Arcs on Determinantal Varieties

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Abstract

We study arc spaces and jet schemes of generic determinantal varieties. Using the natural group action, we decompose the arc spaces into orbits, and analyze their structure. This allows us to compute the number of irreducible components of jet schemes, log canonical thresholds, and topological zeta functions.

Introduction

Let $M = \mathbf{A}^{rs}$ denote the space of $r \times s$ matrices, and assume that $r \leq s$. Let $D^k \subset M$ be the *generic determinantal variety* of rank k, that is, the subvariety of M whose points correspond to matrices of rank at most k. The purpose of this paper is to analyze the structure of arc spaces and jet schemes of generic determinantal varieties.

Arc spaces and jet schemes have attracted considerable attention in recent years. They were introduced to the field by J. F. Nash [Nas95], who noticed for the fist time their connection with resolution of singularities. A few years later, M. Kontsevich introduced motivic integration [Kon95, DL99], popularizing the use of the arc space. And starting with the work of M. Mustaţă, arc and jets have become a standard tool in birational geometry, mainly because of their role in formulas for controlling discrepancies [Mus01, Mus02, EMY03, ELM04, EM06, dFEI08].

But despite their significance from a theoretical point of view, arc spaces are often hard to compute in concrete examples. The interest in Nash's conjecture led to the study of arcs in isolated surface singularities [LJ90, Nob91, LJR98, Plé05, PPP06, LJR08]. Quotient singularities are analyzed from the point of view of motivic integration in [DL02]. We also understand the situation for monomial ideals [GS06, Yue07b] and for toric varieties [Ish04]. But beyond these cases very little is known about the geometry of the arc space of a singular algebraic variety. The purpose of this article is to analyze in detail the geometric structure of arc spaces and jet schemes of generic determinantal varieties, giving a new family of examples for which the arc space is well understood.

Recall that arcs and jets are higher order analogues of tangent vectors. Given a variety X defined over $\mathbf{C}$, an arc of X is a $\mathbf{C}[[t]]$-valued point of X, and an n-jet is a $\mathbf{C}[t]/(t^{n+1})$ -valued point. A 1-jet is the same as a tangent vector. Just as in the case of the tangent space, arcs on X can be identified with the closed points of a scheme X_∞, which we call the *arc space* of X [Nas95, Voj07]. Similarly, n-jets give rise to the *n-th jet scheme* of X, which we denote by X_n (see Section 1 for more details).

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For the space of matrices M, the arc space M_∞ and the jet scheme M_n can be understood set-theoretically as the spaces of matrices with entries in the rings $\mathbf{C}[[t]]$ and $\mathbf{C}[t]/(t^{n+1})$, respectively. D^k_∞ and D^k_n are contained in M_∞ and M_n, and their equations are obtained by "differentiating" the $k \times k$ minors of a matrix of independent variables. We approach the study of D^k_∞ and D^k_n with three goals in mind: understand the *topology* of D^k_n, compute *log canonical thresholds* for the pairs (M, D^k), and compute *topological zeta functions* for (M, D^k).

0.1 Irreducible components of jet schemes

The topology of the jet schemes D^k_n is intimately related to the *generalized Nash problem.*

Given an irreducible family of arcs $\mathscr{C} \subset M_\infty$, we can consider $\nu_{\mathscr{C}}$, the order of vanishing along a general element of $\mathscr{C}$. The function $\nu_{\mathscr{C}}$ is almost a discrete valuation, the only problem being that it takes infinite value on those functions vanishing along all the arcs in $\mathscr{C}$. If there are no such functions, we call the family *fat* [Ish08], and we see that irreducible fat families of arcs give rise to discrete valuations.

Conversely, given a divisorial valuation ν over M, any isomorphism from $\mathbf{C}[[t]]$ to the completion of the valuation ring produces a non-closed point of M_∞. The closure of any these points can be easily seen to give an irreducible fat family of arcs inducing ν.

Among all closed irreducible fat families of arcs inducing a given divisorial valuation, there exist a maximal one with respect to the order of containment, known as the *maximal divisorial set* (see Section 1 for details). In this way we get a bijection between divisorial valuations and maximal divisorial sets in the arc space, and we can use the topology in the arc space to give structure to the set of divisorial valuations. More concretely, the containment of maximal divisorial sets induces a partial order on valuations. The understanding of this order is known as the *generalized Nash problem* [Ish08].

There are other ways to define orders in the set of divisorial valuations. For example, thinking of valuations as functions on $\mathscr{O}_M$, we can partially order them by comparing their values. In dimension two, the resolution process also gives an order. It can be shown that the order induced by the arc space is different from any previously known order [Ish08], but beyond that, not much is known about the generalized Nash problem. A notable exception is the case of of toric valuations on toric varieties, which was studied in detail in [Ish04].

Determining the irreducible components for D^k_n is essentially equivalent to computing minimal elements among those valuations over M that satisfy certain contact conditions with respect to D^k. In Section 4 we solve the generalized Nash problem for *invariant* divisorial valuations, and use this to prove the following theorem.

Theorem A. *Let D^k be the determinantal variety of matrices of size $r \times s$ and rank at most k, where $k < r \leq s$. Let D^k_n be the n-th jet scheme of D^k. If $k = 0$ or $k = r-1$, the jet scheme D^k_n is irreducible. Otherwise the number of irreducible components of D^k_n is*

$$n + 2 - \left\lceil \frac{n+1}{k+1} \right\rceil.$$

Jet schemes for determinantal varieties were previously studied in [KS05a, KS05b, Yue07a]. Up to now, the approach has always been to use techniques from commutative algebra, performing a careful study of the defining equations. This has been quite successful for ranks 1 and $r-1$, especially for square matrices, but the general case seems too complex for these methods.

Our approach is quite different in nature: we focus on the natural group action. This is a technique that already plays a central role in Ishii's study of the arc spaces of toric varieties [Ish04]. Consider the group $G = \mathrm{GL}_r \times \mathrm{GL}_s$, which acts on the space of matrices M via change of basis. The rank of a matrix is the unique invariant for this action, the orbit closures being precisely the determinantal varieties D^k. The assignments sending a variety X to its arc space X_∞ and its jet schemes X_n are functorial. Since G is an algebraic group and its action on M is rational, we see that G_∞ and G_n are also groups, and that they act on M_∞ and M_n, respectively. Determinantal varieties are G-invariant, hence their arc spaces are G_∞-invariant and their jet schemes are G_n-invariant. The main observation is that most questions regarding components and dimensions of jet schemes and arc spaces of determinantal varieties can be reduced to the study of orbits in M_∞ and M_n.

Orbits in the arc space M_∞ are easy to classify. As a set, M_∞ is just the space of matrices with coefficients in $\mathbf{C}[[t]]$, and G_∞ acts via change of basis over the ring $\mathbf{C}[[t]]$. Gaussian elimination allows us to find representatives for the orbits: each of them contains a unique diagonal matrix of the form $\mathrm{diag}(t^{\lambda_1}, \ldots, t^{\lambda_r})$, where $\infty \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$, and the sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ determines the orbit. In Section 3 we see how to decompose arc spaces and jet schemes of determinantal varieties as unions of these orbits. Once this is done, the main difficulty to determine irreducible components is the understanding of the generalized Nash problem for orbits closures in M_∞. This is the purpose of the following theorem, which is proven in the article as Theorem 4.7.

Theorem B (Nash problem for invariant valuations)**.** *Consider two sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ and $\lambda' = (\lambda'_1 \geq \cdots \geq \lambda'_r \geq 0)$, and let $\mathscr{C}_\lambda$ and $\mathscr{C}_{\lambda'}$ be the corresponding orbits in the arc space M_∞. Then the closure of $\mathscr{C}_\lambda$ contains $\mathscr{C}_{\lambda'}$ if and only if*

$$\lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-k} \quad \leq \quad \lambda'_r + \lambda'_{r-1} + \cdots + \lambda'_{r-k} \qquad \forall k \in \{0, \ldots, r\}.$$

Sequences of the form $(\infty \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ are closely related to partitions (the only difference being the possible presence of infinite terms), and the order that appears in the above theorem is a modification of a well-known order on the set of partitions: the order of domination (see Section 2). Since the poset of partitions is well understood, one has very explicit information about the structure of the poset of orbits in the arc space. This allows us to compute minimal elements among some interesting families of orbits, leading to the proof of Theorem A (see Section 4).

0.2 Log canonical thresholds

Mustaţă's formula [Mus01, ELM04, dFEI08] allows us to compute log discrepancies for divisorial valuations by computing codimensions of the appropriate sets in the arc space. In the case at hand, the most natural valuations one can look at are the invariant divisorial valuations. In Section 5 we see that the maximal divisorial sets corresponding to these valuations are precisely the orbit closures in M. Hence computing log discrepancies gets reduced to computing codimensions of orbits. This explains the relevance of the following result, which appears in Section 5 as Proposition 5.5.

Theorem C (Log discrepancies of invariant valuations)**.** *Consider a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ and let $\mathscr{C}_\lambda$ be the corresponding orbit in the arc space M_∞. Then the codimension of $\mathscr{C}_\lambda$ in M_∞ is*

$$\mathrm{codim}(\mathscr{C}_\lambda, M_\infty) = \sum_{i=1}^{r} \lambda_i (s - r + 2i - 1).$$

Once these codimensions are known, one can compute log canonical thresholds for pairs involving determinantal varieties. The following result appears in Section 5 as Theorem 5.7.

Theorem D. *Let M be the space of matrices of size $r \times s$, and D^k the subvariety of M containing matrices of rank at most k. The log canonical threshold of the pair (M, D^k) is*

$$\operatorname{lct}(M, D^k) = \min_{i=0\dots k} \frac{(r-i)(s-i)}{k+1-i}.$$

We should note that the previous result is not new. Log resolutions for generic determinantal varieties are now classical objects. They are essentially spaces of complete collineations, obtained by blowing up D^k along D^0, D^1, $\dots$, D^{k-1}, in this order [Sem51, Tyr56, Vai84, Lak87]. It is possible to use these resolutions to compute log canonical thresholds, and this was done by A. Johnson in her Ph.D. thesis [Joh03]. In fact she is able to compute all the multiplier ideals $\mathscr{J}(M, c \cdot D^k)$. Our method does not need any knowledge about the structure of these log resolutions.

0.3 Topological zeta function

Using our techniques, we are able to understand orbits in M_∞ quite explicitly. In Section 6 we compute motivic volumes of orbits, and this allows us to determine topological zeta functions for determinantal varieties (for square matrices).

Theorem E. *Let $M = \mathbf{A}^{r^2}$ be the space of square $r \times r$ matrices, and let D^k be the subvariety of matrices of rank at most k. Then the topological zeta function of the pair (M, D^k) is given by*

$$Z^{\text{top}}_{D^k}(s) = \prod_{\zeta \in \Omega} \frac{1}{1 - s\zeta^{-1}}$$

where Ω is the set of poles:

$$\Omega = \left\{ -\frac{r^2}{k+1}, \quad -\frac{(r-1)^2}{k}, \quad -\frac{(r-2)^2}{k-1}, \quad \dots, \quad -(r-k)^2 \right\}.$$

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1 Arc spaces and motivic integration

We briefly review in this section the basic theory of arc spaces and motivic integration, as these tools will be used repeatedly. Most of these results are well-known. We have gathered them mainly from [DL98], [DL99], [ELM04], [Ish08], [Vey06], and [dFEI08]. We direct the reader to those papers for more details and proofs.

We will always work with varieties and schemes defined over the complex numbers. When we use the word scheme, we do not necessarily assume that it is of finite type.

1.1 Arcs and jets

Given a variety X, we consider the following functors from the category of $\mathbf{C}$-algebras to the category of sets:

$$F_X^\infty(A) = \mathrm{Hom}\big(\mathrm{Spec}\, A[[t]], X\big), \qquad F_X^n(A) = \mathrm{Hom}\big(\mathrm{Spec}\, A[t]/(t^{n+1}), X\big).$$

Both of these functors are representable by schemes, which we denote by X_∞ and X_n respectively. X_∞ is known as the *arc space* of X and X_n as the *n-th jet scheme* of X. The natural projections $\psi_n : X_\infty \to X_n$ are known as *truncation maps*.

