbors of the point(s) of Q_{i+1}. Therefore, by the remark that follows Observation 3.2, the left side of the rectangle(s) corresponding to the point(s) of Q_{i+1} coincides with the right side of the rectangle(s) corresponding to the point(s) of Q_i. If Q_i consists of two points then we have two rectangles whose union is a rectangle split by a horizontal segment. The argument is similar for a descending block.

Therefore, such blocks in the graph of ρ will be also called ascending (respectively, descending) F-blocks. Fig. 24 shows a floorplan partition with maximal F-blocks denoted by bold lines, and the F-blocks in the corresponding permutation $R(P)$ (the graph of $S(P)$ is also shown).

Let $F_1, F_2, \dots$ be all the maximal F-blocks in the graph of ρ (ordered from left to right). For $i \ge 1$, let $[y_i, y_i']$ be the interval of values $\rho(j)$ occurring in F_i, and define $d_i := +$ if F_i is descending, $d_i := -$ otherwise (d_i is left undefined if F_i has size 1 or 2). The *F-structure* of ρ is the sequence $\hat{F}_1, \hat{F}_2, \dots$, where $\hat{F}_i = ([y_i, y_i'], d_i)$. For example, the F-structure of the permutation in Fig. 24 is

$$([7,9],+)\,,\quad ([1])\,,\quad ([6])\,,\quad ([13])\,,\quad ([10])\,,\quad ([2,5],+)\,,\quad ([11,12])\,.$$

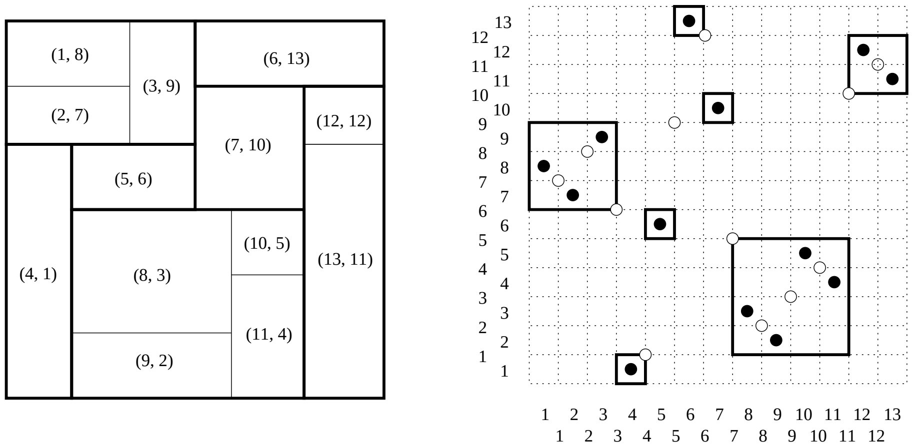


Figure 24: Maximal F-blocks in floorplan partitions and in permutations.

Theorem 4.6. *Let P_1 and P_2 be two floorplan partitions with n segments. Then $S(P_1) = S(P_2)$ if and only if $R(P_1)$ and $R(P_2)$ have the same F-structure.*

In other words, $S(P_1) = S(P_2)$ if and only if $R(P_1)$ and $R(P_2)$ may be obtained from each other by replacing some F-blocks $F_1, F_2, \dots$ with, respectively, F-blocks $F'_1, F'_2, \dots$, where F_i is S-equivalent to F'_i for all i.

The "if" direction is easy to prove. Assume $R(P_1)$ and $R(P_2)$ have the same F-structure. In view of the way one obtains $S(P)$ from $R(P)$ (Theorem 4.2), there holds $S(P_1) = S(P_2)$. Observe in particular that inside a maximal F-block of $R(P)$, the points of $S(P)$ are organized on the diagonal (in the ascending case) or the anti-diagonal (in the descending case).

In order to prove the "only if" direction, we will first relate, for a point of $S(P)$, the fact of being inside a maximal F-block to the property of being weak. (Recall that a point N_i in the graph of $S(P)$ is weak if it has at most one neighbor in each of the directions NW, NE, SE, SW, and strong otherwise.) If a maximal F-block of $R(P)$ occupies the area $[x, x'] \times [y, y']$, then the point $N_i = (i, j)$ is inside this block if $x \le i < x'$ and $y \le j < y'$. For example, in Fig. 24 six points in the graph of $S(P)$ (the white points in the combined diagram) are inside a maximal F-block: $(1, 7)$, $(2, 8)$, $(8, 2)$, $(9, 3)$, $(10, 4)$ and $(12, 11)$. Observe that the notion of "being inside" a maximal F-block is *a priori* relative to $R(P)$. However, the following proposition shows that it is an intrinsic notion, depending on $S(P)$ only.

Proposition 4.7. *Let N_i be a point in the graph of $\sigma = S(P)$. Then N_i is inside a maximal F-block of $R(P)$ if and only if it is a weak point of $S(P)$.*

Proof. Let $N_i = (i, j)$ be inside a maximal F-block of $R(P)$. Assume for the sake of contradiction that N_i is strong, and for instance, has several NE-neighbors. Let N_k be the leftmost NE-neighbor of N_i, and let N_ℓ be the lowest NE-neighbor of N_i. If $k > i + 1$, then $\sigma(k - 1) < \sigma(i)$, and, therefore, $i, k - 1, k, \ell$ form a forbidden pattern 2-14-3. Similarly, if $\sigma(\ell) > j + 1$, then we have a forbidden pattern. Therefore, $k = i + 1$ and $\sigma(\ell) = j + 1$ (Fig. 25). Note also that $\sigma(i + 1) > j + 1$ and $\sigma^{-1}(j + 1) > i + 1$. Since the points of $S(P)$ inside an F-block are either on the diagonal or the anti-diagonal of this block, N_i is the highest (and rightmost) point of $S(P)$ inside the maximal F-block that contains it, and this F-block is of ascending type. In particular, either $\rho(i + 1) = j + 1$, or $\rho(i) = j + 1$ and $\rho(i + 1) = j$.

Since $\rho(i + 1) \le j + 1$ and $\sigma(i + 1) \ge j + 2$, then $\rho(i + 2) \ge j + 3$. Symmetrically, $\rho^{-1}(j + 2) \ge i + 3$. But then the position of the point $(\rho^{-1}(j + 2), j + 2)$ is not compatible with the position of N_{i+1}: by Theorem 4.2, there cannot be a point of ρ located to the right of $\rho(i + 2)$ and in the rows between those of $\rho(i + 1)$ and N_{i+1}. Hence N_i cannot have several NE-neighbors. Symmetric statements hold for the other directions, and N_i is a weak point.

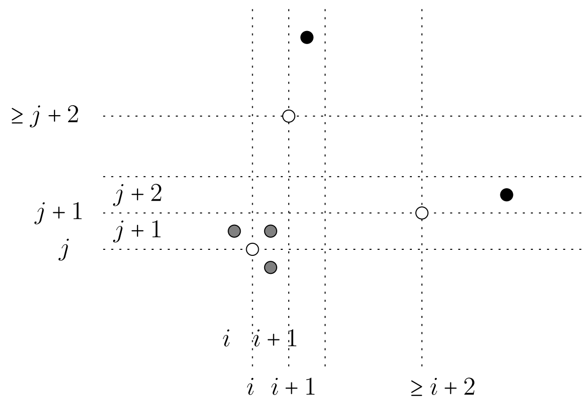


Figure 25: Some points of the combined diagram of ρ and σ. The grey points represent the two possibilities $\rho(i+1) = j+1$, or $\rho(i) = j+1$ and $\rho(i+1) = j$.

Now let $N_i = (i, j)$ be a point of the graph of σ, not inside a maximal F-block. Assume without loss of generality that ρ has an ascent at i: $\rho(i) \le j < \rho(i+1)$ and (see Theorem 4.2) $\rho^{-1}(j) \le i < \rho^{-1}(j+1)$. We shall show that N_i has several SW-neighbors or several NE-neighbors.

