

A proposal of a UCN experiment to check an earthquake waves model

Vladimir K.Ignatovich

*Frank Laboratory of Neutron Physics,
Joint Institute for Nuclear research,
Dubna, Moscow region 141980, Russia**

(Ω Dated: November 12, 2010)

Abstract

Elastic waves with transverse polarization inside incidence plane can create longitudinal surface wave (LSW) after reflection from a free surface. At a critical incidence angle this LSW accumulates energy density, which can be orders of magnitude higher than energy density of the incident transverse wave. A specially arranged vessel for storage of ultracold neutrons (UCN) can be used to verify this effect.

I. INTRODUCTION

It is known¹⁻⁶ that reflection from an interface of an elastic wave of a given mode in isotropic and anisotropic media is accompanied with mode conversion. In⁷ it was noticed that at grazing angles smaller than some critical angle the mode conversion can result in creation of a longitudinal surface wave (LSW), which is alike to those appearing at earthquakes. The most prominent property of LSW is that critical angle it can accumulate an energy density orders of magnitude larger than that of the incident plain wave.

In the next section the theory of elastic waves in isotropic media is presented according to⁷, and in the third section an experiment with ultracold neutrons (UCN) (see, for example⁸) is proposed to observe effect of LSW on UCN storage.

II. ELASTIC WAVES IN ISOTROPIC MEDIA

The main element of the theory of elastic waves⁹ is a displacement vector $\mathbf{u}(\mathbf{r}, t)$ of a material point at a position \mathbf{r} at a time t . Its Cartesian coordinates $u_i(\mathbf{r}, t)$ obey the Newtonian equation of motion

$$\rho \frac{\partial^2}{\partial t^2} u_i(\mathbf{r}, t) = \frac{\partial}{\partial x_j} \sigma_{ij}(\mathbf{r}, t), \quad (1)$$

where ρ is the mass density, x_j are components of the position vector \mathbf{r} , σ_{ij} is a stress tensor

$$\sigma_{ij} = c_{ijkl} u_{kl}, \quad (2)$$

which is proportional to the deformation or strain tensor u_{ij}

$$u_{jk} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right), \quad (3)$$

and coefficients of proportionality c_{ijkl} in (2) form a tensor which in isotropic media looks

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{kj} + \delta_{ik} \delta_{lj}), \quad (4)$$

where δ_{ij} is the Kronecker delta function, and λ and μ are Lamé elastic constants.

With the stress tensor (2), (4) the Newtonian equation of motion (1) for the displacement vector becomes

$$\rho \ddot{u}_i = \nabla_j \sigma_{ij} = \mu [\Delta u_i + \nabla_i (\nabla \cdot \mathbf{u})] + \lambda \nabla_i (\nabla \cdot \mathbf{u}). \quad (5)$$

Usually the displacement vector $\mathbf{u}(\mathbf{r}, t)$ is represented as a sum of two parts $\mathbf{u}(\mathbf{r}, t) = \nabla\varphi + \nabla \times \boldsymbol{\psi}$, where φ is a scalar and $\boldsymbol{\psi}$ is a vector potentials. This tradition was broken in ⁷, and it helped to find new unexpected effects. The solution of (5) is represented in the form of a complex plane wave

$$\mathbf{u}(\mathbf{r}, t) = u_0 \mathbf{A} \exp(i\mathbf{k}\mathbf{r} - i\omega t), \quad (6)$$

where vector \mathbf{A} is a unit polarization vector, and u_0 is the wave amplitude with dimension of length. After substitution of \mathbf{u} from (6) into (5) one obtains an equation for \mathbf{A} :

$$\rho\omega^2 \mathbf{A} = \mu k^2 \mathbf{A} + (\lambda + \mu) \mathbf{k}(\mathbf{k} \cdot \mathbf{A}). \quad (7)$$

Because of the linearity of the equation the amplitude u_0 of (6) does not matter and for some time will be omitted.

