CONGRUENCES FOR ANDREWS' SPT-FUNCTION MODULO POWERS OF 5, 7 AND 13

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Abstract. Congruences are found modulo powers of 5, 7 and 13 for Andrews' smallest parts partition function $\operatorname{spt}(n)$. These congruences are reminiscent of Ramanujan's partition congruences modulo powers of 5, 7 and 11. Recently, Ono proved explicit Ramanujan-type congruences for $\operatorname{spt}(n)$ modulo ℓ for all primes $\ell \geq 5$ which were conjectured earlier by the author. We extend Ono's method to handle the powers of 5, 7 and 13 congruences. We need the theory of weak Maass forms as well as certain classical modular equations for the Dedekind eta-function.

1. Introduction

Andrews [2] defined the function $\operatorname{spt}(n)$ as the number of smallest parts in the partitions of n. He related this function to the second rank moment. He also proved some surprising congruences mod 5, 7 and 13. Namely, he showed that

$$\operatorname{spt}(n) = np(n) - \frac{1}{2}N_2(n), \tag{1.1}$$

where $N_2(n)$ is the second rank moment function [3] and $p(n)$ is the number of partitions of n, and he proved that

$$\operatorname{spt}(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$

$$\operatorname{spt}(7n+5) \equiv 0 \pmod{7}, \tag{1.3}$$

$$\operatorname{spt}(13n+6) \equiv 0 \pmod{13}. \tag{1.4}$$

Bringmann [8] studied analytic, arithmetic and asymptotic properties of the generating function for the second rank moment as a quasi-weak Maass form. Further congruence properties of Andrews' spt-function were found by the author [13], Folsom and Ono [11] and Ono [19]. In particular, Ono [19] proved that if $\left(\frac{1-24n}{\ell}\right) = 1$ then

$$\operatorname{spt}(\ell^2 n - \tfrac{1}{24}(\ell^2-1)) \equiv 0 \pmod{\ell}, \tag{1.5}$$

for any prime $\ell \geq 5$. This amazing result was originally conjectured by the author(i). Earlier special cases were observed by Tina Garrett [14] and her students.

We prove some suprising congruences for $\operatorname{spt}(n)$ modulo powers of 5, 7 and 13. For a, b, $c \geq 3$,

$$\operatorname{spt}(5^a n + \delta_a) + 5\operatorname{spt}(5^{a-2}n + \delta_{a-2}) \equiv 0 \pmod{5^{2a-3}}, \tag{1.6}$$

$$\operatorname{spt}(7^b n + \lambda_b) + 7\operatorname{spt}(7^{b-2}n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2)\rfloor}}, \tag{1.7}$$

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(i)The congruence (1.5) was first conjectured by the author in a Colloquium given at the University of Newcastle, Australia on July 17, 2008.

$$\mathrm{spt}(13^c n + \gamma_c) - 13\,\mathrm{spt}(13^{c-2}n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}, \tag{1.8}$$

where δ_a, λ_b and γ_c are the least nonnegative residues of the reciprocals of 24 mod 5^a, 7^b and 13^c respectively. This together with (1.2)–(1.4) implies that

$$\mathrm{spt}(5^a n + \delta_a) \equiv 0 \pmod{5^{\lfloor \frac{a+1}{2} \rfloor}}, \tag{1.9}$$

$$\mathrm{spt}(7^b n + \lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+1}{2} \rfloor}}, \tag{1.10}$$

$$\mathrm{spt}(13^c n + \gamma_c) \equiv 0 \pmod{13^{\lfloor \frac{c+1}{2} \rfloor}}, \tag{1.11}$$

for a, b, $c \ge 1$. These congruences are reminiscent of Ramanujan's partition congruences for powers of 5, 7 and 11:

$$p(5^a n + \delta_a) \equiv 0 \pmod{5^a}, \tag{1.12}$$

$$p(7^b n + \lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+2}{2} \rfloor}}, \tag{1.13}$$

$$p(11^c n + \varphi_c) \equiv 0 \pmod{11^c}, \tag{1.14}$$

for all a, b, $c \ge 1$. Here φ_c is the reciprocal of 24 mod 11^c. The congruences mod powers of 5 and 7 were proved by Watson [22], although many of the details had been worked out earlier by Ramanujan in an unpublished manuscript. The powers of 11 congruence was proved by Atkin [6].

Following Ono [19], we define

$$\mathbf{a}(n) := 12\,\mathrm{spt}(n) + (24n-1)p(n), \tag{1.15}$$

for $n \ge 0$, and define

$$\alpha(z) := \sum_{n \ge 0} \mathbf{a}(n) q^{n - \frac{1}{24}}, \tag{1.16}$$

where as usual $q = \exp(2\pi i z)$ and $\Im(z) > 0$. We note that $\mathrm{spt}(0) = 0$ and $p(0) = 1$. Bringmann [8] showed that $\alpha(24z)$ is the holomorphic part of a weight $\frac{3}{2}$ weak Maass form. Using this observation and the idea of using the weight $\frac{3}{2}$ Hecke operator $T(\ell^2)$ to annihilate the nonholomorphic part enabled Ono [19] to prove the general congruence (1.5). We use a similar idea. Instead of a Hecke operator we use Atkin's $U(\ell)$ operator to annihilate the nonholomorphic part.

We show that

$$\mathbf{a}(5^a n + \delta_a) + 5\,\mathbf{a}(5^{a-2}n + \delta_{a-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}(5a-7) \rfloor}}, \tag{1.17}$$

$$\mathbf{a}(7^b n + \lambda_b) + 7\,\mathbf{a}(7^{b-2}n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2) \rfloor}}, \tag{1.18}$$

$$\mathbf{a}(13^c n + \gamma_c) - 13\,\mathbf{a}(13^{c-2}n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}, \tag{1.19}$$

for all a, b, $c \ge 3$. We note that (1.17) is a stronger congruence than (1.6). The congruences (1.6)–(1.7) follow from (1.17)–(1.18) and Ramanujan's partition congruences for powers of 5 and 7 that were first proved by Watson [22]. The congruence (1.8) follows easily from (1.19).

Let $\ell \ge 5$ be prime. In Section 2 we use results of Bringmann [8] to show how Atkin's $U(\ell)$ operator can be used to annihilate the nonholomorphic part of the weight $\frac{3}{2}$ weak Maass form that corresponds to the function $\alpha(24z)$, and prove that the function

$$\alpha_\ell(z) := \sum_{n=0}^{\infty} \left(\mathbf{a}(\ell n - \tfrac{1}{24}(\ell^2 - 1)) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n}{\ell}\right) \right) q^{n - \frac{\ell}{24}} \tag{1.20}$$

is a weakly holomorphic weight $\frac{3}{2}$ modular form on $\Gamma_0(\ell)$. Here χ_{12} is the character given below in (2.2), and we note $\mathbf{a}(n) = 0$ if n is not a nonnegative integer. We determine the multiplier of this form and exact information about the orders at cusps. See Theorem 2.2. This enables us to prove identities such as

$$\alpha_5(z) = \sum_{n=0}^{\infty} \left(\mathbf{a}(5n-1) + 5\,\mathbf{a}\left(\frac{n}{5}\right)\right) q^{n-\frac{5}{24}} = \frac{5}{4}\frac{(5E_2(5z) - E_2(z))}{\eta(5z)} \left(125\frac{\eta(5z)^6}{\eta(z)^6} - 1\right), \tag{1.21}$$

where $E_2(z)$ is the usual quasimodular Eisenstein series of weight 2, and $\eta(z)$ is the Dedekind eta-function. We then use Watson's [22] and Atkin's [7] method of modular equations to prove the congruences (1.17)–(1.19). These details are carried out in Section 3. In Section 4 we improve some results in [13] and [9] on $\mathrm{spt}(\ell n - \frac{1}{24}(\ell^2-1))$ and $N_2(\ell n - \frac{1}{24}(\ell^2-1))$ modulo ℓ.

2. The Atkin operator U_ℓ^*

In this section we prove that the function $\alpha_\ell(z)$, which is defined in (1.20) is a weakly holomorphic weight $\frac{3}{2}$ modular form on $\Gamma_0(\ell)$ when $\ell \ge 5$ is prime. The proof uses results of Bringmann [8] and the idea of using the Atkin operator U_ℓ to annihilate the nonholomorphic part of a certain weak Maass form.

Following Bringmann [8] and Ono [19] we define

$$\mathcal{M}(z) := \alpha(24z) - \frac{3i}{\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)\,d\tau}{(-i(\tau+z))^{\frac{3}{2}}}, \tag{2.1}$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function and $\alpha(z)$ is defined in (1.16). Then $\mathcal{M}(z)$ is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_0(576)$ with Nebentypus χ_{12} where

$$\chi_{12}(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Let

$$\mathcal{N}(z) = -\frac{3i}{\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)\,d\tau}{(-i(\tau+z))^{\frac{3}{2}}} = \frac{3}{\pi\sqrt{2}} \int_{y}^{\infty} \frac{\eta(24(-x+it))\,dt}{(y+t)^{3/2}}, \tag{2.3}$$

where $z = x + iy$, $y > 0$, so that

$$\mathcal{M}(z) = \alpha(24z) + \mathcal{N}(z). \tag{2.4}$$

We define

$$\mathcal{A}(z) := \mathcal{M}\left(\frac{z}{24}\right). \tag{2.5}$$

The following theorem follows in a straightforward way from the work of Bringmann [8].

Theorem 2.1.

$$\mathcal{A}\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^{3/2}}{\nu_\eta(A)}\mathcal{A}(z),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, *and* $\nu_\eta(A)$ *is the eta-multiplier.*

Remark. When defining $z^{3/2}$ we use the principal branch; i.e. for $z = re^{i\theta}$, $r > 0$, $-\pi \le \theta < \pi$, we take $z^{3/2} = r^{3/2}e^{3i\theta/2}$.

