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# IRREGULAR TIME DEPENDENT OBSTACLES

PETER LINDQVIST AND MIKKO PARVIAINEN

Abstract. We study the obstacle problem for the Evolutionary p-Laplace Equation when the obstacle is discontinuous and without regularity in the time variable. Two quite different procedures yield the same solution.

## 1. Introduction

Our objective is the obstacle problem for the Evolutionary $p$-Laplace Equation in the slow diffusion case $p > 2$. The appearing functions are forced to lie almost everywhere above a given function, the obstacle $\psi$. Our emphasis is on very irregular obstacles. Then some uniqueness and convergence results, known in the stationary case, are no longer valid in the parabolic theory. Thus some precaution is called for.

The weak solutions and weak supersolutions of the Evolutionary $p$-Laplace Equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

are *a priori* required to belong to the Sobolev space $L^p(0,T;W^{1,p}(\Omega))$. Therefore it is natural to treat the obstacle problem under the assumption that the obstacle $\psi$ belongs to the same space. Needless to say, when it comes to the basic theory, it is very important that no further assumptions be imposed on the obstacle. However, the natural

$$\textsf{Assumption:}\quad \psi \in L^p(0,T;W^{1,p}(\Omega))$$

does not include any requirements about the time derivative $\frac{\partial\psi}{\partial t}$. Neither must $\psi$ be continuous. Indeed, for instance rather irregular discontinuous functions of the type $\psi(x,t) = \psi(t)$ belong to this space. The variational problem is difficult to handle under this general assumption. In the literature, so far as we know, extra conditions about the "missing" time derivative or other devices to control the time behavior are always present. In the present work, we carefully avoid such additional regularity assumptions, but for convenience we require that the obstacle $\psi$ is bounded and of compact support.

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Given a general obstacle $\psi$, belonging to the natural space mentioned above, we will define the solution of the obstacle problem in two different ways:

- the least solution $w^*$. This comes from the pointwise infimum of weak supersolutions lying above the obstacle almost everywhere.

- the variational solution $v$. The obstacle $\psi$ is approximated by time convolutions $\psi_\varepsilon$ and these act as obstacles. The limit of the solutions of the approximating obstacle problems is the variational solution $v$.

We prove that the least solution and the variational solution coincide (Theorem 4.10). Since $w^*$ is unique by its definition, it follows that also the variational solution is unique. The uniqueness of $v$ is, as it were, difficult to achieve without evoking $w^*$. Furthermore, the variational inequality

$$\int_0^T\int_\Omega\Big(|\nabla v|^{p-2}\nabla v\cdot\nabla(\phi-v)+(\phi-v)\frac{\partial\phi}{\partial t}\Big)\,dx\,dt \geq \frac{1}{2}\int_\Omega|\phi(x,T)-v(x,T)|^2dx \tag{1.1}$$

holds for all *smooth* $\phi$, $\phi\geq\psi$ a.e. and $\phi=\psi$ on the parabolic boundary[1]. The same holds for $w^*$, since $v=w^*$. However, in the presence of an irregular obstacle, the above variational inequality also can have "false solutions": uniqueness fails at this level[2]. Therefore the procedure with the convolutions $\psi_\varepsilon$ is decisive; the $\psi_\varepsilon$'s capture the time behavior of their limit $\psi$.

We seize the opportunity to mention the celebrated Lavrentiev phenomenon. If the obstacle $\psi$ is not upper semicontinuous, one cannot always reach the least solution by using merely *continuous* weak supersolutions $u$ satisfying $u\geq\psi$. Neither can one in the construction of the variational solution, restrict oneself to approximants satisfying $\psi_j\geq\psi$ almost everywhere. See section 5. This excludes some easy definitions.

We emphasize that this is not the theory about *thin obstacles*, where the functions are forced to lie above the obstacle at *each* point. Our inequalities are usually valid only almost everywhere and no finer theory about capacities is used. —It has not escaped our notice that the results suggest a generalization to other equations of the same structural type. Also the wider range $p>2n/(n+2)$ of exponents could be included.

[1] The reader may notice that, strictly speaking not even the obstacle $\psi$ itself, is always admissible as a test function in (1.1).

[2] A counterexample is presented in section 5

## 2. Preliminaries

We consider the domain

$$\Omega_T = \Omega \times (0, T),$$

where $\Omega$ is a regular and bounded domain in $\mathbf{R}^n$, for example a ball will do. Its parabolic boundary is

$$\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T]).$$

Let

$$B = B_R(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < R\}$$

denote the ball of radius $r$ centered at $x$. The space-time cylinders

$$Q = Q_r(x, t) = B_r(x) \times (t - r^p, t + r^p).$$

are convenient for some limit procedures.

As usual, $W^{1,p}(\Omega)$ denotes the Sobolev space of those real-valued functions $f$ that together with their distributional first partial derivatives $\partial f / \partial x_i$, $i = 1, 2, \dots, n$, belong to $L^p(\Omega)$. We use the norm

$$\|f\|_{W^{1,p}(\Omega)} = \left( \int_\Omega (|f|^p + |\nabla f|^p)\, dx \right)^{1/p}.$$

The Sobolev space $W_0^{1,p}(\Omega)$ with zero boundary values is the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm.

The Sobolev space

$$L^p(0, T; W^{1,p}(\Omega)),$$

consists of all functions $u(x, t)$ such that $u(x, t)$ belongs to $W^{1,p}(\Omega)$ for almost every $0 < t < T$, $u(x, t)$ is measurable as a mapping from $(0, T)$ to $W^{1,p}(\Omega)$, and the norm

$$\left( \iint_{\Omega_T} (|u(x,t)|^p + |\nabla u(x,t)|^p)\, dx\, dt \right)^{1/p}$$

is finite. The definition of the space $L^p(0, T; W_0^{1,p}(\Omega))$ is analogous.

**Definition 2.1.** A function $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a *weak supersolution* to the $p$-parabolic equation, if

$$\iint_{\Omega_T} \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) \mathrm{d}x\, \mathrm{d}t \geq 0 \tag{2.2}$$

for every $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$. It is a *weak subsolution*, if the integral is non-positive. A function $u$ is a *weak solution* if it is both a super- and a subsolution, that is,

$$\iint_{\Omega_T} \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) \mathrm{d}x\, \mathrm{d}t = 0 \tag{2.3}$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

By parabolic regularity theory, a continuous representative of a weak solution always exists. It is here called a *p-parabolic function.* For the theory of weak solutions the reader may consult [DiB93] and [WZYL01].

We shall use the regularizations

$$w^*(x,t) = \operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} w(y,s) = \lim_{r\to 0}\Big(\operatorname*{ess\,inf}_{Q_r(x,t)\cap\Omega_T} w\Big)$$

and

$$\hat{w}(x,t) = \liminf_{(y,s)\to(x,t)} w(y,s) = \lim_{r\to 0}\Big(\inf_{Q_r(x,t)} w\Big).$$

Both are lower semicontinuous.

The lower semicontinuity of $w^*$ follows from the definition in a straightforward manner: Fix $(x,t)\in\Omega_T$. Then for every $\varepsilon>0$, we may choose a radius $r>0$ such that $Q_r(x,t)\subset\Omega_T$ and

$$\left| w^*(x,t) - \operatorname*{ess\,inf}_{Q_r(x,t)} w \right| \le \varepsilon.$$

Choose $(y,s)\in Q_r(x,t)$ and observe that for all small enough $\rho>0$, we have $Q_\rho(y,s)\subset Q_r(x,t)$. Thus,

$$w^*(y,s) \ge \operatorname*{ess\,inf}_{Q_r(x,t)} w \ge w^*(x,t)-\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this leads to

$$\liminf_{(y,s)\to(x,t)} w^*(y,s) \ge w^*(x,t),$$

which proves the assertion. The proof at the boundary is analogous.

According to [Kuu09] the ess lim inf-regularization of a weak supersolution coincides with the original function almost everywhere, and thus *every weak supersolution has a lower semicontinuous representative.*

Let us now introduce the *obstacle* $\psi$. In this section it is only assumed to be a measurable function satisfying the inequality $0\le\psi\le L$ in $\Omega_T$.

**Definition 2.4.** Let $\psi$ be the obstacle and consider the class

$$\mathcal{S}_\psi = \{u \,:\, u \text{ is ess lim inf-regularized weak supersolution,} \\ u\ge\psi \text{ a.e. in } \Omega_T\}.$$

Define the function

$$w(x,t) = \inf_u u(x,t),$$

where the infimum is taken over the whole class $\mathcal{S}_\psi$. We say that its regularization $w^*(x,t)$ is the *least solution* to the obstacle problem[3].