The assignment $X \mapsto X_\infty$ is functorial: each morphism $f : X' \to X$ induces by composition a morphism $f_\infty : X'_\infty \to X_\infty$, and $(g \circ f)_\infty = f_\infty \circ g_\infty$. As a consequence, if G is a group scheme, so is G_∞, and if X has an action by G, the arc space X_∞ has an action by G_∞. Analogous statements hold for the jet schemes.

1.2 Contact loci and valuations

A constructible subset $\mathscr{C} \subset X_\infty$ is called *thin* if one can find a proper subscheme $Y \subset X$ such that $\mathscr{C} \subset Y_\infty$. Constructible subsets which are not thin are called *fat*. A *cylinder* in X_∞ is a set of the form $\psi_n^{-1}(C)$, for some constructible set $C \subset X_n$. On a smooth variety, cylinders are fat, but in general a cylinder might be contained in S_∞, where $S = \mathrm{Sing}(X) \subset X$ is the singular locus.

An arc $\alpha \in X_\infty$ induces a morphism $\alpha : \mathrm{Spec}\, K[[t]] \to X$, where K is the residue field of α. Given an ideal $\mathscr{I} \subset \mathscr{O}_X$, its pull-back $\alpha^*(\mathscr{I}) \subset K[[t]]$ is of the form (t^e), where e is either a non-negative integer or infinity (by convention $t^\infty = 0$). We call e the *order of contact* of α along $\mathscr{I}$ and denote it by $\mathrm{ord}_\alpha(\mathscr{I})$. Given a collection of ideals $I = (\mathscr{I}_1, \ldots, \mathscr{I}_r)$ and a multi-index $\mu = (m_1, \ldots, m_r) \in \mathbf{Z}_{\geq 0}$ we introduce the *contact locus*:

$$\mathrm{Cont}^{=\mu}(I) = \{\, \alpha \in X_\infty \;:\; \mathrm{ord}_\alpha(\mathscr{I}_j) = m_j \text{ for all } j \,\},$$
$$\mathrm{Cont}^{\mu}(I) = \{\, \alpha \in X_\infty \;:\; \mathrm{ord}_\alpha(\mathscr{I}_j) \geq m_j \text{ for all } j \,\}.$$

Notice that contact loci are cylinders.

Let $\mathscr{C} \subset X_\infty$ be an irreducible fat set. Then $\mathscr{C}$ contains a generic point $\gamma \in \mathscr{C}$ which we interpret as a morphism $\gamma : \mathrm{Spec}\, K[[t]] \to X$, where K is the residue field of γ. Let η be the generic point of $\mathrm{Spec}\, K[[t]]$. Since $\mathscr{C}$ is fat, $\gamma(\eta)$ is the generic point of X, and we get an inclusion of fields

$$\mathbf{C}(X) \to K((t)).$$

The composition of this inclusion with the canonical valuation on $K((t))$ is a valuation on $\mathbf{C}(X)$, which we denote by $v_{\mathscr{C}}$. In this way we obtain a map from the set of fat irreducible subsets of X_∞ to the set of valuation of $\mathbf{C}(X)$ defined over X:

$$\{\, \mathscr{C} \subseteq X_\infty \;:\; \mathscr{C} \text{ irreducible fat} \,\} \quad \longrightarrow \quad \{\text{ discrete valuations over } X \,\}.$$

This map is always surjective: for a discrete valuation v of $\mathbf{C}(X)$ defined over X, the completion of the discrete valuation ring $\mathscr{O}_v$ is isomorphic to a power series ring $k_v[[t]]$. But it is far from being injective. For example, different choices of uniformizing parameter in a discrete valuation ring give rise to different arcs.

A valuation v of $\mathbf{C}(X)$ is called *divisorial* if it is of the form $q \cdot \mathrm{val}_E$, where q is a positive integer and E is a prime divisor on a variety X' birational to X. An irreducible fat set $\mathscr{C} \subset X_\infty$ is said to be *divisorial* if the corresponding valuation $\nu_{\mathscr{C}}$ is divisorial. In [Ish08] it is shown that the union of all divisorial sets corresponding to a given valuation ν is itself a divisorial set defining ν (in fact it is an irreducible component of a contact locus). These unions are called *maximal divisorial sets.* There is a one to one correspondence between divisorial valuations and maximal divisorial sets. This gives an inclusion

$$\{ \text{ divisorial valuations over } X \} \quad \hookrightarrow \quad \{ \mathscr{C} \subseteq X_\infty \ : \ \mathscr{C} \text{ irreducible fat } \}.$$

Through this inclusion, the topology on the arc space X_∞ gives structure to the set of divisorial valuations. For example, given two valuations ν and ν' with corresponding maximal divisorial sets $\mathscr{C}$ and $\mathscr{C}'$, we say that ν *dominates* ν' if $\mathscr{C} \supseteq \mathscr{C}'$. The *generalized Nash problem* consists in understanding the relation of domination among divisorial valuations.

1.3 Discrepancies

Let X be a variety of dimension n. The *Nash blowing-up* of X, denoted $\widehat{X}$, is defined as the closure of X_{reg} in $\mathbf{P}_X(\Omega^n_X)$; it is equipped with a tautological line bundle $\mathscr{O}_{\mathbf{P}_X(\Omega^n_X)}(1)|_{\widehat{X}}$, which we denote by $\widehat{K}_X$ and call the *Mather canonical line bundle* of X. When X is smooth, $X = \widehat{X}$ and $K_X = \widehat{K}_X$.

When Y is a smooth variety and $f : Y \to X$ is a birational morphism that factors through the Nash blowing-up, we define the *relative Mather canonical divisor* of f as the unique effective divisor supported on the exceptional locus of f and linearly equivalent to $K_Y - \widehat{K}_X$; we denote it by $\widehat{K}_{Y/X}$.

Let ν be a divisorial valuation of X. Then we can find a smooth variety Y and a birational map $Y \to X$ factoring through the Nash blowing-up of X, such that $\nu = q \cdot \text{val}_E$ for some prime divisor $E \subset Y$. We define the *Mather discrepancy* of X along ν as

$$\hat{k}_\nu(X) = q \cdot \text{ord}_E(\widehat{K}_{Y/X}).$$

This definition is independent of the choice of resolution Y.

In the smooth case, Mustaţă showed that we can compute discrepancies using the arc space [Mus01, ELM04]. This is generalized to arbitrary varieties in [dFEI08] via the use of Mather discrepancies. More precisely, given a divisorial valuation ν with multiplicity q, let $\mathscr{C}_\nu \subset X_\infty$ be the corresponding maximal divisorial set. Then

$$\text{codim}(\mathscr{C}_\nu, X_\infty) = \hat{k}_\nu(X) + q.$$

1.4 Motivic integration

Let $\mathscr{M}_0$ be the Grothendieck ring of algebraic varieties over $\mathbf{C}$. In [DL99], the authors introduce a certain completion of a localization of $\mathscr{M}_0$, which we denote by $\mathscr{M}$. Also, for each variety X over $\mathbf{C}$, they define a measure μ_X on X_∞ with values in $\mathscr{M}$. This measure is known as the *motivic measure* of X. The following properties hold for $\mathscr{M}$ and the measures μ_X:

- There is a canonical ring homomorphism $\mathscr{M}_0 \to \mathscr{M}$. In particular, for each variety X one can associate an element $[X] \in \mathscr{M}$, and the map $X \mapsto [X]$ is additive (meaning that $[X] = [Y] + [U]$, where $Y \subset X$ is a closed subvariety and $U = X \setminus Y$).
- The element $[\mathbf{A}^1] \in \mathscr{M}$ has a multiplicative inverse. We write $\mathbf{L} = [\mathbf{A}^1]$.
- Both the Euler characteristic and the Hodge-Deligne polynomial, considered as ring homomorphisms with domain $\mathscr{M}_0$, extend to homomorphisms

$$\chi : \mathscr{M} \to \mathbf{R}, \qquad E : \mathscr{M} \to \mathbf{Z}((u, v)),$$

where $\chi(\mathbf{L}) = 1$ and $E(\mathbf{L}) = uv$.

- Constructible sets in X_∞ are μ_X-measurable. In particular, thin sets, fat sets, cylinders, and contact loci are all measurable.
- If X is smooth, $\mu_X(X_\infty) = [X]$.
- A thin set has measure zero.
- Let $\mathscr{C} \subset X_\infty$ be a cylinder in X_∞. Then the truncations $\psi_n(\mathscr{C}) \subset X_n$ are of finite type, so they define elements $[\psi_n(\mathscr{C})] \in \mathscr{M}$. Then
$$\mu_X(\mathscr{C}) = \lim_{n\to\infty} [\psi_n(\mathscr{C})] \cdot \mathbf{L}^{-nd}$$
where d is the dimension of X. Furthermore, if $\mathscr{C}$ does not intersect $(X_{\text{sing}})_\infty$, then $[\psi_n(\mathscr{C})] \cdot \mathbf{L}^{-nd}$ stabilizes for n large enough.
- Given an ideal $\mathscr{I} \subset \mathscr{O}_X$, we define a function $|\mathscr{I}|$ on X_∞ with values on $\mathscr{M}$ via
$$|\mathscr{I}|(\alpha) = \mathbf{L}^{-\operatorname{ord}_\alpha(\mathscr{I})} \qquad \alpha \in X_\infty.$$
Notice that $\operatorname{ord}_\beta(\mathscr{I}) = \infty$ if and only if $\beta \in \operatorname{Zeroes}(\mathscr{I})_\infty$, so $|\mathscr{I}|$ is only defined up to a measure zero set. Then $|\mathscr{I}|$ is μ_X-integrable and
$$\int_{X_\infty} |\mathscr{I}|\, d\mu_X = \sum_{p=0}^{\infty} [\operatorname{Cont}^{=p}(\mathscr{I})] \cdot \mathbf{L}^{-p}.$$
- Let $f : Y \to X$ be a birational map factoring through the Nash blowing-up of X, and assume Y smooth. Let $\operatorname{Jac}(f)$ be the ideal of the relative Mather canonical divisor $\widehat{K}_{Y/X}$. Then $(f_\infty)^*(\mu_X) = |\operatorname{Jac}(f)| \cdot \mu_Y$; in other words, for a measurable set $\mathscr{C} \subset X_\infty$, and a μ_X-integrable function φ,
$$\int_{\mathscr{C}} \varphi\, d\mu_X = \int_{f_\infty^{-1}(\mathscr{C})} (\varphi \circ f_\infty)\, |\operatorname{Jac}(f)|\, d\mu_Y.$$
This is known as the *change of variables formula* for motivic integration.
- Given a subscheme $Y \subset X$ with ideal $\mathscr{I} \subset \mathscr{O}_X$, the *motivic Igusa zeta function* of the pair (X, Y) is defined as
$$Z_Y(s) = \int_{X_\infty} |\mathscr{I}|^s\, d\mu_X = \sum_{p=0}^{\infty} [\operatorname{Cont}^{=p}(\mathscr{I})] \cdot \mathbf{L}^{-sp}.$$
In this expression, $\mathbf{L}^{-s}$ is to be understood as a formal variable, so $Z_Y(s) \in \mathscr{M}[[\mathbf{L}^{-s}]]$. It is shown in [DL98] that $Z_Y(s)$ is a rational function. More precisely, $Z_Y(s)$ belongs to the subring of $\mathscr{M}[[\mathbf{L}^{-s}]]$ generated by $\mathscr{M}$ and the elements of the form $\frac{\mathbf{L}^{-sa}}{\mathbf{L}^{b} - \mathbf{L}^{-sa}}$, where a and b are positive integers.
- The motivic Igusa zeta function specializes to the topological zeta function in the following way. Formally expanding $\mathbf{L}^{-s}$ and the denominators in $Z_Y(s)$ as power series in $(\mathbf{L}-1)$, one gets a well defined element $\tilde{Z}_Y(s) \in \mathscr{M}(s)[[\mathbf{L}-1]]$. Using the Euler characteristic map $\chi : \mathscr{M} \to \mathbf{R}$ and considering the quotient by the ideal generated by $\mathbf{L}-1$ we obtain an element $Z_Y^{\text{top}}(s) \in \mathbf{R}(s)$. The rational function $Z_Y^{\text{top}}(s)$ is known as the *topological zeta function* for the pair (X, Y).