For a given propagation direction $\boldsymbol{\kappa} = \mathbf{k}/k$ one can introduce two orthonormal vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$, which are perpendicular to $\boldsymbol{\kappa}$, and the orthonormal basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)} \equiv \boldsymbol{\kappa}$, in which the polarization vector \mathbf{A} looks

$$\mathbf{A} = \alpha^{(1)} \mathbf{e}^{(1)} + \alpha^{(2)} \mathbf{e}^{(2)} + \alpha^{(3)} \mathbf{e}^{(3)}. \quad (8)$$

After multiplication of (7) by each of the $\mathbf{e}^{(i)}$ one obtains three equations for the coordinates:

$$(\rho\omega^2 - \mu k^2) \alpha^{(1,2)} = 0, \quad (\rho\omega^2 - [\lambda + 2\mu] k^2) \alpha^{(3)} = 0. \quad (9)$$

They are independent and give three solutions: two for shear modes $\mathbf{A}^{1,2} = \mathbf{e}^{(1,2)}$ with speed $c_t = \sqrt{\mu/\rho}$ and wave number $k^{(1,2)} = \omega/c_t$, and one longitudinal mode $\mathbf{A}^3 = \boldsymbol{\kappa}$ with speed $c_l = \sqrt{(\lambda + 2\mu)/\rho}$ and wave number $k^3 = \omega/c_l$.

Consider now reflection from a free surface at $z = 0$ of the wave (6) propagating in the medium at $z < 0$. In this case it is convenient to choose basis vectors $\mathbf{e}^{(1,2)} \perp \boldsymbol{\kappa}$ in such a way, that $\mathbf{e}^{(1)}$ is perpendicular to the incidence plain, and $\mathbf{e}^{(2)}$ lies in it. The most interesting is reflection of the shear mode $\mathbf{A}^{(2)} = \mathbf{e}^{(2)}$. The displacement vector, when the incident wave is of this mode, looks

$$\mathbf{u}(\mathbf{r}, t) = \exp(i\mathbf{k}_{\parallel}\mathbf{r}_{\parallel} - i\omega t) \left[\mathbf{A}^{(2)} e^{ik_{\perp}^{(2)}z} + r^{(22)} \mathbf{A}_R^{(2)} e^{-ik_{\perp}^{(2)}z} + r^{(32)} \mathbf{A}_R^{(3)} e^{-ik_{\perp}^{(3)}z} \right], \quad (10)$$

where $r^{(j2)}$ are reflection amplitudes of the mode 2 with transformation into mode $j = 2, 3$, and $k_{\perp}^{(j)} = \sqrt{k^{(j)2} - k_{\parallel}^2}$. It is important to notice that $k_{\perp}^{(3)} < k_{\perp}^{(2)}$, therefore reflected

longitudinal wave propagates in nonspecular direction. Its grazing angle $\varphi^{(3)}$ is less than $\varphi^{(2)}$ of the incident wave. Therefore at some critical angle $\varphi_c^{(2)} = \arccos(c_t/c_l)$ the longitudinal wave starts to propagate along the surface, and at $\varphi^{(2)} < \varphi_c^{(2)}$ the component $k_{\perp}^{(3)}$ becomes imaginary, i.e. the longitudinal reflected wave becomes LSW. This is the most interesting phenomenon which, is a model of the earthquake waves, because the most devastating effect of earthquakes results from vibrations along the surface.

Let's find the energy of the LSW¹⁰. To do that we need to know reflection amplitudes. They are found from the boundary condition, which requires the stress vector \mathbf{T} with components $T_j = \sigma_{jl}n_l$, where n_j are components of a unit vector \mathbf{n} along the normal, to be zero at $z = 0$. For stress tensor (2), (4) the stress vector for the displacement $\mathbf{u}(\mathbf{r}, t)$ is

$$\mathbf{T}(\mathbf{u}(\mathbf{r}, t)) = \lambda \mathbf{n}(\nabla \cdot \mathbf{u}) + \mu[\nabla(\mathbf{u} \cdot \mathbf{n}) + (\mathbf{n} \cdot \nabla)\mathbf{u}]. \quad (11)$$