The least solution always exists and is unique. If $u_1,u_2\in\mathcal{S}_\psi$, then also their pointwise minimum $\min\{u_1,u_2\}$ belongs to $\mathcal{S}_\psi$, cf. for example Lemma 3.2. in [KKP10]. Therefore Choquet's well known topological lemma is applicable.

[3]In Potential Theory, $w^*$ is often called the *balayage.*

**Lemma 2.5** (Choquet)**.** *Let $w$ be as above. There exists a decreasing sequence of functions in $\mathcal{S}_\psi$ converging pointwise to a function $u$ such that*

$$\hat{u}(x,t) = \hat{w}(x,t)$$

*at every point in $\Omega_T$.*

Next we recall Theorem 4.3 from [KLP10], based on Theorem 6 in [LM07], [Sim87], and Theorem 5.3. in [KKP10]. An essential ingredient in the proof is that a Radon measure is assigned to every weak supersolution.

**Theorem 2.6.** *Let $u_i$ be a bounded sequence of weak supersolutions in $\Omega_T$. Then there exist a weak supersolution $u$ and a subsequence, still denoted by $u_i$, such that*

$$u_i \to u, \quad \nabla u_i \to \nabla u \quad a.e. \ in \quad \Omega_T.$$

In Lemma 2.8, we will show that the least solution $w^*$ to the obstacle problem is a weak supersolution. The proof is based on Choquet's lemma and the above convergence result. Since Choquet's lemma is formulated for lim inf-regularizations, while the definition of a least solution uses the ess lim inf-regularization, we show that for the infimum $w$ these coincide.

**Lemma 2.7.** *For the least solution it holds everywhere that*

$$w^* = \hat{w}.$$

*Proof.* Clearly $\hat{w} \le w^*$, and it remains to show that $w^* \le \hat{w}$. First, notice that $w^* \le w$. Indeed,

$$w^* = \operatorname{ess\,lim\,inf} w \le \operatorname{ess\,lim\,inf} u = u$$

for each admissible ess lim inf-regularized $u$, hence $w^* \le \inf\{u\} = w$. Using this and the semicontinuity of $w^*$, we obtain

$$w^* \le \liminf w^* \le \liminf w = \hat{w}. \qquad \square$$

**Theorem 2.8.** *The least solution $w^*$ with the obstacle $\psi$ is a weak supersolution. Furthermore, $w = w^*$ almost everywhere.*

*Proof.* By Lemma 2.5, there exists a decreasing sequence in $\mathcal{S}_\psi$ converging to a function $u$ so that

$$\hat{u}(x,t) = \hat{w}(x,t)$$

at each point. By Theorem 2.6 one can pass to the limit under the integral sign in (2.2), whence the limit $u$ is a weak supersolution. It follows that

$$u^* = u$$

almost everywhere. The proof of Lemma 2.7 also applies to $u$ and thus, $\hat{u} = u^*$ and $\hat{w} = w^*$. Clearly, $u \geq w$. It follows that

$$\hat{w} = \hat{u} = u^* = u \geq w \geq \hat{w}$$

almost everywhere, and since $w^* = \hat{w}$, this implies that $w = w^*$ almost everywhere. □

## 3. Continuous obstacles

In this section we consider *continuous* obstacles. However, we do not assume that the obstacle has a time derivative.

We prove that if the obstacle is continuous, so is $w^*$, and that $w^*$ is even $p$-parabolic in the set where the obstacle does not hinder. For the elliptic case, see [Kil89]. In the proof, we use a so-called Poisson modification.

**Definition 3.1.** Let $Q \Subset \Omega_T$ and let $w$ be a bounded and $\operatorname{ess\,lim\,inf}$-regularized supersolution. We define its *Poisson modification* with respect to $Q$ as

$$w_P(x,t) = \begin{cases} w, & \text{in} \quad \Omega_T \setminus Q \\ v, & \text{in} \quad Q, \end{cases}$$

where

$$v(\xi) = \sup\{h(\xi) \,:\, h \in C(\overline{Q}) \text{ is } p\text{-parabolic and } h \leq w \text{ on } \partial_p Q\}.$$

As shown in Section 4.6. in [KL96], $w_P$ is $p$-parabolic in $Q$. Obviously, $w_P$ is lower semicontinuous. Always, $w_P \leq w$ by the Comparison Principle.

**Theorem 3.2.** *Let $\psi \in C(\overline{\Omega}_T)$. The least solution $w^*$ with the obstacle $\psi$ is continuous up to the boundary, and $w^* = \psi$ at $\partial_p \Omega_T$. Moreover, $w^*$ is $p$-parabolic in the open set $\{w^* > \psi\}$.*

*Proof.* Since $w^* = \hat{w}$, we can work with $\hat{w}$. Since $\hat{w}$ is lower semicontinuous, it remains to show that $\hat{w}$ is upper semicontinuous. To establish this, fix $(x_0, t_0) \in \Omega_T$ and observe that by the lower semicontinuity of $\hat{w}$ and the continuity of $\psi$, there exists a cylinder $Q = Q(x_0, t_0) \Subset \Omega_T$ such that

$$\hat{w} + \varepsilon \geq \psi(x_0, t_0) + \frac{\varepsilon}{2} \geq \psi \quad \text{on} \quad \overline{Q}.$$

Notice also that $\hat{w} + \varepsilon$ is a supersolution. Let $w_P$ be the Poisson modification of $\hat{w}$ in $Q$. Since $w_P + \varepsilon$ is $p$-parabolic in $Q$ and $w_P + \varepsilon \geq \psi(x_0, t_0) + \frac{\varepsilon}{2}$ at $\partial_p Q$, it follows by comparison that

$$w_P + \varepsilon \geq \psi(x_0, t_0) + \frac{\varepsilon}{2} \geq \psi \quad \text{in} \quad Q,$$

and hence,

$$w_P + \varepsilon \geq \psi \quad \text{in} \quad \Omega_T.$$

Thus $w_P + \varepsilon$ an admissible test function in $\mathcal{S}_\psi$. This implies that

$$\hat{w} \le w_P + \varepsilon$$

in $\Omega_T$. Hence

$$\begin{aligned}\limsup_{(y,s)\to(x_0,t_0)} \hat{w}(y,s) &\le \lim_{(y,s)\to(x_0,t_0)} w_P(y,s) + \varepsilon \\ &= w_P(x_0,t_0) + \varepsilon \le \hat{w}(x_0,t_0) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\hat{w}$ is upper semicontinuous at $(x_0, t_0)$ and, as it is also lower semicontinuous, it is continuous at the point $(x_0, t_0)$.

To see that $w^*$ is continuous up to the boundary, we use a barrier argument as in [KL96]. Let $(x_0, t_0) \in \partial_p \Omega$. Since the boundary is regular, there exists a closed $n+1$-dimensional ball

$$\{(x,t) : |x - x'|^2 + (t - t')^2 \le R_0^2\}$$

in the complement that intersects the closure $\overline{\Omega}_T$ exactly at $(x_0, t_0)$. Then the function

$$f(x,t) = e^{-\alpha R_0^2} - e^{-\alpha R^2}, \quad R = \sqrt{|x - x'|^2 + (t - t')^2}$$

with a suitable constant $\alpha > 0$ is a supersolution. The function $f$ takes the value 0 at $(x_0, t_0)$ and is positive in $\overline{\Omega}_T \setminus \{(x_0, t_0)\}$. Then for any $\varepsilon$ there exists $\lambda > 0$ such that

$$\varepsilon + \psi(x_0, t_0) + \lambda f(x,t)$$

is a supersolution and is greater than or equal to $\psi(x,t)$ on $\overline{\Omega}_T$. By comparison

$$\psi(x,t) \le w^*(x,t) \le \varepsilon + \psi(x_0,t_0) + \lambda f(x,t).$$

Since $\varepsilon > 0$ is arbitrary, this implies that $w^*$ is continuous up to the boundary, and that $w^* = \psi$ on $\partial_p \Omega_T$. Observe that the calculation omitted above is delicate: in general, supersolutions cannot be multiplied by constants.

