CONGRUENCES FOR ANDREWS' SPT-FUNCTION MODULO 32760 AND EXTENSION OF ATKIN'S HECKE-TYPE PARTITION CONGRUENCES

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Dedicated to the memory of A.J. (Alf) van der Poorten, my former teacher

Abstract. New congruences are found for Andrews' smallest parts partition function $\mathrm{spt}(n)$. The generating function for $\mathrm{spt}(n)$ is related to the holomorphic part $\alpha(24z)$ of a certain weak Maass form $\mathcal{M}(z)$ of weight $\frac{3}{2}$. We show that a normalized form of the generating function for $\mathrm{spt}(n)$ is an eigenform modulo 72 for the Hecke operators $T(\ell^2)$ for primes $\ell > 3$, and an eigenform modulo p for $p = 5$, 7 or 13 provided that $(\ell, 6p) = 1$. The result for the modulus 3 was observed earlier by the author and considered by Ono and Folsom. Similar congruences for higher powers of p (namely 5^6, 7^4 and 13^2) occur for the coefficients of the function $\alpha(z)$. Analogous results for the partition function were found by Atkin in 1966. Our results depend on the recent result of Ono that $\mathcal{M}_\ell(z/24)$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ for the full modular group where

$$\mathcal{M}_\ell(z) = \mathcal{M}(z)|T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)\mathcal{M}(z).$$

1. Introduction

Andrews [1] defined the function $\mathrm{spt}(n)$ as the number of smallest parts in the partitions of n. He related this function to the second rank moment and proved some surprising congruences mod 5, 7 and 13. Rank and crank moments were introduced by A. O. L. Atkin and the author [2]. Bringmann [6] studied analytic, asymptotic and congruence properties of the generating function for the second rank moment as a quasi-weak Maass form. Further congruence properties of Andrews' spt-function were found by the author [10], [11], Folsom and Ono [8] and Ono [12]. In particular, Ono [12] proved that if $\left(\frac{1-24n}{\ell}\right) = 1$ then

$$\mathrm{spt}(\ell^2 n - \tfrac{1}{24}(\ell^2-1)) \equiv 0 \pmod{\ell}, \tag{1.1}$$

for any prime $\ell \geq 5$. This amazing result was originally conjectured by the author[(i)]. Earlier special cases were observed by Tina Garrett [9] and her students. Recently the author [11] has proved the following congruences for powers of 5, 7 and 13. For a, b, $c \geq 3$,

$$\mathrm{spt}(5^a n + \delta_a) + 5\,\mathrm{spt}(5^{a-2}n + \delta_{a-2}) \equiv 0 \pmod{5^{2a-3}}, \tag{1.2}$$

$$\mathrm{spt}(7^b n + \lambda_b) + 7\,\mathrm{spt}(7^{b-2}n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2)\rfloor}}, \tag{1.3}$$

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(i)The congruence (1.1) was first conjectured by the author in a Colloquium given at the University of Newcastle, Australia on July 17, 2008.

$$\operatorname{spt}(13^c n+\gamma_c) - 13\operatorname{spt}(13^{c-2}n+\gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}, \tag{1.4}$$

where δ_a, λ_b and γ_c are the least nonnegative residues of the reciprocals of 24 mod 5^a, 7^b and 13^c respectively.

As in [12], [11] we define

$$\mathbf{a}(n) := 12\operatorname{spt}(n) + (24n-1)p(n), \tag{1.5}$$

for $n \ge 0$, and define

$$\alpha(z) := \sum_{n\ge 0} \mathbf{a}(n) q^{n-\frac{1}{24}}, \tag{1.6}$$

where as usual $q = \exp(2\pi i z)$ and $\Im(z) > 0$. We note that $\operatorname{spt}(0) = 0$ and $p(0) = 1$. Bringmann [6] showed that $\alpha(24z)$ is the holomorphic part of the weight $\frac{3}{2}$ weak Maass form $\mathcal{M}(z)$ on $\Gamma_0(576)$ with Nebentypus χ_{12} where

$$\mathcal{M}(z) := \alpha(24z) - \frac{3i}{\pi\sqrt{2}}\int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)\, d\tau}{(-i(\tau+z))^{\frac{3}{2}}}, \tag{1.7}$$

$\eta(z) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function, the function $\alpha(z)$ is defined in (1.6), and

$$\chi_{12}(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Ono [12] showed that for $\ell \ge 5$ prime, the operator

$$T(\ell^2) - \chi_{12}(\ell)\ell(1+\ell) \tag{1.9}$$

annihilates the nonholomorphic part of $\mathcal{M}(z)$, and the function $\mathcal{M}_\ell(z/24)$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ for the full modular group where

$$\mathcal{M}_\ell(z) = \mathcal{M}(z) | T(\ell^2) - \chi_{12}(\ell)(1+\ell)\mathcal{M}(z) = \alpha(24z) | T(\ell^2) - \chi_{12}(\ell)(1+\ell)\alpha(24z). \tag{1.10}$$

In fact he obtained

Theorem 1.1 (Ono [12])**.** *If $\ell \ge 5$ is prime then the function*

$$\mathcal{M}_\ell(z/24)\,\eta(z)^{\ell^2} \tag{1.11}$$

is an entire modular form of weight $\frac{1}{2}(\ell^2+3)$ for the full modular group $\Gamma(1)$.

Applying this theorem Ono obtained

$$\mathcal{M}_\ell(z) \equiv 0 \pmod{\ell}. \tag{1.12}$$

The congruence (1.1) then follows easily.

Folsom and Ono [8] sketched the proof of the following

Theorem 1.2 (Folsom and Ono)**.** *If $\ell \ge 5$ is prime then*

$$\operatorname{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\operatorname{spt}(n) + \ell\operatorname{spt}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \chi_{12}(\ell)\,(1+\ell)\operatorname{spt}(n) \pmod{3}, \tag{1.13}$$

where

$$s_\ell = \frac{1}{24}(\ell^2-1). \tag{1.14}$$

This result was observed earlier by the author. In this paper we prove a much stronger result.