Proof. We note that

$$\sum_{n=0}^{\infty}(24n-1)p(n)q^{n-\frac{1}{24}} = -\frac{E_2(z)}{\eta(z)}, \tag{2.6}$$

where $E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$ is a quasi-modular form that satisfies

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2E_2(z) - \frac{6iz}{\pi}c(cz+d). \tag{2.7}$$

Using (2.7) and Corollary 4.3 and Lemma 4.4 in [8],

$$\mathcal{M}\left(-\frac{1}{z}\right) = \frac{-(-iz)^{3/2}}{48\sqrt{6}}\mathcal{M}\left(\frac{z}{576}\right),$$

and hence

$$\mathcal{A}\left(-\frac{1}{z}\right) = -(-iz)^{3/2}\mathcal{A}(z) = e^{\pi i/4}z^{3/2}\mathcal{A}(z).$$

Therefore,

$$\mathcal{A}(Sz) = \frac{z^{3/2}}{\nu_\eta(S)}\mathcal{A}(z),$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From (1.16), (2.3) and (2.4)

$$\begin{aligned}
\mathcal{M}(z+\tfrac{1}{24}) &= e^{-\pi i/12}\mathcal{M}(z),\\
\mathcal{N}(z+\tfrac{1}{24}) &= e^{-\pi i/12}\mathcal{N}(z),\\
\mathcal{A}(z+1) &= e^{-\pi i/12}\mathcal{A}(z),\\
\mathcal{A}(Tz) &= \frac{1}{\nu_\eta(T)}\mathcal{A}(z),
\end{aligned}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since S, T generate $\mathrm{SL}_2(\mathbb{Z})$ the result follows. ☐

In what follows $\ell \geq 5$ is prime. We let d_ℓ denote the least nonnegative residue of the reciprocal of 24 mod ℓ so that $24d_\ell \equiv 1 \pmod{\ell}$. We define

$$r_\ell := \frac{24d_\ell - 1}{\ell}, \qquad r_\ell^* := \frac{24d_\ell + \ell^2 - 1}{24\ell}, \qquad s_\ell := \frac{(\ell^2-1)}{24}. \tag{2.8}$$

so that

(2.9)

$$\alpha_\ell(z) := \sum_{n=-r_\ell^*}\left(\mathbf{a}(\ell n + d_\ell) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n+r_\ell^*}{\ell}\right)\right)q^{n+\frac{r_\ell}{24}} = \sum_{n=0}^{\infty}\left(\mathbf{a}(\ell n - s_\ell) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n}{\ell}\right)\right)q^{n-\frac{\ell}{24}}.$$

For a function $G(z)$ we define the Atkin-type operator U_ℓ^* by

$$U_\ell^*(G) := \frac{1}{\ell}\sum_{k=0}^{\ell-1}G\left(\frac{z+24k}{\ell}\right), \tag{2.10}$$

so that

$$\alpha_\ell(z) = U_\ell^*(\alpha) - \chi_{12}(\ell)\,\ell\,\alpha(\ell z).$$

The usual Atkin operator U_ℓ is defined by

$$U_\ell(G) := \frac{1}{\ell}\sum_{k=0}^{\ell-1} G\left(\frac{z+k}{\ell}\right). \tag{2.11}$$

We need U_ℓ^* since $\alpha(z)$ has fractional powers of q, and we note that

$$U_\ell^*(G) = U_\ell(G^*)(z/24),$$

where $G^*(z) = G(24z)$. For a congruence subgroup Γ we let $M_k(\Gamma)$ denote the space of entire modular forms of weight k with respect to the group Γ, and we let $M_k(\Gamma,\chi)$ denote the space of entire modular forms of weight k and character χ with respect to the group Γ. Then

Theorem 2.2. *If $\ell \ge 5$ is prime, then*

$$G_\ell(z) := \alpha_\ell(z)\,\frac{\eta^{2\ell}(z)}{\eta(\ell z)} \in M_{\ell+1}(\Gamma_0(\ell)). \tag{2.12}$$

In other words, the function $G_\ell(z)$ is an entire modular form of weight $\ell+1$ with respect to the group $\Gamma_0(\ell)$.

Proof. We assume $\ell \ge 5$ is prime. We divide the proof into four parts:

(i) $U_\ell^*(\mathcal{A}) - \ell\,\chi_{12}(\ell)\,\mathcal{A}(\ell z) = \alpha_\ell(z)$ and $G_\ell(z)$ is holomorphic for $\Im(z) > 0$.

(ii) $G_\ell(Az) = (cz+d)^{\ell+1}G_\ell(z)$ for all $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(\ell)$.

(iii) $G_\ell(z)$ is holomorphic at $i\infty$.

(iv) $G_\ell(z)$ is holomorphic at the cusp 0.

Part (i). It is well-known (and an easy exercise) to show that

$$U_\ell(\eta(24z)) = \chi_{12}(\ell)\,\eta(24z). \tag{2.13}$$

Using (2.3) and (2.13) we easily find that

$$U_\ell(\mathcal{N}(z)) = \ell\,\chi_{12}(\ell)\mathcal{N}(z).$$

It follows that

$$U_\ell(\mathcal{M}) - \ell\,\chi_{12}(\ell)\,\mathcal{M}(\ell z)$$

is holomorphic for $\Im(z) > 0$. By replacing z by $\frac{z}{24}$ we see that

$$U_\ell^*(\mathcal{A}) - \ell\,\chi_{12}(\ell)\,\mathcal{A}(\ell z) = U_\ell^*(\alpha) - \ell\,\chi_{12}(\ell)\,\alpha(\ell z) = \alpha_\ell(z)$$

and it is clear that $G_\ell(z)$ is holomorphic for $\Im(z) > 0$.

Part (ii). Now let $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(\ell)$. We must show that

$$G_\ell(Az) = (cz+d)^{\ell+1}G_\ell(z).$$

Since it is well-known that

$$\left(\frac{\eta^\ell(z)}{\eta(\ell z)}\right)^2 \in M_{\ell-1}(\Gamma_0(\ell)),$$

it suffices to show that

$$\alpha_\ell(Az)\,\eta(\ell Az) = (cz+d)^2\alpha_\ell(z)\eta(\ell z).$$

We need to show that

$$f_\ell(Az) = (cz+d)^2 f_\ell(z), \tag{2.14}$$

$$g_\ell(Az) = (cz+d)^2 g_\ell(z), \tag{2.15}$$

where

$$f_\ell(z) = U_\ell^*(\mathcal{A})\,\eta(\ell z), \qquad g_\ell(z) = \mathcal{A}(\ell z)\,\eta(\ell z).$$

Let

$$A^* = \begin{pmatrix} a & \ell b \\ c/\ell & d \end{pmatrix}.$$

Then $A^* \in \mathrm{SL}_2(\mathbb{Z})$ and (2.15) follows from Theorem 2.1 and the fact that

$$\mathcal{A}(\ell A z)\,\eta(\ell A z) = \mathcal{A}(A^* z)\,\eta(A^* z).$$

Now,

$$f_\ell(z) = U_\ell^*(\mathcal{A})\eta(\ell z) = U_\ell^*(\mathcal{A}(z)\eta(\ell^2 z)).$$

We define

$$F_\ell(z) := \mathcal{A}(z)\eta(\ell^2 z) = \mathcal{A}(z)\eta(z)\,\frac{\eta(\ell^2 z)}{\eta(z)}. \tag{2.16}$$

Using Theorem 2.1 and the fact that $\frac{\eta(\ell^2 z)}{\eta(z)}$ is a modular function on $\Gamma_0(\ell^2)$ we have

$$F_\ell(Cz) = (c_1 z + d_1)^2 F_\ell(z),$$

for $C = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_0(\ell^2)$.

Now for $0 \le k \le \ell - 1$, let

$$B_k = \begin{pmatrix} 1 & 24k \\ 0 & \ell \end{pmatrix}$$

so that

$$f_\ell(z) = U_\ell^*(F_\ell(z)) = \frac{1}{\ell}\sum_{k=0}^{\ell-1} F_\ell(B_k z).$$

Since $A \in \Gamma_0(\ell)$, $(a,\ell)=1$ and we can choose unique $0 \le k^* \le \ell-1$ such that

$$24ak^* \equiv b + 24kd \pmod{\ell}.$$

Then

$$B_k A = A_k^* B_{k^*},$$

where $A_k^* \in \Gamma_0(\ell^2)$. We have

$$f_\ell(Az) = \frac{1}{\ell}\sum_{k=0}^{\ell-1} F_\ell(B_k A z) = \frac{1}{\ell}\sum_{k^*=0}^{\ell-1} F_\ell(A_k^* B_{k^*} z) = \frac{(cz+d)^2}{\ell}\sum_{k^*=0}^{\ell-1} F_\ell(B_{k^*} z) = (cz+d)^2 f_\ell(z),$$

which is (2.14).

Part (iii). First we note that r_ℓ^* is a positive integer. We have

$$G_\ell(z) = \alpha_\ell(z)\,\frac{\eta^{2\ell}(z)}{\eta(\ell z)} = \sum_{n=-r_\ell^*} \left(\mathbf{a}(\ell n + d_\ell) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n+r_\ell^*}{\ell}\right)\right) q^{n+r_{\ell^*}}\,\frac{E(q)^{2\ell}}{E(q^\ell)}$$

where

$$E(q) = \prod_{n=1}^{\infty}(1-q^n).$$

We see that $G_\ell(z)$ is holomorphic at $i\infty$.

Part (iv). We need to find $G_\ell\left(\frac{-1}{\ell z}\right)$.

$$U_\ell^*(\mathcal{A}) = \frac{1}{\ell}\sum_{k=0}^{\ell-1}\mathcal{A}\left(\frac{z+24k}{\ell}\right) = \frac{1}{\ell}\mathcal{A}\left(\frac{z}{\ell}\right) + \frac{1}{\ell}\sum_{k=1}^{\ell-1}\mathcal{A}\left(\frac{z+24k}{\ell}\right) = \frac{1}{\ell}\mathcal{A}\left(\frac{z}{\ell}\right) + \frac{1}{\ell}\sum_{k=1}^{\ell-1}\mathcal{A}(B_k z).$$

For each $1 \le k \le \ell-1$ choose $1 \le k^* \le \ell-1$ such that $576kk^* \equiv -1 \pmod{\ell}$. Then

$$B_k S = C_k B_{k^*},$$

where

$$C_k = \begin{pmatrix} 24k & \frac{-1-576kk^*}{\ell} \\ \ell & -24k^* \end{pmatrix} \in \Gamma_0(\ell).$$

Then

$$\mathcal{A}(B_k S z) = \mathcal{A}(C_k B_{k^*} z) = z^{3/2}\left(\frac{-24k^*}{\ell}\right)e^{\pi i\ell/4}\mathcal{A}(B_{k^*}z),$$

by Theorem 2.1 since

$$\nu_\eta(C_k) = \left(\frac{-24k^*}{\ell}\right)e^{-\pi i\ell/4},$$

by [17, p.51]. Define

$$S_\ell = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}.$$

By Theorem 2.1,

$$\mathcal{A}\left(\frac{1}{\ell}S_\ell z\right) = e^{\pi i/4}(z\ell^2)^{3/2}\mathcal{A}(\ell^2 z),$$
$$\mathcal{A}\left(\ell S_\ell z\right) = e^{\pi i/4}z^{3/2}\mathcal{A}(z).$$

.