Finally, we show that $\hat{w}$ is $p$-parabolic in $\{\hat{w} > \psi\}$. Indeed, for each $(x_0, t_0) \in \{\hat{w} > \psi\}$, there exists $\lambda > 0$ and a cylinder $Q = Q(x_0, t_0) \Subset \{\hat{w} > \psi\}$ such that

$$\hat{w} > \lambda > \psi$$

in $Q$. But now for the Poisson modification $\hat{w}_P$ of $\hat{w}$ in $Q$, we have

$$\hat{w} \ge \hat{w}_P > \lambda > \psi.$$

This implies that $w_P = \hat{w}$ since $\hat{w}$ was the infimum, and thus $\hat{w}$ is $p$-parabolic in $Q$. $\square$

Next we define a *variational solution,* first for a continuous obstacle. Under assumptions on the time derivative of the obstacle, the existence of a variational solution is treated in [AL83] and [BDM]. See also [KS].

Let $\psi \in C(\overline{\Omega}_T)$ and define the class $\mathcal{F}_\psi$ consisting of all functions $v \in C(\overline{\Omega}_T)$ such that

$$v \in L^p(0,T;W^{1,p}(\Omega)), \quad v = \psi \text{ on } \partial_p\Omega_T \quad \text{and} \quad v \geq \psi \text{ in } \Omega_T.$$

**Definition 3.3.** A function $v \in \mathcal{F}_\psi$ is a *variational solution* to the obstacle problem if

$$\begin{aligned}\iint_{\Omega_T} \left( |\nabla v|^{p-2} \nabla v \cdot \nabla(\phi - v) + (\phi - v)\frac{\partial \phi}{\partial t}\right) \mathrm{d}x\, \mathrm{d}t \\ \geq \frac{1}{2}\int_\Omega |\phi(x,T) - v(x,T)|^2 \,\mathrm{d}x\end{aligned} \tag{3.4}$$

for all $\phi \in C^\infty(\Omega_T)$ in $\mathcal{F}_\psi$ such that $\frac{\partial \phi}{\partial t} \in L^q(\Omega_T)$, $q = p/(p-1)$.

By an approximation procedure, we can extend the admissible class of test functions to include all continuous $\phi \in L^p(0,T;W^{1,p}(\Omega))$ in $\mathcal{F}_\psi$ such that $\frac{\partial \phi}{\partial t} \in L^q(\Omega_T)$, $q = p/(p-1)$.

For a smooth variational solution $v$, integration by parts implies

$$\begin{aligned}\int_0^T \int_\Omega (\phi - v)\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_\Omega |\phi(x,T) - v(x,T)|^2 \,\mathrm{d}x \\ + \int_0^T\int_\Omega (\phi - v)\frac{\partial v}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and thus (3.4) can be written as

$$\iint_{\Omega_T} \left( |\nabla v|^{p-2} \nabla v \cdot \nabla(\phi - v) + (\phi - v)\frac{\partial v}{\partial t}\right) \mathrm{d}x\, \mathrm{d}t \geq 0. \tag{3.5}$$

Next we show that the least solution satisfies Definition 3.3, and thus, for a continuous obstacle, this gives us the existence of a variational solution.

Below, we use the standard mollification

$$u_\sigma(x,t) = \int_{\mathbf{R}} u(x, t-s)\zeta_\sigma(s)\,\mathrm{d}s \tag{3.6}$$

with Friedrichs' mollifier

$$\zeta_\sigma(s) = \begin{cases} \frac{C}{\sigma}e^{-\sigma^2/(\sigma^2 - s^2)}, & |s| < \sigma \\ 0, & |s| \geq \sigma, \end{cases}$$

where the constant $C$ is chosen so that $\int_{-\infty}^{\infty} \zeta_\sigma(s)\,\mathrm{d}s = 1$. Let $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$ and choose $\sigma < \operatorname{dist}(\operatorname{spt}(\varphi), \Omega \times \{0, T\})$. We insert $\varphi_\sigma$ into (2.2), change variables, and apply Fubini's theorem to obtain

$$\iint_{\Omega_T} \left( \left(|\nabla u|^{p-2}\nabla u\right)_\sigma \cdot \nabla \varphi + \frac{\partial u_\sigma}{\partial t}\varphi \right) \mathrm{d}x\,\mathrm{d}t \geq 0 \tag{3.7}$$

for the weak supersolution $u$. The analogous formula with equality holds for weak solutions.

**Theorem 3.8.** *Let $\psi \in C_0(\Omega_T)$. Then the least solution $w^*$ is also a variational solution. In other words, $w^*$ satisfies the variational inequality*

$$\iint_{\Omega_T} \left( |\nabla w^*|^{p-2} \nabla w^* \cdot \nabla(\phi - w^*) + (\phi - w^*)\frac{\partial \phi}{\partial t} \right) \mathrm{d}x\,\mathrm{d}t \geq \frac{1}{2}\int_\Omega |\phi(x,T) - w^*(x,T)|^2 \,\mathrm{d}x$$

*for all $\phi \in C^\infty(\Omega_T)$ in $\mathcal{F}_\psi$ such that $\frac{\partial \phi}{\partial t} \in L^q(\Omega_T)$, $q = p/(p-1)$.*

*Proof.* First, observe that $w^* = \psi$ on $\partial_p\Omega_T$ by Theorem 3.2, and $w^* \in L^p(0,T;W_0^{1,p}(\Omega))$, cf. Lemma 4.3. Denote by $\chi^h_{0,T}$ a continuous, piecewise linear approximation of a characteristic function such that

$$\begin{cases} \frac{\partial \chi^h_{0,T}}{\partial t} = 1/h, & \text{if } h < t < 2h, \\ \chi^h_{0,T} = 1, & \text{if } 2h < t < T - 2h, \\ \frac{\partial \chi^h_{0,T}}{\partial t} = -1/h, & \text{if } T - 2h < t < T - h, \\ \chi^h_{0,T} = 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

and let $\phi$ be the test function in the theorem. Then an approximation argument justifies the use of

$$\varphi = \chi^h_{0,T}(\phi_\sigma - w^*_\sigma)_+ = \chi^h_{0,T}\max(\phi_\sigma - w^*_\sigma\,, 0)$$

as a test function in (3.7), so that

$$\iint_{\Omega_T} \left( \left( |\nabla w^*|^{p-2} \nabla w^* \right)_\sigma \cdot \chi^h_{0,T} \nabla(\phi_\sigma - w^*_\sigma)_+ + \frac{\partial w^*_\sigma}{\partial t}\chi^h_{0,T}(\phi_\sigma - w^*_\sigma)_+ \right) \mathrm{d}x\,\mathrm{d}t \geq 0.$$

By adding the integral of $-\frac{\partial \phi_\sigma}{\partial t}\chi^h_{0,T}(\phi_\sigma - w^*_\sigma)_+$ to both sides and integrating by parts, we get

$$\iint_{\Omega_T} \left( \left( |\nabla w^*|^{p-2} \nabla w^* \right)_\sigma \cdot \chi^h_{0,T} \nabla(\phi_\sigma - w^*_\sigma)_+ + \frac{1}{2}((\phi_\sigma - w^*_\sigma)_+)^2 \frac{\partial \chi^h_{0,T}}{\partial t} \right) \mathrm{d}x\,\mathrm{d}t \geq -\iint_{\Omega_T} \frac{\partial \phi_\sigma}{\partial t}\chi^h_{0,T}(\phi_\sigma - w^*_\sigma)_+ \,\mathrm{d}x\,\mathrm{d}t.$$

Letting first $\sigma \to 0$ and then $h \to 0$, we get

$$\iint_{\Omega_T} \left( |\nabla w^*|^{p-2} \nabla w^* \cdot \nabla(\phi - w^*)_+ + \frac{\partial \phi}{\partial t}(\phi - w^*)_+ \right) \mathrm{d}x\,\mathrm{d}t \geq \frac{1}{2}\int_\Omega (\phi(x,T) - w^*(x,T))_+^2 \,\mathrm{d}x. \tag{3.10}$$

Next we perform a similar calculation, using the fact that $w^*$ is $p$-parabolic in the open set $U = \Omega_T \cap \{\phi < w^*\}$. This time we use the test function $\chi^h_{0,T}(\phi_\sigma - w^*_\sigma)_- = \chi^h_{0,T}\min(\phi_\sigma - w^*_\sigma, 0)$. Since $\phi$ is smooth, we can choose a decreasing sequence of smooth functions $\phi^i$ converging to $\phi$ so that

$$\{\phi^i - w^* < 0\} \Subset U.$$

For a fixed index $i$, we can choose $\sigma > 0$ so small that also

$$\{(\phi^i - w^*)_\sigma < 0\} \Subset U.$$

A similar calculation as the previous one implies, since $w^*$ is $p$-parabolic in $U$,

$$\begin{aligned}\int_U \Bigg( \Big( |\nabla w^*|^{p-2} \nabla w^* \Big)_\sigma &\cdot \chi^h_{0,T} \nabla(\phi^i_\sigma - w^*_\sigma)_- \\ &+ \frac{1}{2}((\phi^i_\sigma - w^*_\sigma)_-)^2 \frac{\partial \chi^h_{0,T}}{\partial t} \Bigg) \,\mathrm{d}x\,\mathrm{d}t \\ = -\int_U \frac{\partial \phi^i_\sigma}{\partial t} \chi^h_{0,T}(\phi^i_\sigma - w^*_\sigma)_- \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