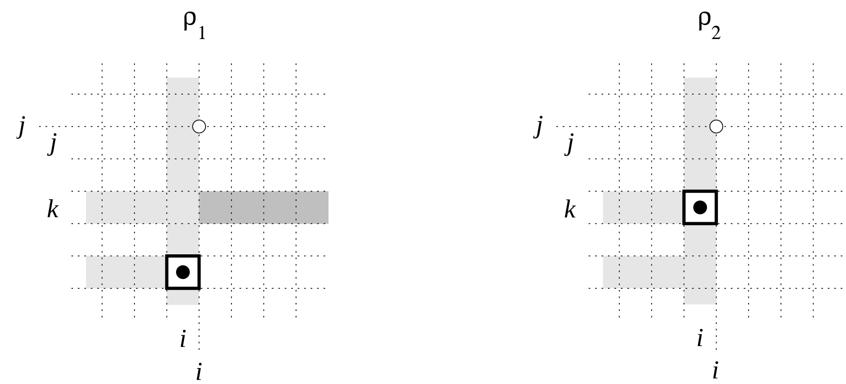
First, if $\rho(i) = j$ and $\rho(i+1) = j+1$, then M_i and M_{i+1} form an F-block, and N_i is inside this block. Therefore, we may assume without loss of generality that $\rho(i) \ne j$; hence $\rho(i) < j$ and $\rho^{-1}(j) < i$. Then it follows from the definition of Baxter permutations that $\rho(i-1) \le j$ (otherwise, there is an occurrence of 2-41-3 at positions $\rho^{-1}(j), i-1, i, \rho^{-1}(j+1)$). Consequently, we have $\sigma(i-1) \le j-1$. Symmetrically, $\rho^{-1}(j-1) \le i$ and $\sigma^{-1}(j-1) \le i-1$. There are two possibilities: either $\sigma(i-1) < j-1$ and $\sigma^{-1}(j-1) < i-1$, or $\sigma(i-1) = j-1$. In the former case, N_{i-1} and $N_{\sigma^{-1}(j-1)}$ are two SW-neighbors of N_i. In the latter case, we have $\rho(i-1) = j$ and $\rho(i) = j-1$. By a similar argument we obtain that either N_{i+1} and $N_{\sigma^{-1}(j+1)}$ are two NW-neighbors of N_i, or $\rho(i+1) = j+2$ and $\rho(i+2) = j+1$. Thus, N_i has at least two SW-neighbors or at least two NE-neighbors, unless $\rho(i-1) = j$, $\rho(i) = j-1$, $\rho(i+1) = j+2$, and $\rho(i+2) = j+1$. However, in the latter case M_{i-1}, M_i, M_{i+1} and M_{i+2} form an F-block, and N_i is inside this block, which contradicts our initial assumption. □

Now we can prove the "only if" direction in Theorem 4.6. Recall that the "if" direction is easy.

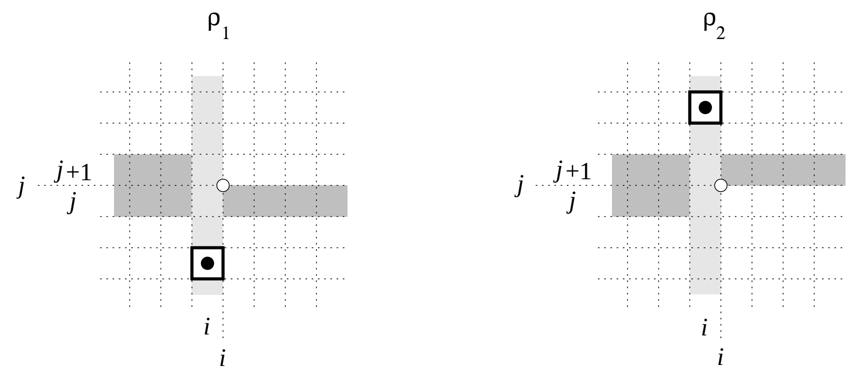
Proof of Theorem 4.6 (the "only if" direction). Let σ be a (2-14-3, 3-41-2)-avoiding permutation of size n. Let $\mathcal{B}$ be the set of Baxter permutations whose S-permutation (described by Theorem 4.2) is σ. By Proposition 4.7, for each point N_i of the graph of σ it is determined uniquely whether it is inside an F-block, or not. Moreover, points that lie inside an F-block are organized along its diagonal or anti-diagonal, depending on whether the block is ascending or descending. It follows that the location of all non-trivial F-blocks in the graph of ρ, for $\rho \in \mathcal{B}$, and their type (ascending or descending, for blocks of size at least 3), are also determined uniquely. It remains to show that the location of the trivial F-blocks (that is, F-blocks of size 1) is also determined uniquely by σ.

Assume for the sake of contradiction that there are $\rho_1, \rho_2 \in \mathcal{B}$ with trivial F-blocks in the ith column such that $\rho_1(i) \ne \rho_2(i)$. Let i be the minimal number for which this happens. This means that the F-structures of ρ_1 and ρ_2 coincide to the left of the ith column. Denote $j = \sigma(i)$. By symmetry, there are, essentially, two cases: (1) $\rho_1(i) < \rho_2(i) \le j$; (2) $\rho_1(i) \le j < \rho_2(i)$.

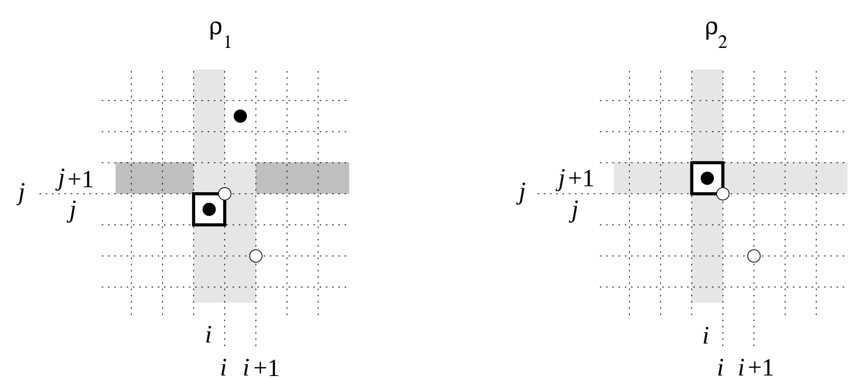
In the first case (illustrated below), denote $k = \rho_2(i)$. Consider $\rho_1^{-1}(k)$. By assumption, $\rho_1^{-1}(k) \ne i$. Since $\rho_1(i) < k$ and $\sigma(i) = j \ge k$, we have $\rho_1^{-1}(k) < i$ by Theorem 4.2. However, this is impossible since $\rho_2^{-1}(k) = i$, and the F-structures of ρ_1 and ρ_2 coincide to the left of the ith column.



Consider the second case. Since $\rho_1(i) \leq j$ and $\sigma(i) = j$, the areas $[1, i] \times \{j+1\}$ and $[i+1, n] \times \{j\}$ are empty in the graph of ρ_1. Similarly, since $\rho_2(i) \geq j+1$, the areas $[1, i] \times \{j\}$ and $[i+1, n] \times \{j+1\}$ are empty in the graph of ρ_2. Since the F-structures of ρ_1 and ρ_2 coincide in $[1, i-1] \times [1, n]$ the areas $[1, i-1] \times \{j, j+1\}$ are empty in the graphs of both permutations. Given that rows cannot be empty, this forces $\rho_1(i) = j$ and $\rho_2(i) = j+1$.



Assume without loss of generality that $\sigma(i+1) < j$. Since $\rho_1(i) = j$ and $\sigma(i) = j$, we have, by Theorem 4.2, $\rho_1(i+1) \geq j+1$. It is impossible that $\rho_1(i+1) = j+1$ since otherwise the point $(i, \rho_1(i))$ would not form a trivial F-block. Thus, $\rho_1(i+1) > j+1$. Now, since $\sigma(i+1) < j$, the area $[i+2, n] \times \{j+1\}$ is empty in the graph of ρ_1. The area $[1, i-1] \times \{j+1\}$ is empty in the graph of ρ_2, since $\rho_2(i) = j+1$. Since the F-structures of ρ_1 and ρ_2 coincide in $[1, i-1] \times [1, n]$, the area $[1, i-1] \times \{j+1\}$ is also empty in the graph of ρ_1. Since $\rho_1(i) = j$ and $\rho_1(i+1) > j+1$, we have a contradiction: the whole row $j+1$ is empty in the graph of ρ_1.



Thus, we have proved that all $\rho \in \mathcal{P}$ have the same F-structure. □

5 Enumeration of (2-14-3, 3-41-2)-avoiding permutations

In this section we enumerate (2-14-3, 3-41-2)-avoiding permutations and, thus, S-equivalence classes of floor-plan partitions. We first describe the shape of the generating tree of (2-14-3, 3-41-2)-avoiding permutations obtained by adding/deleting their rightmost value. This tree gives functional equations defining the generating function of these permutations, which we solve in Section 5.2. The solution involves the generating function of Baxter permutations (Theorem 5.3), and suggests that another connection between the two classes, different from the one described in Section 4, exists.