Substitution of (10) in it gives boundary condition in the form

$$\mathbf{B}^{(2)} + r^{(22)}\mathbf{B}_r^{(2)} + r^{(32)}\mathbf{B}_r^{(3)} = 0, \quad (12)$$

where for every plane wave $\mathbf{A} \exp(i\mathbf{k}\mathbf{r})$ the vector \mathbf{B} is defined as

$$\mathbf{B} = \lambda \mathbf{n}(\mathbf{k} \cdot \mathbf{A}) + \mu[\mathbf{k}(\mathbf{A} \cdot \mathbf{n}) + \mathbf{A}(\mathbf{n} \cdot \mathbf{k})], \quad (13)$$

$\mathbf{B}^{(2)}$ is related to the incident wave with $\mathbf{k}^{(2)} = \mathbf{k}_{\parallel} + \mathbf{n}k_{\perp}^{(2)}$, and $\mathbf{B}_r^{(j)}$ ($j = 2, 3$) are related to the reflected waves with $\mathbf{k}^{(j)} = \mathbf{k}_{\parallel} - \mathbf{n}k_{\perp}^{(j)}$.

Multiplication of (12) by two unit vectors: \mathbf{n} and $\boldsymbol{\tau} = \mathbf{k}_{\parallel}/|\mathbf{k}_{\parallel}|$ gives a system of two equations for determination of $r^{(j2)}$. Their solution is

$$r^{(32)} = \frac{k^{(3)}}{k^{(2)}} \frac{4k_{\perp}^{(2)}k_{\parallel}(k^{(2)2} - 2k_{\parallel}^2)}{4k_{\perp}^{(3)}k_{\perp}^{(2)}k_{\parallel}^2 + (k^{(2)2} - 2k_{\parallel}^2)^2}, \quad r^{(22)} = \frac{4k_{\perp}^{(3)}k_{\perp}^{(2)}k_{\parallel}^2 - (k^{(2)2} - 2k_{\parallel}^2)^2}{4k_{\perp}^{(3)}k_{\perp}^{(2)}k_{\parallel}^2 + (k^{(2)2} - 2k_{\parallel}^2)^2}, \quad (14)$$

and it is seen that imaginary $k_{\perp}^{(3)}$ makes $|r^{(22)}| = 1$, which means the total reflection of the mode 2.

Because of energy conservation the energy flux density of the incident wave along the normal to the interface must be equal to the sum of energy flux densities of the reflected waves. The energy flux is a real quantity, and with our complex solutions it is defined¹⁰ as

$$J_i = -\frac{1}{2} \left[\sigma_{il}^* \frac{du_l}{dt} + \sigma_{il} \frac{du_l^*}{dt} \right] = -\text{Re} \left[\sigma_{il}^* \frac{du_l}{dt} \right], \quad (15)$$

where $\text{Re}(F)$ means real part of F , and $*$ means complex conjugation. After substitution of all the quantities we obtain

$$\mathbf{J} = \text{Re}(\rho\omega c_t^2 u_0^2 [E \mathbf{A}^* (\mathbf{k}^* \cdot \mathbf{A}) + \mathbf{k}^*]), \quad (16)$$

where $E = 1 + \lambda/\mu$, and the amplitude u_0 of the oscillations is taken into account in (6). Expression (16) is valid for real and complex wave vectors and polarizations. The absolute value of this flux for the incident transverse wave is

$$J = \rho\omega c_t^2 u_0^2 k. \quad (17)$$

The flux of the LSW along the surface with account of its factor $r^{(32)}$ is

$$J_S^3(z) = \rho\omega u_0^2 |r^{(32)}|^2 c_t^2 k_{\parallel} \exp(2K_l z). \quad (18)$$

The ratio $Q = J_S^3(z=0)/J$ of two fluxes after substitution of $r^{(32)}$ from (14) is

$$Q = \frac{k_{\parallel}}{k^{(2)}} \frac{c_t^2}{c_l^2} |r^{(23)}|^2 = \frac{k_{\parallel}}{k^{(2)}} \frac{16k_{\perp}^{(2)2} k_{\parallel}^2 (k^{(2)2} - 2k_{\parallel}^2)^2}{16K_l^2 k_{\perp}^{(2)2} k_{\parallel}^4 + (k^{(2)2} - 2k_{\parallel}^2)^4}, \quad (19)$$

or¹²

$$Q = \frac{\cos \varphi \sin^2(4\varphi)}{4(\cos^2 \varphi - \cos^2 \varphi_c) \sin^2(2\varphi) \cos^2(\varphi) + \cos^4(2\varphi)}, \quad (20)$$