As first $\sigma \to 0$, then $h \to 0$ and finally $i \to \infty$, we obtain

$$\begin{aligned}\int_U \Big( |\nabla w^*|^{p-2} \nabla w^* \cdot \nabla(\phi - w^*)_- + \frac{\partial \phi}{\partial t}(\phi - w^*)_- \Big) \,\mathrm{d}x\,\mathrm{d}t \\ = \frac{1}{2} \int_\Omega (\phi(x,T) - w^*(x,T))^2_- \,\mathrm{d}x.\end{aligned} \tag{3.11}$$

Together (3.10) and (3.11) prove the claim. □

We recall the convenient convolution

$$u_\varepsilon(x,t) = \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} u(x,s) \,\mathrm{d}s, \tag{3.12}$$

which is expedient for our purpose; see for example [Nau84], [BDGO97], and [KL06]. It has the following properties.

**Lemma 3.13.** *(i) If $u \in L^p(\Omega_T)$, then*

$$||u_\varepsilon||_{L^p(\Omega_T)} \le ||u||_{L^p(\Omega_T)},$$

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{u - u_\varepsilon}{\varepsilon} \in L^p(\Omega_T),$$

*and*

$$u_\varepsilon \to u \quad in \quad L^p(\Omega_T) \quad as \quad \varepsilon \to 0.$$

*(ii) If $\nabla u \in L^p(\Omega_T)$, then $\nabla u_\varepsilon = (\nabla u)_\varepsilon$ componentwise,*

$$||\nabla u_\varepsilon||_{L^p(\Omega_T)} \le ||\nabla u||_{L^p(\Omega_T)},$$

*and*

$$\nabla u_\varepsilon \to \nabla u \quad in \quad L^p(\Omega_T) \quad as \quad \varepsilon \to 0.$$

(iii) *Furthermore, if $u^k \to u$ in $L^p(\Omega_T)$, then also*

$$u^k_\varepsilon \to u_\varepsilon \quad and \quad \frac{\partial u^k_\varepsilon}{\partial t} \to \frac{\partial u_\varepsilon}{\partial t}$$

*in $L^p(\Omega_T)$.*

(iv) *If $\nabla u^k \to \nabla u$ in $L^p(\Omega_T)$, then $\nabla u^k_\varepsilon \to \nabla u_\varepsilon$ in $L^p(\Omega_T)$.*

(v) *Analogous results hold for the weak convergence in $L^p(\Omega_T)$.*

(vi) *Finally, if $\varphi \in C(\overline{\Omega}_T)$, then*

$$\varphi_\varepsilon(x,t) + e^{-\frac{t}{\varepsilon}}\varphi(x,0) \to \varphi(x,t)$$

uniformly *in $\Omega_T$ as $\varepsilon \to 0$.*

Next we show that a variational solution is unique for a continuous compactly supported obstacle.

**Theorem 3.14.** *Let $\psi \in C_0(\Omega_T)$. The variational solution in Definition 3.3 with this obstacle is unique.*

*Proof.* Suppose that $u$ and $v$ are two solutions. They are continuous. We sum up

$$\iint_{\Omega_T} \left( |\nabla u|^{p-2}\nabla u \cdot \nabla(\phi - u) + (\phi - u)\frac{\partial \phi}{\partial t} \right) \mathrm{d}x\,\mathrm{d}t \geq \frac{1}{2}\int_\Omega |\phi(x,T) - u(x,T)|^2 \,\mathrm{d}x$$

and

$$\iint_{\Omega_T} \left( |\nabla v|^{p-2}\nabla v \cdot \nabla(\phi - v) + (\phi - v)\frac{\partial \phi}{\partial t} \right) \mathrm{d}x\,\mathrm{d}t \geq \frac{1}{2}\int_\Omega |\phi(x,T) - v(x,T)|^2 \,\mathrm{d}x.$$

We end up with

$$\iint_{\Omega_T} \Big( |\nabla v|^{p-2}\nabla v \cdot \nabla(v - \phi) - |\nabla u|^{p-2}\nabla u \cdot \nabla(\phi - u) \Big) \mathrm{d}x\,\mathrm{d}t \leq 2\iint_{\Omega_T} \left( \phi - \frac{u+v}{2} \right) \frac{\partial \phi}{\partial t} \,\mathrm{d}x\,\mathrm{d}t. \tag{3.15}$$

If we could choose the test function $\phi$ equal to $(u+v)/2$, the desired result would follow easily from the structure of the left-hand member. However, this function is not admissible, since its time derivative is not guaranteed. We modify it by utilizing convolution (3.12), and use the test function

$$\phi = \left( \frac{u+v}{2} + \alpha\eta(x) \right)_\varepsilon ,$$

where $\eta \in C_0^\infty(\Omega)$ $\eta \geq 0$ and $\eta = 1$ on $\operatorname{spt}\psi$. Here $\alpha > 0$ is given and $0 < \varepsilon < \varepsilon(\alpha)$, where $\varepsilon(\alpha)$ is so small that

$$\phi \geq (\psi + \alpha\eta)_\varepsilon \geq \psi$$

in $\Omega_T$. Now

$$\frac{\partial \phi}{\partial t} = \frac{1}{\varepsilon}\left[\left(\frac{u+v}{2} + \alpha\eta\right) - \left(\frac{u+v}{2} + \alpha\eta\right)_\varepsilon\right]$$

and so we obtain

$$\begin{aligned}
&\iint_{\Omega_T} \left(\phi - \frac{u+v}{2}\right)\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_{\Omega_T} \left(\phi - \left(\frac{u+v}{2} + \alpha\eta\right)\right)\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \alpha \iint_{\Omega_T} \eta \frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\frac{1}{\varepsilon}\iint_{\Omega_T} \left[\left(\frac{u+v}{2} + \alpha\eta\right) - \left(\frac{u+v}{2} + \alpha\eta\right)_\varepsilon\right]^2 \mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \alpha \iint_{\Omega_T} \eta(x)\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq 0 + \alpha \int_\Omega \eta(x)\left(\frac{u+v}{2} + \alpha\eta\right)_\varepsilon (x,T)\,dx.
\end{aligned}$$

Now we can safely let $\varepsilon \to 0$ after which we also let $\alpha \to 0$. The result is that

$$\frac{1}{2}\iint_{\Omega_T} (|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u)\cdot(\nabla v - \nabla u)\,\mathrm{d}x\,\mathrm{d}t \leq 0.$$

The integrand is non-negative and zero only for $\nabla v = \nabla u$. Since $u$ and $v$ have the same boundary values, they coincide. □

**Corollary 3.16.** *For the obstacle $\psi \in C_0(\Omega_T)$, the variational solution coincides with the least solution. In particular, the variational solution is a weak supersolution.*

*Proof.* According to Theorem 3.7 the least solution $w^*$ is also a variational solution. But there is only one variational solution according to the theorem. □

The corollary can be modified to include the case $\psi \in C^\infty(\overline{\Omega}_T)$. For a different approach to a continuous obstacle problem, see [KKS09].

**Corollary 3.17.** *Let $v_1,\ v_2$ be the variational solutions with the obstacles $\psi_1, \psi_2 \in C_0(\Omega_T)$. If $\psi_1 \leq \psi_2$, then $v_1 \leq v_2$.*

*Proof.* By the previous corollary they are the least solutions: $v_1 = w_1^*$ and $v_2 = w_2^*$. By Theorem 2.8 these are weak supersolutions. Since $v_2 \geq \psi_2 \geq \psi_1$, we must have $w_1^* \leq v_2$, as $w_1^*$ is the least one. □

## 4. Irregular obstacle

In this section we treat the irregular obstacle with

$$\textsf{Assumption:}\quad \begin{aligned}&\psi \in L^p(0,T;W^{1,p}(\Omega)),\\ &\operatorname{spt}\psi \Subset \Omega_T, \quad 0 \le \psi \le L.\end{aligned}$$

The simplifying effect of the compactness assumption is not fully utilized: the benefit for us comes from the zero region near the lateral boundary $\partial\Omega \times [0,T]$.