Theorem 1.3. (i) *If $\ell \geq 5$ is prime then*

$$\mathrm{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\mathrm{spt}(n) + \ell\,\mathrm{spt}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \chi_{12}(\ell)\,(1+\ell)\,\mathrm{spt}(n) \pmod{72}. \tag{1.15}$$

(ii) *If $\ell \geq 5$ is prime, $t = 5$, 7 or 13 and $\ell \neq t$ then*

$$\mathrm{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\mathrm{spt}(n) + \ell\,\mathrm{spt}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \chi_{12}(\ell)\,(1+\ell)\,\mathrm{spt}(n) \pmod{t}. \tag{1.16}$$

Of course this implies the

Corollary 1.4. *If ℓ is prime and $\ell \notin \{2,3,5,7,13\}$ then*

$$\mathrm{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\mathrm{spt}(n) + \ell\,\mathrm{spt}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \chi_{12}(\ell)\,(1+\ell)\,\mathrm{spt}(n) \pmod{32760}. \tag{1.17}$$

This congruence modulo $32760 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ is the congruence referred in the title of this paper.

In 1966, Atkin [4] found a similar congruence for the partition function.

Theorem 1.5 (Atkin)**.** *Let $t = 5$, 7, or 13, and $c = 6$, 4, or 2 respectively. Suppose $\ell \geq 5$ is prime and $\ell \neq t$. If $\left(\frac{1-24n}{t}\right) = -1$, then*

$$\ell^3\, p(\ell^2 n - s_\ell) + \ell\chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)p(n) + p\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \gamma_t\, p(n) \pmod{t^c}, \tag{1.18}$$

where γ_t is an integral constant independent of n.

We find that there is a corresponding result for the function $\mathbf{a}(n)$ defined in (1.5).

Theorem 1.6. *Let $t = 5$, 7, or 13, and $c = 6$, 4, or 2 respectively. Suppose $\ell \geq 5$ is prime and $\ell \neq t$. If $\left(\frac{1-24n}{t}\right) = -1$, then*

$$\mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\mathbf{a}(n) + \ell\,\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \chi_{12}(\ell)\,(1+\ell)\,\mathbf{a}(n) \pmod{t^c}. \tag{1.19}$$

In Section 2 we prove Theorem 1.3. The method involves reviewing the action of weight $-\frac{1}{2}$ Hecke operators $T(\ell^2)$ on the function $\eta(z)^{-1}$ and doing a careful study of the action of weight $\frac{3}{2}$ Hecke operators on the function $\frac{d}{dz}\eta(z)^{-1}$ modulo 5, 7, 13, 27 and 32. In Section 3 we prove Theorem 1.6. The method involves extending Atkin's [4] on modular functions to weight two modular forms on $\Gamma_0(t)$ for $t = 5$, 7 and 13. The proof of both Theorems 1.3 and 1.6 depend on Ono's Theorem 1.1.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Atkin [3] showed essentially that applying certain weight $-\frac{1}{2}$ Hecke operators $T(\ell^2)$ to the function $\eta(z)^{-1}$ produces a function with the same multiplier system as $\eta(z)^{-1}$ and thus $\eta(z)$ times this function is a certain polynomial (depending on ℓ) of Klein's modular invariant $j(z)$. We review Ono's [13] recent explicit form for these polynomials. Although our proof does not depend on Ono's result it is quite useful for computational purposes. The action of the corresponding weight $\frac{3}{2}$ Hecke operators on $\frac{d}{dz}\eta(z)^{-1}$ can be given in terms of the same polynomials. See Theorem 2.3 below. To finish the proof of the theorem we need to make a careful study of the action of these operators modulo 5, 7, 13, 27 and 32.

For $\ell \geq 5$ prime we define

$$Z_\ell(z) = \sum_{n=-s_\ell}^{\infty}\left(\ell^3\, p(\ell^2 n - s_\ell) + \ell\chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)p(n) + p\left(\frac{n+s_\ell}{\ell^2}\right)\right) q^{n-\frac{1}{24}}. \tag{2.1}$$

Proposition 2.1 (Atkin [4]). *The function $Z_\ell(z)\,\eta(z)$ is a modular function on the full modular group $\Gamma(1)$.*

It follows that $Z_\ell(z)\,\eta(z)$ is a polynomial in $j(z)$, where $j(z)$ is Klein's modular invariant

$$j(z) := \frac{E_4(z)^2}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots, \tag{2.2}$$

$E_2(z)$, $E_4(z)$, $E_6(z)$ are the usual Eisenstein series

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \qquad E_4(z) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \qquad E_6(z) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{2.3}$$

$\sigma_k(n) = \sum_{d|n} q^k$, and $\Delta(z)$ is Ramanujan's function

$$\Delta(z) := \eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{2.4}$$

In a recent paper, Ono [13] has found a nice formula for this polynomial. We define

$$E(q) := \prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}\eta(z), \tag{2.5}$$

and a sequence of polynomials $A_m(x) \in \mathbb{Z}[x]$ by

$$\begin{aligned}\sum_{m=0}^{\infty} A_m(x)q^m &= E(q)\,\frac{E_4(z)^2E_6(z)}{\Delta(z)}\,\frac{1}{j(z)-x} \\ &= 1 + (x-745)q + (x^2 - 1489x + 160511)q^2 + \cdots.\end{aligned} \tag{2.6}$$

Theorem 2.2 (Ono [13]). *For $\ell \ge 5$ prime*

$$Z_\ell(z)\,\eta(z) = \ell\,\chi_{12}(\ell) + A_{s_\ell}(j(z)), \tag{2.7}$$

where $Z_\ell(z)$ is given in (2.1), and s_ℓ is given in (1.14).

We define a sequence of polynomials $C_\ell(x) \in \mathbb{Z}[x]$ by

$$\begin{aligned}C_\ell(x) &:= \ell\,\chi_{12}(\ell) + A_{s_\ell}(x), \\ &= \sum_{n=0}^{s_\ell} c_{n,\ell}x^n,\end{aligned} \tag{2.8}$$

so that

$$Z_\ell(z)\,\eta(z) = C_\ell(j(z)). \tag{2.9}$$

We define

$$d(n) := (24n-1)\,p(n), \tag{2.10}$$

so that

$$\sum_{n=0}^{\infty} d(n)q^{24n-1} = q\frac{d}{dq}\frac{1}{\eta(24z)} = -\frac{E_2(24z)}{\eta(24z)}, \tag{2.11}$$

and

$$\mathbf{a}(n) = 12\mathrm{spt}(n) + d(n). \tag{2.12}$$

For $\ell \geq 5$ prime we define

$$\Xi_\ell(z) = \sum_{n=-s_\ell}^{\infty} \left(d(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell} \right) - 1 - \ell \right) d(n) + \ell\, d\left(\frac{n+s_\ell}{\ell^2} \right) \right) q^{n-\frac{1}{24}}. \tag{2.13}$$

We then have the following analogue of Theorem 2.2.