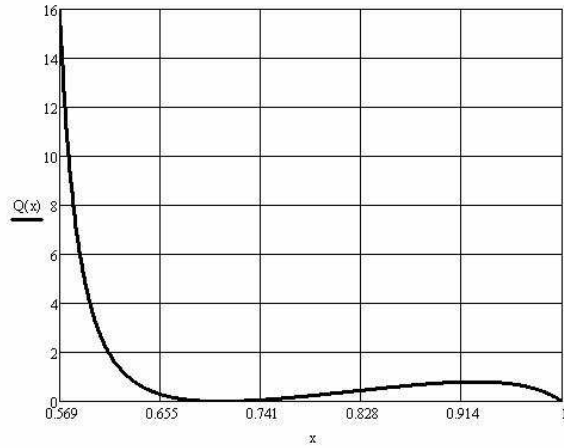


FIG. 1: Dependence of $Q(x)$ on $x = \cos \varphi$ calculated for $E \approx 2.3$ like in stainless steel. The left point on the abscise axis corresponds to critical $\cos \varphi_c = c_t/c_l \approx 0.57$, where $Q = Q_c = 16$. In the case of $E = 1.2$ the similar calculations give $\cos \varphi_c = c_t/c_l = 0.674$ and $Q_c = 323.616$.

where $\varphi \equiv \varphi^{(2)}$. Dependence of this function on $\cos \varphi$ is shown in Fig. 1. We see that the highest energy density is accumulated in longitudinal surface wave, when φ is slightly less than φ_c . The higher is the ratio c_t/c_l , the smaller is the critical grazing angle and the higher is accumulation of the LSW energy density near the critical angle. In the case of stainless steel the ratio c_t/c_l is near 0.56, and $Q(\varphi_c) \approx 16$.

Polarization of LSW is directed along the wave vector $\mathbf{k}^{(3)} = (\mathbf{k}_{\parallel}, iK_{\perp})$, but this vector has an imaginary normal component K_{\perp} . It means that vibrations in LSW have also normal component, which is shifted by phase $\pi/2$ with respect to vibrations along the surface. However near the critical angle the component K_{\perp} is small and vibrations along the normal can be neglected.

III. AN EXPERIMENT WITH UCN TO OBSERVE LSW

Scheme of the proposed experiment with UCN is shown in Fig. 2. The floor of the storage box for UCN is a thick basement. Several vertical thin plates are attached to it. And transducers of shear waves with polarization in the vertical plane generate ultrasound waves going to the storage floor at different angles. The UCN can be stored in the box or

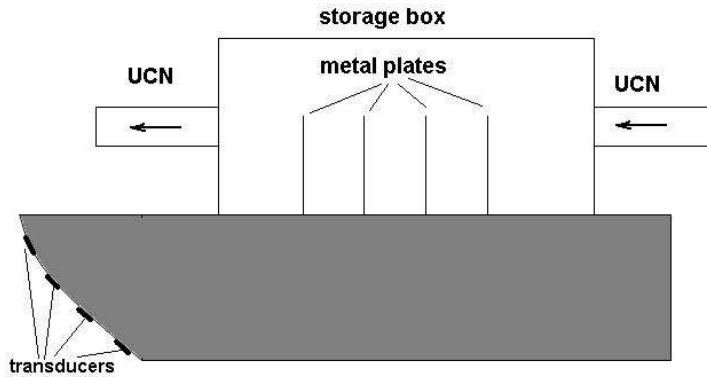


FIG. 2: Scheme of an experiment with UCN sensitive to LSW.

continuously flow through the box, and dependence of the storage time or transmission of the box on angle of incidence of the ultrasound wave is measured. If the grazing angle of the incident ultrasound wave is close and slightly below the critical one, the LSW created on the surface will make oscillating the vertical plates, and the UCN in the box will be heated at every collision with the plates and continuously increase their energy up to the limit after

which they will go away through the box walls. It decreases the storage time or transmission of the box.