The least solutions are well defined in this generality, but there is a difficulty. On the one hand, the variational definition fails to guarantee uniqueness, if only smooth test functions are admissible, see Section 5. On the other hand, complications with time derivatives prevent us from using all the test functions from the regularity class the obstacle belongs to. Nevertheless, an approximation with variational solutions with suitable smooth obstacles turns out to give exactly the unique least solution, Theorem 4.14.

However, first we discuss a convergence result in the elliptic theory, Proposition 4.2. The parabolic counterpart to the proposition is not a simple one.

For $\psi \in W^{1,p}(\Omega)$, we define the class

$$\mathcal{K}_\psi = \{\phi \in W^{1,p}(\Omega) \,:\, \phi \ge \psi \text{ a.e. in } \Omega,\ \phi - \psi \in W_0^{1,p}(\Omega)\}.$$

Then $v \in \mathcal{K}_\psi$ is a variational solution to the elliptic obstacle problem, if

$$\int_\Omega |\nabla v|^{p-2}\,\nabla v \cdot \nabla(\phi - v)\,\mathrm{d}x \ge 0 \tag{4.1}$$

for every $\phi \in \mathcal{K}_\psi$. The variational solution agrees with the least solution: $v = w^*$ a.e. in this case, see for example [HKM93, Theorem 9.26.]. Our approximative definition coincides with the least solution in the elliptic case. Notice that we do not demand $\phi$ to be continuous now. The approximants are pretty arbitrary in the next proposition.

**Proposition 4.2** (Elliptic case)**.** *Let $v_{\psi_j} \in \mathcal{K}_{\psi_j}$ denote the variational solution with the obstacle $\psi_j$. If $\psi_j \to \psi$ in $W^{1,p}(\Omega)$, then*

$$v_{\psi_j} \to v_\psi \quad \textit{in} \quad W^{1,p}(\Omega),$$

*where $v_\psi$ is the variational solution with $\psi$ as an obstacle.*

*Proof.* Use the test functions[4]

$$\phi_j = v_\psi + \psi_j - \psi \in \mathcal{K}_{\psi_j}, \quad \phi = v_{\psi_j} + \psi - \psi_j \in \mathcal{K}_\psi$$

to prove this. See also Theorem 1.4 in Li–Martio [LM94]. □

Let us leave the elliptic case and return to the parabolic situation.

[4]Such a test function is out of the question in the parabolic case, because of complications with the time derivative.

**Lemma 4.3.** *Let* $\psi \in L^p(0,T;W^{1,p}(\Omega))$, $\operatorname{spt}\psi \Subset \Omega_T$, $0 \le \psi \le L$, *and let* $w^*$ *be the least solution with the obstacle* $\psi$. *Then* $w^*$ *is p-parabolic in* $\Omega_T \setminus \operatorname{spt}\psi$ *and* $w^* \in L^p(0,T;W_0^{1,p}(\Omega))$.

*Proof.* The first part of the proof is similar to the end of the proof of Theorem 3.2

To prove the *global* integrability of $w^*$, we show that $w^*$ coincides with the solution to a boundary value problem near the lateral boundary. To this end, we choose a smooth open set $D \subset \mathbf{R}^n$ such that $\operatorname{spt}\psi \Subset D \times (t_1,t_2)$. We solve the Evolutionary p-Laplace Equation (2.3) in $(\Omega \setminus \overline{D}) \times (0,T)$ with the boundary values

$$\begin{cases} u &= w^* \quad \text{on} \quad \partial D \times (0,T) \\ u &= 0 \quad \text{on} \quad (\Omega \setminus \overline{D}) \times \{0\} \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T). \end{cases}$$

The continuity of $u$ and $w^*$ in $(\overline{\Omega} \setminus D) \times (0,T)$ and the "elliptic" comparison principle, Proposition 3 in [LM07] or Lemma 4.5 in [KKP10], imply that the set $\{u > w^* + \varepsilon\}$ is empty for any $\varepsilon > 0$. Thus $u \le w^* + \varepsilon$, and since $\varepsilon > 0$ was arbitrary, it follows that

$$u = w^* \quad \text{in} \quad (\Omega \setminus \overline{D}) \times (0,T).$$

This implies the claim. □

Below we will use the averaged inequality with the convolution (3.12), cf. [KL06]. The averaged equation for a weak supersolution $u$ in $\Omega_T$ is the following

$$\begin{aligned} \iint_{\Omega_T} \left( \left( |\nabla u|^{p-2} \nabla u \right)_\varepsilon \cdot \nabla \varphi - u_\varepsilon \frac{\partial \varphi}{\partial t} \right) \mathrm{d}x\,\mathrm{d}t \\ + \int_\Omega u_\varepsilon(x,T)\varphi(x,T)\,\mathrm{d}x \\ \ge \int_\Omega u(x,0) \left( \frac{1}{\varepsilon} \int_0^T \varphi(x,s) e^{-s/\varepsilon}\,\mathrm{d}s \right) \mathrm{d}x \end{aligned} \tag{4.4}$$

valid for all test functions $\varphi \ge 0$ vanishing on the parabolic boundary $\partial_p \Omega_T$. To see this, we observe that the definition of a supersolution gives us

$$\begin{aligned} \int_s^T \int_\Omega \Big( & |\nabla u(x,t-s)|^{p-2} \nabla u(x,t-s) \cdot \nabla \varphi(x,t) \\ & - u(x,t-s) \frac{\partial \varphi}{\partial t}(x,t) \Big)\,\mathrm{d}x\,\mathrm{d}t \; + \; \int_\Omega u(x,T-s)\varphi(x,T)\,\mathrm{d}x \\ \ge \int_\Omega & u(x,0)\varphi(x,s)\,\mathrm{d}x, \end{aligned}$$

when $0 \le s \le T$. Notice that $(x,t-s) \in \overline{\Omega}_T$. To obtain (4.4) we multiply the above inequality by $e^{-s/\varepsilon}/\varepsilon$, integrate over $[0,T]$ with respect to $s$, and finally change the order of integration to obtain. Upon integration by parts we see that for a supersolution $u \in L^p(0,T;W^{1,p}(\Omega))$ inequality (4.4) implies

$$\begin{aligned}\iint_{\Omega_T} \left( \left( |\nabla u|^{p-2} \nabla u \right)_\varepsilon \cdot \nabla \varphi + \frac{\partial u_\varepsilon}{\partial t} \varphi \right) \mathrm{d}x \, \mathrm{d}t \\ \geq \int_\Omega u(x,0) \left( \frac{1}{\varepsilon} \int_0^T \varphi(x,s) e^{-s/\varepsilon} \, \mathrm{d}s \right) \mathrm{d}x\end{aligned} \tag{4.5}$$

for every $\varphi \in C(\overline{\Omega}_T) \cap C^\infty(\Omega_T)$, $\varphi \geq 0$, vanishing on the parabolic boundary $\partial_p \Omega_T$.

We will use only the simpler version

$$\iint_{\Omega_T} \left( \left( |\nabla u|^{p-2} \nabla u \right)_\varepsilon \cdot \nabla \varphi + \frac{\partial u_\varepsilon}{\partial t} \varphi \right) \mathrm{d}x \, \mathrm{d}t \geq 0 \tag{4.6}$$

valid for $u \geq 0$ and $\varphi$ vanishing on $\partial_p \Omega_T$.

By approximating an irregular obstacle $\psi$ by the mollified obstacles $\psi_\varepsilon$ and solving the corresponding variational problems, we arrive at the least solution as a limit. This is the content of Theorem 4.14. However, arbitrary smooth approximations to the obstacle will not work; we use convolutions. The key observation in the proof of Theorem 4.14 is that we can, without affecting the limit of the approximation, replace the obstacle by the least supersolution above the obstacle. We start with an auxiliary result.

**Lemma 4.7.** *Suppose that $\psi^u, \psi^v \in L^p(0,T;W_0^{1,p}(\Omega))$ and define $\psi_\varepsilon^u, \psi_\varepsilon^v$ as in formula* (3.12). *Let $u$ and $v$ be the variational solutions with $\psi_\varepsilon^u$ and $\psi_\varepsilon^v$. If $\psi_\varepsilon^u \geq \psi_\varepsilon^v$, then $u \geq v$ almost everywhere.*

*Proof.* First we extend $\psi^u$ and $\psi^v$ by zero outside $\Omega$. Then we mollify the obstacles $\psi_\varepsilon^u$ and $\psi_\varepsilon^v$ in space using the standard Friedrichs' mollifier with parameter $\sigma$.