Hence, if we define

(2.17) $$H_\ell(z) := U_\ell^*(\mathcal{A}) - \ell\chi_{12}\mathcal{A}(\ell z),$$

then

$$H_\ell(S_\ell z) = \ell z^{3/2}e^{\pi i/4}\left(\mathcal{A}(\ell^2 z) + \frac{1}{\sqrt{\ell}}e^{\pi i(\ell-1)/4}\sum_{k=1}^{\ell-1}\left(\frac{-24k}{\ell}\right)\mathcal{A}\left(z+\tfrac{24k}{\ell}\right) - \chi_{12}(\ell)\mathcal{A}(z)\right).$$

Replacing z by $24z$ gives

$$H_\ell(S_\ell 24z) = \ell(24z)^{3/2}e^{\pi i/4}\left(\mathcal{M}(\ell^2 z) + \frac{1}{\sqrt{\ell}}\chi_{12}(\ell)\epsilon_\ell^3\sum_{k=1}^{\ell-1}\left(\frac{-k}{\ell}\right)\mathcal{M}\left(z+\tfrac{k}{\ell}\right) - \chi_{12}(\ell)\mathcal{M}(z)\right),$$

since

$$e^{\pi i(\ell-1)/4}\left(\frac{24}{\ell}\right) = \chi_{12}(\ell)\epsilon_\ell^3.$$

Here

$$\epsilon_\ell = \begin{cases} 1 & \text{if } \ell \equiv 1 \pmod 4, \\ i & \text{if } \ell \equiv 3 \pmod 4. \end{cases}$$

By [21, p.451] we have

$$\begin{aligned} H_\ell(S_\ell 24z) &= \ell(24z)^{3/2} e^{\pi i/4} \left(\mathcal{M}|T(\ell^2) - \chi_{12}(\ell)\mathcal{M}(z) - U_{\ell^2}(\mathcal{M})\right), \\ &= \ell(24z)^{3/2} e^{\pi i/4} \left((\mathcal{M}|T(\ell^2) - \chi_{12}(\ell)(1+\ell)\mathcal{M}(z)) - (U_{\ell^2}(\mathcal{M}) - \ell\chi_{12}(\ell)\mathcal{M}(z))\right), \end{aligned}$$

where $T(\ell^2)$ is the Hecke operator which acts on harmonic Maass forms of weight $\frac{3}{2}$, and was used by Ono [19]. When the form is meromorphic it corresponds to the usual Hecke operator as described by Shimura [21]. Ono [19] showed that function

$$\mathcal{M}_\ell(z) = \mathcal{M}|T(\ell^2) - \chi_{12}(\ell)(1+\ell)\mathcal{M}(z))$$

is a weakly holomorphic modular form. In fact, he showed that

$$\mathcal{F}_\ell(z) := \eta(z)^{\ell^2} \mathcal{M}_\ell(z/24) \tag{2.18}$$

is a weight $(l^2+3)/2$ entire modular form on $\mathrm{SL}_2(\mathbb{Z})$. See [19, Theorem 2.2]. We also note that the function

$$U_{\ell^2}(\mathcal{M}) - \ell\chi_{12}(\ell)\mathcal{M}(z) = U_\ell\left(U_\ell(\mathcal{M}) - \ell\chi_{12}(\ell)\mathcal{M}(\ell z)\right)$$

is holomorphic for $\Im(z) > 0$ by the remarks in Part (i). Thus we find that

(2.19)

$$G_\ell\left(\frac{-1}{\ell z}\right) = -(iz\ell)^{\ell+1} \frac{E(q^\ell)^{2\ell}}{E(q)} \left(\sum_{n=-s_\ell}^{\infty} \left(\chi_{12}(\ell)\mathbf{a}(n)\left(\left(\frac{1-24n}{\ell}\right) - 1\right) + \ell\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right)\right) q^{n+2s_\ell}\right),$$

where $s_\ell = \frac{\ell^2-1}{24}$. It follows that $G_\ell(z)$ is holomorphic at the cusp 0. $\square$

Since $G_\ell(z) \in M_{\ell+1}(\Gamma_0(\ell))$, the function $z^{-\ell-1} G_\ell\left(\frac{-1}{\ell z}\right) \in M_{\ell+1}(\Gamma_0(\ell))$ by [4, Lemma 1]. Thus if we define

$$\beta_\ell(z) := \sum_{n=-s_\ell}^{\infty} \left(\chi_{12}(\ell)\mathbf{a}(n)\left(\left(\frac{1-24n}{\ell}\right) - 1\right) + \ell\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right)\right) q^{n-\frac{1}{24}}, \tag{2.20}$$

then the proof of Part (iv) of Theorem 2.2 yields

Corollary 2.3. *If $\ell \geq 5$ is prime, then*

$$J_\ell(z) := \beta_\ell(z)\, \frac{\eta^{2\ell}(\ell z)}{\eta(z)} \in M_{\ell+1}(\ell). \tag{2.21}$$

We illustrate the case $\ell = 5$. For ℓ prime we define

$$\mathcal{E}_{2,\ell}(z) := \frac{1}{\ell - 1}\left(\ell E_2(\ell z) - E_2(z)\right). \tag{2.22}$$

It is well-known that $\mathcal{E}_{2,\ell}(z) \in M_2(\Gamma_0(\ell))$. By [16, Theorem 3.8] $\dim M_6(\Gamma_0(5)) = 3$, and it can be shown that

$$\{\mathcal{E}_{2,5}(z)\frac{\eta(5z)^{10}}{\eta(z)^2}, \mathcal{E}_{2,5}(z)\eta(5z)^4\eta(z)^4, \mathcal{E}_{2,5}(z)\frac{\eta(z)^{10}}{\eta(5z)^2}\}$$

is a basis. We find that

$$G_5(z) = 5\,\mathcal{E}_{2,5}(z)\left(125\,\eta(5z)^4\eta(z)^4 - \frac{\eta(z)^{10}}{\eta(5z)^2}\right),$$

and

$$J_5(z) = 5\,\mathcal{E}_{2,5}(z)\left(\frac{\eta(5z)^{10}}{\eta(z)^2} - \eta(5z)^4\eta(z)^4\right).$$

Thus

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(5n-1) + 5\,\mathbf{a}\left(\frac{n}{5}\right)\right) q^{n-\frac{5}{24}} = 5\,\frac{\mathcal{E}_{2,5}(z)}{\eta(5z)}\left(125\,\frac{\eta(5z)^6}{\eta(z)^6} - 1\right), \tag{2.23}$$

and

$$\sum_{n=-1}^{\infty}\left(-\mathbf{a}(n)\left(\left(\frac{1-24n}{5}\right) - 1\right) + 5\,\mathbf{a}\left(\frac{n+1}{25}\right)\right) q^{n-\frac{1}{24}} = 5\,\frac{\mathcal{E}_{2,5}(z)}{\eta(z)}\left(1 - \frac{\eta(z)^6}{\eta(5z)^6}\right). \tag{2.24}$$

3. The Congruences

In this section we derive explicit formulas for the generating functions of

$$\mathbf{a}(\ell^a n + d_{\ell,a}) - \chi_{12}(\ell)\,\ell\,\mathbf{a}(\ell^{a-2}n + d_{\ell,a-2}), \tag{3.1}$$

when $\ell = 5$, 7, and 13. Here $24d_{\ell,a} \equiv 1 \pmod{\ell^a}$. The presentation of the identities is analogous to those of the partition function as given by Hirschhorn and Hunt [15] and the author [12]. In each case we start by using Theorem 2.2 to find identities for $\alpha_\ell(z)$. This basically gives the initial case $a = 1$. Then we use Watson's [22] and Atkin's [7] method of modular equations to do the induction step and study the arithmetic properties of the coefficients in these identities. The main congruences (1.6)-(1.8) then follow in a straightforward way.

3.1. The SPT-function modulo powers of 5.

Theorem 3.1. *If $a \geq 1$ then*

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a-1}n - t_a) + 5\,\mathbf{a}(5^{2a-3}n - t_{a-1})\right) q^{n-\frac{5}{24}} = \frac{\mathcal{E}_{2,5}(z)}{\eta(5z)}\sum_{i\geq 0} x_{2a-1,i}Y^i, \tag{3.2}$$

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}n - t_a) + 5\,\mathbf{a}(5^{2a-2}n - t_{a-1})\right) q^{n-\frac{1}{24}} = \frac{\mathcal{E}_{2,5}(z)}{\eta(z)}\sum_{i\geq 0} x_{2a,i}Y^i, \tag{3.3}$$

where

$$t_a = \frac{1}{24}(5^{2a}-1), \qquad Y(z) = \frac{\eta(5z)^6}{\eta(z)^6},$$

$$\vec{x}_1 = (x_{1,0}, x_{1,1}, \cdots) = (-5, 5^4, 0, 0, 0, \cdots),$$

and for $a \geq 1$

$$\vec{x}_{a+1} = \begin{cases} \vec{x}_a A, & a \text{ odd},\\ \vec{x}_a B, & a \text{ even}.\end{cases} \tag{3.4}$$

Here $A = (a_{i,j})_{i\geq 0, j\geq 0}$ and $B = (a_{i,j})_{i\geq 0, j\geq 0}$ are defined by

$$a_{i,j} = m_{6i,i+j}, \qquad b_{i,j} = m_{6i+1,i+j}, \tag{3.5}$$

where the matrix $M = (m_{i,j})_{i,j\geq 0}$ is defined as follows: The first five rows of M are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 5^3 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 4\cdot 5^2 & 5^5 & 0 & 0 & 0 & \cdots\\ 0 & 9\cdot 5 & 9\cdot 5^4 & 5^7 & 0 & 0 & \cdots\\ 0 & 2\cdot 5 & 44\cdot 5^3 & 14\cdot 5^6 & 5^9 & 0 & \cdots \end{pmatrix}$$

and for $i \geq 5$, $m_{i,0} = 0$ *and for* $j \geq 1$,

$$m_{i,j} = 25\, m_{i-1,j-1} + 25\, m_{i-2,j-1} + 15\, m_{i-3,j-1} + 5\, m_{i-4,j-1} + m_{i-5,j-1}. \tag{3.6}$$

Lemma 3.2. *If* n *is a positive integer then there are integers* c_m $(\lceil \frac{n}{5} \rceil \leq m \leq n)$ *such that*

$$U_5(\mathcal{E}_{2,5} Z^n) = \mathcal{E}_{2,5} \sum_{m=\lceil \frac{n}{5} \rceil}^{n} c_m Y^m,$$

where

$$Z(z) = \frac{\eta(25z)}{\eta(z)}, \qquad Y(z) = \frac{\eta(5z)^6}{\eta(z)^6}. \tag{3.7}$$

Proof. We need the following dimension formulas which follow from [10] and [16, Theorem 3.8]. For k even,

$$\dim M_k(\Gamma_0(5)) = 2 \left\lfloor \frac{k}{4} \right\rfloor + 1,$$
$$\dim M_k(\Gamma_0(5), \left(\frac{\cdot}{5}\right)) = k - 2 \left\lfloor \frac{k}{4} \right\rfloor.$$

Let n be a positive integer. Then

$$U_5(\mathcal{E}_{2,5} Z^n) = U_5 \left(\mathcal{E}_{2,5}(z) \left(\frac{\eta(5z)^5}{\eta(z)} \right)^n \left(\frac{\eta(25z)}{\eta(5z)^5} \right)^n \right) = U_5 \left(\mathcal{E}_{2,5}(z) \left(\frac{\eta(5z)^5}{\eta(z)} \right)^n \right) \left(\frac{\eta(5z)}{\eta(z)^5} \right)^n.$$

When n is even the function

$$\mathcal{E}_{2,5}(z) \left(\frac{\eta(5z)^5}{\eta(z)} \right)^n$$

belongs to the space $M_{2n+2}(\Gamma_0(5))$, which has as a basis

$$\left\{ \mathcal{E}_{2,5}(z) \eta(z)^{5n-6m} \eta(5z)^{6m-n} \,,\ 0 \leq m \leq n \right\}.$$