Theorem 2.3. *For $\ell \geq 5$ prime we have*

$$\begin{aligned} \ell\, \Xi_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} = &- \sum_{n=0}^{s_\ell} c_{n,\ell}\, E_4(z)^{3n-1}\, \Delta(z)^{s_\ell - n} \left(24nE_6(z) + E_4(z)E_2(z) \right) \\ &+ \chi_{12}(\ell)\ell(1+\ell)E_2(z)\, \Delta(z)^{s_\ell}, \end{aligned} \tag{2.14}$$

where the coefficients $c_{n,\ell}$ are defined by (2.6) and (2.8).

Proof. Suppose $\ell \geq 5$ is prime. In equation (2.9) we replace z by $24z$, apply the operator $q\frac{d}{dq}$ and replace z by $\frac{1}{24}z$ to obtain

$$\ell\, \Xi_\ell(z)\, \eta(z) = 24C'_\ell(j(z))\, q\frac{d}{dq}(j(z)) + (\chi_{12}(\ell)\ell(1+\ell) - C_\ell(j(z))\, E_2(z) \tag{2.15}$$

The result then follows easily from the identities

(2.16)

$$j(z)\, \Delta(z) = E_4(z)^3, \qquad q\frac{d}{dq}(\Delta(z)) = \Delta(z)\, E_2(z), \quad \text{and} \qquad q\frac{d}{dq}(j(z))\, \Delta(z) = -E_4(z)^2 E_6(z),$$

which we leave as an easy exercise. ☐

We are now ready to prove Theorem 1.3. A standard calculation gives the following congruences.

$$E_4(z)^3 - 720\, \Delta(z) \equiv 1 \pmod{65520}, \quad \text{and} \qquad E_2(z) \equiv E_4(z)^2 E_6(z) \pmod{65520}. \tag{2.17}$$

We now use (2.17) to reduce (2.15) modulo 65520.

(2.18)

$$\begin{aligned} &\ell\, \Xi_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} \\ &\equiv - \sum_{n=0}^{s_\ell} c_{n,\ell}\, E_4(z)^{3n-1}\, \Delta(z)^{s_\ell - n} \big(24nE_6(z)(E_4(z)^3 - 720\, \Delta(z)) + E_4(z)^3 E_6(z) \big) \\ &\qquad\quad + \chi_{12}(\ell)\ell(1+\ell)\, E_4(z)^2 E_6(z)\, \Delta(z)^{s_\ell} \pmod{65520} \\ &\equiv - \sum_{n=0}^{s_\ell} (24n+1) c_{n,\ell}\, E_4(z)^{3n+2}\, E_6(z)\, \Delta(z)^{s_\ell - n} \\ &\quad + \sum_{n=0}^{s_\ell} 720 \cdot 24n c_{n,\ell}\, E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell - n + 1} + \chi_{12}(\ell)\ell(1+\ell)\, E_4(z)^2 E_6(z)\, \Delta(z)^{s_\ell} \pmod{65520} \\ &\equiv (720\, c_{1,\ell} - c_{0,\ell} + \chi_{12}(\ell)\ell(1+\ell))\ E_4(z)^2\, E_6(z)\, \Delta(z)^{s_\ell} \\ &\quad + \sum_{n=1}^{s_\ell - 1} (720 \cdot 24(n+1) c_{n+1,\ell} - (24n+1) c_{n,\ell})\ E_4(z)^{3n+2}\, E_6(z)\, \Delta(z)^{s_\ell - n} \\ &\quad - (24s_\ell + 1) c_{s_\ell} E_4(z)^{3s_\ell + 2}\, E_6(z) \pmod{65520}. \end{aligned}$$

We define

$$\mathcal{A}_\ell(z) := \sum_{n=-s_\ell}^{\infty} \left(\mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell}\right) - 1 - \ell \right) \mathbf{a}(n) + \ell\, \mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right) \right) q^{n-\frac{1}{24}} \tag{2.19}$$

and

$$\mathcal{S}_\ell(z) := \sum_{n=1}^{\infty} \left(\operatorname{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell}\right) - 1 - \ell \right) \operatorname{spt}(n) + \ell \operatorname{spt}\left(\frac{n+s_\ell}{\ell^2}\right) \right) q^{n-\frac{1}{24}}, \tag{2.20}$$

so that

$$\mathcal{A}_\ell(z) = 12\, \mathcal{S}_\ell(z) + \Xi_\ell(z) = \mathcal{M}_\ell(z/24). \tag{2.21}$$

By Theorem 1.1 and equation (1.10) we see that the function

$$\ell \mathcal{A}_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} \in M_{\frac{1}{2}(\ell^2+3)}(\Gamma(1)), \tag{2.22}$$

the space of entire modular forms of weight $\frac{1}{2}(\ell^2+3)$ on $\Gamma(1)$. Since $\frac{1}{2}(\ell^2+3) = 2 + 12 s_\ell$ the set

$$\{E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell - n} \,:\, 1 \le n \le s_\ell\} \tag{2.23}$$

is a basis. Hence there are integers $b_{n,\ell}$ $(1 \le n \le s_\ell)$ such that

$$\mathcal{A}_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} = \sum_{n=1}^{s_\ell} b_{n,\ell} E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell-n}. \tag{2.24}$$

Using (2.17) we find that

$$\begin{aligned} \mathcal{A}_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} \equiv{}& -720 b_{1,\ell}\, E_4(z)^2\, E_6(z)\, \Delta(z)^{s_\ell} \\ & + \sum_{n=1}^{s_\ell-1} (b_{n,\ell} - 720 b_{n+1,\ell})\, E_4(z)^{3n+2}\, E_6(z)\, \Delta(z)^{s_\ell-n} \\ & + b_{s_\ell,\ell} E_4(z)^{3s_\ell+2}\, E_6(z) \pmod{65520}. \end{aligned} \tag{2.25}$$