We solve the variational obstacle problem in $\Omega \times (0,T)$ with $\psi_{\varepsilon,\sigma}^u, \psi_{\varepsilon,\sigma}^v \in C^\infty(\overline{\Omega}_T)$. Since the obstacles are smooth and ordered, we conclude from Corollary 3.16 that $u^\sigma, v^\sigma$ are weak supersolutions and

$$v^\sigma \leq u^\sigma \tag{4.8}$$

almost everywhere. The corollary is formulated for $C_0$-obstacles, but it can be modified to the present setting as well. Alternatively, according to [AL83], [BDM], variational solutions $u^\sigma$ $v^\sigma$ exist, attain the boundary values in $L^p(0,T;W_0^{1,p}(\Omega))$ prescribed by the obstacles, and have time derivatives in the dual space. Thus $u^\sigma, v^\sigma$ turn out to be supersolutions, and we can use $u^\sigma + (v^\sigma - u^\sigma)_+$ as a test function for $u^\sigma$ and $v^\sigma - (v^\sigma - u^\sigma)_+$ for $v^\sigma$ to deduce the same result.

Next we establish the needed convergence results. Observe that

$$\iint_{\Omega_T} \Big( |\nabla u^\sigma|^{p-2} \nabla u^\sigma \cdot \nabla(\psi^u_{\varepsilon,\sigma} - u^\sigma) \\ + (\psi^u_{\varepsilon,\sigma} - u^\sigma)\frac{\partial \psi^u_{\varepsilon,\sigma}}{\partial t}\Big) \,\mathrm{d}x\,\mathrm{d}t \geq 0 \tag{4.9}$$

gives us the global estimate

$$\iint_{\Omega_T} |\nabla u^\sigma|^p \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega_T} \left|\nabla \psi^u_{\varepsilon,\sigma}\right|^p \,\mathrm{d}x\,\mathrm{d}t + C\iint_{\Omega_T} \left|\frac{\partial \psi^u_{\varepsilon,\sigma}}{\partial t}\right| \mathrm{d}x\,\mathrm{d}t.$$

This uniform bound with respect to $\sigma$ implies that a subsequence of $u^\sigma$ converges weakly in $L^p(0,T;W^{1,p}(\Omega))$ to some limit $\tilde{u}$. Furthermore, Theorem 2.6 gives us a pointwise convergence of $u^\sigma$ and $\nabla u^\sigma$ to $\tilde{u}$ and $\nabla\tilde{u}$. This is enough to pass to a limit under the integral sign in (4.9). It follows that $\tilde{u}$ is a weak supersolution.

Since $\psi^u_{\varepsilon,\sigma} - u^\sigma \in L^p(0,T;W^{1,p}_0(\Omega))$ we deduce that

$$\psi^u_\varepsilon - \tilde{u} \in L^p(0,T;W^{1,p}_0(\Omega)).$$

This is enough for using the uniqueness from Theorem 6.1 in [BDM] to conclude that $\tilde{u}$ is the unique variational solution with the obstacle $\psi^\varepsilon_u$. In other words $\tilde{u} = u$. We complete the proof by combining this result and (4.8). □

The previous proof contains the following result.

**Corollary 4.10.** *Let $\psi \in L^p(0,T;W^{1,p}(\Omega))$ and define $\psi_\varepsilon$ as in formula* (3.12)*. Then the variational solution $u$ with the obstacle $\psi_\varepsilon$ is a supersolution.*

The next theorem shows that, if the obstacle itself is a supersolution, then the approximation gives the same supersolution at the limit.

**Theorem 4.11.** *Let $w \in L^p(0,T;W^{1,p}(\Omega))$, $0 \leq w \leq L$, be a weak supersolution and define $w_\varepsilon$ as in formula* (3.12)*. Let $v^\varepsilon$ be the variational solutions with the mollified obstacles $w_\varepsilon$. Then, passing to a subsequence if necessary,*

$$\nabla v^\varepsilon \to \nabla w \quad \textit{in} \quad L^p(\Omega_T),$$

$$v^\varepsilon \to w, \quad \nabla v^\varepsilon \to \nabla w \quad \textit{a.e. in} \quad \Omega_T.$$

*Proof.* By Corollary 4.10, $v^\varepsilon$ is a weak supersolution and further $0 \leq v^\varepsilon \leq L$. According to Theorem 2.6, there exists a subsequence, still denoted by $v^\varepsilon$, and a limit $v$ such that

$$v^\varepsilon \to v, \quad \nabla v^\varepsilon \to \nabla v \quad \text{a.e. in} \quad \Omega_T.$$

Thus we have to show that $v = w$ almost everywhere. To this end, observe that the obstacle $w_\varepsilon$ is an admissible test function for $v^\varepsilon$ and write

$$\iint_{\Omega_T}\left(|\nabla v^\varepsilon|^{p-2}\,\nabla v^\varepsilon\cdot\nabla(w_\varepsilon-v^\varepsilon)+(w_\varepsilon-v^\varepsilon)\frac{\partial w_\varepsilon}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t\geq 0.$$

On the other hand, since $w\geq 0$ is a weak supersolution and $v^\varepsilon\geq w_\varepsilon$, we have by (4.6) that

$$\iint_{\Omega_T}\Big(\big(|\nabla w|^{p-2}\,\nabla w\big)_\varepsilon\cdot\nabla(v^\varepsilon-w_\varepsilon)+(v^\varepsilon-w_\varepsilon)\frac{\partial w_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t\geq 0..$$

Since $v^\varepsilon$ takes the boundary values on the parabolic boundary $\partial_p\Omega_T$ in a suitable sense an approximation argument justifies our use of $v^\varepsilon-w_\varepsilon$ as a test function in (4.6).

We sum up the inequalities to obtain

$$\iint_{\Omega_T}\big(|\nabla v^\varepsilon|^{p-2}\,\nabla v^\varepsilon-\big(|\nabla w|^{p-2}\,\nabla w\big)_\varepsilon\big)\cdot\nabla(w_\varepsilon-v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\geq 0. \tag{4.12}$$

Next we aim at passing to the limit under the integral sign in order to deduce that $v^\varepsilon\to w$. We write

$$\begin{aligned}\iint_{\Omega_T}&\big(|\nabla v^\varepsilon|^{p-2}\,\nabla v^\varepsilon-|\nabla w_\varepsilon|^{p-2}\,\nabla w_\varepsilon\big)\cdot\nabla(v^\varepsilon-w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq\iint_{\Omega_T}\big(\big(|\nabla w|^{p-2}\,\nabla w\big)_\varepsilon-|\nabla w_\varepsilon|^{p-2}\,\nabla w_\varepsilon\big)\cdot\nabla(v^\varepsilon-w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq\frac{\alpha^p}{p}\iint_{\Omega_T}|\nabla(v^\varepsilon-w_\varepsilon)|^p\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{q\alpha^q}\iint_{\Omega_T}|\big(|\nabla w|^{p-2}\nabla w\big)_\varepsilon-|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon|^q\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where Young's inequality was used for $\alpha>0$ and $q=p/(p-1)$. The integrand in the left-hand side is greater than

$$2^{2-p}|\nabla(v^\varepsilon-w_\varepsilon)|^p$$

and we fix $\alpha$ so small that the integral of this minorant can absorb the first integral on the right-hand side. In other words

$$\begin{aligned}\iint_{\Omega_T}&|\nabla(v^\varepsilon-w_\varepsilon)|^p\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C(p)\iint_{\Omega_T}|\big(|\nabla w|^{p-2}\nabla w\big)_\varepsilon-|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon|^q\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

As $\varepsilon\to 0$ the majorant vanishes and we arrive at

$$\iint_{\Omega_T}|\nabla(v-w)|^p\,\mathrm{d}x\,\mathrm{d}t\leq\lim_{\varepsilon\to 0}\iint_{\Omega_T}|\nabla(v^\varepsilon-w_\varepsilon)|^p\,\mathrm{d}x\,\mathrm{d}t=0, \tag{4.13}$$

where Fatou's lemma was used.

It follows that $\nabla v = \nabla w$ a.e. in $\Omega_T$. We assure that $w - v \in L^p(0,T;W_0^{1,p}(\Omega))$ similarly as at the end of the proof of Lemma 4.7, and the proof is complete. □

From the previous theorem we can deduce that the variational solutions with the mollified obstacles converge to the least solution.