This follows from the dimension formula. We note that

$$\operatorname{ord}\left(\mathcal{E}_{2,5}(z) \eta(z)^{5n-6m} \eta(5z)^{6m-n}; i\infty \right) = m.$$

The operator U_5 preserves the space $M_{2n+2}(\Gamma_0(5))$. It follows that there are integers c_m $(\lceil \frac{n}{5} \rceil \leq m \leq n)$ such that

$$U_5(\mathcal{E}_{2,5} Z^n) = \mathcal{E}_{2,5}(z) \sum_{m=\lceil \frac{n}{5} \rceil}^{n} c_m\, \eta(z)^{5n-6m} \eta(5z)^{6m-n} \left(\frac{\eta(5z)}{\eta(z)^5} \right)^n = \mathcal{E}_{2,5}(z) \sum_{m=\lceil \frac{n}{5} \rceil}^{n} c_m Y^m.$$

When n is odd the proof is similar except this time one needs to work in the space $M_{2n+2}(\Gamma_0(5), \left(\frac{\cdot}{5}\right))$. ◻

Corollary 3.3.

$$U_5(\mathcal{E}_{2,5}) = \mathcal{E}_{2,5} \tag{3.8}$$

$$U_5(\mathcal{E}_{2,5} Z) = 5^3\, \mathcal{E}_{2,5} Y \tag{3.9}$$

$$U_5(\mathcal{E}_{2,5} Z^2) = 5^2\, \mathcal{E}_{2,5}(4Y + 5^3 Y^2) \tag{3.10}$$

$$U_5(\mathcal{E}_{2,5} Z^3) = 5\, \mathcal{E}_{2,5}(9Y + 9 \cdot 5^3 Y^2 + 5^6 Y^3) \tag{3.11}$$

$$U_5(\mathcal{E}_{2,5} Z^4) = 5\, \mathcal{E}_{2,5}(2Y + 44 \cdot 5^2 Y^2 + 14 \cdot 5^5 Y^3 + 5^8 Y^4). \tag{3.12}$$

Proof. Equation (3.8) is elementary. It also follows from the fact that $\dim M_2(\Gamma_0(5)) = 1$. Equations (3.9)–(3.12) follow from Lemma 3.2 and straightforward calculation. ◻

We need the 5th order modular equation that was used by Watson to prove Ramanujan's partition congruences for powers of 5.

$$Z^5 = \left(25Z^4 + 25Z^3 + 15Z^2 + 5Z + 1\right) Y(5z). \tag{3.13}$$

Lemma 3.4. *For $i \geq 0$*

$$U_5(\mathcal{E}_{2,5}Z^i) = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i}{5} \rceil}^{i} m_{i,j} Y^j,$$

where $Z = Z(z)$, $Y = Y(z)$ are defined in (3.7), and the $m_{i,j}$ are defined in Theorem 3.1.

Proof. The result holds for $0 \leq i \leq 4$ by Corollary 3.3. By (3.13) we have

$$U_5(\mathcal{E}_{2,5}Z^i) = \left(25U_5(\mathcal{E}_{2,5}Z^{i-1}) + 25U_5(\mathcal{E}_{2,5}Z^{i-2}) + 15U_5(\mathcal{E}_{2,5}Z^{i-3}) + 5U_5(\mathcal{E}_{2,5}Z^{i-4}) + U_5(\mathcal{E}_{2,5}Z^{i-5})\right) Y(z),$$

for $i \geq 5$. The result follows by induction on i using the recurrence (3.6). ☐

Lemma 3.5. *For $i \geq 0$,*

$$U_5(\mathcal{E}_{2,5}Y^i) = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i}{5} \rceil}^{5i} a_{i,j} Y^j, \tag{3.14}$$

$$U_5(\mathcal{E}_{2,5}ZY^i) = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i+1}{5} \rceil}^{5i+1} b_{i,j} Y^j, \tag{3.15}$$

where the $a_{i,j}$, $b_{i,j}$ are defined in (3.5).

Proof. Suppose $i \geq 0$. By Lemma 3.4

$$\begin{aligned} U_5(\mathcal{E}_{2,5}Y^i) &= U_5(\mathcal{E}_{2,5}Z^{6i}Y(5z)^{-i}) = Y^{-i}U_5(\mathcal{E}_{2,5}Z^{6i}) \\ &= Y^{-i}\mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{6i}{5} \rceil}^{6i} m_{6i,j} Y^j \\ &= \mathcal{E}_{2,5}(z) \sum_{j\geq\lceil \frac{i}{5} \rceil}^{5i} m_{6i,i+j} Y^j = \mathcal{E}_{2,5}(z) \sum_{j\geq\lceil \frac{i}{5} \rceil}^{5i} a_{i,j} Y^j, \end{aligned}$$

which is (3.14). Similarly

$$\begin{aligned} U_5(\mathcal{E}_{2,5}ZY^i) &= U_5(\mathcal{E}_{2,5}Z^{6i+1}Y(5z)^{-i}) = Y^{-i}U_5(\mathcal{E}_{2,5}Z^{6i+1}) \\ &= Y^{-i}\mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{6i+1}{5} \rceil}^{6i+1} m_{6i+1,j} Y^j \\ &= \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i+1}{5} \rceil}^{5i+1} m_{6i+1,i+j} Y^j = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i+1}{5} \rceil}^{5i+1} b_{i,j} Y^j, \end{aligned}$$

which is (3.15). ☐

Proof of Theorem 3.1. We proceed by induction. The case $a = 1$ of (3.2) is (2.23). We now suppose $a \geq 1$ is fixed and (3.2) holds. Thus

$$E(q^5) \sum_{n=0}^{\infty} \left(\mathbf{a}(5^{2a-1}n - t_a) + 5\,\mathbf{a}(5^{2a-3}n - t_{a-1})\right) q^n = \mathcal{E}_{2,5}(z) \sum_{i\geq 0} x_{2a-1,i} Y^i.$$

We now apply the U_5 operator to both sides and use Lemma 3.5.

$$E(q)\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}n-t_a)+5\,\mathbf{a}(5^{2a-2}n-t_{a-1})\right)q^n=\sum_{i\geq 0}x_{2a-1,i}U_5(\mathcal{E}_{2,5}(z)Y^i)$$

$$=\mathcal{E}_{2,5}(z)\sum_{i\geq 0}x_{2a-1,i}\sum_{j\geq 0}a_{i,j}Y^j=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}\left(\sum_{i\geq 0}x_{2a-1,i}a_{i,j}\right)Y^j=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}x_{2a,j}Y^j.$$

We obtain (3.3) by dividing both sides by $\eta(z)$.

Now again suppose a is fixed and (3.3) holds. Multiplying both sides by $\eta(25z)$ gives

$$E(q^{25})\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}n-t_a)+5\,\mathbf{a}(5^{2a-2}n-t_{a-1})\right)q^{n+1}=\mathcal{E}_{2,5}(z)\sum_{i\geq 0}x_{2a,i}ZY^i.$$

We apply the U_5 operator to both sides.

$$E(q^5)\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}(5n-1)-t_a)+5\,\mathbf{a}(5^{2a-2}(5n-1)-t_{a-1})\right)q^n=\sum_{i\geq 0}x_{2a,i}U_5(\mathcal{E}_{2,5}(z)ZY^i).$$

Using Lemma 3.5 and the fact that $t_{a+1}=5^{2a}+t_a$ we have

$$E(q^5)\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a+1}n-t_{a+1})+5\,\mathbf{a}(5^{2a-1}n-t_a)\right)q^n=\mathcal{E}_{2,5}(z)\sum_{i\geq 0}x_{2a,i}\sum_{j\geq 0}b_{i,j}Y^j$$

$$=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}\left(\sum_{i\geq 0}x_{2a,i}b_{i,j}\right)Y^j=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}x_{2a+1,j}Y^j.$$

We obtain (3.2) with a replaced by $a+1$ after dividing both sides by $\eta(5z)$. This completes the proof of the theorem. ☐

Throughout this section we will make repeated use of the following lemma which we leave as an exercise.

Lemma 3.6. *Suppose x, y, $n\in\mathbb{Z}$ and $n>0$. Then*

$$\left\lfloor\frac{x}{n}\right\rfloor+\left\lfloor\frac{y}{n}\right\rfloor\geq\left\lfloor\frac{x+y-n+1}{n}\right\rfloor. \tag{3.16}$$

For any prime ℓ we let $\pi(n)=\pi_\ell(n)$ denote the exact power of ℓ that divides n.

Then

Lemma 3.7.

$$\pi_5(m_{i,j})\geq\lfloor\tfrac{1}{2}(5j-i+1)\rfloor,$$

where the matrix $M=(m_{i,j})_{i,j\geq 0}$ is defined in Theorem 3.1.

Proof. First we verify the result for $0\leq i\leq 4$. The result is easily proven for $i\geq 5$ using the recurrence (3.6). ☐

Corollary 3.8.

$$\pi_5(a_{i,j})\geq\lfloor\tfrac{1}{2}(5j-i+1)\rfloor,\qquad \pi_5(b_{i,j})\geq\lfloor\tfrac{1}{2}(5j-i)\rfloor,$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (3.5).

Lemma 3.9. *For $b \geq 2$, and $j \geq 1$,*

$$\pi_5(x_{2b-1,j}) \geq 5b - 6 + \max(0, \lfloor \tfrac{1}{2}(5j-7) \rfloor), \tag{3.17}$$

$$\pi_5(x_{2b,j}) \geq 5b - 4 + \lfloor \tfrac{1}{2}(5j-5) \rfloor. \tag{3.18}$$

Proof. A calculation gives

$$\begin{aligned}\vec{x}_3 &= (x_{3,0}, x_{3,1}, x_{3,2}, \cdots) \\ &= (0, 669303124 \cdot 5^4, 3328977476 \cdot 5^{11}, 366098988268 \cdot 5^{14}, 201318006648837 \cdot 5^{15}, 1618593700646527 \cdot 5^{18}, \\ &\quad 6370852555263938 \cdot 5^{21}, 2900024541422883 \cdot 5^{25}, 4237895677971369 \cdot 5^{28}, 21327793208615511 \cdot 5^{30}, \\ &\quad 15532659183030861 \cdot 5^{33}, 8481639849706179 \cdot 5^{36}, 3564573506915806 \cdot 5^{39}, 1175454967692313 \cdot 5^{42}, \\ &\quad 1542192101361916 \cdot 5^{44}, 325171329708596 \cdot 5^{47}, 55431641829564 \cdot 5^{50}, 1532152033009 \cdot 5^{54}, 171561318777 \cdot 5^{57}, \\ &\quad 77490966671 \cdot 5^{59}, 5598792206 \cdot 5^{62}, 318906274 \cdot 5^{65}, 2799863 \cdot 5^{69}, \\ &\quad 91379 \cdot 5^{72}, 10439 \cdot 5^{74}, 149 \cdot 5^{77}, 5^{80}, 0, \cdots),\end{aligned}$$

$$\pi_5(\vec{x}_3) = (\infty, 4, 11, 14, 15, 18, 21, 25, 28, 30, 33, 36, 39, 42, 44, 47, 50, 54, 57, 59, 62, 65, 69, 72, 74, 77, 80, \infty, \infty, \cdots),$$

and (3.17) holds for $b = 2$. Now suppose $b \geq 2$ is fixed and (3.17) holds. By (3.4)

$$x_{2b,j} = \sum_{i \geq 1} x_{2b-1,i} a_{i,j}.$$

Then using Corollary 3.8

$$\pi_5(x_{2b,1}) \geq \min(\{5b-4\} \cup \{5b-6+\lfloor \tfrac{1}{2}(5i-7) \rfloor + \lfloor (\tfrac{1}{2}(6-i) \rfloor \,:\, 2 \leq i \leq 5\}) = 5b-4,$$

and (3.18) holds for $j = 1$. Suppose $j \geq 2$. Then

$$\begin{aligned}\pi_5(x_{2b,j}) &\geq \min_{1 \leq i \leq 5j} (\pi_5(x_{2b-1,i}) + \pi_5(a_{i,j})) \\ &\geq \min_{2 \leq i \leq 5j} (\pi_5(x_{2b-1,1}) + \pi_5(a_{1,j}), (\pi_5(x_{2b-1,i}) + \pi_5(a_{i,j})) \\ &\geq \min(\{5b-6+\lfloor \tfrac{1}{2}(5j) \rfloor\} \cup \{5b-6+\lfloor \tfrac{1}{2}(5i-7) \rfloor) + \lfloor \tfrac{1}{2}(5j-i+1) \rfloor \,:\, 2 \leq i \leq 5j\}).\end{aligned}$$