By (2.18), (2.21) and (2.24) we deduce that there are integers $a_{n,\ell}$ $(0 \le n \le s_\ell)$ such that

$$12\, \ell\, \mathcal{S}_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} \equiv \sum_{n=0}^{s_\ell} a_{n,\ell}\, E_4(z)^{3n+2}\, E_6(z)\, \Delta(z)^{s_\ell-n} \pmod{65520}. \tag{2.26}$$

It follows that

$$12\, \ell\, \mathcal{S}_\ell(z) \equiv 0 \pmod{65520}, \tag{2.27}$$

since

$$\begin{aligned} \operatorname{ord}_{i\infty} (12\, \ell\, \mathcal{S}_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell}) &= s_\ell + 1, \\ 0 \le \operatorname{ord}_{i\infty} \left(E_4(z)^{3n+2}\, E_6(z)\, \Delta(z)^{s_\ell-n} \right) &\le s_\ell, \\ E_4(z)^{3n+2}\, E_6(z)\, \Delta(z)^{s_\ell-n} &= q^{s_\ell-n} + \cdots, \end{aligned} \tag{2.28}$$

for $0 \le n \le s_\ell$ and all functions have integral coefficients. Since $65550 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, the congruence (2.27) implies Part (ii) of Theorem 1.3. To prove Part (i) we need to work a little harder. We note that the congruence (2.27) does imply

$$\mathcal{S}_\ell(z) \equiv 0 \pmod{12}. \tag{2.29}$$

We need to show this congruence actually holds modulo 72.

First we show the congruence holds modulo 8 by studying $\Xi_\ell(z)$ modulo 32. We need the congruences,

$$E_2(z) \equiv E_4(z)\, E_6(z) + 16\Delta(z) \pmod{32}, \quad \text{and} \quad E_4(z)^2 \equiv 1 \pmod{32}, \tag{2.30}$$

which are routine to prove. We proceed as in the proof of (2.18) to find that

$$\begin{aligned} &\ell\, \Xi_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} \\ &\equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell} - 16c_{1,\ell})\ E_2(z)\, \Delta(z)^{s_\ell} \\ &\quad - \sum_{n=1}^{s_\ell - 1} ((24n+1)c_{n,\ell} + 16c_{n+1,\ell})\ E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell - n} \\ &\quad - (24s_\ell + 1)c_{s_\ell} E_4(z)^{3s_\ell - 1}\, E_6(z) \pmod{32}. \end{aligned} \tag{2.31}$$

By (2.31), (2.21) and (2.24) we deduce that there are integers $a'_{n,\ell}$ $(0 \le n \le s_\ell)$ such that

$$12\,\ell\,\mathcal{S}_\ell(z)\,\eta(z)\,\Delta(z)^{s_\ell} \equiv \sum_{n=1}^{s_\ell} a'_{n,\ell}\, E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell - n} + a'_{0,\ell}\, E_2(z)\, \Delta(z)^{s_\ell} \pmod{32}. \tag{2.32}$$

Arguing as before, it follows that

$$12\mathcal{S}_\ell(z) \equiv 0 \pmod{32}, \quad \text{and} \quad \mathcal{S}_\ell(z) \equiv 0 \pmod{8}. \tag{2.33}$$

To complete the proof, we need to study $\Xi_\ell(z)$ modulo 27. We need the congruences,

$$E_2(z) \equiv E_4(z)^5 + 18\Delta(z) \pmod{27} \quad \text{and} \quad E_6(z) \equiv E_4(z)^6 \pmod{27}, \tag{2.34}$$

which are routine to prove. We proceed as in the proof of (2.18) and (2.31) to find that

$$\begin{aligned} &\ell\, \Xi_\ell(z)\, \eta(z)\, \Delta(z)^{s_\ell} \\ &\equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell} - 18c_{1,\ell})\ E_2(z)\, \Delta(z)^{s_\ell} \\ &\quad - \sum_{n=1}^{s_\ell - 1} ((24n+1)c_{n,\ell} + 18c_{n+1,\ell})\ E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell - n} \\ &\quad - (24s_\ell + 1)c_{s_\ell} E_4(z)^{3s_\ell - 1}\, E_6(z) \pmod{27}. \end{aligned} \tag{2.35}$$

By (2.35), (2.21) and (2.24) we deduce that there are integers $a''_{n,\ell}$ $(0 \le n \le s_\ell)$ such that

$$12\,\ell\,\mathcal{S}_\ell(z)\,\eta(z)\,\Delta(z)^{s_\ell} \equiv \sum_{n=1}^{s_\ell} a''_{n,\ell}\, E_4(z)^{3n-1}\, E_6(z)\, \Delta(z)^{s_\ell - n} + a''_{0,\ell}\, E_2(z)\, \Delta(z)^{s_\ell} \pmod{27}. \tag{2.36}$$

Arguing as before, it follows that

$$12\mathcal{S}_\ell(z) \equiv 0 \pmod{27}, \quad \text{and} \quad \mathcal{S}_\ell(z) \equiv 0 \pmod{9}. \tag{2.37}$$

The congruences (2.33) and (2.37) give (1.15) and this completes the proof of Theorem 1.3.

3. Proof of Theorem 1.6

In this section we prove Theorem 1.6. Atkin [4] proved Theorem 1.5 by constructing certain special modular functions on $\Gamma_0(t)$ and $\Gamma_0(t^2)$ for $t = 5$, 7 and 13. We attack the problem by extending Atkin's results to the corresponding weight 2 case.

Let $GL_2^+(\mathbb{R})$ denote the group of all real 2×2 matrices with positive determinant. $GL_2^+(\mathbb{R})$ acts on the complex upper half plane $\mathcal{H}$ by linear fractional transformations. We define the slash operator for modular forms of integer weight. Let $k \in \mathbb{Z}$. For a function $f : \mathcal{H} \longrightarrow \mathbb{C}$ and $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ we define

$$f(z) \mid_k L = f \mid_k L = f \mid L = (\det L)^{\frac{k}{2}} (cz+d)^{-k} f(Lz). \tag{3.1}$$

Let $\Gamma' \subset \Gamma(1)$ (a subgroup of finite index). We say $f(z)$ is a *weakly holomorphic modular form* of weight k on Γ' if $f(z)$ is holomorphic on the upper half plane $\mathcal{H}$, $f(z) \mid_k L = f(z)$ for all L in Γ', and $f(z)$ has at most polar singularities in the local variables at the cusps of the fundamental region of Γ'. We say $f(z)$ is a *weakly holomorphic modular function* if it is a weakly holomorphic modular form of weight 0. We say $f(z)$ is an *entire modular form* of weight k on Γ' if it is a *weakly holomorphic modular form* that is holomorphic at the cusps of the fundamental region of Γ'. We denote the space of entire modular forms of weight k on Γ' by $M_k(\Gamma')$.