**Theorem 4.14.** *Let $\psi \in L^p(0,T;W^{1,p}(\Omega))$, $\operatorname{spt}\psi \Subset \Omega_T$, $0 \le \psi \le L$, and let $u^\varepsilon$ be the variational solutions with the mollified obstacles $\psi_\varepsilon$. Let $w^*$ denote the least solution with the obstacle $\psi$. Then*

$$u^\varepsilon \to w^*, \quad \nabla u^\varepsilon \to \nabla w^* \quad \textit{a.e. in} \quad \Omega_T.$$

*Proof.* By Corollary 4.10, $u^\varepsilon$ is a weak supersolution and $0 \le u^\varepsilon \le L$. Theorem 2.6 yields a subsequence, still denoted by $u^\varepsilon$, and a limit $u$ such that

$$u^\varepsilon \to u, \quad \nabla u^\varepsilon \to \nabla u \quad \text{a.e. in} \quad \Omega_T$$

as $\varepsilon \to 0$. The function $u$ is a weak supersolution, and we may even assume it to be ess lim inf-regularized. Since $\psi_\varepsilon \to \psi$, $u \ge \psi$ almost everywhere, and so we conclude that

$$w^* \le u,$$

because $w^*$ is the least solution.

Let $v^\varepsilon$ be the variational solutions with the mollified obstacles $w^*_\varepsilon$. Since $w^* \ge \psi$, also $w^*_\varepsilon \ge \psi_\varepsilon$. Due to the assumption $\operatorname{spt}\psi \subset \Omega_T$, we see by Lemma 4.3 that $w^* \in L^p(0,T;W_0^{1,p}(\Omega))$. By the previous lemma

$$v^\varepsilon \to w^*, \quad \nabla v^\varepsilon \to \nabla w^* \quad \text{a.e. in} \quad \Omega_T$$

as $\varepsilon \to 0$, at least for a subsequence. But now $w^*_\varepsilon \ge \psi_\varepsilon$ implies that $v^\varepsilon \ge u^\varepsilon$ almost everywhere according to Lemma 4.7. Thus by passing to a limit, we have

$$w^* \ge u$$

almost everywhere. Thus $u = w^*$ almost everywhere. □

We could also have taken a slightly different approach, and used the mollification (3.12) in time and then a mollification analogous to (3.6) in space. The space mollifications are well defined also near the lateral boundary as we extend the functions by zero outside $\Omega$. A good point in this approach is that, since the mollified obstacles are in $C^\infty$, Lemma 4.7 is immediate. Observe also that, in this approach, we do not assume that the obstacle is in the Sobolev space. Thus for example a characteristic function is an admissible obstacle.

**Theorem 4.15.** *Let $\psi$ be a measurable function such that $\operatorname{spt}\psi \Subset \Omega_T$, $0 \le \psi \le L$, and let $u^{\varepsilon,\sigma}$ be the solutions to the variational obstacle* *problems with the time and space mollified obstacles $(\psi_\varepsilon)_\sigma$. Let $w^*$ denote the least solution with the obstacle $\psi$. Then*

$$u^{\varepsilon,\sigma} \to w^*, \quad \nabla u^{\varepsilon,\sigma} \to \nabla w^* \quad a.e.\ in \quad \Omega_T.$$

## 5. Special cases

First, we consider the possibility to extend Definition 3.3 directly to the irregular case. Needless to say, the variational inequality (1.1) makes sense without the assumption that the obstacle is continuous. However, the time derivative of the test function is present, and thus we might be led to use smooth or, at least, continuous test functions. We encounter a difficulty. It turns out that such a restriction on the admissible test functions destroys the uniqueness property if the obstacle is too irregular: there are too few test functions to detect the "true solution".

To illustrate this, we consider the elliptic obstacle problem. Let $\psi \in W^{1,p}(\Omega)$ and recall

$$\mathcal{K}_\psi = \{\phi \in W^{1,p}(\Omega) \,:\, \phi \geq \psi \text{ a.e. in } \Omega,\ \phi - \psi \in W_0^{1,p}(\Omega)\}.$$

Then $w \in \mathcal{K}_\psi$ is a solution to the elliptic obstacle problem if

$$\int_\Omega |\nabla w|^{p-2}\, \nabla w \cdot \nabla(\phi - w)\, \mathrm{d}x \geq 0 \tag{5.1}$$

for every $\phi \in \mathcal{K}_\psi$.

Let us begin our discussion with the simplest relevant special case, the Dirichlet integral. Thus $p = 2$, the equation is linear and stationary. Even here the so-called Lavrentiev Phenomenon, described in [KL95], enters and will destroy the uniqueness, if continuity is imposed on the admissible functions. Fix a function $\psi \in W^{1,2}(\Omega)$ and consider the class

$$\mathcal{K}_\psi = \{\phi \in W^{1,2}(\Omega) \,:\, \phi \geq \psi \text{ a.e. in } \Omega,\ \phi - \psi \in W_0^{1,2}(\Omega)\}$$

of admissible functions. If $\psi$ itself is a superharmonic function, say $\psi = u$, it solves the obstacle problem: for all $\phi \in \mathcal{K}_u$

$$\int_\Omega |\nabla u|^2\, dx \leq \int_\Omega |\nabla \phi|^2\, dx,$$

or equivalently

$$\int_\Omega \nabla u \cdot (\nabla \phi - \nabla u)\, dx \geq 0.$$

According to [KL95] there exists a superharmonic function $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla u|^2\, dx < \inf_\phi \int_\Omega |\nabla \phi|^2\, dx,$$

where we restrict ourselves to *continuous* functions $\phi$ in $\mathcal{K}_u$. Notice that the inequality is strict. Thus the true minimum cannot be reached via continuous admissible functions. This is an instance of the Lavrentiev Phenomenon. From now on $u$ denotes this function.

There exists another superharmonic function $w$ ($w \geq u$ everywhere and $w \neq u$ in a subset of positive measure) such that

$$\int_\Omega |\nabla w|^2\,dx = \inf_\phi \int_\Omega |\nabla \phi|^2\,dx,$$

where the infimum is taken over all $\phi \in C(\Omega) \cap \mathcal{K}_u$. Also a.e.

$$w = \widehat{\inf v}, \tag{5.2}$$

where the infimum is taken over all *continuous* superharmonic functions $v$ such that $v \geq u$ a.e. in $\Omega$.

Now

$$\int_\Omega \nabla u \cdot \nabla(\phi - u)\,dx \geq 0$$

for all $\phi \in \mathcal{K}_u$ and *a fortiori* for all $\phi \in C(\Omega) \cap \mathcal{K}_u$. We also have

$$\int_\Omega \nabla w \cdot \nabla(\phi - w)\,dx \geq 0$$

for all $\phi \in \mathcal{K}_w$. We claim that this also holds for all $\phi \in C(\Omega) \cap \mathcal{K}_u$, where the class of test functions is now defined using $u$. To see this, notice that

$$\begin{aligned}
&\int_\Omega \nabla w \cdot \nabla(\phi - w)\,dx \\
&\quad = \int_\Omega \nabla w \cdot \nabla(\max(\phi, w) - w)\,dx + \int_\Omega \nabla w \cdot \nabla(\min(\phi, w) - w)\,dx \\
&\quad \geq 0 + \int_{\{\phi < w\}} \nabla w \cdot \nabla(\phi - w)\,dx.
\end{aligned}$$

The set $\{\phi < w\}$ is open, and in any case $\phi \geq u$. Therefore one can conclude that $w$, in fact, is a harmonic function in this open set. To see this, fix a point in this set. In a sufficiently small ball centered at this point, we can replace $w$ by the harmonic function with the boundary values $w$ on the sphere (this is given by Poisson's integral) without touching $\phi$; the local Poisson modification lies above $u$. If we now perform the same construction on each of the continuous superharmonic functions, the infimum of which appears in (5.2), we notice that locally $w$ is the limit of harmonic functions. Thus the last integral is zero. This proves the claim.

The consequence of this construction is that the variational inequality

$$\int_\Omega \nabla v \cdot \nabla(\phi - v)\,dx \geq 0$$

has (at least) two solutions in the class $\mathcal{K}_u$, if merely *continuous* functions $\phi$ in $\mathcal{K}_u$ are admissible. The solutions exhibited are $u$ and $w$. However, if $\phi$ runs through the whole class $\mathcal{K}_u$, then $u$ is the unique solution.

The same phenomenon occurs for the problem

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(\phi - v)\, dx \geq 0.$$

Using an obstacle of the form $u(x,t) = u(x)$ we get a counterexample to uniqueness for the parabolic case, if the admissible functions are required to be continuous.

In the light of the previous calculation, testing with smooth functions is insufficient to obtain uniqueness even in the elliptic case. On the other hand, (3.4) does not make sense if the test functions have poor regularity in the time direction. This is the difficulty.