Now

$$5b-6+\lfloor \tfrac{1}{2}(5j) \rfloor = 5b-4+\lfloor \tfrac{1}{2}(5j-4) \rfloor.$$

If $2 \leq i \leq 5j$, then using Lemma 3.6 we have

$$\begin{aligned}5b-6+\lfloor \tfrac{1}{2}(5i-7) \rfloor) + \lfloor \tfrac{1}{2}(5j-i+1) \rfloor &\geq 5b-6+\lfloor \tfrac{1}{2}(5j+4i-7) \rfloor \\ &\geq 5b-6+\lfloor \tfrac{1}{2}(5j+1) \rfloor = 5b-4+\lfloor \tfrac{1}{2}(5j-3) \rfloor\end{aligned}$$

and (3.18) holds. Now suppose $b \geq 2$ is fixed and (3.18) holds. By (3.4)

$$x_{2b+1,j} = \sum_{i \geq 1} x_{2b,i} b_{i,j}.$$

We observe that $\pi_5(b_{1,1}) = \pi_5(500) = 3$. Then using Corollary 3.8

$$\pi_5(x_{2b+1,1}) \geq \min(\{5b-1\} \cup \{5b-4+\lfloor \tfrac{1}{2}(5i-4) + \lfloor (\tfrac{1}{2}(5-i) \rfloor \,:\, 2 \leq i \leq 4\}) = 5b-1,$$

and (3.17) holds for $j = 1$ with b replaced by $b+1$. Suppose $j \geq 2$. Then

$$\begin{aligned}\pi_5(x_{2b+1,j}) &\geq \min_{1 \leq i \leq 5j-1} (\pi_5(x_{2b,i}) + \pi_5(b_{i,j})) \\ &\geq \min_{2 \leq i \leq 5j-1} (\pi_5(x_{2b,1}) + \pi_5(b_{1,j}), (\pi_5(x_{2b,i}) + \pi_5(b_{i,j})) \\ &\geq \min(\{5b-4+\lfloor \tfrac{1}{2}(5j-1) \rfloor\} \cup \{5b-4+\lfloor \tfrac{1}{2}(5i-4) \rfloor) + \lfloor \tfrac{1}{2}(5j-i) \rfloor \,:\, 2 \leq i \leq 5j-1\}).\end{aligned}$$

Now

$$5b-4+\lfloor\tfrac{1}{2}(5j-1)\rfloor = 5b-1+\lfloor\tfrac{1}{2}(5j-7)\rfloor.$$

If $2\le i\le 5j-1$, then again using Lemma 3.6 we have

$$\begin{aligned}5b-4+\lfloor\tfrac{1}{2}(5i-4)\rfloor)+\lfloor\tfrac{1}{2}(5j-i)\rfloor &\ge 5b-4+\lfloor\tfrac{1}{2}(5j+4i-5)\rfloor\\ &\ge 5b-4+\lfloor\tfrac{1}{2}(5j+3)\rfloor = 5b-1+\lfloor\tfrac{1}{2}(5j-3)\rfloor\end{aligned}$$

and (3.17) holds with b replaced by $b+1$. Lemma 3.9 follows by induction. ☐

Corollary 3.10. *For* $b\ge 2$,

(3.19) $$\mathbf{a}(5^{2b-1}n+\delta_{2b+1})+5\,\mathbf{a}(5^{2b-3}n+\delta_{2b-3})\equiv 0 \pmod{5^{5b-6}},$$

(3.20) $$\mathbf{a}(5^{2b}n+\delta_{2b})+5\,\mathbf{a}(5^{2b-2}n+\delta_{2b-2})\equiv 0 \pmod{5^{5b-4}}.$$

For $a\ge 1$,

(3.21) $$\operatorname{spt}(5^{a+2}n+\delta_{a+2})+5\operatorname{spt}(5^{a}n+\delta_{a})\equiv 0 \pmod{5^{2a+1}},$$

(3.22) $$\operatorname{spt}(5^{a}n+\delta_{a})\equiv 0 \pmod{5^{\lfloor\frac{a+1}{2}\rfloor}}.$$

Proof. The congruences (3.19)–(3.20) follow from Theorem 3.1 and Lemma 3.9. Let

$$\operatorname{dp}(n)=(24n-1)p(n).$$

Then

(3.23) $$\operatorname{dp}(5^{a}n+\delta_{a})\equiv 0 \pmod{5^{2a}},$$

by (1.12). The congruence (3.21) follows from (3.19)–(3.20), and (3.23). Andrews' congruence (1.2) implies that (3.22) holds for $a=1$, 2. The general result follows by induction using (3.21). ☐

We note that when $a=0$ there is a stronger congruence than (3.21). We prove that

(3.24) $$\operatorname{spt}(25n-1)+5\operatorname{spt}(n)\equiv 0 \pmod{25}.$$

We have calculated

$$\begin{aligned}\vec{x}_2 &= (x_{2,0},x_{2,1},x_{2,2},\cdots)\\ &= (-5^1, 63\cdot 5^6, 104\cdot 5^9, 189\cdot 5^{11}, 24\cdot 5^{14}, 5^{17}, 0,\cdots).\end{aligned}$$

Thus

(3.25)
$$\sum_{n=0}^{\infty}\left(\mathbf{a}(25n-1)+5\,\mathbf{a}(n)\right)q^{n-\frac{1}{24}}$$
$$=5\frac{\mathcal{E}_{2,5}(z)}{\eta(z)}\left(-1+63\cdot 5^5\frac{\eta^6(5z)}{\eta^6(z)}+104\cdot 5^8\frac{\eta^{12}(5z)}{\eta^{12}(z)}+189\cdot 5^{10}\frac{\eta^{18}(5z)}{\eta^{18}(z)}+24\cdot 5^{13}\frac{\eta^{24}(5z)}{\eta^{24}(z)}+5^{16}\frac{\eta^{30}(5z)}{\eta^{30}(z)}\right),$$

and

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(25n-1)+5\,\mathbf{a}(n)\right)q^{n-\frac{1}{24}}\equiv 20\frac{E_2(z)}{\eta(z)} \pmod{25}.$$

But from (2.6) we see that

$$\sum_{n=0}^{\infty}\left(dp(25n-1)+5\,dp(n)\right)q^{n-\frac{1}{24}}\equiv 20\frac{E_2(z)}{\eta(z)} \pmod{25},$$

and

$$12\sum_{n=0}^{\infty}\left(\mathrm{spt}(25n-1)+5\,\mathrm{spt}(n)\right)q^{n-\frac{1}{24}}$$
$$=\sum_{n=0}^{\infty}\left(\mathbf{a}(25n-1)+5\,\mathbf{a}(n)\right)q^{n-\frac{1}{24}}-\sum_{n=0}^{\infty}\left(dp(25n-1)+5\,dp(n)\right)q^{n-\frac{1}{24}}\equiv 0 \pmod{25},$$

which gives (3.24).

3.2. **The SPT-function modulo powers of 7.**

Theorem 3.11. *If $a\geq 1$ then*

$$\sum_{n=0}^{\infty}\big(\mathbf{a}(7^{2a-1}n-u_a)+7\,\mathbf{a}(7^{2a-3}n-u_{a-1})\big)\,q^{n-\frac{7}{24}}=\frac{\mathcal{E}_{2,7}(z)}{\eta(7z)}\sum_{i\geq 0}x_{2a-1,i}Y^i, \tag{3.26}$$

$$\sum_{n=0}^{\infty}\big(\mathbf{a}(7^{2a}n-u_a)+7\,\mathbf{a}(7^{2a-2}n-u_{a-1})\big)\,q^{n-\frac{1}{24}}=\frac{\mathcal{E}_{2,7}(z)}{\eta(z)}\sum_{i\geq 0}x_{2a,i}Y^i, \tag{3.27}$$

where

$$u_a=\frac{1}{24}(7^{2a}-1),\qquad Y(z)=\frac{\eta(7z)^4}{\eta(z)^4},$$
$$\vec{x}_1=(x_{1,0},x_{1,1},\cdots)=(-7,3\cdot 7^3,7^5,0,0,\cdots),$$

and for $a\geq 1$

$$\vec{x}_{a+1}=\begin{cases}\vec{x}_aA, & a\ \text{odd},\\ \vec{x}_aB, & a\ \text{even}.\end{cases}$$

Here $A=(a_{i,j})_{i\geq 0,j\geq 0}$ and $B=(a_{i,j})_{i\geq 0,j\geq 0}$ are defined by

$$a_{i,j}=m_{4i,i+j},\qquad b_{i,j}=m_{4i+1,i+j}, \tag{3.28}$$

where the matrix $M=(m_{i,j})_{i,j\geq 0}$ is defined as follows: The first seven rows of M are defined so that

$$U_7(\mathcal{E}_{2,7}Z^i)=\sum_{j=\lceil\frac{2i}{7}\rceil}^{2i}m_{i,j}Y^j\qquad(0\leq i\leq 6),$$

where

$$Z(z)=\frac{\eta(49z)}{\eta(z)}.$$

and for $i\geq 7$, $m_{i,0}=0$, $m_{i,1}=0$, and for $j\geq 2$,

(3.29)
$$m_{i,j}=49\,m_{i-1,j-1}+35\,m_{i-2,j-1}+7\,m_{i-3,j-1}+343\,m_{i-1,j-2}+343\,m_{i-2,j-2}+147\,m_{i-3,j-2}$$
$$+49\,m_{i-4,j-2}+21\,m_{i-5,j-2}+7\,m_{i-6,j-2}+m_{i-7,j-2}.$$

The proof of the following lemma is analogous to that of Lemma 3.2.