Suppose that $t \geq 5$ is prime. We need

$$W_t = W = \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad V_a = \begin{pmatrix} a & \lambda \\ t & a' \end{pmatrix}, \quad B_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},$$
$$T_{b,t} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad Q_{b,t} = \begin{pmatrix} 1/t & b/t \\ 0 & 1 \end{pmatrix},$$

where for $1 \leq a \leq t-1$, a' is uniquely defined by $1 \leq a' \leq t-1$, and $a'a - \lambda t = 1$. We have

$$B_t\, R^{at} = W_t\, V_a\, T_{-a'/t} \tag{3.2}$$
$$R^{at}\, W_t = W_{t^2}\, Q_{a,t}. \tag{3.3}$$

We define

$$\Phi_t(z) = \Phi(z) := \frac{\eta(z)}{\eta(t^2 z)}. \tag{3.4}$$

Then $\Phi_t(z)$ is a modular function of $\Gamma_0(t)$,

$$\Phi_t(z) \mid W_{t^2} = t\, \Phi_t(z)^{-1} \qquad ([4, (24)]), \tag{3.5}$$

and

$$\Phi_t(z) \mid R^{at} = \sqrt{t}\, e^{\pi i (t-1)/4}\, e^{-\pi i a' t/12} \left(\frac{a'}{t}\right) \frac{\eta(z)}{\eta(z - a'/t)} \qquad ([4, (25)]). \tag{3.6}$$

Although $E_2(z)$ is not a modular form, it well-known that

$$\mathcal{E}_{2,t}(z) := \frac{1}{t-1} \left(t\, E_2(tz) - E_2(z) \right), \tag{3.7}$$

is an entire modular form of weight 2 on $\Gamma_0(t)$ and

$$\mathcal{E}_{2,t}(z) \mid W_t = -\mathcal{E}_{2,t}(z). \tag{3.8}$$

Proposition 3.1. *Suppose $t \geq 5$ is prime, $K(z)$ is a weakly holomorphic modular function on $\Gamma_0(t)$, and*

$$S(z) = \mathcal{E}_{2,t}(tz)\, K^*(tz) \frac{\eta(z)}{\eta(t^2 z)} - \chi_{12}(t)\, \eta(z) \sum_{n=m}^{\infty} \left(\frac{1-24n}{t} \right) \beta_t(n) q^{n - \frac{1}{24}}, \tag{3.9}$$

where

$$\mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)} = \sum_{n=m}^{\infty} \beta_t(n) q^{n - \frac{1}{24}}, \tag{3.10}$$

and

$$K^*(z) = K(z) \mid W_t. \tag{3.11}$$

Then $S(z)$ is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t)$.

Proof. Suppose $t \ge 5$ is prime and $K(z)$, $K^*(z)$, $S(z)$ are defined as in the statement of the proposition. The function

$$H(z) := \mathcal{E}_{2,t}(tz)\, \Phi_t(z)\, K^*(tz) \tag{3.12}$$

is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t^2)$. As in [4, Lemma1] the function

$$S_1(z) := \sum_{a=0}^{t-1} H(z) \mid R^{at} \tag{3.13}$$

is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t)$. Utilizing (3.2), (3.6), (3.8) and the evaluation of a quadratic Gauss sum [5, (1.7)] we find that

(3.14)

$$S_1(z) = \mathcal{E}_{2,t}(tz)\, K^*(tz) \frac{\eta(z)}{\eta(t^2 z)} - \frac{1}{\sqrt{t}}\, e^{\pi i (t-1)/4}\, \eta(z) \sum_{a'=1}^{t-1} e^{-\pi i a' t/12} \left(\frac{a'}{t}\right) \mathcal{E}_{2,t}(z - a'/t)\, \frac{K(z - a'/t)}{\eta(z - a'/t)}$$

Here we have also used the fact that

$$\mathcal{E}_{2,t}(z) \mid R^{at} = -\frac{1}{t}\, \mathcal{E}_{2,t}(z - a'/t), \tag{3.15}$$

where $a\,a' \equiv 1 \pmod{t}$. Hence

$$S_1(z) = S(z).$$

This gives the result. □

We illustrate Proposition 3.1 with two examples:

$$S(z) = \mathcal{E}_{2,5}(z) \left(\frac{\eta(z)}{\eta(5z)}\right)^6 \qquad (K(z) = 1 \text{ and } t = 5) \tag{3.16}$$

and

$$S(z) = \mathcal{E}_{2,7}(z) \left(\left(\frac{\eta(z)}{\eta(7z)}\right)^8 + 3 \left(\frac{\eta(z)}{\eta(7z)}\right)^4\right) \qquad (K(z) = 1 \text{ and } t = 7). \tag{3.17}$$

Corollary 3.2. *Suppose $t \ge 5$ is prime and $S(z)$, $K(z)$ and the sequence $\beta_t(n)$ are defined as in Proposition 3.1. Then*

$$S(z) \mid W_t = -\eta(tz) \sum_{tn - s_t \ge m} \beta_t(tn - s_t) q^{n - \frac{t}{24}}. \tag{3.18}$$

Proof. The result follows easily from (3.3), (3.5) and (3.8). □

We illustrate the corollary by applying W to both sides of the equations (3.16)–(3.17):