2 Partitions

In order to enumerate orbits in the arc space of determinantal varieties, it will be convenient to use the language of partitions. In fact, we will consider a slight generalization of the concept of partition, where we allow terms of infinite size and an infinite number of terms (we call these objects pre-partitions). In this section we recall some basic facts about partitions that will be needed in the rest of the article, and extend them to the case of pre-partitions. Most of the results are well known. For a detailed account of the theory of partitions we refer the reader to [dCEP80] and [Ful97].

2.1 Definitions

Let $\mathbf{N}$ denote the set of non-negative integers, and consider $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. We extend the natural order on $\mathbf{N}$ to $\overline{\mathbf{N}}$ by setting $\infty > n$ for any $n \in \mathbf{N}$. We also set $\infty + n = \infty$ for any $n \in \overline{\mathbf{N}}$.

A *pre-partition* is an infinite non-increasing sequence of elements of $\overline{\mathbf{N}}$. Given a pre-partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the elements λ_i are known as the *terms* of λ. The first term λ_1 is called the *maximal term* or *co-length* of λ. If all the terms of λ are non-zero, we say that λ has infinite length; otherwise, the largest integer i such that $\lambda_i \neq 0$ is called the *length* of λ. If a pre-partition λ has length no bigger than ℓ, we will often denote λ by the finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

A *partition* is a finite non-increasing sequence of positive integers. A partition can be naturally identified with a pre-partition of finite length and finite co-length.

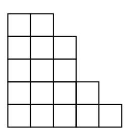
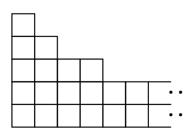
Given a pre-partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we define

$$\lambda_i^* = \sup\left\{ j \,:\, \lambda_j \geq i \right\} \in \overline{\mathbf{N}}.$$

Then $\lambda_i^* \geq \lambda_{i+1}^*$, and we obtain a new pre-partition λ^*, known as the *conjugate* pre-partition of λ. It follows from the definition that $\lambda^{**} = \lambda$, that the length of λ^* is the co-length of λ, and that the co-length of λ^* is the length of λ. In particular, the conjugate of a partition is also a partition.

2.2 Diagrams

It will be helpful to visualize pre-partitions as Young diagrams (sometimes also known as Ferrers diagrams). A Young diagram is a graphical representation of a pre-partition; it is a collection of boxes, arranged in left-justified rows, with non-increasing row sizes. To each pre-partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ there is a unique Young diagram whose i-th row has size λ_i. For example:

$(5,4,3,3,2) \quad = $  $\qquad (\infty,\infty,4,2,1) \quad = $ 

The length of a partition corresponds to the height of the associated diagram, whereas the co-length corresponds to the width. The diagram of the conjugate pre-partition is obtained from the original diagram by switching rows with columns. More concretely, if T denotes the diagram associated to a pre-partition λ, the terms λ_i of the pre-partition give the row sizes of T, and the terms λ_i^* of the conjugate pre-partition give the column sizes of T.

2.3 Posets of partitions

Given two pre-partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, we say that λ is *contained* in μ, and denote it by $\lambda \subseteq \mu$, if $\lambda_i \le \mu_i$ for all i. Containment of pre-partitions corresponds to containment of the associated diagrams. In particular, $\lambda \subseteq \mu$ if and only if $\lambda^* \subseteq \mu^*$.

If λ and μ are pre-partitions with finite co-length, we say that μ *dominates* λ, denoted by $\lambda \le \mu$, if

$$\lambda_1 + \lambda_2 + \dots + \lambda_i \quad \le \quad \mu_1 + \mu_2 + \dots + \mu_i$$

for all positive integers i. If λ and μ have finite length, we say that μ *co-dominates* λ, denoted $\lambda \trianglelefteq \mu$, if

$$\lambda_i + \lambda_{i+1} + \dots \quad \le \quad \mu_i + \mu_{i+1} + \dots$$

for all positive integers i (notice that the sums above have only a finite number of terms because the pre-partitions have finite length). It is shown in [dCEP80, Prop. 1.1] that the conditions of domination and co-domination of pre-partitions can be expressed in terms of the conjugates. More precisely, we have:

$$\lambda \le \mu \quad \Longleftrightarrow \quad \lambda^* \trianglelefteq \mu^* \quad \Longleftrightarrow \quad \lambda^*_i + \lambda^*_{i+1} + \dots \quad \le \quad \mu^*_i + \mu^*_{i+1} + \dots \qquad \forall i,$$

$$\lambda \trianglelefteq \mu \quad \Longleftrightarrow \quad \lambda^* \le \mu^* \quad \Longleftrightarrow \quad \lambda^*_1 + \lambda^*_2 + \dots + \lambda^*_i \quad \le \quad \mu^*_1 + \mu^*_2 + \dots + \mu^*_i \qquad \forall i.$$

The three relations (containment, domination and co-domination) define partial orders. We are mostly interested in the order of co-domination. Given a positive integer r, we denote by $\overline{\Lambda}_r$ (respectively Λ_r) the poset of pre-partitions (resp. partitions) of length at most r with the order of co-domination. By $\Lambda_{r,n}$ we denote the poset of partitions of length at most r and co-length at most n. It can be shown that $\overline{\Lambda}_r$, Λ_r and $\Lambda_{r,n}$ are all latices.

2.4 Adjacencies

In Section 4 we will need to have a good understanding of the structure of the posets $\overline{\Lambda}_r$. For our purposes, it will be enough to determine the adjacencies in $\overline{\Lambda}_r$.

Let λ and μ be two different pre-partitions in $\overline{\Lambda}_r$ such that $\lambda \trianglelefteq \mu$. We say that λ and μ are *adjacent* (or that μ *covers* λ) if there is no pre-partition ν in $\overline{\Lambda}_r$, different from λ and μ, such that $\lambda \trianglelefteq \nu \trianglelefteq \mu$. Adjacencies in Λ_r were determined in [dCEP80, Prop. 1.2]. They come in three different types, which we call *single removals*, *slips* and *falls*.

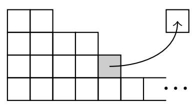
- We say that a pre-partition λ is obtained from μ via a *single removal* if $\lambda_i = \mu_i$ for all $i \neq j$ and $\lambda_j = \mu_j - 1$, where j is the smallest integer such that μ_j is finite. At the level of diagrams, λ is obtained from μ by removing one box in the lowest row of finite size. Notice that this removal can only be done if $\mu_{j+1} < \mu_j$.

- We say that λ is obtained from μ via a *slip* if there exists a positive integer j such that $\lambda_{j+1} = \mu_{j+1} - 1$, $\lambda_j = \mu_j + 1$ and $\lambda_i = \mu_i$ for all $i \notin \{j, j+1\}$. In this case, the diagram of λ is obtained from the diagram of μ by moving a box from row $j+1$ to row j. A slip from row $j+1$ can only happen if $\mu_{j+2} < \mu_{j+1}$, and $\mu_j < \mu_{j-1}$.

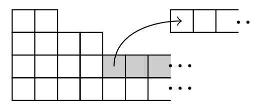
- We say that λ is obtained from μ via a *fall* if μ^* is obtained from λ^* via a slip. In other words, there exist positive integers $j < k$ such that $\lambda_k = \mu_k - 1$, $\lambda_j = \mu_j + 1$, and $\lambda_i = \mu_i$ for all $i \notin \{j, k\}$. A fall from row k to row j can only happen if $\mu_{k+1} < \mu_k$, $\mu_j < \mu_{j-1}$, and $\mu_i = \mu_{i'}$ for all $i, i' \in \{j, j+1, \dots, k\}$. During a fall, a box in the diagram of μ is moved from one column to the next.

Since we are dealing with pre-partitions, we will also need to consider *infinite removals*:

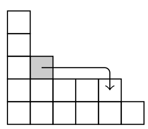
- We say that a pre-partition λ is obtained from μ via an *infinite removal* if $\lambda_i = \mu_i$ for all $i \neq j$ and $\lambda_j < \mu_j = \infty$, where j is the largest integer such that μ_j is infinite. At the level of diagrams, λ is obtained from μ by removing infinitely many boxes in the highest row of infinite size.



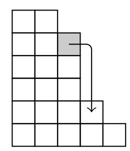
Single removal



Infinite removal



Slip



Fall

In [dCEP80] the authors show that adjacencies in the set of partitions with respect to the order of domination correspond to simple removals, slips and falls. The result for partitions with the order of co-domination follows immediately from the fact that $\lambda \triangleleft \mu \Leftrightarrow \lambda^* \leq \mu^*$. Now consider two pre-partitions $\lambda \triangleleft \mu$ with finite length, and assume they are adjacent. They must have the same number of infinite terms, otherwise the pre-partition ν obtained from λ by adding one box in the lowest finite row verifies $\lambda \triangleleft \nu \triangleleft \mu$. Let $\hat{\lambda}$ and $\hat{\mu}$ be the partitions obtained from λ and μ by removing the infinite terms. Then $\hat{\lambda}$ and $\hat{\mu}$ are adjacent and we can apply the result of [dCEP80] to show that λ can be obtained from μ via a simple removal, a slip, or a fall.

Theorem 2.1. *Let λ and μ be two pre-partitions in $\overline{\Lambda}_r$, and assume that $\lambda \triangleleft \mu$. Then there exists a finite sequence of pre-partitions in $\overline{\Lambda}_r$,*

$$\lambda = \nu^m \triangleleft \cdots \triangleleft \nu^1 \triangleleft \nu^0 = \mu,$$

such that ν^i is obtained from ν^{i-1} via a simple removal, an infinite removal, a slip, or a fall.

Proof. Assume first that there are more infinite terms in μ that in λ. To each infinite row j in μ which is finite in λ we apply an infinite removal, leaving at least λ_j boxes (depending on the particular μ we might need to leave more boxes). This way we obtain a sequence

$$\lambda \triangleleft \nu^{m_0} \triangleleft \cdots \triangleleft \nu^1 \triangleleft \nu^0 = \mu,$$

where λ has the same number of infinite rows as ν^{m_0} and where each ν^i is obtained from ν^{i-1} via an infinite removal. Notice that m_0 is the number of rows which are infinite in μ but finite in λ. Since μ has finite length, m_0 is finite.

Let k be the number of boxes in the finite rows of ν^{m_0}. Then any pre-partition with the same number of infinite rows as ν^{m_0} and co-dominated by ν^{m_0} must use at most k boxes in its finite rows. In particular there are only finitely many such pre-partitions. It follows that we can find a finite sequence

$$\lambda = \nu^m \triangleleft \cdots \triangleleft \nu^{m_0+1} \triangleleft \nu^{m_0}$$

where consecutive terms are adjacent. From the discussion preceding the theorem, we see that v^i can be obtained from v^{i-1} by a simple removal, a slip, or a fall, and the result follows. □

3 Orbit decomposition of the arc space

We start by recalling our basic setup from the introduction. $M = \mathbf{A}^{rs}$ is the space of $r \times s$ matrices with coefficients in $\mathbf{C}$, and we assume that $r \leq s$. The ring of regular functions on M is a polynomial ring on the entries of a generic matrix x:

$$\mathscr{O}_M = \mathbf{C}[x_{11}, \dots, x_{rs}], \qquad x = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1s} \\ x_{21} & x_{22} & \dots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r1} & x_{r2} & \dots & x_{rs} \end{pmatrix}.$$

The generic determinantal variety of matrices of rank at most k is denoted by D^k. The ideal of D^k is generated by the $(k+1) \times (k+1)$ minors of x. It can be shown that generic determinantal varieties are irreducible, and that the singular locus of D^k is D^{k-1} when $0 < k < r$ (D^0 and D^r are smooth). They are also Cohen-Macaulay, Gorenstein, and have rational singularities. For proofs of the previous statements, and a comprehensive account of the theory of determinantal varieties we refer the reader to [BV88].