Lemma 3.12. *If n is a positive integer then there are integers c_m ($\lceil\frac{2n}{7}\rceil\leq m\leq 2n$) such that*

$$U_7(\mathcal{E}_{2,7}Z^n)=\mathcal{E}_{2,7}\sum_{m=\lceil\frac{2n}{7}\rceil}^{2n}c_mY^m,$$

where

$$Z(z) = Z_7(z) = \frac{\eta(49z)}{\eta(z)}, \qquad Y(z) = \frac{\eta(7z)^4}{\eta(z)^4}. \tag{3.30}$$

Corollary 3.13.

$$U_7(\mathcal{E}_{2,7}) = \mathcal{E}_{2,7} \tag{3.31}$$

$$U_7(\mathcal{E}_{2,7}Z) = 7^2\mathcal{E}_{2,7}(3\,Y + 7^2\,Y^2) \tag{3.32}$$

$$U_7(\mathcal{E}_{2,7}Z^2) = 7\mathcal{E}_{2,7}(10\,Y + 27\cdot 7^2\,Y^2 + 10\cdot 7^4\,Y^3 + 7^6\,Y^4) \tag{3.33}$$

$$U_7(\mathcal{E}_{2,7}Z^3) = 7\mathcal{E}_{2,7}(Y + 190\cdot 7\,Y^2 + 255\cdot 7^3\,Y^3 + 104\cdot 7^5\,Y^4 + 17\cdot 7^7\,Y^5 + 7^9\,Y^6) \tag{3.34}$$

$$\begin{aligned} U_7(\mathcal{E}_{2,7}Z^4) = 7^2\mathcal{E}_{2,7}(&82\,Y^2 + 352\cdot 7^2\,Y^3 + 2535\cdot 7^3\,Y^4 + 1088\cdot 7^5\,Y^5 + 230\cdot 7^7\,Y^6 \\ &+ 24\cdot 7^9\,Y^7 + 7^{11}\,Y^8) \end{aligned} \tag{3.35}$$

$$\begin{aligned} U_7(\mathcal{E}_{2,7}Z^5) = 7\mathcal{E}_{2,7}(&114\,Y^2 + 253\cdot 7^3\,Y^3 + 4169\cdot 7^4\,Y^4 + 3699\cdot 7^6\,Y^5 + 11495\cdot 7^7\,Y^6 \\ &+ 2852\cdot 7^9\,Y^7 + 405\cdot 7^{11}\,Y^8 + 31\cdot 7^{13}\,Y^9 + 7^{15}\,Y^{10}) \end{aligned} \tag{3.36}$$

$$\begin{aligned} U_7(\mathcal{E}_{2,7}Z^6) = 7\mathcal{E}_{2,7}(&9\,Y^2 + 736\cdot 7^2\,Y^3 + 27970\cdot 7^3\,Y^4 + 6808\cdot 7^6\,Y^5 + 38475\cdot 7^7\,Y^6 \\ &+ 17490\cdot 7^9\,Y^7 + 33930\cdot 7^{10}\,Y^8 + 5890\cdot 7^{12}\,Y^9 + 629\cdot 7^{14}\,Y^{10} \\ &+ 38\cdot 7^{16}\,Y^{11} + 7^{18}\,Y^{12}) \end{aligned} \tag{3.37}$$

We need the 7th order modular equation that was used by Watson to prove Ramanujan's partition congruences for powers of 7.

$$Z^7 = (1+7\,Z+21\,Z^2+49\,Z^3+147\,Z^4+343\,Z^5+343\,Z^6)\,Y(7z)^2+(7\,Z^4+35\,Z^5+49\,Z^6)\,Y(7z) \tag{3.38}$$

Lemma 3.14. *For $i \ge 0$*

$$U_7(\mathcal{E}_{2,7}Z^i) = \mathcal{E}_{2,7}(z) \sum_{j=\lceil \frac{2i}{7}\rceil}^{2i} m_{i,j} Y^j,$$

where $Z = Z(z)$, $Y = Y(z)$ are defined in (3.30), and the $m_{i,j}$ are defined in Theorem 3.11.

Lemma 3.15. *For $i \ge 0$,*

$$U_7(\mathcal{E}_{2,7}Y^i) = \mathcal{E}_{2,7}(z) \sum_{j=\lceil \frac{i}{7}\rceil}^{7i} a_{i,j} Y^j, \tag{3.39}$$

$$U_7(\mathcal{E}_{2,7}ZY^i) = \mathcal{E}_{2,7}(z) \sum_{j=\lceil \frac{i+2}{7}\rceil}^{7i+2} b_{i,j} Y^j \tag{3.40}$$

where the $a_{i,j}$, $b_{i,j}$ are defined in (3.28).

Let $\pi_7(n)$ denote the exact power of 7 dividing n. Then

Lemma 3.16.

$$\pi_7(m_{i,j}) \ge \lfloor \tfrac{1}{4}(7j - 2i + 3)\rfloor,$$

where the matrix $M = (m_{i,j})_{i,j\ge 0}$ is defined in Theorem 3.11.

Corollary 3.17.

$$\pi_7(a_{i,j}) \ge \lfloor \tfrac{1}{4}(7j - i + 3)\rfloor, \qquad \pi_7(b_{i,j}) \ge \lfloor \tfrac{1}{4}(7j - i + 1)\rfloor,$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (3.28).

Lemma 3.18. *For $b \geq 2$, and $j \geq 1$,*

$$\pi_7(x_{2b-1,j}) \geq 3b-3+\lfloor \tfrac{1}{4}(7j-4)\rfloor. \tag{3.41}$$

$$\pi_7(x_{2b,j}) \geq 3b-1+\lfloor \tfrac{1}{4}(7j-6)\rfloor. \tag{3.42}$$

Corollary 3.19. *For $b \geq 2$,*

$$\mathbf{a}(7^{2b-1}n+\lambda_{2b+1})+7\cdot\mathbf{a}(7^{2b-3}n+\lambda_{2b-3}) \equiv 0 \pmod{7^{3b-3}}, \tag{3.43}$$

$$\mathbf{a}(7^{2b}n+\lambda_{2b})+7\cdot\mathbf{a}(7^{2b-2}n+\lambda_{2b-2}) \equiv 0 \pmod{7^{3b-1}}. \tag{3.44}$$

For $a \geq 1$,

$$\mathrm{spt}(7^{a+2}n+\lambda_{a+2})+7\cdot\mathrm{spt}(7^{a}n+\lambda_{a}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3a+4)\rfloor}}, \tag{3.45}$$

$$\mathrm{spt}(7^{a}n+\lambda_{a}) \equiv 0 \pmod{7^{\lfloor \frac{a+1}{2}\rfloor}}. \tag{3.46}$$

We note that (3.45) also holds for $a=0$. The proof of the congruence

$$\mathrm{spt}(49n-2)+7\cdot\mathrm{spt}(n) \equiv 0 \pmod{49}. \tag{3.47}$$

is analogous to the proof of (3.24).

3.3. **The SPT-function modulo powers of 13.**

Theorem 3.20. *If $a \geq 1$ then*

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(13^{2a-1}n-v_a)-13\,\mathbf{a}(13^{2a-3}n-v_{a-1})\right)q^{n-\frac{13}{24}} = \frac{\mathcal{E}_{2,13}(z)}{\eta(13z)}\sum_{i\geq 0}x_{2a-1,i}Y^i, \tag{3.48}$$

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(13^{2a}n-v_a)-13\,\mathbf{a}(13^{2a-2}n-v_{a-1})\right)q^{n-\frac{1}{24}} = \frac{\mathcal{E}_{2,13}(z)}{\eta(z)}\sum_{i\geq 0}x_{2a,i}Y^i, \tag{3.49}$$

where

$$v_a = \frac{1}{24}(13^{2a}-1), \qquad Y(z) = \frac{\eta(13z)^2}{\eta(z)^2},$$

$$\vec{x}_1 = (x_{1,0},x_{1,1},\cdots) = (13, 11\cdot 13^2, 108\cdot 13^3, 190\cdot 13^4, 140\cdot 13^5, 54\cdot 13^6, 11\cdot 13^7, 13^8, 0,0,0,\cdots),$$

and for $a \geq 1$

$$\vec{x}_{a+1} = \begin{cases} \vec{x}_a A, & a \text{ odd},\\ \vec{x}_a B, & a \text{ even}.\end{cases}$$

Here $A=(a_{i,j})_{i\geq 0,j\geq 0}$ and $B=(a_{i,j})_{i\geq 0,j\geq 0}$ are defined by

$$a_{i,j} = m_{2i,i+j}, \qquad b_{i,j} = m_{2i+1,i+j}, \tag{3.50}$$

where the matrix $M=(m_{i,j})_{i\ge -12, j\ge -6}$ *is defined as follows: The first 13 rows of* M *are*

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 13^6 & 0 & 0 & \cdots\\
0 & 82\cdot 13 & 456\cdot 13^2 & 360\cdot 13^3 & 126\cdot 13^4 & 18\cdot 13^5 & 13^6 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 13^5 & 0 & 0 & \cdots\\
0 & 0 & 18\cdot 13 & -36\cdot 13^2 & -40\cdot 13^3 & -14\cdot 13^4 & -13^5 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 13^4 & 0 & 0 & \cdots\\
0 & 0 & 0 & -14\cdot 13 & -12\cdot 13^2 & 0 & 13^4 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 13^3 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 4\cdot 13 & 6\cdot 13^2 & 13^3 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 13^2 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & -13^2 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 13 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & -13 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots
\end{pmatrix}$$

and for $m_{k,\ell}=0$ *for* $k\ge 1$ *and* $-6\le \ell\le 0$*; and for* $i\ge 1$ *and* $j\ge 1$*,*

$$m_{i,j}=\sum_{r=1}^{13}\sum_{s=\lfloor\frac{1}{2}(r+2)\rfloor}^{7}\psi_{r,s}m_{i-r,j-s}, \tag{3.51}$$

where $\Psi=(\psi_{r,s})_{1\le r\le 13, 1\le s\le 7}$ *is the matrix*

$$\Psi=\begin{pmatrix}
11\cdot 13 & 36\cdot 13^2 & 38\cdot 13^3 & 20\cdot 13^4 & 6\cdot 13^5 & 13^6 & 13^6\\
0 & -204\cdot 13 & -346\cdot 13^2 & -222\cdot 13^3 & -74\cdot 13^4 & -13^6 & -13^6\\
0 & 36\cdot 13 & 126\cdot 13^2 & 102\cdot 13^3 & 38\cdot 13^4 & 7\cdot 13^5 & 7\cdot 13^5\\
0 & 0 & -346\cdot 13 & -422\cdot 13^2 & -184\cdot 13^3 & -37\cdot 13^4 & -3\cdot 13^5\\
0 & 0 & 38\cdot 13 & 102\cdot 13^2 & 56\cdot 13^3 & 13^5 & 15\cdot 13^4\\
0 & 0 & 0 & -222\cdot 13 & -184\cdot 13^2 & -51\cdot 13^3 & -5\cdot 13^4\\
0 & 0 & 0 & 20\cdot 13 & 38\cdot 13^2 & 13^4 & 19\cdot 13^3\\
0 & 0 & 0 & 0 & -74\cdot 13 & -37\cdot 13^2 & -5\cdot 13^3\\
0 & 0 & 0 & 0 & 6\cdot 13 & 7\cdot 13^2 & 15\cdot 13^2\\
0 & 0 & 0 & 0 & 0 & -13^2 & -3\cdot 13^2\\
0 & 0 & 0 & 0 & 0 & 13 & 7\cdot 13\\
0 & 0 & 0 & 0 & 0 & 0 & -13\\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{3.52}$$

The proof of the following lemma is analogous to that of Lemma 3.2.