(3.19)

$$\sum_{n=1}^{\infty} \beta_5(5n-1) q^{n - \frac{5}{24}} = 5^3\, \frac{\mathcal{E}_{2,5}(z)}{\eta(5z)} \left(\frac{\eta(5z)}{\eta(z)}\right)^6 \qquad (K(z) = 1 \text{ and } t = 5)$$

and

(3.20)
$$\sum_{n=1}^{\infty}\beta_7(7n-2)q^{n-\frac{7}{24}} = 7^2\,\frac{\mathcal{E}_{2,7}(z)}{\eta(7z)}\left(3\left(\frac{\eta(7z)}{\eta(z)}\right)^4 + 7^2\left(\frac{\eta(7z)}{\eta(z)}\right)^8\right) \qquad (K(z)=1 \text{ and } t=7).$$

For t and $K(z)$ as in Proposition 3.1 we define

$$\Psi_{t,K}(z) = \Psi_t(z) = \mathcal{E}_{2,t}(tz)\,\frac{K^*(tz)}{\eta(t^2z)} - \chi_{12}(t)\sum_{n=m}^{\infty}\left(\frac{1-24m}{t}\right)\beta_t(n)q^{n-\frac{1}{24}} - \sum_{t^2n-s_t\geq m}\beta_t(t^2n-s_t)q^{n-\frac{1}{24}}, \tag{3.21}$$

where $K^*(z)$ and the sequence $\beta_t(n)$ is defined in (3.10)–(3.11). We have the following analogue of 2.1.

Corollary 3.3. *The function $\Psi_{t,K}(z)\,\eta(z)$ is a weakly holomorphic modular form of weight 2 on the full modular group $\Gamma(1)$.*

Proof. Let $S(z)$ be defined as in (3.9), so that $S(z)$ is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t)$. By [14, Lemma 7], the function

$$S(z) + S(z)\ \mid\ W_t\ \mid\ U \tag{3.22}$$

is a modular form of weight 2 on $\Gamma(1)$. Here $U = U_t$ is the Atkin operator

$$g(z)\ \mid\ U_t = \frac{1}{t}\sum_{a=0}^{t-1} g\left(\frac{z+a}{t}\right). \tag{3.23}$$

The result then follows from applying the U-operator to equation (3.18). ∎

We illustrate the $K(z)=1$ case of Corollary 3.3 with two examples:

$$\Psi_5(z) = \frac{E_4(z)^2\,E_6(z)}{\eta(z)^{25}} \tag{3.24}$$

and

$$\Psi_7(z) = \frac{1}{\eta(z)^{49}}\left(E_4(z)^5\,E_6(z) - 745E_4(z)^2\,E_6(z)\,\Delta(z)\right). \tag{3.25}$$

We need a weight 2 analogue of [4, Lemma 3]. For $t = 5$, 7 or 13 the genus of $\Gamma_0(t)$ is zero, and a Hauptmodul is

$$G_t(z) := \left(\frac{\eta(z)}{\eta(tz)}\right)^{24/(t-1)}. \tag{3.26}$$

This function satisfies

$$G_t\left(\frac{-1}{tz}\right) = t^{12/(t-1)}G_t(z)^{-1}. \tag{3.27}$$

Proposition 3.4. *Suppose $t = 5$, 7 or 13, and let m be any negative integer such that $24m \not\equiv 1 \pmod{t}$. Suppose constants k_j $(1 \leq j \leq -m)$ are chosen so that*

$$\beta_t(n) = 0, \qquad \textit{for } m+1 \leq n \leq -1, \tag{3.28}$$

where

$$K(z) = G_t(z)^{-m} + \sum_{k=1}^{-m-1} k_j G_t(z)^j \tag{3.29}$$

and

$$\sum_{n=m}^{\infty}\beta_t(n)q^{n-\frac{1}{24}} = \mathcal{E}_{2,t}(z)\frac{K(z)}{\eta(z)}. \tag{3.30}$$

Then

$$\beta_t(n)=0, \qquad \text{for} \quad \left(\frac{1-24n}{t}\right) = -\left(\frac{1-24m}{t}\right). \tag{3.31}$$

Proof. Suppose $t = 5$, 7 or 13, and m is a negative integer such that $24m \not\equiv 1 \pmod{t}$. Suppose $K(z)$ is chosen so that (3.28) holds. Let $S(z)$ be defined as in (3.9), and define

$$B(z) := S(z) + \chi_{12}(t)\left(\frac{1-24m}{t}\right)\mathcal{E}_{2,t}(z)\,K(z), \tag{3.32}$$

so that

$$B(z) \mid W_t = S^*(z) - \chi_{12}(t)\left(\frac{1-24m}{t}\right)\mathcal{E}_{2,t}(z)\,K^*(z), \tag{3.33}$$

where

$$S^*(z) = S(z) \mid W_t. \tag{3.34}$$

Since $24m \not\equiv 1 \pmod{t}$, we see that

$$\operatorname{ord}_0(S(z)) = \operatorname{ord}_{i\infty}(S^*(z)) > 0. \tag{3.35}$$

From (3.27) and (3.29) we see that

$$\operatorname{ord}_0(K(z)) = \operatorname{ord}_{i\infty}(K^*(z)) > 0 \tag{3.36}$$

and hence

$$\operatorname{ord}_0(B(z)) > 0. \tag{3.37}$$

Now

$$\operatorname{ord}_{i\infty}\left(\mathcal{E}_{2,t}(tz)\,K^*(tz)\frac{\eta(z)}{\eta(t^2z)}\right) \ge t - \frac{1}{24}(t^2-1) > 0, \tag{3.38}$$

for $t = 5$, 7, 13. By construction the coefficient of q^m in $B(z)$ is zero and so (3.28), (3.38) imply that

$$\operatorname{ord}_{i\infty}(B(z)) \ge 0. \tag{3.39}$$

Therefore $B(z)$ is an entire modular form of weight 2 and hence a multiple of $\mathcal{E}_{2,t}(z)$ since there are no nontrivial cusp forms of weight 2 on $\Gamma_0(t)$ for $t = 5$, 7 or 13 by [7]. This implies that $B(z)$ is identically zero by (3.37). Hence

$$\frac{B(z)}{E(q)} = q^{-s_t}\mathcal{E}_{2,t}(tz)\,K^*(tz)\frac{1}{E(q^{t^2})} - \chi_{12}(t)\sum_{n=m}^{\infty}\left(\left(\frac{1-24n}{t}\right) - \left(\frac{1-24m}{t}\right)\right)\beta_t(n)q^n = 0. \tag{3.40}$$

Since $-24s_t - 1 \equiv 0 \pmod{t}$, this implies that $\beta_t(n) = 0$ whenever $\left(\frac{1-24n}{t}\right) = -\left(\frac{1-24m}{t}\right)$. □

We illustrate Proposition 3.4 with two examples:

$$\begin{aligned}\sum_{n=-2}^{\infty}\beta_5(n)q^n &= \frac{\mathcal{E}_{2,5}(z)}{E(q)}\,\left(G_5(z)^2 + 5\,G_5(z)\right) \\ &= q^{-2} + 1 - 379\,q^3 + 625\,q^4 + 869\,q^5 - 20125\,q^8 + 23125\,q^9 + 25636\,q^{10} - 329236\,q^{13} + \cdots .\end{aligned} \tag{3.41}$$

In this example, $t = 5$ and $m = -2$, and we see that $\beta_5(n) = 0$ for $n \equiv 1, 2 \pmod 5$. In our second example, $t = 7$ and $m = -1$.