We denote by G the group $\mathrm{GL}_r \times \mathrm{GL}_s$. It acts naturally on M via change of basis:

$$(g, h) \cdot A = g A h^{-1}, \qquad (g, h) \in G, \quad A \in M.$$

The rank of a matrix is the only invariant for this action, and the determinantal varieties are the orbit closures (their ideals being the only invariant prime ideals of $\mathscr{O}_M$).

The group G is a reductive algebraic group. In particular it is an algebraic variety, and we can consider its arc space G_∞ and its jet schemes G_n. The action of G on M and D^k induces actions at the level of arc spaces and jet schemes:

$$\begin{aligned} G_\infty \times M_\infty &\to M_\infty, \qquad & G_\infty \times D^k_\infty &\to D^k_\infty, \\ G_n \times M_n &\to M_n, \qquad & G_n \times D^k_n &\to D^k_n. \end{aligned}$$

In this section we classify the orbits associated to all of these actions.

As a set, the arc space M_∞ contains matrices of size $r \times s$ with entries in the power series ring $\mathbf{C}[[t]]$. Analogously, the group $G_\infty = (\mathrm{GL}_r)_\infty \times (\mathrm{GL}_s)_\infty$ is formed by pairs of square matrices with entries in $\mathbf{C}[[t]]$ which are invertible, that is, their determinant is a unit in $\mathbf{C}[[t]]$. Orbits in M_∞ correspond to similarity classes of matrices over the ring $\mathbf{C}[[t]]$, and we can easily classify these using the fact that $\mathbf{C}[[t]]$ is a principal ideal domain.

Definition 3.1 (Orbit associated to a partition)**.** Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \overline{\Lambda}_r$ be a pre-partition with length at most r. Consider the following diagonal matrix in M_∞:

$$\delta_\lambda \quad = \quad \begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & t^{\lambda_2} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & t^{\lambda_r} \end{pmatrix}$$

(we use the convention that $t^\infty = 0$). Then the G_∞-orbit of the matrix δ_λ is called the orbit in M_∞ *associated* to the pre-partition λ, and it is denoted by $\mathscr{C}_\lambda$.

Proposition 3.2 (Orbits in M_∞)**.** *Every G_∞-orbit of M_∞ is of the form $\mathscr{C}_\lambda$ for some pre-partition $\lambda \in \overline{\Lambda}_r$. An orbit $\mathscr{C}_\lambda$ is contained in D^k_∞ if and only if the associated pre-partition λ contains at least $r-k$ leading infinities, i.e. $\lambda_1 = \dots = \lambda_{r-k} = \infty$. In particular, $M_\infty \setminus D^{r-1}_\infty$ is the union of the orbits corresponding to regular partitions, and the orbits in $D^k_\infty \setminus D^{k-1}_\infty$ are in bijection with Λ_k. Moreover, the orbit corresponding to the empty partition $(0,0,\dots)$ is the arc space $\left(M \setminus D^{r-1}\right)_\infty$.*

Proof. As mentioned above, M_∞ is the set of $r \times s$-matrices with coefficients in the ring $\mathbf{C}[[t]]$, and the group G_∞ acts on M_∞ via row and column operations, also with coefficients in $\mathbf{C}[[t]]$. Using Gaussian elimination and the fact that $\mathbf{C}[[t]]$ is a principal ideal domain, we see that each G_∞-orbit in M_∞ contains a diagonal matrix, where the diagonal entries are powers of t or zeroes. Think of the diagonal zeroes as powers t^∞. After permuting columns and rows, we can assume that the exponents of these powers form a weakly decreasing sequence when read from the upper-left corner to the lower-right corner. Moreover, the usual structure theorems for finitely generated modules over principal ideal domains guarantee that each orbit contains a unique diagonal matrix in this form. This shows that the set of G_∞-orbits in M_∞ is in bijection with $\overline{\Lambda}_r$.

The ideal defining D^k in M is generated by the minors of size $(k+1)\times(k+1)$. Let $\lambda \in \overline{\Lambda}_r$ be a pre-partition of length at most r and consider δ_λ as in Definition 3.1. The $(k+1)\times(k+1)$ minors of δ_λ are either zero or of the form $\prod_{i\in I} t^{\lambda_i}$, where I is a subset of $\{1,\dots,r\}$ with $k+1$ elements. For all of the minors to be zero, we need at least $r-k$ infinities in the set $\{\lambda_1,\dots,\lambda_r\}$ (recall that $t^\infty = 0$). In other words, δ_λ is contained in D^k_∞ if and only if λ contains $r-k$ leading infinities.

The variety D^k is invariant under the action of G, so D^k_∞ is invariant under the action of G_∞. In particular the orbit $\mathscr{C}_\lambda$ is contained in D^k_∞ if and only if δ_λ is. The rest of the proposition follows immediately. ◻

Proposition 3.3 (Orbits and contact loci)**.** *The contact locus* $\operatorname{Cont}^p(D^k)$ *is invariant under the action of G_∞, and the orbits contained in* $\operatorname{Cont}^p(D^k)$ *correspond to those pre-partitions $\lambda \in \overline{\Lambda}_r$ whose last $k+1$ terms add up to at least p:*

$$\lambda_{r-k} + \dots + \lambda_r \geq p.$$

Proof. The truncations maps from the arc space to the jet schemes are in fact natural transformations of functors. This means that we have the following natural diagram:

$$\begin{array}{ccccc} G_\infty & \times & M_\infty & \longrightarrow & M_\infty \\ \downarrow & & \downarrow & & \downarrow \\ G_n & \times & M_n & \longrightarrow & M_n \end{array}$$

Since D^k is G-invariant, D^k_n is G_n-invariant, so $\operatorname{Cont}^p(D^k)$ (the inverse image of D^k_{p-1} under the truncation map) is G_∞-invariant. In particular, an orbit $\mathscr{C}_\lambda$ is contained in $\operatorname{Cont}^p(D^k)$ if and only if its base point δ_λ is. The order of vanishing of $\mathscr{I}_{D^k}$ along δ_λ is $\lambda_{r-k} + \dots + \lambda_r$ (recall that $\mathscr{I}_{D^k}$ is generated by the minors of size $(k+1)\times(k+1)$ and that $\lambda_1 \geq \dots \geq \lambda_r$). Hence δ_λ belongs to $\operatorname{Cont}^p(D^k)$ if and only if $\lambda_{r-k} + \dots + \lambda_r \geq p$, and the proposition follows. ◻

Proposition 3.4 (Orbits are cylinders)**.** *Let $\lambda \in \overline{\Lambda}_r$ be an pre-partition, and let $\mathscr{C}_\lambda$ be the associated orbit in M_∞. If λ is a partition, $\mathscr{C}_\lambda$ is a cylinder of M_∞. More generally, let $r-k$ be the number of infinite terms of λ. Then $\mathscr{C}_\lambda$ is a cylinder of D^k_∞.*

Proof. Assume that λ is an pre-partition with $r-k$ leading infinities, and consider the following cylinders in M_∞:

$$A_\lambda = \mathrm{Cont}^{\lambda_r}(D^0) \cap \mathrm{Cont}^{\lambda_r+\lambda_{r-1}}(D^1) \cap \mathrm{Cont}^{\lambda_r+\cdots+\lambda_{r-k}}(D^k),$$
$$B_\lambda = \mathrm{Cont}^{\lambda_r+1}(D^0) \cup \mathrm{Cont}^{\lambda_r+\lambda_{r-1}+1}(D^1) \cup \mathrm{Cont}^{\lambda_r+\cdots+\lambda_{r-k}+1}(D^k).$$

By Propositions 3.2 and 3.3, we know that

$$\mathscr{C}_\lambda = (A_\lambda \setminus B_\lambda) \cap D^k_\infty.$$

Hence $\mathscr{C}_\lambda$ is a cylinder in D^k_∞, as required. □

Remark 3.5. The proof of Proposition 3.4 tells us that we can express all orbits in M_∞ with contact conditions with respect to the determinantal varieties. In particular, if we know the order of contact of an arc with respect to all determinantal varieties, we know which orbit it belongs to. Moreover, this is also true not only for closed points, but for all schematic points of M_∞. It follows that every point of M_∞ (closed or not) is contained in an orbit generated by a closed point.

We now study the jet schemes G_n and M_n. As in the case of the arc space, elements in G_n and M_n are given by matrices, but now the coefficients lie in the ring $\mathbf{C}[t]/(t^{n+1})$.

Definition 3.6. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \Lambda_{r,n+1}$ be a partition with length at most r and co-length at most $n+1$. Then the diagonal matrix δ_λ considered in Definition 3.1 gives an element of the jet scheme M_n. The G_n-orbit of δ_λ is called the orbit of M_n *associated* to λ and it is denoted by $\mathscr{C}_{\lambda,n}$.

Proposition 3.7 (Orbits in M_n)**.** *Every G_n-orbit of M_n is of the form $\mathscr{C}_{\lambda,n}$ for some partition $\lambda \in \Lambda_{r,n+1}$. An orbit $\mathscr{C}_{\lambda,n}$ is contained in D^k_n if and only if the associated partition contains at least $r-k$ terms equal to $n+1$. In particular, the set of orbits in $D^k_n \setminus D^{k-1}_n$ is in bijection with $\Lambda_{k,n}$.*

Proof. This can be proven in the same way as Proposition 3.2. The only difference is that we now work with a principal ideal ring $\mathbf{C}[t]/(t^{n+1})$, as opposed to with the principal ideal domain $\mathbf{C}[[t]]$, but the domain condition played no role in the proof of Proposition 3.2. Alternatively, one can notice that $\mathbf{C}[t]/(t^{n+1})$ is a quotient of $\mathbf{C}[[t]]$, so modules over $\mathbf{C}[t]/(t^{n+1})$ correspond to modules over $\mathbf{C}[[t]]$ with the appropriate annihilator, and one can reduce the problem of classifying G_n-orbits in M_n to classifying G_∞-orbits in M_∞ with bounded exponents. □

Definition 3.8. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a pre-partition, and let n be a nonnegative integer. Then the *truncation* of λ to level n is the partition $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots)$ where

$$\overline{\lambda}_i = \min(\lambda_i, n).$$

Proposition 3.9 (Truncation of orbits)**.** *Let $\lambda \in \overline{\Lambda}_r$ be a pre-partition, and let $\overline{\lambda}$ be its truncation to level $n+1$. Then the image of $\mathscr{C}_\lambda$ under the natural truncation map $M_\infty \to M_n$ is $\mathscr{C}_{\overline{\lambda},n}$. Conversely, fix a partition $\overline{\lambda} \in \Lambda_{r,n+1}$, and let $\Gamma \subset \overline{\Lambda}_r$ be the set of pre-partitions whose truncation to level $n+1$ is $\overline{\lambda}$. Then the inverse image of $\mathscr{C}_{\overline{\lambda},n}$ under the truncation map is the union of the orbits of M_∞ corresponding to the pre-partitions in Γ.*

Proof. Notice that $\delta_{\overline{\lambda}} \in M_n$ is the truncation of $\delta_\lambda \in M_\infty$. Then the fact that the truncation of $\mathscr{C}_\lambda = G_\infty \cdot \delta_\lambda$ equals $\mathscr{C}_{\overline{\lambda},n} = G_n \cdot \delta_{\overline{\lambda}}$ is an immediate consequence of the fact that the truncation map is a natural transformation of functors (see the proof of Proposition 3.3). Conversely, if λ and λ' have different truncations, the G_n-orbits $\mathscr{C}_{\overline{\lambda},n}$ and $\mathscr{C}_{\overline{\lambda}',n}$ are different, so $\mathscr{C}_{\lambda'}$ is not in the fiber of $\mathscr{C}_{\overline{\lambda},n}$. □

4 Orbit poset and irreducible components of jet schemes

After obtaining a classification of the orbits of the action of $G_\infty = (\mathrm{GL}_r)_\infty \times (\mathrm{GL}_s)_\infty$ on M_∞ and D^k_∞, we start the study of their geometry. The first basic question is the following: how are these orbits positioned with respect to each other inside the arc space M_∞? We can make this precise by introducing the notion of orbit poset.