Lemma 3.21. *If* n *is a positive integer then there are integers* c_m $(\lceil\frac{7n}{13}\rceil\le m\le 7n)$ *such that*

$$U_{13}(\mathcal{E}_{2,13}Z^n)=\mathcal{E}_{2,13}\sum_{m=\lceil\frac{7n}{13}\rceil}^{7n}c_mY^m,$$

where

$$Z(z)=Z_{13}(z)=\frac{\eta(169z)}{\eta(z)},\qquad Y(z)=\frac{\eta(13z)^2}{\eta(z)^2}. \tag{3.53}$$

We need a version for Lemma 3.21 when n is negative.

Lemma 3.22. *If n is a nonnegative integer then there are integers c_m $(-6n \le m \le n - \lceil \frac{6n}{13} \rceil)$ such that*

$$U_{13}(\mathcal{E}_{2,13} Z^{-n}) = \mathcal{E}_{2,13} \sum_{m=-6n}^{n-\lceil \frac{6n}{13} \rceil} c_m Y^{-m}.$$

Proof. The proof is analogous to Lemma 3.21. The main difference is that we write

$$U_{13}(\mathcal{E}_{2,13} Z^{-n}) = U_{13}\left(\mathcal{E}_{2,13}(z)\left(\eta(z)\eta^{11}(13z)\right)^n\right)\left(\eta^{11}(z)\eta(13z)\right)^{-n},$$

and use the fact that $\mathcal{E}_{2,13}(z)\left(\eta(z)\eta^{11}(13z)\right)^n \in M_{2+6n}(\Gamma_0(13), \left(\frac{\cdot}{13}\right)^n)$. □

Corollary 3.23.

$$\begin{aligned}
U_{13}(\mathcal{E}_{2,13}) &= \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-1}) &= -13\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-2}) &= 13\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-3}) &= -13^2\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-4}) &= 13^2\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-5}) &= 13\, \mathcal{E}_{2,13}(4\, Y^{-2} + 6 \cdot 13\, Y^{-1} + 13^2)\\
U_{13}(\mathcal{E}_{2,13} Z^{-6}) &= 13^3\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-7}) &= 13\, \mathcal{E}_{2,13}(-14\, Y^{-3} - 12 \cdot 13\, Y^{-2} + 13^3)\\
U_{13}(\mathcal{E}_{2,13} Z^{-8}) &= 13^4\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-9}) &= 13\, \mathcal{E}_{2,13}(18\, Y^{-4} - 36 \cdot 13\, Y^{-3} - 40 \cdot 13^2\, Y^{-2} - 14 \cdot 13^3\, Y^{-1} - 13^4)\\
U_{13}(\mathcal{E}_{2,13} Z^{-10}) &= 13^5\, \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13} Z^{-11}) &= 13\, \mathcal{E}_{2,13}(82\, Y^{-5} + 456 \cdot 13\, Y^{-4} + 360 \cdot 13^2\, Y^{-3} + 126 \cdot 13^3\, Y^{-2} + 18 \cdot 13^4\, Y^{-1} + 13^5)\\
U_{13}(\mathcal{E}_{2,13} Z^{-12}) &= 13^6\, \mathcal{E}_{2,13}
\end{aligned}$$

We need the 13th order modular equation that was used by Atkin and O'Brien [5] to study properties of $p(n)$ modulo powers of 13. Lehner [18] derived this equation earlier.

$$Z^{13}(z) = \sum_{r=1}^{13} \sum_{s=\lfloor \frac{1}{2}(r+2) \rfloor}^{7} \psi_{r,s} Y^s(13z)\, Z^{13-r}(z), \tag{3.54}$$

where the matrix $\Psi = (\psi_{i,j})$ is given in (3.52), and $Y(z)$, $Z(z)$ are given in (3.53). The modular equation and the matrix Ψ are given explicitly in Appendix C in [5]

Lemma 3.24. *For $i \ge 0$*

$$U_{13}(\mathcal{E}_{2,13} Z^i) = \mathcal{E}_{2,13}(z) \sum_{j=\lceil \frac{7i}{13} \rceil}^{7i} m_{i,j} Y^j,$$

where $Z = Z(z)$, $Y = Y(z)$ are defined in (3.53), and the $m_{i,j}$ are defined in Theorem 3.20.

Lemma 3.25. *For* $i \geq 0$,

$$U_{13}(\mathcal{E}_{2,13}Y^i) = \mathcal{E}_{2,13}(z) \sum_{j=\lceil \frac{i}{13} \rceil}^{13i} a_{i,j} Y^j, \tag{3.55}$$

$$U_{13}(\mathcal{E}_{2,13}ZY^i) = \mathcal{E}_{2,13}(z) \sum_{j=\lceil \frac{i+7}{13} \rceil}^{13i+7} b_{i,j} Y^j \tag{3.56}$$

where the $a_{i,j}$, $b_{i,j}$ *are defined in (3.50).*

Let $\pi_{13}(n)$ denote the exact power of 13 dividing n. Then

Lemma 3.26. *For* $i,\, j \geq 0$,

$$\pi_{13}(m_{i,j}) \geq \lfloor \tfrac{1}{14}(13j - 7i + 13) \rfloor, \tag{3.57}$$

where the matrix $M = (m_{i,j})$ *is defined in Theorem 3.20.*

Proof. As noted in [5] we observe that

$$\pi_{13}(\psi_{r,s}) \geq \lfloor \tfrac{1}{14}(13s - 7r + 13) \rfloor, \tag{3.58}$$

for all $1 \leq t \leq 13$ and $1 \leq s \leq 13$. We verify the result for $0 \leq i \leq 12$ by direct computation using the recurrence (3.51). We use (3.58), the recurrence (3.51) and Lemma 3.6 to prove the general result by induction. □

Corollary 3.27.

$$\pi_{13}(a_{i,j}) \geq \lfloor \tfrac{1}{14}(13j - i + 13) \rfloor, \qquad \pi_{13}(b_{i,j}) \geq \lfloor \tfrac{1}{14}(13j - i + 6) \rfloor,$$

where the $a_{i,j}$, $b_{i,j}$ *are defined by (3.50).*

We provide more complete details for the proof of the following lemma since congruences for the spt-function modulo 13 are stronger than those for the partition function.

Lemma 3.28.

$$\pi_{13}(x_{2,0}) = 1, \tag{3.59}$$

$$\pi_{13}(x_{2,j}) \geq 3 + \lfloor \tfrac{1}{14}(13j) \rfloor \qquad \textit{for } j \geq 1 \tag{3.60}$$

$$\pi_{13}(x_{2b-1,j}) \geq 2b - 2 + \lfloor \tfrac{1}{14}(13j - 10) \rfloor \qquad \textit{for } b \geq 2, \textit{ and } j \geq 1 \tag{3.61}$$

$$\pi_{13}(x_{2b,j}) \geq 2b - 1 + \lfloor \tfrac{1}{14}(13j) \rfloor \qquad \textit{for } b \geq 2, \textit{ and } j \geq 1. \tag{3.62}$$

Proof. We have calculated $\vec{x}_2$ and verified (3.59)–(3.60). We note that $x_{2,j} = 0$ for $j > 91$. Now,

$$x_{3,j} = \sum_{i \geq 0} x_{2,i} b_{i,j},$$

and we note that $x_{3,0} = 0$. We have

$$\pi_{13}(x_{2,0}b_{0,j}) = 1 + \pi_{13}(b_{0,j}) \geq 2 + \lfloor \tfrac{1}{14}(13j - 8) \rfloor$$

by Corollary 3.27. For $i \geq 1$

$$\begin{aligned} \pi_{13}(x_{2,i}b_{i,j}) = \pi_{13}(x_{2,i}) + \pi_{13}(b_{i,j}) &\geq 3 + \lfloor \tfrac{1}{14}(13i) \rfloor + \lfloor \tfrac{1}{14}(13j - i + 6) \rfloor \\ &\geq 3 + \lfloor \tfrac{1}{14}(13j + 12i - 7) \rfloor \geq 2 + \lfloor \tfrac{1}{14}(13j - 9) \rfloor, \end{aligned}$$

again by Corollary 3.27. It follows that

$$\pi_{13}(x_{3,j}) \geq 2 + \lfloor \tfrac{1}{14}(13j - 9) \rfloor,$$

and (3.61) holds for $b = 2$. Now supposed $b \geq 2$ is fixed and that (3.61) holds. We have

$$x_{2b,j} = \sum_{i\geq 1} x_{2b-1,i} a_{i,j}.$$

Now

$$\pi_{13}(x_{2b-1,1}a_{1,j}) = \pi_{13}(x_{2b-1,1}) + \pi_{13}(a_{1,j}) \geq 2b-2+\pi_{13}(a_{1,j}) \geq 2b-1+\lfloor \tfrac{1}{14}(13j)\rfloor,$$

by a direct calculation noting that $a_{1,j} = 0$ for $j > 13$. For $i \geq 2$

$$\begin{aligned}\pi_{13}(x_{2b-1,i}a_{i,j}) &= \pi_{13}(x_{2b-1,i}) + \pi_{13}(a_{i,j}) \geq 2b-2+\lfloor \tfrac{1}{14}(13i-10)\rfloor + \lfloor \tfrac{1}{14}(13j-i+13)\rfloor \\ &\geq 2b-2+\lfloor \tfrac{1}{14}(13j+12i-10)\rfloor \geq 2b-1+\lfloor \tfrac{1}{14}(13j)\rfloor,\end{aligned}$$

again by Corollary 3.27. It follows that

$$\pi_{13}(x_{2b,j}) \geq 2b-1+\lfloor \tfrac{1}{14}(13j)\rfloor,$$

and (3.62) holds. For $i \geq 1$

Again suppose $b \geq 2$ is fixed, and that (3.62) holds. We have

$$x_{2b+1,j} = \sum_{i\geq 1} x_{2b,i} b_{i,j}.$$

For $i \geq 1$

$$\begin{aligned}\pi_{13}(x_{2b,i}b_{i,j}) &= \pi_{13}(x_{2b,i}) + \pi_{13}(b_{i,j}) \geq 2b-1+\lfloor \tfrac{1}{14}(13i)\rfloor + \lfloor \tfrac{1}{14}(13j-i+6)\rfloor \\ &\geq 2b-1+\lfloor \tfrac{1}{14}(13j+12i-8)\rfloor \geq 2b+\lfloor \tfrac{1}{14}(13j-10)\rfloor,\end{aligned}$$

again by Corollary 3.27. It follows that

$$\pi_{13}(x_{2b+1,j}) \geq 2b+\lfloor \tfrac{1}{14}(13j-10)\rfloor,$$

and (3.61) holds with b replaced by $b+1$. Lemma 3.28 follows by induction. □

Corollary 3.29. *For $c \geq 2$,*

$$\mathbf{a}(13^c n + \gamma_c) - 13\cdot \mathbf{a}(13^{c-2}n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}. \tag{3.63}$$

For $a \geq 1$,

$$\operatorname{spt}(13^{a+2}n + \gamma_{a+2}) - 13\cdot \operatorname{spt}(13^a n + \gamma_a) \equiv 0 \pmod{13^{a+1}}, \tag{3.64}$$

$$\operatorname{spt}(13^a n + \gamma_a) \equiv 0 \pmod{13^{\lfloor \frac{a+1}{2}\rfloor}}. \tag{3.65}$$

We note that (3.63) holds when $c = 2$ by taking $\gamma_0 = 1$. Also when $a = 0$ the congruence (3.64) has a stronger form. The proof of the congruence

$$\operatorname{spt}(169n-7) - 13\cdot \operatorname{spt}(n) \equiv 0 \pmod{169}. \tag{3.66}$$

is analogous to the proof of (3.24).