(3.42)
$$\sum_{n=-1}^{\infty} \beta_7(n)q^n = \frac{\mathcal{E}_{2,7}(z)}{E(q)}\, G_7(z)$$
(3.43)
$$= q^{-1} + 1 - 15\,q^2 + 49\,q^5 - 24\,q^6 + 88\,q^7 - 311\,q^9 + 392\,q^{12} - 182\,q^{13} + 811\,q^{14} - 1886\,q^{16} + \cdots.$$

In this example we see that $\beta_7(n) = 0$ for $n \equiv 1, 3, 4 \pmod 7$.

The function

(3.44)
$$\frac{E_4(z)^2\,E_6(z)}{\Delta(z)} = \frac{E_6(z)}{E_4(z)}\, j(z)$$
$$= q^{-1} - 196884\,q - 42987520\,q^2 - 2592899910\,q^3 - 80983425024\,q^4 - 1666013203000\,q^5 + \cdots$$

is a modular form of weight 2 on $\Gamma(1)$. As a modular form on $\Gamma_0(t)$ it has a simple pole at $i\infty$ and a pole of order t at $z = 0$. When $t = 5$, 7 or 13, it is straightforward to show that there are integers $a_{j,t}$ $(-1 \le j \le t)$ such that

$$\frac{E_6(z)}{E_4(z)}\, j(z) = \mathcal{E}_{2,t}(z) \sum_{j=-1}^{t} a_{j,t} G_t(z)^j. \tag{3.45}$$

For example,

(3.46)
$$\frac{E_6(z)}{E_4(z)}\, j(z) = \mathcal{E}_{2,5}(z)\, \big(G_5(z) - 3^2 \cdot 5^5 \cdot 7\, G_5(z)^{-1} - \cdot 2^3 \cdot 5^8 \cdot 13\, G_5(z)^{-2} - \cdot 3^3 \cdot 5^{10} \cdot 7\, G_5(z)^{-3}$$
$$-3 \cdot 2^3 \cdot 5^{13}\, G_5(z)^{-4} - 5^{16}\, G_5(z)^{-5}\big).$$

Reducing (3.45) mod t^c we obtain a weight 2 analogue of [4, Lemma 4].

Lemma 3.5. *We have*

$$\frac{E_6(z)}{E_4(z)}\, j(z) \equiv \mathcal{E}_{2,5}(z)\, \big(G_5(z) + 2 \cdot 31 \cdot 5^5\, G_5(z)^{-1}\big) \pmod{5^8}, \tag{3.47}$$

$$\frac{E_6(z)}{E_4(z)}\, j(z) \equiv \mathcal{E}_{2,7}(z)\, G_7(z) \pmod{7^4}, \tag{3.48}$$

$$\frac{E_6(z)}{E_4(z)}\, j(z) \equiv \mathcal{E}_{2,13}(z)\, G_{13}(z) \pmod{13^2}. \tag{3.49}$$

We also need [4, Lemma 4].

Lemma 3.6 (Atkin [4])**.** *[Atkin* [4]*] We have*

$$j(z) \equiv G_5(z) + 750 + 3^2 \cdot 7 \cdot 5^5\, G_5(z)^{-1} \pmod{5^8}, \tag{3.50}$$

$$j(z) \equiv G_7(z) + 748 \pmod{7^4}, \tag{3.51}$$

$$j(z) \equiv G_{13}(z) + 70 \pmod{13^2}. \tag{3.52}$$

Remark 3.7. In equation (3.50) we have corrected a misprint in [4, Lemma 4].

To handle the $(t, c) = (5, 6)$ case of Theorem 1.6 we will need

Lemma 3.8.

$$\frac{E_6(z)}{E_4(z)}\, j(z) \equiv \mathcal{E}_{2,5}(z)\left(G_5(z) + 2\cdot 5^5\, G_5(z)^{-1}\right) \pmod{5^6}, \tag{3.53}$$

$$\frac{E_6(z)}{E_4(z)}\, j(z)^2 \equiv \mathcal{E}_{2,5}(z)\left(2\cdot 3\cdot 5^3\, G_5(z) + G_5(z)^2\right) \pmod{5^6}, \tag{3.54}$$

and

$$\frac{E_6(z)}{E_4(z)}\, j(z)^a \equiv \mathcal{E}_{2,5}(z)\left(\varepsilon_{1,a}\, G_5(z)^{a-2} + \varepsilon_{2,a}\, G_5(z)^{a-1} + G_5(z)^a\right) \pmod{5^6}, \tag{3.55}$$

for $a \ge 3$, where $\varepsilon_{1,a}$, $\varepsilon_{2,a}$ are integers satisfying

$$\varepsilon_{1,a} \equiv 0 \pmod{5^5} \quad \text{and} \qquad \varepsilon_{2,a} \equiv 0 \pmod{5^3}. \tag{3.56}$$

Proof. The result can be proved from Lemmas 3.5 and 3.6, some calculation and an easy induction argument. ☐

We need bases for $M_{2+12s_\ell}(\Gamma_0(t))$ for $t = 5$, 7, 13. The following result follows from [7] and by checking the modular forms involved are holomorphic at the cusps $i\infty$ and 0.