Definition 4.1 (Orbit poset)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be two G_∞-orbits in M_∞. We say that $\mathscr{C}$ *dominates* $\mathscr{C}'$, and denote it by $\mathscr{C}' \leq \mathscr{C}$, if $\mathscr{C}'$ is contained in the Zariski closure of $\mathscr{C}$. The relation of dominance defines a partial order on the set of G_∞-orbits of M_∞. The pair $(M_\infty/G_\infty, \leq)$ is known as the *orbit poset* of M_∞.

Our goal is to prove that the bijection that maps a pre-partition to its associated orbit in M_∞ is in fact an order-reversing isomorphism between Λ_r and the orbit poset. At this stage it is not hard to show that one of the directions of this bijection reverses the order.

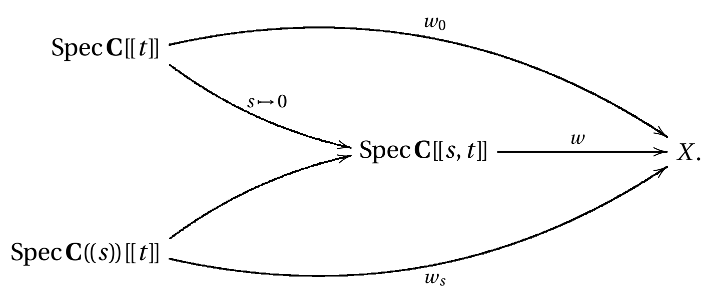
Proposition 4.2 (Domination of orbits implies co-domination of partitions)**.** *Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be two pre-partitions of length at most r, and let $\mathscr{C}_\lambda$ and $\mathscr{C}_{\lambda'}$ be the associated orbits in M_∞. If $\mathscr{C}_\lambda$ dominates $\mathscr{C}_{\lambda'}$, then λ' co-dominates λ:*

$$\mathscr{C}_\lambda \geq \mathscr{C}_{\lambda'} \quad \Longrightarrow \quad \lambda \lhd \lambda'.$$

Proof. From Proposition 3.3 we get that $\mathscr{C}_\lambda \subset \mathrm{Cont}^p(D^k)$, where $p = \lambda_r + \cdots + \lambda_{r-k}$. Since a contact locus is always Zariski closed, if $\mathscr{C}_\lambda$ dominates $\mathscr{C}_{\lambda'}$ we also know that $\mathscr{C}_{\lambda'} \subset \mathrm{Cont}^p(D^k)$. Again by Proposition 3.3, this gives $\lambda'_r + \cdots + \lambda'_{r-k} \geq p$, as required. □

We now proceed to prove the converse to Proposition 4.2. Given two pre-partitions $\lambda, \lambda' \in \overline{\Lambda}_r$ with $\lambda \lhd \lambda'$, we need to show that the closure of $\mathscr{C}_\lambda$ contains $\mathscr{C}_{\lambda'}$. We will exhibit this containment by producing a "path" in the arc space M_∞ whose general point is in $\mathscr{C}_\lambda$ but specializes to a point in $\mathscr{C}_{\lambda'}$. These types of "paths" are known as *wedges.*

Definition 4.3 (Wedge)**.** Let X be a scheme over $\mathbf{C}$. A *wedge* w on X is a morphism of schemes $w : \operatorname{Spec} \mathbf{C}[[s,t]] \to X$. Given a wedge w, one can consider the diagram



The map w_0 is known as the *special arc* of w, and w_s as the *generic arc* of w.

Lemma 4.4. *Let w be a wedge on M and let $\mathscr{C}_0$ be the G_∞-orbit in M_∞ of the special arc w_0 of w. Assume that there is a G_∞-orbit $\mathscr{C}_s$ in M_∞ containing the generic arc w_s of w. Then $\mathscr{C}_s$ dominates $\mathscr{C}_0$.*

Proof. From the hypothesis, the closure of $\mathscr{C}_s$ contains w_0 (because w_0 is in the closure of w_s). But the closure of an orbit is invariant, so $\overline{\mathscr{C}_s}$ must contain $\mathscr{C}_0 = G_\infty \cdot w_0$. □

Lemma 4.5. *Let λ and μ be two pre-partitions in $\overline{\Lambda}_r$ and assume that λ is obtained from μ via a removal (simple or infinite). Then $\mathscr{C}_\lambda$ dominates $\mathscr{C}_\mu$.*

Proof. Let i be the index such that $\lambda_i < \mu_i$, and consider the following wedge on M:

$$w = \begin{pmatrix} 0 &&&&&& \\ & \ddots &&&&& \\ && 0 &&&& \\ &&& st^{\lambda_i} + t^{\mu_i} &&& \\ &&&& t^{\lambda_{i+1}} && \\ &&&&& \ddots & \\ &&&&&& t^{\lambda_r} \end{pmatrix}.$$

The special arc of w is δ_μ. The generic arc w_s only differs from δ_λ by the presence of the unit $s + t^{\mu_i - \lambda_i}$ on row i. Therefore w_s and δ_λ have the same contact with respect to all the determinantal varieties, and the proof of Proposition 3.4 shows that w_s is contained in $\mathscr{C}_\lambda$. Now we can apply Lemma 4.4 with $\mathscr{C}_0 = \mathscr{C}_\mu$ and $\mathscr{C}_s = \mathscr{C}_\lambda$, and the result follows. □

Lemma 4.6. *Let λ and μ be two pre-partitions in $\overline{\Lambda}_r$ and assume that λ is obtained from μ via a slip or a fall. Then $\mathscr{C}_\lambda$ dominates $\mathscr{C}_\mu$.*

Proof. Let $i < j$ be the indices such that $\lambda_i = \mu_i + 1$, $\lambda_j = \mu_j - 1$ and $\mu_k = \lambda_k$ for $k \neq i, j$. Consider the following wedge:

$$w = \begin{pmatrix} t^{\lambda_1} &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && t^{\lambda_{i-1}} &&&&&&&& \\ &&& \alpha & 0 & \cdots & 0 & \beta &&& \\ &&& 0 & t^{\lambda_{i+1}} & \cdots & 0 & 0 &&& \\ &&& \vdots & \vdots & \ddots & \vdots & \vdots &&& \\ &&& 0 & 0 & \cdots & t^{\lambda_{j-1}} & 0 &&& \\ &&& \gamma & 0 & \cdots & 0 & \delta &&& \\ &&&&&&&& t^{\lambda_{j+1}} && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& t^{\lambda_r} \end{pmatrix}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} st^{\lambda_i} + t^{\lambda_i - 1} & t^{\lambda_i - 1} \\ st^{\lambda_j} & st^{\lambda_j} + t^{\lambda_j + 1} \end{pmatrix}.$$

Notice that

$$\operatorname{ord}_t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \lambda_j, \qquad \operatorname{ord}_t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\Bigg|_{s=0} = \lambda_j + 1, \qquad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = t^{\lambda_i + \lambda_j}\left(1 + st + s^2\right).$$

From these equations, we see that w_0 and δ_μ have the same order of contact with respect to all determinantal varieties, and the proof of Proposition 3.4 tells us that w_0 is contained in $\mathscr{C}_\mu$. Analogously, the equations above show that w_s is contained in $\mathscr{C}_\lambda$. Lemma 4.4 gives the result. □

Theorem 4.7 (Orbit poset = Pre-partition poset)**.** *The map that sends a pre-partition $\lambda \in \overline{\Lambda}_r$ of length at most r to the associated orbit $\mathscr{C}_\lambda$ in M_∞ is an order-reversing isomorphism between $\overline{\Lambda}_r$ and the orbit poset:*

$$\mathscr{C}_\lambda \geq \mathscr{C}_\mu \iff \lambda \lhd \mu \iff \lambda_{r-i} + \cdots + \lambda_r \leq \mu_{r-i} + \cdots + \mu_r \quad \forall i.$$

Proof. The theorem follows from Proposition 4.2, Lemma 4.5, Lemma 4.6, and Theorem 2.1. ☐

In the remainder of the section we use Theorem 4.7 to compute the number of irreducible components of the jet schemes of generic determinantal varieties.

Notation 4.8. As it is customary in the theory of partitions, we write $\lambda = (d_1^{a_1} \dots d_j^{a_j})$ to denote the pre-partition that has a_i copies of d_i. For example $(\infty, \infty, 5, 3, 3, 3, 2, 1, 1) = (\infty^2\, 5^1\, 3^3\, 2^1\, 1^2)$.

Proposition 4.9. *Recall that $D^k \subset M$ denotes the determinantal variety of matrices of size $r \times s$ and rank at most k. Assume that $0 < k < r-1$, and let $\mathscr{C}$ be an irreducible component of $\mathrm{Cont}^p(D^k) \subset M_\infty$. Then $\mathscr{C}$ contains a unique dense G_∞-orbit $\mathscr{C}_\lambda$. Moreover, λ is a partition (contains no infinite terms) and $\lambda = (d^{a+r-k}\, e^1)$ where*

$$p = (a+1)\,d + e, \qquad 0 \leq e < d,$$

and either $e = 0$ and $0 \leq a \leq k$ or $e > 0$ and $0 \leq a < k$. Conversely, for any partition as above, its associated orbit is dense in an irreducible component of $\mathrm{Cont}^p(D^k)$.

Example 4.10. When $r = 8$, $k = 6$, and $p = 5$, the partitions given by the proposition are

$$(5,5), \quad (4,4,1), \quad (3,3,2), \quad (2,2,2,1), \quad (1,1,1,1,1,1).$$

When $r = 5$, $k = 3$, and $p = 5$, we only get

$$(5,5), \quad (4,4,1), \quad (3,3,2), \quad (2,2,2,1).$$

Proof. By Theorem 4.7 and Proposition 3.3, computing the irreducible components of $\mathrm{Cont}^p(D^k)$ is equivalent to computing the minimal elements (with respect to the order of co-domination) among all pre-partitions $\lambda \in \overline{\Lambda}_r$ such that

$$\lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-k} \geq p.$$

Let Σ be the set of such partitions. To find minimal elements in Σ it will be useful to keep in mind the structure of the adjacencies in $\overline{\Lambda}_r$ discussed in Section 2.4

First notice that all minimal elements in Σ must be partitions. Indeed, given an element $\lambda \in \Sigma$, truncating all infinite terms of λ to a high enough number produces another element of Σ. Moreover, if $\lambda \in \Sigma$ is minimal, we must have $\lambda_1 = \lambda_2 = \cdots = \lambda_{r-k}$. If this were not the case, we could consider the partition λ' such that $\lambda'_1 = \cdots = \lambda'_{r-k} = \lambda_{r-k}$, and $\lambda'_i = \lambda_i$ for $i > r-k$. Then λ' would also be in Σ, but $\lambda' \lhd \lambda$, contradicting the fact that λ is minimal. It is also clear that minimal elements of Σ must verify $\lambda_{r-k} + \cdots + \lambda_r = p$. In fact, if a partition in Σ does not verify this, we can decrease the last terms of the partition an still remain in Σ.

So far we know that the minimal elements in Σ are partitions that verify $\lambda_1 = \cdots = \lambda_{r-k}$ and $\lambda_{r-k} + \cdots + \lambda_r = p$. Note that we assume $0 < k < r-1$, so for any two partitions λ, λ' with the previous properties, if $\lambda_{r-k} \neq \lambda'_{r-k}$, then λ and λ' are not comparable.