5.1 A generating tree

Let us first observe that deleting the rightmost value of a (2-14-3, 3-41-2)-avoiding permutation τ, and normalizing the resulting sequence so as to obtain a permutation σ, gives another (2-14-3, 3-41-2)-avoiding permutation. This allows us to display (2-14-3, 3-41-2)-avoiding permutations as the nodes of a *generating tree*, rooted at the permutation 1, and in which the father of a permutation is obtained by deleting its rightmost value (and normalizing).

Conversely, let σ be a (2-14-3, 3-41-2)-avoiding permutation of $[n]$. We wish to construct a (2-14-3, 3-41-2)-avoiding permutation τ of $[n+1]$, taking $\tau(j) = \sigma(j)$ for $1 \le j \le n$, choosing $\tau(n+1) \in \{0.5, 1.5, \ldots, n+0.5\}$, and then normalizing τ so that it becomes a permutation of $[n+1]$. A value $i \in \{0.5, 1.5, 2.5, \ldots, n+0.5\}$ is *admissible* if choosing $\tau(n+1) = i$ results in a (2-14-3, 3-41-2)-avoiding permutation.

Observation 5.1. *A value $i \in \{0.5, 1.5, 2.5, \ldots, n+0.5\}$ is not admissible if and only if there exist $1 \le a < b < n$ such that $\sigma(b) < \sigma(a) = i - 0.5 < \sigma(b+1)$ or $\sigma(b) > \sigma(a) = i + 0.5 > \sigma(b+1)$.*

Graphically, this means the following. Consider the graph of σ and add the point $(n+1, i)$. Consider two segments: the first, connecting $(n+1, i)$ to $(\sigma^{-1}(i-0.5), i-0.5)$; the second, connecting $(n+1, i)$ to $(\sigma^{-1}(i+0.5), i+0.5)$. The value i is not admissible if and only if at least one of these segments intersects a segment with the same sign of the slope that connects $(b, \sigma(b))$ and $(b+1, \sigma(b+1))$ (for some $1 < b < n$). Fig. 26 demonstrates this for $\sigma = 24135$: the values $i = 1.5, 2.5, 4.5$, denoted by $\times$, are not admissible (the forbidden configurations are indicated); the values $i = 0.5, 3.5, 5.5$, denoted by $\circ$, are admissible, resulting in $\tau = 352461$, $\tau = 251364$, $\tau = 241356$, respectively.

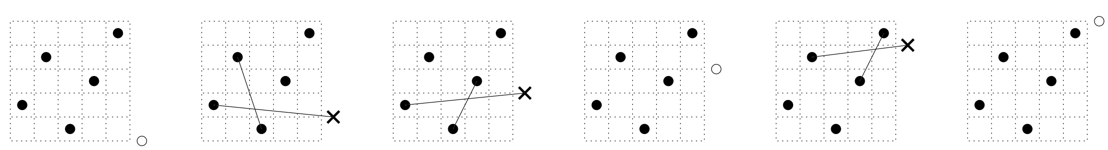


Figure 26: Admissible and inadmissible values.

Observe also that 0.5 and $n + 0.5$ are always admissible.

We are interested in how the number of admissible values changes as we add a new value to the right of σ. As we shall see, this depends on whether or not $|\sigma(n) - \sigma(n-1)|$ equals 1; moreover, if $|\sigma(n) - \sigma(n-1)| > 1$, it also depends on whether $\sigma(n) - \sigma(n-1)$ is positive or negative. Assign to σ a *type*, which is either a triple $(k, m; S)$ or a quadruple $(k, m; L, \pm)$, as follows:

- k is the number of admissible values that are smaller than $\sigma(n)$;
- m is the number of admissible values that are larger than $\sigma(n)$;
- if $|\sigma(n) - \sigma(n-1)| = 1$, then the type is $(k, m; S)$ (where S stands for *small*);
- if $|\sigma(n) - \sigma(n-1)| > 1$, then the type is $(k, m; L, -)$ if $\sigma(n) < \sigma(n-1)$, and $(k, m; L, +)$ otherwise (where L stands for *large*).

For example, $\sigma = 24135$ (the permutation form Fig. 26) is of the type $(2,1;L,+)$.

For the only permutation of $\{1\}$, there holds $(k,m) = (1,1)$, and it will be convenient to assign to this permutation the type $(1,1;S)$.

Proposition 5.2. *The generating tree for* (2-14-3, 3-41-2)*-avoiding permutations is isomorphic to the tree that has root* $(1,1;S)$ *and for which the labels of the children of a node are given by the following rewriting rule:*

- $(k,m;S) \rightsquigarrow$
  $(1,m+1;L,-),(2,m+1;L,-),\ldots,(k-2,m+1;L,-),(k-1,m+1;L,-),(k,m+1;S);$
  $(k+1,m;S),(k+1,m-1;L,+),(k+1,m-2;L,+),\ldots,(k+1,2;L,+),(k+1,1;L,+).$
- $(k,m;L,-) \rightsquigarrow$
  $(1,m+1;L,-),(2,m+1;L,-),\ldots,(k-2,m+1;L,-),(k-1,m+1;L,-),(k,m+1;S);$
  $(k,m;L,+),(k,m-1;L,+),\ldots,(k,2;L,+),(k,1;L,+).$
- $(k,m;L,+) \rightsquigarrow$
  $(1,m;L,-),(2,m;L,-),\ldots,(k-1,m;L,-),(k,m;L,-);$
  $(k+1,m;S),(k+1,m-1;L,+),(k+1,m-2;L,+),\ldots,(k+1,2;L,+),(k+1,1;L,+).$

Proof. The root is $(1,1;S)$ by convention, and it is easily checked that its children are $(1,2;S)$ and $(2,1;S)$. Observe also that taking the complement[6] of a permutation replaces the type $(k,m;S)$ by $(m,k;S)$, and the type $(k,m;L,\pm)$ by $(m,k;L,\mp)$. Due to this symmetry, and the symmetry of the rewriting rules, it suffices to prove them when σ ends with a descent.

Case 1*:* σ *is of the type* $(k,m;S)$.

First, note that the values $\sigma(n)-0.5$ and $\sigma(n)+0.5$ are admissible. For $\sigma(n)+0.5$ this is clear in view of Observation 5.1 (recall that, by assumption, $\sigma(n-1) = \sigma(n)+1$). If $\sigma(n)-0.5$ were not admissible, then $(n+1,\sigma(n)-0.5)$ would form a forbidden configuration with some three points, the rightmost of these being $(n-1,\sigma(n-1))$, and it is easy to see that the point $(n,\sigma(n))$ would also form a forbidden configuration with the same three points.

Let $i_1, i_2, \ldots, i_k$ be the admissible values below $\sigma(n)$, ordered from below; let $j_1, j_2, \ldots, j_m$ be the admissible values above $\sigma(n)$, ordered from above. We have just seen that $i_k = \sigma(n)-0.5$ and $j_m = \sigma(n)+0.5$. Choosing $\tau(n+1) = i_k$ or j_m results in a permutation of type $(k',m';S)$; Choosing $\tau(n+1) = i_\alpha$ with $\alpha < k$ gives a permutation of type $(k',m';L,-)$, and finally, choosing $\tau(n+1) = j_\beta$ with $\beta < m$ gives a permutation of type $(k',m';L,+)$.

Let us now discuss the values of k' and m'. We claim that if τ is obtained by adding i_α, with $\alpha \le k$, then $(k',m') = (\alpha, m+1)$, while if τ is obtained by adding j_β, with $\beta \le m$, then $(k',m') = (k+1,\beta)$. The argument is illustrated in Fig. 27(2) (a point denoted by an asterisk $\circledast$ may be admissible or not).

In the former case ($\tau_{n+1} = i_\alpha$) all the admissible values of σ below i_α, and i_α itself, remain admissible in τ, while the forbidden values remain forbidden (since they would give in τ the same forbidden configuration as they give in σ). Therefore, we have α admissible values below the rightmost point in τ, and $k' = \alpha$. The values above $\sigma(n)$ are admissible in τ if and only if they are admissible in σ. Among the values between $i_\alpha + 1$ and $\tau(n)-0.5$, only $\tau(n)-0.5$ is admissible in τ: as we saw above, $\sigma(n)-0.5$ is admissible in σ, and it is not hard to see that (once incremented by 1) it remains admissible in τ; however, all other values in this interval form a forbidden configuration with $(n,\tau(n))$, $(n+1,\tau(n+1))$ and some fourth point. Since $\tau(n)-0.5$ is above $\tau(n+1)$, there are $m' := m+1$ admissible values above the rightmost point in τ.

The case where $\tau(n+1) = j_\beta$ is similar; see Fig. 27(3).