Lemma 3.9. *Suppose $t = 5$, 7 or 13 and $\ell > 3$ is prime. Then*

$$\dim M_{2+12s_\ell}(\Gamma_0(t)) = 1 + (1+t)\, s_\ell, \tag{3.57}$$

and the set

$$\{\mathcal{E}_{2,t}(z)\, \Delta(z)^{s_\ell}\, G_t(z)^a \,:\, -ts_\ell \le a \le s_\ell\} \tag{3.58}$$

is a basis for $M_{2+12s_\ell}(\Gamma_0(t))$.

We are now ready to prove Theorem 1.6. We have two cases:

Case 1. In the first case we assume that $(t,c) = (5,5)$, $(7,4)$ or $(13,2)$. Suppose $\ell > 3$ is prime and $\ell \ne t$. By (2.24) we have

$$\mathcal{A}_\ell(z) = \sum_{n=1}^{s_\ell} b_{n,\ell} \frac{E_6(z)}{E_4(z)}\, \frac{j(z)^n}{\eta(z)}. \tag{3.59}$$

By Theorem 1.1, equation (2.21) and Lemma 3.9 we have

$$\mathcal{A}_\ell(z) = \sum_{n=-ts_\ell}^{s_\ell} d_{n,\ell} \frac{\mathcal{E}_{2,t}(z)}{\eta(z)}\, G_t(z)^n, \tag{3.60}$$

for some integers $d_{n,\ell}$ ($-ts_\ell \le n \le s_\ell$). Now let

$$K(z) = \sum_{n=1}^{s_\ell} d_{n,\ell} G_t(z)^n. \tag{3.61}$$

By using Lemmas 3.5 and 3.6 to reduce equation (3.59) modulo t^c and comparing the result with (3.60) we deduce that

$$\mathcal{A}_\ell(z) \equiv \mathcal{E}_{2,t}(z)\, \frac{K(z)}{\eta(z)} \pmod{t^c} \tag{3.62}$$

and that

$$d_{n,\ell} \equiv 0 \pmod{t^c}, \tag{3.63}$$

for $-ts_\ell \le n \le 0$. By examining (3.60) we see that

$$\mathcal{A}_\ell(z) = \mathcal{E}_{2,t}(z)\,\frac{K(z)}{\eta(z)} + O(q^{-\frac{1}{24}}). \tag{3.64}$$

So if we let

$$\mathcal{E}_{2,t}(z)\,\frac{K(z)}{\eta(z)} = \sum_{n=-s_\ell}^{\infty} \beta_{t,\ell}(n) q^{n-\frac{1}{24}},, \tag{3.65}$$

then (3.62) may be rewritten as

$$\mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\left(\frac{1-24n}{\ell}\right) - 1 - \ell\right)\mathbf{a}(n) + \ell\,\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \beta_{t,\ell}(n) \pmod{t^c} \tag{3.66}$$

and from (3.64) we have

$$\mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\left(\frac{1-24n}{\ell}\right) - 1 - \ell\right)\mathbf{a}(n) + \ell\,\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right) = \beta_{t,\ell}(n) \tag{3.67}$$

for $-s_\ell \le n \le -1$. Equation (3.67) implies that

$$\beta_{t,\ell}(-s_\ell) = -\ell \tag{3.68}$$

and

$$\beta_{t,\ell}(n) = 0, \tag{3.69}$$

for $-s_\ell \le n \le -1$. We can now apply Proposition 3.4 with $m = -s_\ell$ since $1 - 24m = \ell^2$ and $t \ne \ell$. Hence

$$\beta_{t,\ell}(n) = 0, \quad \text{provided} \quad \left(\frac{1-24n}{t}\right) = -1. \tag{3.70}$$

This gives Theorem 1.6 when $(t, c) = (5, 5)$, $(7, 4)$ or $(13, 2)$ by (3.66).

Case 2. We consider the remaining case $(t, c) = (5, 6)$ and assume $\ell > 5$ is prime. We proceed as in Case 1. This time when we use Lemma 3.8 to reduce (2.24) modulo 5^6 we see that the only extra term occurs when $n = 1$. We find that with $K(z)$ as before we have

$$\mathcal{A}_\ell(z) \equiv \mathcal{E}_{2,t}(z)\,\frac{K(z)}{\eta(z)} + b_{1,5}\cdot 2\cdot 5^5\,\frac{\mathcal{E}_{2,5}(z)}{\eta(z)}\,G_5(z)^{-1} \pmod{5^6}. \tag{3.71}$$

All that remains is to show that

$$b_{1,5} \equiv 0 \pmod{5}, \tag{3.72}$$

since then (3.62) actually holds modulo 5^6 and the rest of the proof proceeds as in Case 1. Since $E_4(z) \equiv 1 \pmod 5$ we may reduce (2.18) modulo 5 to obtain

$$\ell\,\Xi_\ell \equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell})\,\frac{E_6(z)}{E_4(z)}\,\frac{1}{\eta(z)} - \sum_{n=1}^{s_\ell}(24n+1)c_{n,\ell}\,\frac{E_6(z)}{E_4(z)}\,\frac{j(z)^n}{\eta(z)} \pmod{5}. \tag{3.73}$$

But

$$\mathcal{S}_\ell(z) \equiv 0 \pmod{5}, \tag{3.74}$$

by Theorem 1.3 (ii). Hence

$$\ell\mathcal{A}_\ell(z) \equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell})\,\frac{E_6(z)}{E_4(z)}\,\frac{1}{\eta(z)} - \sum_{n=1}^{s_\ell}(24n+1)c_{n,\ell}\,\frac{E_6(z)}{E_4(z)}\,\frac{j(z)^n}{\eta(z)} \pmod{5} \tag{3.75}$$

and we see that $b_{1,5}$ the coefficient of $\frac{E_6(z)}{E_4(z)}\frac{j(z)}{\eta(z)}$ is divisible by 5 as required. This completes the proof of Theorem 1.6.

We close by illustrating Theorem 1.6 when $t = 5$ and $\ell = 7$. In this case the theorem predicts that

$$\mathbf{a}(49n-2) - \left(\frac{1-24n}{7}\right)\mathbf{a}(n) + 7\mathbf{a}\left(\frac{n+2}{49}\right) \equiv -8\,\mathbf{a}(n) \pmod{5^6},$$

when $n \equiv 1, 2 \pmod 5$. When $n = 1$ this says

$$149077845 \equiv -280 \pmod{5^6},$$

which is easy to check.

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