Pick a minimal element $\lambda \in \Sigma$, and write $d = \lambda_{r-k}$. Let ℓ be the length of λ. The proposition will follow if we show that the sequence $(\lambda_{r-k}, \lambda_{r-k+1}, \dots, \lambda_\ell)$ is of the form $(d, \dots, d, e)$ for some $0 \leq e < d$. But this is clear from the analysis of the adjacencies in $\overline{\Lambda}_r$ given in Section 2.4. Consider the Young diagram Γ

associated to λ. The longest row of Γ has length d. If there are two rows, say $i < j$, with length less than d, then we must have $r - k < i$ and we can move one box from row j to row i (via a sequence of falls and slips) and obtain a partition still in Σ but co-dominated by λ. This contradicts the fact that λ is minimal, and we see that λ must have the form given in the proposition. □

Proposition 4.11. *Assume that $k = 0$ or $k = r - 1$. Then $\mathrm{Cont}^p(D^k) \subset M_\infty$ is irreducible and contains a unique dense orbit $\mathscr{C}_\lambda$, where $\lambda = (p^{r-k})$.*

Proof. Form Proposition 3.3, the orbits $\mathscr{C}_\lambda$ contained in $\mathrm{Cont}^p(D^0)$ are the ones that verify $\lambda_r \geq p$. It is clear that the minimal partition of this type is (p^r). Analogously, $\mathrm{Cont}^p(D^{r-1})$ contains orbits whose associated partitions verify $\lambda_1 + \cdots + \lambda_r \geq p$, and the minimal one among these is (p^1). □

Theorem 4.12. *If $k = 0$ or $k = r - 1$, the contact locus $\mathrm{Cont}^p(D^k) \subset M_\infty$ is irreducible. Otherwise, the number of irreducible components of $\mathrm{Cont}^p(D^k) \subset M_\infty$ is*

$$p + 1 - \left\lceil \frac{p}{k+1} \right\rceil.$$

Proof. The first assertion follows directly from Proposition 4.11. For the second one, we need to count the number of partitions that appear in Proposition 4.9. Recall that these were partitions of the form $\lambda^d = (d^{a+r-k}, e^1)$ of length at most r such that $p = (a+1)d + e$ and $0 \leq e < d$. Since d ranges from 0 to p, we have at most $p + 1$ such partitions. But as we decrease d, the length of λ^d increases, possibly surpassing the limit r. Therefore the number of allowed partitions is $p + 1 - d_0$, where d_0 is the smallest integer such that λ^{d_0} has length no greater than r.

If d divides p, the length of λ^d is $(\frac{p}{d} - 1 + r - k)$. Otherwise it is $(\lfloor \frac{p}{d} \rfloor + r - k)$. In either case, the length is no greater that r if and only if $d \geq \lceil \frac{p}{k+1} \rceil$. Hence $d_0 = \lceil \frac{p}{k+1} \rceil$, and the theorem follows. □

Corollary 4.13. *It $k = 0$ or $k = r - 1$, the jet scheme D^k_n is irreducible. Otherwise, the number of irreducible components of D^k_n is*

$$n + 2 - \left\lceil \frac{n+1}{k+1} \right\rceil.$$

Proof. The contact locus $\mathrm{Cont}^{n+1}(D^k)$ is the inverse image of the jet scheme D^k_n under the truncation map $M_\infty \to M_n$. Since M is smooth, this truncation map is surjective, so D^k_n has the same number of components as $\mathrm{Cont}^{n+1}(D^k)$. Now the result follows directly from Theorem 4.12. □

5 Discrepancies and log canonical thresholds

In this section we compute discrepancies for all invariant divisorial valuations over M and over D^k, and use it to give formulas for log canonical thresholds involving determinantal varieties. We start with a proposition that determines all possible invariant maximal divisorial sets in terms of orbits in the arc space.

Proposition 5.1 (Divisorial sets = Orbit closures, M)**.** *Let ν be a G-invariant divisorial valuation over M, and let $\mathscr{C}$ be the associated maximal divisorial set in M_∞. Then there exists a unique partition $\lambda \in \Lambda_r$ of length at most r whose associated orbit $\mathscr{C}_\lambda$ is dense in $\mathscr{C}$. Conversely, the closure of $\mathscr{C}_\lambda$, where λ is a partition, is a maximal divisorial set associated to an invariant valuation.*

Proof. Recall from Section 1.2 (or see [Ish08]) that $\mathscr{C}$ is the union of the fat sets of M_∞ that induce the valuation v. Therefore, since v is G-invariant, $\mathscr{C}$ is G_∞-invariant and can be written as a union of orbits. Note that the thin orbits of M_∞ are all contained in D_∞^{r-1}, and that $\mathscr{C}$ is itself fat, so $\mathscr{C}$ must contain a fat orbit. Let $\Sigma \subset \Lambda_r$ be the set of partitions indexing fat orbits contained in $\mathscr{C}$. For $\mu \in \Sigma$ we denote by v_μ the valuation induced by $\mathscr{C}_\mu$. Then, for $f \in \mathscr{O}_M$ we have:

$$v(f) \quad = \quad \min_{\gamma \in \mathscr{C}} \{\mathrm{ord}_\gamma(f)\} \quad = \quad \min_{\mu \in \Sigma} \min_{\gamma \in \mathscr{C}_\mu} \{\mathrm{ord}_\gamma(f)\} \quad = \quad \min_{\mu \in \Sigma} \{v_\mu(f)\}.$$

As a consequence, since v_μ is determined by its value on the ideals $\mathscr{I}_{D^0},\ldots,\mathscr{I}_{D^{r-1}}$, the same property holds for v. Let $\lambda = (\lambda_1,\ldots,\lambda_r)$ be such that $v(\mathscr{I}_{D^k}) = \lambda_r + \cdots + \lambda_{r-k}$. From the fact that $\mathscr{I}_{D^k}\mathscr{I}_{D^{k-2}} \subset \mathscr{I}^2_{D^{k-1}}$ we deduce that $\lambda_k \geq \lambda_{k+1}$, and we get a partition $\lambda \in \Lambda_r$ whose associated orbit $\mathscr{C}_\lambda$ induces the valuation v (so $\lambda \in \Sigma$). The proposition follows if we show that $\mathscr{C}_\lambda$ is dense in $\mathscr{C}$.

Consider $\mu \in \Sigma$. Since $\mathscr{C}_\mu \subset \mathscr{C}$, we know that $v_\mu \geq v$, and we get that

$$\mu_r + \cdots + \mu_{r-k} \;=\; v_\mu(\mathscr{I}_{D^k}) \;\geq\; v(\mathscr{I}_{D^k}) \;=\; \lambda_r + \cdots + \lambda_{r-k}.$$

Hence $\lambda \lhd \mu$, and Theorem 4.7 tells us that $\mathscr{C}_\mu$ is contained in the closure of $\mathscr{C}_\lambda$, as required. □

Notation 5.2. For the purpose of the next proposition it will be convenient to introduce the following notation. Fix positive integers $k < r$. Given a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell) \in \Lambda_k$ of length at most k, denote by $\lambda^+ = (\infty,\ldots,\infty,\lambda_1,\ldots,\lambda_\ell) \in \overline{\Lambda}_r$ the pre-partition obtained by adjoining $r-k$ infinities.

Proposition 5.3 (Divisorial sets = Orbit closures, D^k)**.** *Let v be a G-invariant divisorial valuation over D^k, and let $\mathscr{C}$ be the associated maximal divisorial set in D^k_∞. Then there exists a unique partition $\lambda \in \Lambda_k$ such that the orbit $\mathscr{C}_{\lambda^+}$ is dense in $\mathscr{C}$. Conversely, the closure of $\mathscr{C}_{\lambda^+}$, where $\lambda \in \Lambda_k$, is a maximal divisorial set in D^k_∞ associated to a G-invariant divisorial valuation.*

Proof. Analogous to the proof of 5.1. □

We now proceed to compute discrepancies for invariant divisorial valuations. These are closely related to the codimensions of the corresponding maximal divisorial sets, which by the previous propositions are just given by orbit closures. Since orbits are cylinders, their codimension can be computed by looking at the corresponding orbit in a high enough jet scheme. But jet schemes are of finite type, so orbits have a finite dimension that can be computed via the codimension of the corresponding stabilizer. For this reason, we will try to understand the structure of the different stabilizers in the jet schemes G_n.

Recall from Definition 3.1 that $\mathscr{C}_\lambda$ is the orbit containing the following matrix:

$$\delta_\lambda = \begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & t^{\lambda_2} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & t^{\lambda_r} \end{pmatrix}.$$

This matrix defines an element of the jet scheme M_n as long as n is greater than the co-length of λ; the corresponding G_n-orbit in M_n is denoted by $\mathscr{C}_{\lambda,n}$. The following proposition determines the codimension of the stabilizer of δ_λ in the jet group G_n.

Proposition 5.4. *Let $\lambda \in \Lambda_r$ be a partition of length at most r, and let n be a positive integer greater than the highest term of λ. Let $H_{\lambda,n}$ denote the stabilizer of δ_λ in the group G_n. Then*

$$\mathrm{codim}(H_{\lambda,n}, G_n) = (n+1)rs - \sum_{i=1}^{r} \lambda_i(s - r + 2i - 1).$$

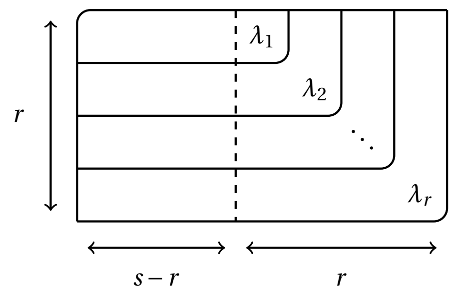
Proof. Pick $(g,h) \in G_n = (\mathrm{GL}_r)_n \times (\mathrm{GL}_s)_n$. Then:

$$(g,h) \in H_{\lambda,n} \iff g\cdot\delta_\lambda\cdot h^{-1} = \delta_\lambda \iff g\cdot\delta_\lambda = \delta_\lambda\cdot h \iff$$

$$\begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ 0 & \cdots & 0 & t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \end{pmatrix} = \begin{pmatrix} t^{\lambda_1}* & \cdots & t^{\lambda_1}* & t^{\lambda_1}* & t^{\lambda_1}* & \cdots & t^{\lambda_1}* \\ t^{\lambda_2}* & \cdots & t^{\lambda_2}* & t^{\lambda_2}* & t^{\lambda_2}* & \cdots & t^{\lambda_2}* \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_r}* & \cdots & t^{\lambda_r}* & t^{\lambda_r}* & t^{\lambda_r}* & \cdots & t^{\lambda_r}* \end{pmatrix}.$$

This equality of matrices gives one equation of the form $t^{a(i,j)}* = t^{b(i,j)}*$ for each entry (i,j) in a $r\times s$ matrix. We have $a(i,j) = \lambda_{j-s+r}$ and $b(i,j) = \lambda_i$ (assume $\lambda_j = \infty$ for $j<0$).