Case 2*:* σ *is of the type* $(k,m;L,-)$.

If a point below $\sigma(n)$ is added, the situation is similar to that from the first case. In particular, the value just below $\tau(n)$ is admissible. If a point above $\sigma(n)$ is added, no value between $\tau(n)$ and $\tau(n+1)$ is admissible (the lowest of them is not admissible because it was not admissible in σ). Note also that since the value just above $\sigma(n)$ is not admissible, no "small" permutation is obtained here. See Fig. 28. ☐

[6]The *complement* of a permutation $\sigma = \sigma(1),\ldots,\sigma(n)$ is the permutation $n+1-\sigma(1),\ldots,n+1-\sigma(n)$.

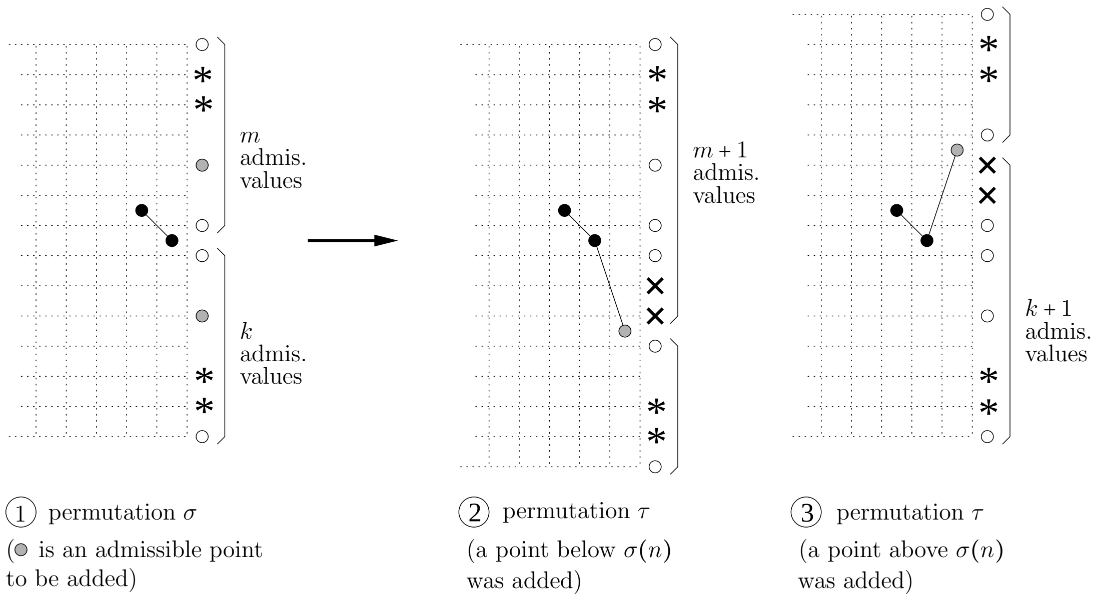


Figure 27: Proof of the rewriting rule for $(k, m; S)$.

5.2 Enumeration

We introduce for (2-14-3, 3-41-2)-avoiding permutations three generating functions corresponding to the three kinds of types occurring in Proposition 5.2. These series involve three variables: t keeps track of the size of the permutation, while x and y respectively keep track of the number of admissible values below and above the rightmost value (the numbers k and m of Proposition 5.2). Let $S(t; x, y)$ be the generating function of permutations of type $(*, *; S)$. Let $L_+(t; x, y)$ (respectively, $L_-(t; x, y)$) be the generating function of permutations of type $(*, *; L, +)$ (respectively, $(*, *; L, -)$). The rules of Proposition 5.2 translate into the following equations:

$$\begin{aligned} S(x,y) &= txy + t(x+y)S(x,y) + txL_+(x,y) + tyL_-(x,y), \\ L_+(x,y) &= \frac{tx}{1-y}\big(yS(x,1) + yL_+(x,1) - S(x,y) - L_+(x,y)\big) + \frac{ty}{1-y}\big(L_-(x,1) - L_-(x,y)\big), \\ L_-(x,y) &= \frac{ty}{1-x}\big(xS(1,y) + xL_+(1,y) - S(x,y) - L_-(x,y)\big) + \frac{tx}{1-x}\big(L_+(1,y) - L_+(x,y)\big). \end{aligned}$$

The form of these equations suggests to introduce

$$L(x,y) := xL_+(x,y) + yL_-(x,y).$$

This reduces the size of the system to two equations:

$$S(x,y) = txy + t(x+y)S(x,y) + tL(x,y), \tag{1}$$

$$L(x,y) = \frac{t}{1-y}\big(x^2yS(x,1) + xyL(x,1) - x^2S(x,y) - xL(x,y)\big) + \frac{t}{1-x}\big(xy^2S(1,y) + xyL(1,y) - y^2S(x,y) - yL(x,y)\big). \tag{2}$$

We will derive from these equations that (2-14-3, 3-41-2)-avoiding permutations are related to Baxter permutations as follows.

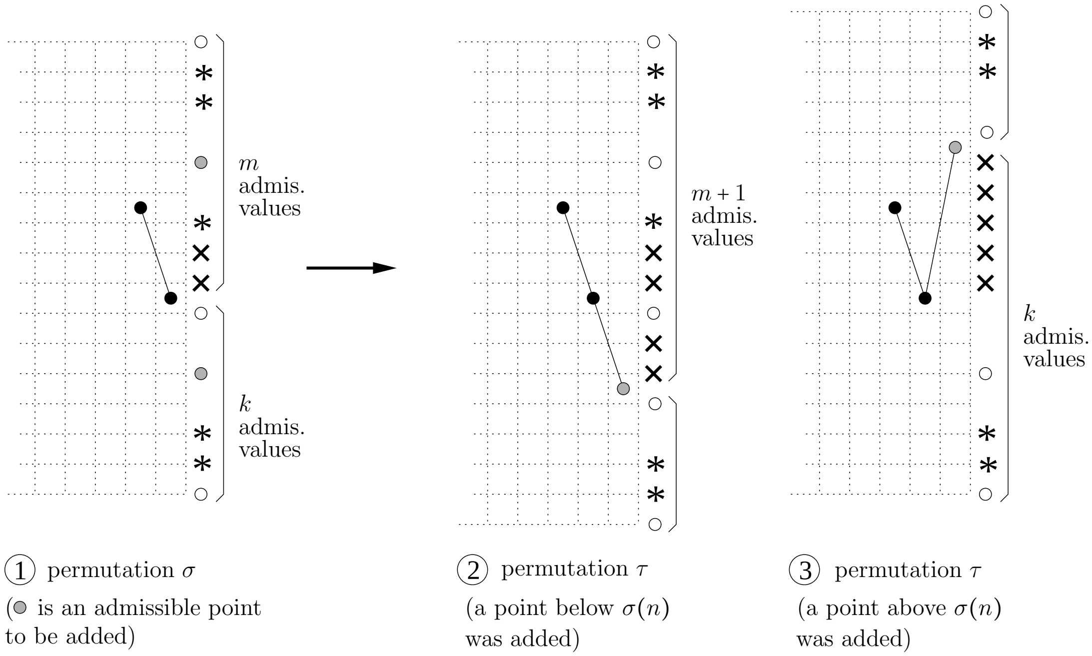


Figure 28: Proof of the rewriting rule for $(k, m; L, -)$.

Theorem 5.3. *The generating function of* $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$*-avoiding permutations is*

$$\sum_{n\geq 1} t^{n-1}(1-t)^n b_n, \tag{3}$$

where

$$b_n = \sum_{m=0}^{n} \frac{2}{n(n+1)^2}\binom{n+1}{m}\binom{n+1}{m+1}\binom{n+1}{m+2}$$

is the number of Baxter permutations of size n*. Therefore, the number of* $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$*-avoiding permutations of size* n *is*

$$\sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i}.$$

More precisely, let $B(s; x, y)$ *be the generating function of (non-empty) Baxter permutations, counted by the size, the number of left-to-right maxima and the number of right-to-left maxima. It is known [25] that*

$$B(s; x, y) = \sum_{n,i,j\geq 1, m\geq 0} s^n x^i y^j \frac{ij}{n(n+1)}\binom{n+1}{m+1}\left(\binom{n-i-1}{m-1}\binom{n-j-1}{n-m-2} - \binom{n-i-1}{m}\binom{n-j-1}{n-m-1}\right).$$

Then the series $S(t; x, y)$ *and* $L(t; x, y)$ *defined above satisfy*

$$\begin{aligned} (1-t)S(t; x, y) &= B(t(1-t); x, y), \\ t(1-t)L(t; x, y) &= (1-t(x+y))B(t(1-t); x, y) - xyt(1-t). \end{aligned}$$

In particular,

$$xy + S(t; x, y) + L(t; x, y) = \frac{1-t(x+y-1)}{t(1-t)} B(t(1-t); x, y),$$

which gives (3) *for* $x = y = 1$.