Next we consider two special cases: upper semicontinuous obstacles, including characteristic functions of compact sets, and lower semicontinuous obstacles.

First, we observe that with the characteristic function $\chi_K$ of a compact set $K$ as an obstacle, $w^*$ is $p$-parabolic and, in particular, continuous in $\Omega_T \setminus \overline{K}$ by Lemma 4.3.

**Lemma 5.3.** *Let $K \subset \Omega_T$ be a compact set, and let $w^*$ be the least solution with the obstacle $\chi_K$. Then $w^*$ is p-parabolic in $\Omega_T \setminus K$. Moreover, $w^* \in L^p(0,T;W^{1,p}(\Omega))$.*

Let us now consider a *lower* semicontinuous obstacle and approximate it pointwise from *below* by smooth functions. Solving the corresponding obstacle problems we obtain the least solution as a limit, cf. Corollary 3.16. Needless to say, this is no surprise.

**Proposition 5.4.** *Suppose that the obstacle $\psi$, $0 \leq \psi \leq L$, is lower semicontinuous in $\overline{\Omega}_T$ and let $\psi_i$ be an increasing sequence of smooth functions so that*

$$\psi_i \to \psi$$

*pointwise. Let $u_i$ be the variational solutions with the obstacles $\psi_i$, and let $w^*$ be the least solution with the obstacle $\psi$. Then*

$$u_i \to w^*, \quad \nabla u_i \to \nabla w^* \text{ a.e. in } \quad \Omega_T.$$

*Proof.* This is a simple consequence of a comparison principle because it implies $u_i \leq w^*$, and on the other hand, clearly for the limit $u$ it holds that $\psi \leq u$. Since by our convergence results $u$ is a supersolution, $w^* \leq u$.

To be more precise, since $\psi_i$ is smooth, it follows that $u_i = \psi_i$ at the boundary of the open set $\{u_i > \psi_i\}$ and $u_i$ is $p$-parabolic in the set $\{u_i > \psi_i\}$. Furthermore, $w^* \geq \hat{\psi}_i = \psi_i$ and, due to the comparison principle, $u_i \leq w^*$ in the set $\{u_i > \psi_i\}$.

The convergence of $u_i$ to some limit $u$ follows from Theorem 2.6. Since the reasoning above was independent of $i$, it follows that $u \leq w^*$ in the whole domain. On the other hand, $u_i$ is an increasing and bounded sequence and, clearly, $u \geq \psi$. Therefore, the limit $u$ is a supersolution above $\psi$. It follows that $w^* = u$ almost everywhere. $\square$

Counterexample: The situation is not symmetric. A similar statement is clearly false for an approximation of an upper semicontinuous obstacle $\psi$ by smooth functions from above, when one uses the variational solutions for the corresponding obstacle problems. To see this, take

$$\psi(x,t) = \begin{cases} 1, & (x,t) \in \Omega \times \{\frac{T}{2}\} \\ 0, & \text{otherwise,} \end{cases}$$

as an obstacle. (Further, one can define $\psi$ as zero near the lateral boundary, so that it has compact support. This has no bearing.) This $\psi = 0$ a.e., so clearly the least solution is identically zero, but an approximation of $\psi$ from above produces a supersolution $u$ that is not identically zero. Indeed, one has the minorant

$$v(x,t) = \begin{cases} 0, & t \leq \frac{T}{2} \\ h(x,t), & t > \frac{T}{2}, \end{cases}$$

where $h$ is the $p$-parabolic function in $\Omega \times (\frac{T}{2}, T)$ with initial values 1 at $t = T/2$ and lateral boundary values 0.

Notice also that both $u$ and $\psi$ satisfy Definition 3.3 when testing with continuous test functions *everywhere* above the obstacle, so clearly uniqueness fails with these test functions. It is $u$ that is the variational solution resulting from the approximation procedure, because it is plain that $\psi_\varepsilon = 0$. Thus it is also the least solution. For the non-uniqueness it was essential to use continuous test functions satisfying $\phi \geq \psi$ at *each* point, although $\psi$ is discontinuous.

The example also shows that the convolutions $\psi_\varepsilon$ cannot be replaced (in Theorem 4.14) by arbitrary smooth obstacles, say $\psi_j$ converging to $\psi$ in the Sobolev space $L^p(0,T;W^{1,p}(\Omega))$.

As we already have pointed out, the theory of thin obstacles is outside the scope of our work, see [Pet06]. However, we include the following considerations. If we strengthen *almost everywhere* in the definition of a least solution to the requirement that the inequalities hold at *each* point, then we can avoid the phenomenon in the counterexample. However, we must restrict ourselves to a *semicontinuous obstacle* in this situation.

Thus we temporarily use the smaller class

$$\mathcal{S}^{\#}_{\psi} = \{u : u \text{ is ess lim inf-regularized weak supersolution,} \\ u \geq \psi \text{ at each point}\}. \tag{5.5}$$

to define the function $w^*_{\#}$. Instead, we then obtain the following result.

**Proposition 5.6.** *Suppose that the obstacle $\psi$, $0 \le \psi \le L$, is upper semicontinuous in $\overline{\Omega}_T$ and define the least solution $w^*_\#$, using* (5.5). *Further, let $\psi_i$ be a decreasing sequence of smooth obstacles so that*

$$\psi_i \to \psi$$

*pointwise. Then for the variational solutions $u_i$ with the obstacles $\psi_i$, it holds that*

$$u_i \to w^*_\#, \quad \nabla u_i \to \nabla w^*_\# \quad \text{a.e. in} \quad \Omega_T.$$

*Proof.* The idea in the proof is to extract, by the definition of the least solution, a decreasing sequence of lower semicontinuous supersolutions converging to $w^*_\#$. By lower semicontinuity of these supersolutions and upper semicontinuity of the obstacle, there exists a continuous obstacle in between. This yields a sequence of continuous solutions, and upon a second approximation procedure by smooth obstacles, we can pass to a sequence of smooth solutions.

Next we work out the details. The proof of Theorem 2.8 yields a sequence $v_i$, $v_i \ge \psi$, of ess lim inf-regularized supersolutions converging almost everywhere to $w^*_\#$. Since $\psi$ is upper semicontinuous and $v_i$ lower semicontinuous, there exists a continuous $\tilde{\psi}_i$ in $\overline{\Omega}_T$ such that

$$\psi \le \tilde{\psi}_i \le v_i$$

as shown in [Hah17]. Denote the continuous least solutions with the obstacles $\tilde{\psi}_i$ by $\tilde{u}_i$. It follows that

$$\tilde{u}_i \to w^*_\#$$

almost everywhere because it immediately follows that $w^*_\# \le \tilde{u}_i \le v_i$. Further, Theorem 2.6 implies the convergence of the gradients.

Remember that $\tilde{u}_i$ is continuous, and choose for every index $i$ a decreasing sequence $\psi^i_j$ of smooth obstacles such that

$$\psi^i_j \to \tilde{u}_i$$

uniformly as $j \to \infty$. Fix $\varepsilon > 0$ and choose a $\psi^i_j$ such that $\tilde{u}_i + \varepsilon \ge \psi^i_j$. Thus $j = j(i, \varepsilon)$. Denote by $u^i_j$ the variational solution with the obstacle $\psi^i_j$. Since $\tilde{u}_i + \varepsilon \ge \psi^i_j$ and $\tilde{u}_i + \varepsilon$ is a continuous supersolution, it follows by comparison that

$$\tilde{u}_i + \varepsilon \ge u^i_j \ge \psi^i_j \ge \tilde{u}_i.$$

By a diagonalization argument, we can extract a subsequence of smooth obstacles so that the related solutions converge to some $u$ such that $w^*_\# + \varepsilon \ge u \ge w^*_\#$ almost everywhere. By letting $\varepsilon \to 0$ via a subsequence $\varepsilon_k$ and diagonalizing once more, we can extract a new subsequence $\psi'_k$ with corresponding solutions $u'_k$, converging to $w^*_\#$ in the sense of the claim.

To finish the proof, it is enough to notice that for any $\delta > 0$ and $\psi_k'$, it holds for all $j$ large enough that $\psi_j \leq \psi_k' + \delta$, where $\psi_j$ refers to the sequence in the statement of the proposition. □

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Peter Lindqvist,
Department of Mathematics
Norwegian University of Science and Technology
N-7491,
Trondheim, Norway
lqvist@math.ntnu.no

Mikko Parviainen
Institute of Mathematics
Aalto University
P.O. Box 111100, FIN-00076 Aalto,
Espoo, Finland
mikko.parviainen@tkk.fi