Each equation of the form $t^a* = t^b*$ gives $(n+1) - \min\{a,b\}$ independent equations on the coefficients of the power series, so it reduces the dimension of the stabilizer by $(n+1)-\min\{a,b\}$. The entries (i,j) for which $\min\{a(i,j), b(i,j)\} = \lambda_k$ form an ⌟-shaped region of the $r\times s$ matrix, as we illustrate in the following diagram:



The region corresponding to λ_i contains $(s-r+2i-1)$ entries, and the result follows. □

Proposition 5.5. *Let $\lambda \in \overline{\Lambda}_r$ be a pre-partition of length at most r, and consider its associated G_∞-orbit $\mathscr{C}_\lambda$ in M_∞. If λ contains infinite terms, $\mathscr{C}_\lambda$ has infinite codimension. If λ is a partition, the codimension is given by:*

$$\operatorname{codim}(\mathscr{C}_\lambda, M_\infty) = \sum_{i=1}^r \lambda_i(s-r+2i-1).$$

Proof. If λ contains infinite terms, $\mathscr{C}_\lambda$ is thin, so it has infinite codimension. Otherwise Proposition 3.9 tells us that $\mathscr{C}_\lambda$ is the inverse image of $\mathscr{C}_{\lambda,n}$ under the truncation map $M_\infty \to M_n$ for n large enough. Since M is smooth, we see that the codimension of $\mathscr{C}_\lambda$ in M_∞ is the same as the codimension of $\mathscr{C}_{\lambda,n}$ in M_n. The dimension of $\mathscr{C}_{\lambda,n}$ is the codimension of the stabilizer of δ_λ in G_n. The result now follows from Proposition 5.4 and the fact that M_n has dimension $(n+1)rs$. □

Corollary 5.6. *Let ν be a G-invariant valuation of M and let $\lambda \in \Lambda_r$ be the unique partition such that $\mathscr{C}_\lambda$ induces ν. Let $k_\nu(M)$ be the discrepancy of M along ν, and let q_ν be the multiplicity of ν. Then*

$$k_\nu(M) + q_\nu = \sum_{i=1}^r \lambda_i(s-r+2i-1).$$

Proof. From Proposition 5.1 we know that the closure of $\mathscr{C}_\lambda$ is the maximal divisorial set associated to ν. Since M is smooth, the log discrepancy $k_\nu(M)+q_\nu$ agrees with the codimension of the associated maximal divisorial set (see Section 1.3). The result now follows from Proposition 5.5. □

Theorem 5.7. *Recall that M denotes the space of matrices of size $r \times s$ and D^k is the variety of matrices of rank at most k. The log canonical threshold of the pair (M, D^k) is*

$$\operatorname{lct}(M, D^k) = \min_{i=0\dots k} \frac{(r-i)(s-i)}{k+1-i}.$$

Proof. We will use Mustaţă's formula (see [ELM04, Cor. 3.2]) to compute log canonical thresholds:

$$\operatorname{lct}(M, D^k) = \min_n \left\{ \frac{\operatorname{codim}(D^k_n, M_n)}{n+1} \right\} = \min_p \left\{ \frac{\operatorname{codim}(\operatorname{Cont}^p(D^k), M_\infty)}{p} \right\}.$$

Let $\Sigma_p \subset \overline{\Lambda}_r$ be the set of pre-partitions of length at most r such that $\lambda_r + \cdots + \lambda_{r-k} = p$. By Propositions 3.3 and 4.9, we have:

$$\operatorname{lct}(M, D^k) = \min_p \min_{\lambda \in \Sigma_p} \left\{ \frac{\operatorname{codim}(\mathscr{C}_\lambda, M_\infty)}{p} \right\}.$$

Consider the following linear function

$$\psi(a_1, \dots, a_r) = \sum_{i=1}^r a_i (s - r + 2i - 1).$$

Then, by Proposition 5.5, we get:

$$\operatorname{lct}(M, D^k) = \min_p \min_{\lambda \in \Sigma_p} \left\{ \frac{\psi(\lambda)}{p} \right\} = \min_p \min_{\lambda \in \Sigma_p} \left\{ \psi\left(\frac{\lambda}{p}\right) \right\}.$$

Let $\Sigma \subset \mathbf{Q}^r$ be the set of tuples $(a_1, \dots, a_r)$ such that $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ and $a_r + \cdots + a_{r-k} = 1$. Then:

$$\operatorname{lct}(M, D^k) = \min_{a \in \Sigma} \left\{ \psi(a) \right\}.$$

The map $\varphi(a_1, \dots, a_r) = (a_1 - a_2, \dots, a_{r-1} - a_r, a_r)$ sends Σ to Σ', where $\Sigma' \subset \mathbf{Q}^r$ is the set of tuples $(b_1, \dots, b_r)$ such that $b_i \geq 0$ and $(k+1) b_r + k b_{r-1} \cdots + b_{r-k} = 1$. Then

$$\operatorname{lct}(M, D^k) = \min_{b \in \Sigma'} \{\xi(b)\},$$

where

$$\xi(b) = \psi(\varphi^{-1}(b)) = \sum_{i=1}^r (b_r + b_{r-1} + \cdots + b_i)(s - r + 2i - 1) = \sum_{j=1}^r b_j\, j\,(s - r + j).$$

Note that in the definition of Σ' the only restriction on the first $r-k-1$ coordinates $b_1, b_2, \dots, b_{r-k-1}$ is that they are nonnegative. Let Σ'' be the subset of Σ' obtained by setting $b_1 = \cdots = b_{r-k-1} = 0$. From the formula for $\xi(b)$ we see that the minimum $\min_{b \in \Sigma'} \{\xi(b)\}$ must be achieved in Σ''. But Σ'' is a simplex and ξ is linear, so the minimum is actually achieved in one of the extremal points of Σ''. These extremal points are:

$$P_{r-k} = (0, \dots, 0, 1, 0, \dots, 0, 0), \quad P_{r-k+1} = (0, \dots, 0, 0, \tfrac{1}{2}, \dots, 0, 0), \quad \dots$$
$$\dots \quad P_{r-1} = (0, \dots, 0, 0, 0, \dots, \tfrac{1}{k}, 0), \quad P_r = (0, \dots, 0, 0, 0, \dots, 0, \tfrac{1}{k+1}).$$

The value of ξ at these points is:

$$\xi(P_{r-i}) = \frac{1}{k+1-i} (r-i)(s-i).$$

Therefore

$$\operatorname{lct}(M, D^k) = \min_{i=0\dots k} \frac{(r-i)(s-i)}{k+1-i},$$

as required. □

6 Motivic integration

In the previous section we computed codimensions of orbits in the arc space M_∞, as a mean to obtain formulas for discrepancies and log canonical thresholds. But a careful look at the proofs shows that we can understand more about the orbits than just their codimensions. As an example of this, in this section we compute the motivic volume of the orbits in the arc space. This allows us to determine topological zeta functions of determinantal varieties.

Throughout this section, we will restrict ourselves to the case of square matrices, i.e. we assume $r = s$.

6.1 Motivic volume of orbits

Before we state the main proposition, we need to recall some notions from the group theory of GL_r: parabolic subgroups, Levi factors, flag manifolds, and the natural way to obtain a parabolic subgroup from a partition.

Definition 6.1. Let $0 < v_1 < v_2 < \cdots < v_j < r$ be integers. A *flag* in $\mathbf{C}^r$ of signature $(v_1, \ldots, v_j)$ is a nested chain $V_1 \subset V_2 \subset \cdots \subset V_j \subset \mathbf{C}^r$ of vector subspaces with $\dim V_i = v_i$. The general linear group GL_r acts transitively on the set of all flags with a given signature. The stabilizer of a flag is known as a *parabolic subgroup* of GL_r. If $P \subset \mathrm{GL}_r$ is a parabolic subgroup, the quotient GL_r / P parametrizes flags of a given signature and it is known as a *flag variety*.

Definition 6.2. Let $\{e_1, \ldots, e_r\}$ be the standard basis for $\mathbf{C}^r$, and let $\lambda = (d_1^{a_1} \ldots d_j^{a_j}) \in \Lambda_r$ be a partition. Write $a_{j+1} = r - \sum_{i=1}^{j} a_i$ and $v_i = a_1 + \cdots + a_i$, and consider the following vector subspaces of $\mathbf{C}^r$:

$$V_i = \mathrm{span}(e_1, \ldots, e_{v_i}), \qquad W_i = \mathrm{span}(e_{v_{i-1}+1}, \ldots, e_{v_i}).$$

We denote by P_λ the stabilizer of the flag $V_1 \subset \cdots \subset V_j$ and call it the *parabolic subgroup of* GL_r *associated to* λ. The group $L_\lambda = \mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_{j+1}}$ embeds naturally in P_λ as the group endomorphisms of W_i, and it is known as the *Levi factor* of the parabolic P_λ.

Example 6.3. Assume $r = 6$ and consider the partition $\lambda = (4,4,4,1,1) = (4^3 1^2)$. Then P_λ and L_λ are the groups of invertible $r \times r$ matrices of the forms

$$P_\lambda : \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \qquad L_\lambda : \begin{pmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Proposition 6.4. *Assume that $r = s$. Let $\lambda \in \Lambda_r$ be a partition of length at most r and consider its associated parabolic subgroup P_λ and Levi factor L_λ. Let μ be the motivic measure in M_∞, and $\mathscr{C}_\lambda$ the orbit in M_∞ associated to λ. If b is the log discrepancy of the valuation induced by $\mathscr{C}_\lambda$, we have:*

$$\mu(\mathscr{C}_\lambda) \quad = \quad \mathbf{L}^{-b}\, [\mathrm{GL}_r / P_\lambda]^2\, [L_\lambda].$$

Proof. Consider n, δ_λ and $H_{\lambda,n} \subset G_n$ as in Proposition 5.4. If $\mathscr{C}_{\lambda,n}$ is the truncation of $\mathscr{C}_\lambda$ to M_n, we know that for n large enough

$$\mu(\mathscr{C}_\lambda) \;=\; \mathbf{L}^{-r^2 n}\, [\mathscr{C}_{\lambda,n}] \;=\; \mathbf{L}^{-r^2 n}\, [G_n]\, [H_{\lambda,n}]^{-1} \;=\; \mathbf{L}^{r^2 n}\, [\mathrm{GL}_r]^2\, [H_{\lambda,n}]^{-1}.$$

At the beginning of the proof of Proposition 5.4 we found the equations defining $H_{\lambda,n}$:

$$(g,h)\in H_{\lambda,n} \quad\Leftrightarrow\quad g\cdot\delta_\lambda\cdot h^{-1}=\delta_\lambda \quad\Leftrightarrow\quad g\cdot\delta_\lambda=\delta_\lambda\cdot h \quad\Leftrightarrow$$

$$\begin{pmatrix} t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \end{pmatrix} = \begin{pmatrix} t^{\lambda_1}* & t^{\lambda_1}* & \cdots & t^{\lambda_1}* \\ t^{\lambda_2}* & t^{\lambda_2}* & \cdots & t^{\lambda_2}* \\ \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_r}* & t^{\lambda_r}* & \cdots & t^{\lambda_r}* \end{pmatrix}. \tag{1}$$

As a variety, G_n can be written as product $G\times\mathfrak{g}^{2n}$, where $\mathfrak{g}\simeq\mathbf{A}^{r^2}$ is the Lie algebra of GL_r. Let $g_{i,j}^{(k)}$ and $h_{i,j}^{(k)}$ be the natural coordinates on $G_n=(\mathrm{GL}_r\times\mathfrak{g})^2$, where $g_{i,j}^{(0)}=g_{i,j}$ and $h_{i,j}^{(0)}=h_{i,j}$ are coordinates for $G=\mathrm{GL}_r\times\mathrm{GL}_r$. Then, for n large enough, the equations in (1) can be expressed as:

$$t^{\lambda_j}\sum_{k=0}^{n} g_{i,j}^{(k)}t^k \quad=\quad t^{\lambda_i}\sum_{k=0}^{n} h_{i,j}^{(k)}t^k \qquad \mod t^{n+1}. \tag{2}$$

Let $H\subset G$ be the truncation of $H_{\lambda,n}$. Then H is the subgroup of G given by those equation in (2) involving only the variables $g_{i,j}$ and $h_{i,j}$; these equations are:

$$g_{i,j}=h_{i,j} \qquad \text{if } \lambda_i=\lambda_j, \tag{3}$$

$$g_{i,j}=0 \qquad \text{if } \lambda_i\neq\lambda_j \text{ and } i<j, \tag{4}$$

$$h_{i,j}=0 \qquad \text{if } \lambda_i\neq\lambda_j \text{ and } i>j. \tag{5}$$

Form (4) and (5), we see that H is a subgroup of $P_\lambda^{\mathrm{op}}\times P_\lambda\subset G$, and (3) tells us that we can obtain H from $P_\lambda^{\mathrm{op}}\times P_\lambda$ by identifying the two copies of the Levi L_λ. Hence $[H]=[P_\lambda]^2\,[L_\lambda]^{-1}$.