Remarks

1. Let $C(s) = \frac{1-\sqrt{1-4s}}{2s}$ be the generating function of Catalan numbers. Since $t(1-t) = s$ if $t = sC(s)$, the above identities can be rewritten in terms of $C(s)$. For instance,

$$B(s;1,1) = sC(s)\left(1 + S(sC(s);1,1) + L(sC(s);1,1)\right).$$

This suggests that a connection between S- and R-permutations, different from the one described in Section 4, exists. It is all the more natural to look for a combinatorial interpretation of these identities that $C(s)$ counts τ-avoiding permutations, for any pattern τ of size 3.

2. In the above expression of $B(s;x,y)$, the variable m counts the number of descents. We do not know if this parameter has a natural counterpart in terms of (2-14-3, 3-41-2)-avoiding permutations.

3. The number b_n of Baxter permutations of size n satisfies $b_n \sim 8^n/n^4$ as $n \to \infty$ (up to a multiplicative constant) [28]. Equivalently, the dominant singularity of $B(s;1,1)$ is at $s = 1/8$, and the singular part of this series is $(1-8t)^3 \log(1-8t)$. By (3), the number of (2-14-3, 3-41-2)-avoiding permutations is thus equivalent to $(4+2\sqrt{2})^n/n^4$.

Proof. Thanks to (1), one can express $L(x,y)$ in terms of $S(x,y)$. By specializing x or y to 1, this gives expressions of $L(x,1)$ and $L(1,y)$ in terms of $S(x,1)$ and $S(1,y)$, respectively. Replacing in (2) all occurrences of the series L by their expressions in terms of S gives an equation that only involves the series S:

$$\begin{aligned}\left((1-x)(1-y) - xyt(1-t)(x+y-2)\right) S(x,y) = \\ txy(1-x)(1-y) + txy(1-t)(1-x)S(x,1) + txy(1-t)(1-y)S(1,y).\end{aligned}$$

The generating function $B(s;x,y) \equiv B(x,y)$ of Baxter permutations is known [10] to be characterized by

$$\left((1-x)(1-y) - sxy(x+y-2)\right) B(x,y) \;=\; sxy(1-x)(1-y) + sxy(1-x)B(x,1) + sxy(1-y)B(1,y).$$

Comparing both equations shows that $(1-t)S(t;x,y) = B(t(1-t);x,y)$.

The proof of the identity that relates $L(t;x,y)$ to $B(t(1-t);x,y)$ is similar. □

For $1 \le n \le 30$, the number of (2-14-3, 3-41-2)-avoiding permutations of $[n]$ is given in the following table.

1	374	929480	4023875702	23320440656376	161762725797343554
2	1668	4803018	22346542912	135126739754922	963907399885885724
6	7744	25274088	125368768090	788061492048436	5769548815574513550
22	37182	135132886	709852110576	4623591001082002	34679563373252224012
88	183666	732779504	4053103780006	27277772831911348	209275178482957838142

6 The case of guillotine partitions

6.1 Guillotine partitions and separable-by-point permutations

In this section we study the restriction of the map S to an important family of partitions called *guillotine partitions* [12, 15, 21, 33].

Definition 6.1. A floorplan partition P is a *guillotine partition* (also called *slicing floorplan* [23]) if either it consists of just one rectangle, or there is a segment in P that extends from one side of the boundary to the opposite side, and splits P into two sub-partitions that are also guillotine.

The restriction of the map R to guillotine partitions induces a bijection between R-equivalence classes of guillotine partitions and *separable* permutations [1]. Here, we characterize permutations that are obtained as S-permutations of guillotine partitions.

A nonempty permutation σ is *separable* if it has size 1, or its graph can be split into two nonempty blocks H_1 and H_2, which are themselves separable. In this case, either all the points in H_1 are to the SW of all the points of H_2 (then σ, as a separable permutation, has an *ascending structure*), or all the points in H_1 are to the NW of all the points of H_2 (then σ, as a separable permutation, has a *descending structure*). Separable permutations are known to coincide with (2-4-1-3, 3-1-4-2)-avoiding permutations [9]. In particular, they form a subclass of Baxter permutations. The number of separable permutations of $[n]$ is the nth *Schröder number* [29, A006318].

Definition 6.2. A permutation σ of $[n]$ is *separable-by-point* if it is empty, or its graph can be split into three blocks H_1, H_2, H_3 such that

– H_2 consists of one point N,

– H_1 and H_3 are themselves separable-by-point (thus, they may be empty), and

– either all the points of H_1 are to the SW of N, and all the points of H_3 are to the NE of N (then σ has an *ascending structure*),

or all the points of H_1 are to the NW of N and all the points of H_3 are to the SE of N (then σ has a *descending structure*).

The letter N for the central block refers to the fact that we have denoted by N_i the point $(i, \sigma(i))$ of an S-permutation σ. Observe also that N necessarily corresponds to a fixed point of σ if σ is ascending, and to a point such that $\sigma(i) = n + 1 - i$ is σ is descending and has size n.

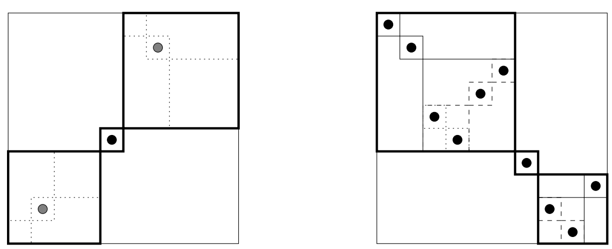


Figure 29: Separable-by-point permutations.

See Fig. 29 for a schematic description and an example of separable-by-point permutations. For $n \leq 3$, all permutations are separable-by-point. It is clear that if a nonempty permutation σ is separable-by-point, then it is separable. The permutations 2143 and 3412 are separable, but not separable-by-point. The following result characterizes separable-by-point permutations in terms of forbidden patterns. In particular, it implies that these permutations are S-permutations.

Proposition 6.3. *Let σ be a permutation of $[n]$. Then σ is separable-by-point if and only if it is* (2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2)*-avoiding.*

Proof. Assume that σ is separable-by-point. In particular, σ is separable, and, therefore, it avoids 2-4-1-3 and 3-1-4-2. Assume for the sake of contradiction that σ contains an occurrence of 2-14-3, corresponding to the points N_i, N_j, N_{j+1} and N_k, and has a minimal size for this property. Then the points forming the pattern must be spread in at least two of the three blocks. This forces σ to have an ascending structure, with N_i and N_j in one block, N_{j+1} and N_k in the following one (because N_j and N_{j+1} are adjacent). But this is impossible as the central block of σ contains a unique point. Similarly one shows that σ avoids 3-41-2.

Conversely, we argue by induction on the size of σ. Let σ be a (2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2)-avoiding permutation of $[n]$. For $n \leq 3$ there is nothing to prove. Let $n \geq 4$. Since σ is (2-4-1-3, 3-1-4-2)-avoiding, it is separable. Assume without loss of generality that σ (as a separable permutation) has an ascending structure: the first block is $[1, i] \times [1, i]$, the second block is $[i+1, n] \times [i+1, n]$ where $1 \le i < n$. If $\sigma(i) \ne i$ and $\sigma(i+1) \ne i+1$, then $\sigma^{-1}(i), i, i+1, \sigma^{-1}(i+1)$ form a forbidden pattern 2-14-3. Thus, $\sigma(i) = i$ or $\sigma(i+1) = i+1$, and one obtains a three-block decomposition of σ by choosing for the central block N one of these two fixed points. The remaining two blocks avoid all four patterns, and, therefore are separable-by-point themselves by the induction hypothesis. It follows that σ is separable-by-point. □

Theorem 6.4. *A partition P is a guillotine partition if and only if $S(P)$ is separable-by-point.*

Proof. Let P be a guillotine partition. We argue by induction on the size of P. If P consists of a single rectangle, then $S(P)$ is the empty permutation, and is separable-by-point. Otherwise, consider a segment that splits P into two rectangles. Assume that this segment is I_i (that is, the ith segment in the $\nwarrow$ order) and that it is vertical. All the segments in the left (respectively, right) part of P are to the left (respectively, right) of I_i, and thus come before (respectively, after) I_i in the $\nwarrow$ and $\swarrow$ orders. Consequently:

- I_i is also the ith segment in the $\swarrow$ order, so that $N_i = (i, i)$,
- by Observation 3.2, all the points of the graph of σ that correspond to segments located to the left (respectively, right) of I_i are to the SW (respectively, NE) of N_i.