4. The spt-function modulo ℓ

In this section we improve on results in [13] and [9] for the spt-function and the second moment rank function modulo ℓ. We let

$$J_\ell(z) = \sum_{n=s_\ell}^{\infty} j_\ell(n)q^n,$$

where $J_\ell(z)$ is defined in (2.21), and define

$$K_\ell(z) := G_\ell(z) + (-1)^{\frac{1}{2}(\ell-1)}\,\ell \sum_{n=\lceil \frac{s_\ell}{\ell}\rceil}^{\infty} j_\ell(\ell n)q^n, \tag{4.1}$$

where $G_\ell(z)$ is defined in (2.12). Then we have

Theorem 4.1. *If $\ell \geq 5$ is prime, then $K_\ell(z)$ is an entire modular form of weight $(\ell+1)$ on the full modular group $SL_2(\mathbb{Z})$.*

Proof. Suppose $\ell \geq 5$ is prime. We utilize Serre's [20, pp.223–224] results on the trace of a modular form on $\Gamma_0(\ell)$. By Theorem 2.2 we know that $G_\ell(z)$ is an entire modular form of weight $(\ell+1)$ on $\Gamma_0(\ell)$. By [20, Lemma 7],

$$\mathrm{Tr}(G_\ell) = G_\ell + \ell^{1-\frac{1}{2}(\ell+1)}\, G_\ell \mid W \mid U \tag{4.2}$$

is an entire modular form of weight $(\ell+1)$ on $\mathrm{SL}_2(\mathbb{Z})$. See [20, pp.223–224] for definition of W, U and the notation used. From (2.19) we find that

$$G_\ell \mid W = (-1)^{\frac{1}{2}(\ell-1)}\,\ell^{\frac{1}{2}(\ell+1)}\,J_\ell. \tag{4.3}$$

From (4.1), (4.2) and (4.3) we see that

$$K_\ell = \mathrm{Tr}(G_\ell)$$

is an entire modular form of weight $(\ell+1)$ on $\mathrm{SL}_2(\mathbb{Z})$. ∎

We observed special cases of the following Corollary in [13, Theorem 6.1].

Corollary 4.2. *Suppose $\ell \geq 5$ is prime. Then*

$$\sum_{n=\lceil \frac{\ell}{24}\rceil}^{\infty} \mathrm{spt}(\ell n - s_\ell)q^{n-\frac{\ell}{24}} \equiv \eta(z)^{r_\ell}\, L_\ell(z) \pmod{\ell} \tag{4.4}$$

for some integral entire modular form $L_\ell(z)$ on the full modular group of weight $\ell + 1 - 12\lceil\frac{\ell}{24}\rceil$, and where r_ℓ and s_ℓ are defined in (2.8).

Proof. Suppose $\ell \geq 5$ is prime. Since

$$(24n-1)\,p(n) \equiv 0 \pmod{\ell},$$

for $24n \equiv 1 \pmod{\ell}$, and using Theorem 4.1 we have

$$\frac{\eta(z)^{2\ell}}{\eta(\ell z)} \sum_{n=0}^{\infty} \mathbf{a}(\ell n - s_\ell)q^{n-\frac{\ell}{24}} \equiv P_\ell(z) \pmod{\ell},$$

for some integral $P_\ell(z) \in M_{\ell+1}(\Gamma(1))$. We note that

$$\mathrm{spt}(\ell n - s_\ell) \neq 0$$

implies that $\ell n - s_\ell \geq 1$ and $n \geq \lceil \frac{\ell}{24} \rceil$. It follows that

$$\frac{\eta(z)^{2\ell}}{\eta(\ell z)} \sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} \operatorname{spt}(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \Delta(z)^c \, L_\ell(z) \pmod{\ell},$$

where $\Delta(z)$ is Ramanujan's function

$$\Delta(z) := \eta(z)^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \tag{4.5}$$

$c = \lceil \frac{\ell}{24} \rceil$ and $L_\ell(z)$ is some integral modular form in $M_{\ell+1-12c}(\Gamma(1))$. Thus

$$\sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} \operatorname{spt}(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \Delta(z)^{c-\ell} \, L_\ell(z) \pmod{\ell},$$

and the result follows since

$$r_\ell = c - \ell.$$

□

We conclude the paper by improving a result in [9] for the second rank moment function. From (1.1)

$$N_2(n) = 2n \, p(n) - 2 \operatorname{spt}(n). \tag{4.6}$$

We note that the analog of Corollary 4.2 holds for the partition function $p(n)$ except the weight is 2 less. See either [13, Theorem 3.4] or [1, Theorem3]. This together with Corollary 4.2 and (4.6) implies

Corollary 4.3. *Suppose $\ell \geq 5$ is prime. Then*

$$\sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} N_2(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \eta(z)^{r_\ell} \left(Q_\ell(z) + L_\ell(z) \right) \pmod{\ell} \tag{4.7}$$

for some integral entire modular forms $Q_\ell(z)$ and $L_\ell(z)$ on the full modular group of weights k and $k+2$ respectively where $k = \ell - 1 - 12\lceil \frac{\ell}{24} \rceil$.

We illustrate Theorem 4.1 and Corollaries 4.2 and 4.3 in the case $\ell = 17$. We find that

$$K_{17}(z) = G_{17}(z) + 17 \sum_{n=1}^{\infty} j_{17}(17n) q^n = -17 \, E_6(z)^3 - 26148 \, \Delta(z) \, E_6(z),$$

$$\sum_{n=0}^{\infty} \operatorname{spt}(17n+5) q^{n+\frac{7}{24}} \equiv 14 \, \eta(z)^7 \, E_6(z) \pmod{17},$$

and

$$\sum_{n=0}^{\infty} N_2(17n+5) q^{n+\frac{7}{24}} \equiv \eta(z)^7 \left(2 \, E_4(z) + 6 \, E_6(z) \right) \pmod{17}.$$

Here $E_4(z)$ and $E_6(z)$ are the usual Eisenstein series

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{4.8}$$

where $\sigma_k(n) = \sum_{d|n} q^k$.

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References

1. S. Ahlgren and M. Boylan, *Arithmetic properties of the partition function*, Invent. Math. **153** (2003), 487–502. URL: http://dx.doi.org/10.1007/s00222-003-0295-6
2. G. E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math. **624** (2008), 133–142. URL: http://dx.doi.org/10.1515/CRELLE.2008.083
3. A. O. L. Atkin and F. G. Garvan, *Relations between the ranks and cranks of partitions*, Ramanujan J. **7** (2003), 343–366. URL: http://dx.doi.org/10.1023/A:1026219901284
4. A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160. URL: http://dx.doi.org/10.1007/BF01359701
5. A. O. L. Atkin, and J. N. O'Brien, *Some properties of* $p(n)$ *and* $c(n)$ *modulo powers of* 13, Trans. Amer. Math. Soc. **126** (1967), 442–459. URL: http://dx.doi.org/10.2307/1994308
6. A. O. L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Math. J. **8** (1967), 14–32. URL: http://dx.doi.org/10.1017/S0017089500000045
7. A. O. L. Atkin, *Ramanujan congruences for* $p_{-k}(n)$, Canad. J. Math. 20 (1968), 67-78; corrigendum, ibid. **21** (1968), 256. URL: http://cms.math.ca/cjm/v20/p67
8. K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J. **144** (2008), 195–233. URL: http://dx.doi.org/10.1215/00127094-2008-035
9. K. Bringmann, F. Garvan and K. Mahlburg, *Partition statistics and quasiharmonic Maass forms*, Int. Math. Res. Not. IMRN, Vol. 2009, No. 1, 63–97. URL: http://dx.doi.org/10.1093/imrn/rnn124
10. H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, Springer Lecture Notes, Vol. 627, 1977, 69–78. URL: http://dx.doi.org/10.1007/BFb0065297
11. A. Folsom and K. Ono, *The spt-function of Andrews*, Proc. Natl. Acad. Sci. USA **105** (2008), 20152–20156. URL: http://mathcs.emory.edu/~ono/publications-cv/pdfs/111.pdf
12. F. G. Garvan, *A simple proof of Watson's partition congruences for powers of* 7, J. Austral. Math. Soc. Ser. A **36** (1984), 316–334. URL: http://dx.doi.org/10.1017/S1446788700025386
13. F. G. Garvan, *Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank*, Int. J. Number Theory **6** (2010), 1–29. URL: http://dx.doi.org/10.1142/S179304211000296X
14. K. C. Garrett, Private communication, October 18, 2007.
15. M. D. Hirschhorn, and D. C. Hunt, *A simple proof of the Ramanujan conjecture for powers of* 5, J. Reine Angew. Math. **326** (1981), 1–17. URL: http://dx.doi.org/10.1515/crll.1981.326.1
16. L. J. P. Kilford, *Modular Forms*, A Classical and Computational Introduction, Imperial College Press, London, 2008.
17. M. I. Knopp, *Modular functions in analytic number theory* Markham Publishing Co., Chicago, Illinois, 1970.
18. J. Lehner, *Further congruence properties of the Fourier coefficients of the modular invariant* $j(\tau)$, Amer. J. Math. **71** (1949), 373–386. URL: http://dx.doi.org/10.2307/2372252
19. K. Ono, *Congruences for the Andrews spt-function*, Proc. Natl. Acad. Sci. USA, to appear. URL: http://mathcs.emory.edu/~ono/publications-cv/pdfs/132.pdf
20. J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques* in "Modular functions of one variable, III", (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 191–268, Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973. URL: http://dx.doi.org/10.1007/978-3-540-37802-0_4
21. G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481. URL: http://dx.doi.org/10.2307/1970831

22. G. N. Watson, *Ramanujans Vermutung Über Zerfällungszahlen*, J. reine angew. Math. **179** (1938), 97–128. URL: http://dx.doi.org/10.1515/crll.1938.179.97

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