From (2) we also see that $H_{\lambda,n}$ is a sub-bundle of $H\times\mathfrak{g}^{2n}$. More precisely, if $\mathfrak{h}$ is the fiber of $H_{\lambda,n}$ over the identity in H, then $\mathfrak{h}\subset\mathfrak{g}^{2n}$ is an affine space and all the fibers of $H_{\lambda,n}$ are isomorphic to $\mathfrak{h}$. The codimension of $\mathfrak{h}$ in $\mathfrak{g}^{2n}$ can be computed with the same method used in the proof of Proposition 5.4:

$$\mathrm{codim}(\mathfrak{h},\mathfrak{g}^{2n})=nr^2-\sum_{i=1}^{r}\lambda_i(2i-1)=nr^2-b,$$

where b is the log discrepancy of the valuation induced by $\mathscr{C}_\lambda$. As a consequence

$$[\mathfrak{h}] \quad=\quad [\mathfrak{g}^{2n}]\,\mathbf{L}^{-nr^2+b} \quad=\quad \mathbf{L}^{nr^2+b},$$

and

$$\begin{aligned}\mu(\mathscr{C}_\lambda) &= \mathbf{L}^{r^2n}\,[\mathrm{GL}_r]^2\,[H_{\lambda,n}]^{-1} = \mathbf{L}^{r^2n}\,[\mathrm{GL}_r]^2\,[H]^{-1}\,[\mathfrak{h}]^{-1} = \mathbf{L}^{-b}\,[\mathrm{GL}_r]^2\,[H]^{-1}\\ &= \mathbf{L}^{-b}\,[\mathrm{GL}_r]^2\,[P_\lambda]^{-2}\,[L_\lambda] = \mathbf{L}^{-b}\,[\mathrm{GL}_r/P_\lambda]^2\,[L_\lambda].\end{aligned}$$

□

6.2 Topological zeta function

Recall from Section 1.4 that the motivic Igusa zeta function for the pair (M,D^k) is defined as

$$Z_{D^k}(s):=\int_{M_\infty}|\mathscr{I}_{D^k}|^s d\mu=\sum_{p=0}^{\infty}\mu\left(\mathrm{Cont}^{=p}D^k\right)\mathbf{L}^{-sp},$$

where μ is the motivic measure on M_∞ and $\mathbf{L}^{-s}$ is considered as a formal variable. The topological zeta function $Z^{\mathrm{top}}_{D^k}(s)$ can be obtained from the Igusa zeta function by formally expanding $Z_{D^k}(s)$ as a power series in $(\mathbf{L}-1)$ and then extracting the constant term (i.e. by specializing $\mathbf{L}$ to $\chi(\mathbf{A}^1)=1$).

Using Propositions 3.3 and 6.4 we can write $Z_{D^k}(s)$ as

$$Z_{D^k}(s) = \sum_{\lambda\in\Lambda_r} \mu(\mathscr{C}_\lambda)\mathbf{L}^{-s(\lambda_r+\cdots+\lambda_{r-k})} = \sum_{\lambda\in\Lambda_r} [\mathrm{GL}_r/P_\lambda]^2[L_\lambda]\mathbf{L}^{-b_\lambda-s(\lambda_r+\cdots+\lambda_{r-k})}, \tag{6}$$

where $b_\lambda = \sum_{i=1}^r \lambda_i(2i-1)$ is the log discrepancy of the valuation induced by $\mathscr{C}_\lambda$. There are only finitely many possibilities for the value of $[\mathrm{GL}_r/P_\lambda]^2[L_\lambda]$, and it will be convenient to group the terms in the sum above accordingly. In order to do so, consider the bijection between Λ_r and $\mathbf{N}^r$ given by

$$\begin{aligned} \lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\in\Lambda_r &\quad\longmapsto\quad a(\lambda)=(\lambda_1-\lambda_2,\lambda_2-\lambda_3,\dots,\lambda_r)\in\mathbf{N}^r,\\ a=(a_1,a_2,\dots,a_r)\in\mathbf{N}^r &\quad\longmapsto\quad \lambda(a)=(a_1+\cdots+a_r,a_2+\cdots+a_r,\dots,a_r)\in\Lambda_r. \end{aligned}$$

For a subset $I\subseteq\{1,\dots,r-1\}$, consider $I^c=\{1,\dots,r-1\}\setminus I$ and define

$$\Omega_I = \{\, a\in\mathbf{N}^r \;:\; (a_i=0\ \forall i\in I) \text{ and } (a_j\neq 0\ \forall j\in I^c)\,\}.$$

Let λ and λ' be two partitions such that both $a(\lambda)$ and $a(\lambda')$ belong to Ω_I for some I. From the definitions of P_λ and L_λ (see Section 6.1) we see that $[\mathrm{GL}_r/P_\lambda]^2[L_\lambda]=[\mathrm{GL}_r/P_{\lambda'}]^2[L_{\lambda'}]$. Hence, given a subset $I\subseteq\{1,\dots,r-1\}$ we can consider $\eta(I)=[\mathrm{GL}_r/P_\lambda]^2[L_\lambda]$, where λ is any partition with $a(\lambda)\in\Omega_I$, and we obtain a well-defined function on the subsets of $\{1,\dots,r-1\}$.

Fix a subset $I\subseteq\{1,\dots,r-1\}$ and a partition λ such that $a(\lambda)\in\Omega_I$. Consider $I_r^c=\{1,\dots,r\}\setminus I=\{i_1,\dots,i_\ell\}$, where $i_j<i_{j+1}$. Set $i_0=0$. Then GL_r/P_λ is the manifold of partial flags of signature $(i_1,\dots,i_\ell)$, and its class in the Grothendieck group of varieties is given by

$$[\mathrm{GL}_r/P_\lambda] = \prod_{j=1}^{\ell}[G(i_{j-1},i_j)] = \prod_{j=1}^{\ell}[G(i_j-i_{j-1},i_j)],$$

where $G(u,v)$ is the Grassmannian of u-dimensional vector subspaces of $\mathbf{C}^v$. Analogously:

$$[L_\lambda] = \prod_{j=1}^{\ell}[\mathrm{GL}_{i_j-i_{j-1}}].$$

If we define $d(I,i_j)=i_j-i_{j-1}$ for $i_j\in I_r^c$, we can write:

$$\eta(I) = [\mathrm{GL}_r/P_\lambda]^2[L_\lambda] = \prod_{i\notin I}[G(d(I,i),i)]^2[\mathrm{GL}_{d(I,i)}]. \tag{7}$$

This shows more explicitly that $\eta(I)$ depends only on I, and not on the particular partition λ in Ω_I. From Equation (6) we obtain:

$$Z_{D^k}(s) = \sum_{I\subseteq\{1,\dots,r-1\}}\left(\eta(I)\sum_{a(\lambda)\in\Omega_I}\mathbf{L}^{-b_\lambda-s(\lambda_r+\cdots+\lambda_{r-k})}\right). \tag{8}$$

Consider

$$\psi(a) = b_{\lambda(a)} + s(\lambda(a)_r+\cdots+\lambda(a)_{r-k}) = \psi_1 a_1+\cdots+\psi_r a_r,$$

where

$$\psi_i = i^2 + s\ \max\{0, k+1+i-r\}. \tag{9}$$

Then

$$\begin{aligned}\sum_{a(\lambda)\in\Omega_I} \mathbf{L}^{-b_\lambda - s(\lambda_r+\cdots+\lambda_{r-k})} &= \sum_{a\in\Omega_I} \mathbf{L}^{-\psi_1 a_1 - \cdots - \psi_r a_r} = \sum_{a\in\Omega_I} \mathbf{L}^{-\sum_{i\notin I}\psi_i a_i} \\ &= \left(\prod_{\substack{i\notin I\\ i\neq r}} \sum_{a_i=1}^{\infty} \mathbf{L}^{-\psi_i a_i}\right)\cdot\left(\sum_{a_r=0}^{\infty} \mathbf{L}^{-\psi_r a_r}\right) = \mathbf{L}^{\psi_r}\cdot\prod_{i\notin I}\frac{\mathbf{L}^{-\psi_i}}{1-\mathbf{L}^{-\psi_i}} = \mathbf{L}^{\psi_r}\cdot\prod_{i\notin I}\frac{1}{\mathbf{L}^{\psi_i}-1}. \end{aligned} \tag{10}$$

Combining Equations (7), (8), and (10), we get:

$$Z_{D^k}(s) = \mathbf{L}^{\psi_r} \sum_{I\subseteq\{1,\ldots,r-1\}} \prod_{i\notin I} \frac{1}{\mathbf{L}^{\psi_i}-1} [G(d(I,i),i)]^2 [\mathrm{GL}_{d(I,i)}]. \tag{11}$$

We will not try to simplify Equation (11) any further. Instead, we will use it to compute the topological zeta function. For this, it is enough to expand each summand in (11) as a power series in $(\mathbf{L}-1)$. We have:

$$\begin{aligned} \mathbf{L}^{\psi_i} &= 1 + O(\mathbf{L}-1), & \mathbf{L}^{\psi_i} - 1 &= \psi_i\cdot(\mathbf{L}-1) + O((\mathbf{L}-1)^2), \\ [\mathrm{GL}_1] &= \mathbf{L}-1, & [\mathrm{GL}_d] &= O((\mathbf{L}-1)^d), \end{aligned}$$

and

$$[G(1,i)] = [\mathbf{P}^{i-1}] = 1 + \mathbf{L} + \mathbf{L}^2 + \cdots + \mathbf{L}^{i-1} = i + O(\mathbf{L}-1).$$

Hence

$$\frac{1}{\mathbf{L}^{\psi_i}-1}[G(d,i)]^2[\mathrm{GL}_d] = O((\mathbf{L}-1)^{d-1}),$$

and

$$\frac{1}{\mathbf{L}^{\psi_i}-1}[G(1,i)]^2[\mathrm{GL}_1] = \frac{i^2}{\psi_i} + O(\mathbf{L}-1).$$

In particular, the only summands in Equation (11) not divisible by $(\mathbf{L}-1)$ are those for which $d(I,i)=1$ for all $i\notin I$. Since $d(I,i)=1$ if and only if $i-1\notin I$, the only significant summand is the one corresponding to $I=\emptyset$. Hence

$$Z_{D^k}(s) = \prod_{i=1}^{r}\frac{i^2}{\psi_i} + O(\mathbf{L}-1).$$

Combining this with Equation (9) we get the topological zeta function:

$$\begin{aligned} Z^{\mathrm{top}}_{D^k}(s) &= \prod_{i=1}^{r}\frac{i^2}{\psi_i} = \prod_{i=1}^{r-k-1}\frac{i^2}{i^2}\prod_{i=r-k}^{r}\frac{i^2}{i^2+s(k+1-i-r)} \\ &= \prod_{i=r-k}^{r}\left(1+s\,\frac{k+1-i-r}{i^2}\right)^{-1} = \prod_{j=0}^{k}\left(1+s\,\frac{k+1-j}{(r-j)^2}\right)^{-1}. \end{aligned}$$

The following theorem summarizes the results of this section.

Theorem 6.5. *Let $M = \mathbf{A}^{r^2}$ be the space of square $r \times r$ matrices, and let D^k be the subvariety of matrices of rank at most k. Then the topological zeta function of the pair (M, D^k) is given by*

$$Z^{\text{top}}_{D^k}(s) = \prod_{\zeta \in \Omega} \frac{1}{1 - s\zeta^{-1}}$$

where Ω is the set of poles:

$$\Omega = \left\{ -\frac{r^2}{k+1}, \quad -\frac{(r-1)^2}{k}, \quad -\frac{(r-2)^2}{k-1}, \quad \ldots, \quad -(r-k)^2 \right\}.$$

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