Thus, we have three blocks with an ascending structure. The blocks H_1 and H_3 are the S-permutations of the two parts of P, which are themselves guillotine: by the induction hypothesis, H_1 and H_3 are separable-by-point. Thus $S(P)$ is separable-by-point with an ascending structure.

Similarly, if I_i is horizontal, we obtain a separable-by-point permutation with a descending structure.

Conversely, assume that $\sigma := S(P)$ is separable-by-point. We will prove by induction on n that P is a guillotine partition.

The claim is clear for $n = 1$. For $n > 1$, assume without loss of generality that σ has an ascending structure. Let $H_2 = \{(i, i)\}$ be the second block in a decomposition of σ. Then for all $j < i$, we have $I_j \leftarrow I_i$, and for all $j > i$, we have $I_i \leftarrow I_j$. Therefore, if I_i is vertical, it has no below- or above-neighbors, and, thus, I_i extends from the lower to the upper side of the boundary. The two sub-partitions of P correspond respectively to the blocks H_1 and H_3: hence they are guillotine by the induction hypothesis. Suppose now that I_i is horizontal. Then we have $\sigma(i-1) = i-1$ (if $i > 1$) and $\sigma(i+1) = i+1$ (if $i < n$), since otherwise I_i has at least two left-neighbors or at least two right-neighbors, which is never the case for a horizontal segment. Assume without loss of generality that $i > 1$. Then another block decomposition of σ is obtained with the central block $H'_2 = \{(i-1, i-1)\}$, corresponding to the vertical segment I_{i-1}. The previous argument then shows that P is guillotine. □

6.2 Enumeration and multidimensional generalization

In this section we enumerate S-equivalence classes of guillotine partitions. The reasoning actually applies for higher dimensional guillotine partitions, and we therefore study the problem in this generality. We first need to define d-dimensional guillotine partitions, and the counterpart of S-equivalence.

Definition 6.5. Let B be a d-dimensional axes-aligned box. A *guillotine partition* of B is either the trivial partition (whose only part is B itself), or a partition obtained by cutting B by a hyperplane which is perpendicular to an axis x_i, $1 \le i \le d$, into two sub-boxes whose partitions are also guillotine.

Fig. 30 shows a guillotine partition of a 3-dimensional box. The intersection of B with a hyperplane that splits B into two sub-boxes is a *primary cut* (like c and c_3 in Fig. 30). We often denote by B the partition as well as the box. We hope this will not cause any confusion.

A *cut* in a guillotine partition is either a primary cut, or (in a recursive manner) a cut in the partition of one of the sub-boxes. Similarly to the planar case, we assume that parallel cuts do not meet. Therefore, a guillotine partition B with n cuts consists of $n+1$ boxes. We say that it has *size* $n+1$, and denote $|B| = n+1$.

A guillotine partition B may have several primary cuts. In this case, these cuts are perpendicular to the same axis x_i. The lowest primary cut (with respect to x_i) is called the *principal cut* of B. The sub-boxes

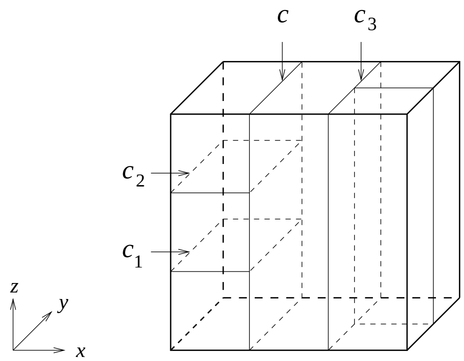


Figure 30: A guillotine partition of a 3-dimensional box.

obtained by cutting B along the principal cut are denoted by B^- (the part of B below the principal cut) and B^+ (the part of B above the principal cut). In Fig. 30, the principal cut of B is c, the principal cut of B^- is c_1 and the principal cut of B^+ is c_3. Notice that the principal cut of B^- is never parallel to that of B.

We now define d order relations between boxes and between cuts, which generalize $\twoheadleftarrow$ and $\twoheaddownarrow$ from the planar case. Their definitions do not involve any notion of neighborhood, but instead, generalize the following characterization of the $\twoheadleftarrow$ order in planar guillotine partitions.

Observation 6.6. *Consider a non-trivial 2-dimensional guillotine partition B, with principal cut c.*

1. *Let K and L be two distinct rectangles in the partition. Then $K \twoheadleftarrow L$ if and only if*
   - *c is vertical, K is in B^-, L is in B^+; or*
   - *K and L are in B^-, and $K \twoheadleftarrow L$ in the partition of B^-; or*
   - *K and L are in B^+, and $K \twoheadleftarrow L$ in the partition of B^+.*
2. *Let u and v be two distinct segments in the partition. Then $u \twoheadleftarrow v$ if and only if*
   - *c is vertical, u is in B^-, v is in B^+; or*
   - *c is vertical, u is in B^-, $v = c$; or*
   - *c is vertical, $u = c$, v is in B^+; or*
   - *u and v are in B^-, and $u \twoheadleftarrow v$ in the partition of B^-; or*
   - *u and v are in B^+, and $u \twoheadleftarrow v$ in the partition of B^+.*

A similar observation holds for the $\twoheaddownarrow$ order.

Definition 6.7. Consider a non-trivial d-dimensional guillotine partition B with principal cut c.

- Let K and L be two distinct boxes in the partition. We say that K is *below* L (equivalently, L is *above* K) *with respect to the axis* x_i $(1 \le i \le d)$, to be denoted by $K \underset{i}{\twoheadleftarrow} L$, if
  - c is perpendicular to x_i, K is in B^-, L is in B^+; or
  - K and L are in B^-, and $K \underset{i}{\twoheadleftarrow} L$ in the partition of B^-; or
  - K and L are in B^+, and $K \underset{i}{\twoheadleftarrow} L$ in the partition of B^+.
- Let u and v be two distinct cuts in the partition. We say that u is *below* v (equivalently, v is *above* u) *with respect to the axis* x_i $(1 \le i \le d)$, to be denoted by $u \underset{i}{\twoheadleftarrow} v$, if

– c is perpendicular to x_i, u is in B^-, v is in B^+; or

– c is perpendicular to x_i, u is in B^-, $v = c$; or

– c is perpendicular to x_i, $u = c$, v is in B^+; or

– u and v are in B^-, and $u \underset{i}{\leftarrow} v$ in the partition of B^-; or

– u and v are in B^+, and $u \underset{i}{\leftarrow} v$ in the partition of B^+.

If two distinct cuts u and v lie respectively in B^- and B^+ (or if one of them is the principal cut c), they are comparable for the order $\underset{i}{\leftarrow}$, where x_i is perpendicular to c, but for no other order $\underset{j}{\leftarrow}$. By induction, it follows that each pair (u, v) of distinct cuts stands in exactly one of the order relations $u \underset{i}{\leftarrow} v$ or $v \underset{i}{\leftarrow} u$ (to be denoted by $u \underset{i}{\leftrightarrows} v$) for a unique i, $1 \le i \le d$.

We now define B-equivalence and C-equivalence of guillotine partitions[7] which generalize the R- and S-equivalences studied in the planar case. Two guillotine partitions B and D, both of size $n+1$, are *C-equivalent* if it is possible to label the cuts of B by $u_1, \dots, u_n$ and the cuts of D by $v_1, \dots, v_n$ in such a way that for all $1 \le j, k \le n$ we have $u_j \underset{i}{\leftarrow} u_k$ if and only if $v_j \underset{i}{\leftarrow} v_k$. Two such labelings are said to be *C-compatible.* We define B-equivalence in a similar way. Lemma 2.6 implies that for 2-dimensional boxes, S-equivalence is indeed equivalent to C-equivalence (and similarly for R- and B-equivalences).

Ackerman *et al.* [2] proved that the number of B-equivalence classes of d-dimensional guillotine partitions of size $n+1$ is

$$\frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}(d-1)^k d^{n-k}.$$

Moreover, B-equivalence classes may be described by *separable multidimensional permutations* [4].

Here, we count C-equivalence classes of guillotine partitions (Theorem 6.11). The counting will be based on the following three lemmas.

Lemma 6.8. *Let $n \ge 2$. Two C-equivalent guillotine partitions B and D of size $n+1$ have their principal cuts in parallel directions.*

Proof. Assume that the principal cut u of B is perpendicular to x_i. Then u is comparable, for the $\underset{i}{\leftarrow}$ order, to any other cut. For any $j \ne i$ however, there exists no cut v that is $\underset{j}{\leftarrow}$ comparable to all other cuts (v would have to be $\underset{j}{\leftarrow}$ comparable to u, which is impossible). Since B and D are C-equivalent, these two properties hold as well for the cuts of D, so that the principal cut of D is perpendicular to x_i as well. □

Lemma 6.9. *Let $n \ge 2$. Let B be a d-dimensional guillotine partition of size $n+1$ such that $|B^-| = 2$. Then there exists a d-dimensional guillotine partition D of size $n+1$ which is C-equivalent to B and satisfies $|D^-| = 1$.*

Proof. Let u be the principal cut of B, and assume that it is perpendicular to x_i. Let v be the only cut of B^-. Then v is perpendicular to x_j for some $j \ne i$. Replace v by a cut w perpendicular to x_i. This gives a new guillotine partition D which is easily seen to be C-equivalent to B. Furthermore, w is the principal cut of D; therefore, $|D^-| = 1$. □

Lemma 6.10. *Let $n \ge 2$. Let B and D be two d-dimensional guillotine partitions of size $n+1$ such that $|B^-| \ne 2$ and $|D^-| \ne 2$. These partitions are C-equivalent if and only if*

– *the principal cuts of B and C are parallel,*

– *the partition B^- is C-equivalent to the partition D^-, and*

[7] B for *boxes*, C for *cuts*.

– *the partition B^+ is C-equivalent to the partition D^+.*

In this case, B^- and D^- have the same size.

Proof. The "if" direction is easily seen by induction on n.

Conversely, assume that B and D are C-equivalent, with C-compatible labelings $u_1, \dots, u_n$ and $v_1, \dots, v_n$. Let u_ℓ be the principal cut of B and v_m the principal cut of D. By Lemma 6.8, u_ℓ and v_m are perpendicular to the same axis x_i. Let us prove that $\ell = m$. Assume for the sake of contradiction that $\ell \neq m$. Since v_m is the principal cut of D, we have $v_\ell \underset{i}{\leftrightarrows} v_m$.

Assume first $v_\ell \underset{i}{\leftarrow} v_m$. Then v_ℓ is in D^-, and the partition D^- is not trivial. We shall prove that v_ℓ is the principal cut of D^-. Indeed, let v_k be the principal cut of D^-. It is perpendicular to x_j where $j \neq i$. Therefore, if $\ell \neq k$, then we have $v_\ell \underset{j}{\leftrightarrows} v_k$. Since the two labelings are C-equivalent, $u_\ell \underset{j}{\leftrightarrows} u_k$. However, since u_ℓ is the principal cut of B, we also have $u_\ell \underset{i}{\leftrightarrows} u_k$, which is a contradiction since $i \neq j$. Hence v_ℓ is the principal cut of D^-. By assumption, $|D^-| \neq 2$. Therefore there exists v_p in D^- such that $p \neq \ell$. We then have $v_\ell \underset{j}{\leftrightarrows} v_p$, so that $u_\ell \underset{j}{\leftrightarrows} u_p$. But there also holds $u_\ell \underset{i}{\leftrightarrows} u_p$, which gives a contradiction.

Assume now $v_m \underset{i}{\leftarrow} v_\ell$. Then $u_m \underset{i}{\leftarrow} u_\ell$, and we can repeat the above argument in B rather than D. This concludes the proof that $m = \ell$. That is, the principal cut of D is v_ℓ.

Clearly, u_p is in B^- if and only if v_p is in D^-. In particular, $|B^-| = |D^-|$ and $|B^+| = |D^+|$. Moreover, any two cuts in B^-, u_p and u_q, stand in the same order as v_p and v_q do in D^-. Therefore, the partition B^- is C-equivalent D^-. Similarly, the partition B^+ is C-equivalent to D^+. □

Theorem 6.11. *Fix $d \geq 2$. Let A_n be the number of C-equivalence classes of d-dimensional guillotine partitions of size $n+1$ (that is, having n cuts). Let $A(t) = \sum_{n\geq 0} A_n t^n$ be the associated generating function. Then*

$$A(t) = \frac{1 - t + (d-1)t^2 - \sqrt{(1 - t + (d-1)t^2)^2 - 4(d-1)t(1-(d-1)t)}}{2(d-1)t}.$$

Equivalently, $A_0 = A_1 = 1$, and for $n \geq 2$,

$$A_n = dA_0 A_{n-1} + (d-1) \sum_{k=2}^{n-1} A_k A_{n-1-k}. \tag{4}$$

Proof. That $A_0 = A_1 = 1$ is clear. Let $n \geq 2$. Lemma 6.8 shows that two partitions of size $n+1$ with their principal cuts in distinct directions cannot be C-equivalent. Therefore, $A_n = dA_n^{(1)}$, where $A_n^{(1)}$ is the number of C-equivalence classes of partitions with n cuts where the principal cut is perpendicular to x_1.

By Lemma 6.10, a partition B such that $|B^-| > 2$ is only equivalent to partitions D such that $|D^-| = |B^-|$. By Lemma 6.9, a partition B such that $|B^-| = 2$ is equivalent to a partition D such that $|D^-| = 1$. In turn, D is only equivalent to partitions E such that $|E'| = 1$ or 2 (by Lemma 6.10). Consequently,

$$A_n^{(1)} = \sum_{\substack{0 \leq k \leq n-1 \\ k \neq 1}} A_{n,k}^{(1)},$$

where $A_{n,k}^{(1)}$ is the number of classes containing a partition B such that $|B^-| = k+1$ (and $|B| = n+1$, and the principal cut is perpendicular to x_1, as in the definition of the numbers $A_n^{(1)}$).

By Lemma 6.10, these classes are in one-to-one correspondence with ordered pairs $(\mathcal{C}_1, \mathcal{C}_2)$ of classes, of respective size $k+1$ and $n-k$, such that the principal cut of $\mathcal{C}_1$ (if it exists, that is, if $k \geq 2$) is not perpendicular to x_1. By Lemma 6.8, the number of choices for $\mathcal{C}_1$ is then $\frac{d-1}{d} A_k$ (for $k \geq 2$). Therefore

$$A_{n,k}^{(1)} = \begin{cases} A_0 A_{n-1}, & \text{if } k = 0; \\ \frac{d-1}{d} A_k A_{n-k-1}, & \text{otherwise.} \end{cases}$$

This yields, for $n \geq 2$,

$$A_n = d \cdot \left(A_0 A_{n-1} + \frac{d-1}{d} \sum_{k=2}^{n-1} A_k A_{n-1-k} \right),$$

which can be rewritten as in the proposition. Equivalently,

$$A_n = dA_{n-1} + (d-1) \sum_{k=0}^{n-1} A_k A_{n-1-k} - (d-1)(A_{n-1} + A_{n-2}), \qquad (5)$$

so that

$$A(t) - 1 - t = dt(A(t) - 1) + (d-1)t(A^2(t) - 1) - (d-1)t(A(t) - 1) - (d-1)t^2 A(t).$$

The expression of $A(t)$ follows. □

Remarks

1. Let us return to the planar case, $d = 2$. The numbers A_n then count S-equivalence classes of planar guillotine partitions of size $n+1$. By Theorem 6.4 and Proposition 6.3, they also count (2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2)-avoiding permutations of $[n]$. The first values are $1, 2, 6, 20, 70, 254, 948, 3618, 14058, 55432$. This sequence [29, A078482] also enumerates irreducible stack sortable permutations, or (2-4-3-1, 3-2-4-1, 2-4-1-3, 3-1-4-2)-avoiding permutations, as found by Atkinson and Stitt [5, Theorem 17]. The associated generating function is

$$A(t) = \frac{1 - t + t^2 - \sqrt{1 - 6t + 7t^2 - 2t^3 + t^4}}{2t}.$$

Using the methods from [17, Sec. VI.4], we can determine the asymptotic behavior of this sequence:

$$A_n \sim \kappa \mu^n n^{-3/2},$$

where κ is a constant, and

$$\mu = \frac{2}{1 - \sqrt{8\sqrt{2} - 11}} \approx 4.5465.$$

2. One can express the numbers A_n as a double sum. We have

$$A(t) = \frac{1 - (d-1)t}{1 - t(1 - (d-1)t)} C\left(\frac{(d-1)t(1 - (d-1)t)}{(1 - t(1 - (d-1)t))^2} \right),$$

where $C(t) = \frac{1 - \sqrt{1-4t}}{2t} = \sum_{n \geq 0} C_n t^n$ is the generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus,

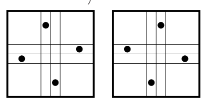
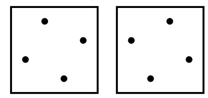
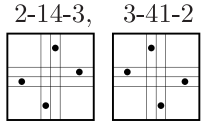
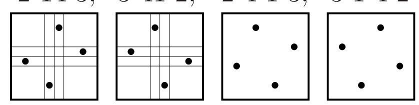
$$\begin{aligned} A(t) &= \sum_{k \geq 0} C_k \frac{(d-1)^k t^k (1 - (d-1)t)^{k+1}}{(1 - t(1 - (d-1)t))^{2k+1}} \\ &= \sum_{k \geq 0} \sum_{j \geq 0} C_k \binom{2k+j}{j} (d-1)^k t^{k+j} (1 - (d-1)t)^{k+1+j}. \end{aligned}$$

Hence, the coefficient of t^n in $A(t)$ is

$$A_n = \sum_{k=0}^{n} \sum_{j=0}^{n-k} (-1)^{n-k-j} C_k \binom{2k+j}{j} \binom{k+1+j}{n-k-j} (d-1)^{n-j}.$$

7 Summary

We conclude by a summary of the results obtained in [1] for R-equivalence classes and in the present paper for S-equivalence classes.

	All floorplan partitions	Planar guillotine partitions
R-equivalence classes	**Forbidden patterns:** 2-41-3, 3-14-2  **Enumerating sequence:** $1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, \ldots$ (Baxter numbers [29, A001181]) **Growth rate:** 8.	**Forbidden patterns:** 2-4-1-3, 3-1-4-2  **Enumerating sequence:** $1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots$ (Schröder numbers [29, A006318]) **Growth rate:** $3 + 2\sqrt{2} \approx 5.8284$.
S-equivalence classes	**Forbidden patterns:** 2-14-3, 3-41-2  **Enumerating sequence:** $1, 2, 6, 22, 88, 374, 1668, 7744, 37182, 183666, \ldots$ **Growth rate:** $4 + 2\sqrt{2} \approx 6.8284$.	**Forbidden patterns:** 2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2  **Enumerating sequence:** $1, 2, 6, 20, 70, 254, 948, 3618, 14058, 55432, \ldots$ ([29, A078482]) **Growth rate:** $\frac{2}{1-\sqrt{8\sqrt{2}-11}} \approx 4.5465$.

References

[1] E. Ackerman, G. Barequet, and R. Y. Pinter. A bijection between permutations and floorplans, and its applications. *Discrete Applied Mathematics*, **154** (2006), 1674 – 1684.

[2] E. Ackerman, G. Barequet, R. Y. Pinter, and D. Romik. The number of guillotine partitions in d dimensions. *Information Processing Letters*, **98** (2006), 162 – 167.

[3] M. H. Albert and M. D. Atkinson. The enumeration of simple permutations. *Journal of Integer Sequences*, **6** (2003), paper 03.4.4.

[4] A. Asinowski and T. Mansour. Separable d-permutations and guillotine partitions. *Annals of Combinatorics*, **14** (2010), 17 – 43.

[5] M. D. Atkinson and T. Stitt. Restricted permutations and the wreath product. *Discrete Mathematics*, **259** (2002), 19 – 36.

[6] G. Baxter. On fixed points of the composite of commuting functions. *Proceedings of the American Mathematical Society*, **15** (1964), 851 – 855.

[7] I. Baybars and C. M. Eastman. Enumerating architectural arrangements by generating their underlying graphs. *Environment and Planning, Series B*, **7** (1980), 289 – 310.

[8] N. Bonichon, M. Bousquet-Mélou, and É. Fusy. Baxter permutations and plane bipolar orientations. *Séminaire Lotharingien de Combinatoire*, **61A** (2010), paper B61Ah, 19 pp. arXiv:0805.4180.

[9] P. Bose, J. F. Buss, and A. Lubiw. Pattern matching for permutations. *Information Processing Letters*, **65** (1998), 277 – 283.

[10] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels, *Electronic Journal of Combinatorics* **9** no. 2 (2003), paper R19.

[11] F. C. Calheiros, A. Lucena, and C. C. de Souza. Optimal rectangular partitions. *Networks*, **41:1** (2003), 51 – 67.

[12] M. Cardei, X. Cheng, X. Cheng, and D.-Z. Du. A tale on guillotine cut. *Proceedings of Novel Approaches to Hard Discrete Optimization*, Ontario, Canada, 2001.

[13] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman. The number of Baxter permutations. *Journal of Combinatorial Theory, Series A*, **24** (1978), 382 – 394.

[14] J. Cousin. *Organisation Topologique de l'Espace Architectural / Topological Organization of Architectural Spaces.* Les Presses de l'Université de Montréal, 1970.

[15] D.-Z. Du, L. Q. Pan, and M. T. Shing. Minimum edge length guillotine rectangular partition. Technical Report MSRI 02418-86, University of Califonia, Berkeley, CA, 1986.

[16] S. Felsner, É. Fusy, M. Noy, and D. Orden. Bijections for Baxter families and related objects. *Journal of Combinatorial Theory, Series A*, to appear. arXiv:0803.1546

[17] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* Cambridge University Press, 2009.

[18] U. Flemming. Wall representation of rectangular dissections and their use in automated space allocation. *Environment and Planning, Series B*, **5** (1978), 215 – 232.

[19] É. Fusy, D. Poulalhon, and G. Schaeffer. Bijective counting of plane bipolar orientations and Schnyder woods. *European Journal of Combinatorics*, bf 30 no. 7 (2009), 1646 – 1658,

[20] S. Gire. *Arbres, permutations à motifs exclus et cartes planaires: quelques problèmes algorithmiques et combinatoires.* PhD thesis, LaBRI, Université Bordeaux 1, 1993.

[21] T. F. Gonzalez and S.-Q. Zheng. Improved bounds for rectangular and guillotine partitions. *Journal of Symbolic Computation*, **7** (1989), 591 – 610.

[22] D. P. La Potin and S. W. Director. Mason: A global floorplanning approach for VLSI design. *IEEE Transactions on CAD of Integrated Circuits and Systems*, **5** (1986), 477 – 489.

[23] T. Lengauer. *Combinatorial Algorithms for Integrated Circuit Layout.* Wiley - Teubner, 1990.

[24] A. Lingas, R. Y. Pinter, R. L. Rivest, and A. Shamir. Minimum edge length rectilinear decomposition of rectilinear figures. *Proceedings of 20th Annual Allerton Conference on Communication, Control and Computing*, University of Illinois Press, Monticello, IL, 1982, 53 – 63.

[25] C. L. Mallows. Baxter permutations rise again. *Journal of Combinatorial Theory, Series A*, **27** (1979), 394 – 396.

[26] C. N. de Meneses and C. C. de Souza. Exact solutions of optimal rectangular partitions via integer programming. *International Journal of Computational Geometry and Applications*, **10** (2000), 477 – 522.

[27] R. L. Rivest. The "PI" (placement and intercorrect) system. *Proceedings of 19th ACM-IEEE Design Automation Conference*, Las Vegas, NV, 1982, 475 – 481.

[28] Z. C. Shen and C. C. N. Chu. Bounds on the number of slicing, mosaic, and general floorplans. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, **22** (2003), 1354 – 1361.

[29] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. http://www.research.att.com/~njas/sequences/.

[30] J. P. Steadman. *Architectural Morphology.* Pion Limited, 1983.

[31] J. P. Steadman. Graph-theoretic representation of architectural arrangement. In *The Architecture of Form*, L. March (ed.), Cambridge University Press, 1976, pp. 94 – 115.

[32] E. Steingrímsson. Generalized permutation patterns — a short survey. In *Permutation Patterns*, S. A. Linton, N. Ruskuc, V. Valter (eds.), LMS Lecture Note Series, Vol. 376, Cambridge University Press, 2010, pp. 137 – 152.
Preprint http://www.math.ru.is/download/St08__Generalized_permutation.pdf

[33] L. J. Stockmeyer. Optimal orientations of cells in slicing floorplan designs. *Information and Control*, **57** (1983), 91 – 101.

[34] H. Watanabe and B. Ackland. FLUTE: An expert floorplanner for VLSI. *IEEE Design and Test of Compututers*, **4**, (1987), 32 – 41.

[35] S. Wimer, I. Koren, and I. Cederbaum. Floorplans, planar graphs, and layouts. *IEEE Transactions on Circuits and Systems*, **35** (1988), 267 – 278.