For estimation of the effect one can consider some material like stainless steel with density $\rho \approx 8 \text{ g/cm}^3$, the speed of transverse waves $c_t \approx 3 \text{ km/s}$ and $c_t/c_l = 0.57$, like that used in Fig. 1. Then ultrasound with $\omega = 1 \text{ MHz}$ and amplitude $u_0 = 0.1 \mu$, has energy density $\rho\omega^2cu^2$ of the the order of 10 W/cm^2 . The speed v_0 of the horizontal vibrations on the box floor surface at critical incidence angle will be near $v_l = 0.4 \text{ m/s}$. It will create vibrations of the vertical plates with the same speed. Every collision of UCN with walls will increase in average their v^2 by $v_l^2 = 0.16 \text{ m}^2/\text{s}^2$. Therefore, after 100 collisions with the walls their energy will be high enough to penetrate the walls and to escape from the storage box.

The dependence of the storage time τ or transmission of the box on cosine of the grazing angle φ is expected to look as shown in fig. 3. Observation of the dip will prove the

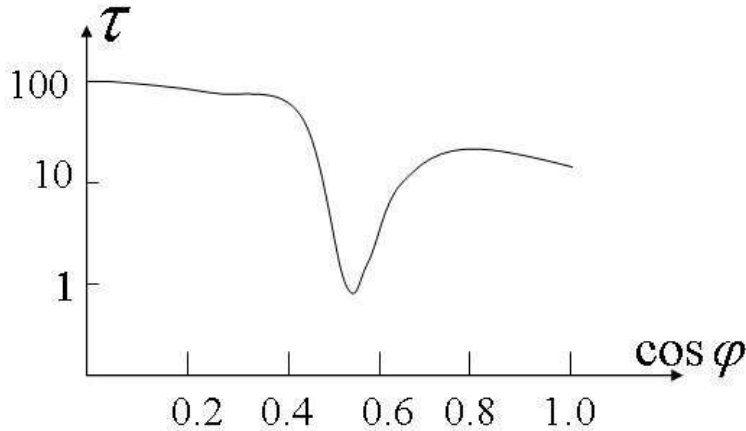


FIG. 3: Qualitative dependence of the storage time τ on $\cos \varphi$, where φ is the grazing angle of the incident sound wave. The position of the dip depends on ratio of c_t/c_l of the material used for basement in Fig. 2. The depth and the width of the dip depend on the power and frequency of the ultrasound.

appearance of the LSW and therefore will prove the validity of the earthquake model.

IV. CONCLUSION

The theory of LSW is presented. Though surface waves were studied extensively¹¹ the longitudinal one was discovered⁷ only because of nonstandard approach to theory of elas-

ticity. The LSW appears at reflection from a free surface of a shear wave with polarization in the incidence plane, when grazing angle of the incident wave is below of the critical one $\varphi_c = \arccos(c_t/c_l)$. Below and near φ_c the energy density of LSW can be orders of magnitude higher than energy density of the incident wave. From the properties of LSW one can conclude that they can have the same devastating effect as earthquakes. Therefore LSW can be considered as a model of earthquake waves. We described here an experiment, which can check the properties of LSW. It is related to storage or transmission of UCN. In the experiment one can see a sharp drop of storage time or transmission of UCN through the storage box when LSW appears on the surface of the storage box floor.

V. HISTORY OF SUBMISSIONS AND REJECTIONS

The paper was submitted to Phys.Rev.Lett. on 07.11. On 10.11 I received a short letter from Senior Assistant editor with words that my paper is not suitable for PRL.

* Electronic address: ignatovi@nf.jinr.ru

¹ W. P. Leung. "Demonstration of shear waves, Lamb waves and Rayleigh waves by mode conversion." Am. J. Phys. **48**, 539-42 (1980).

² P. M. Morse, H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953) Part 1.

³ E. Dieulesaint, D. Royer, *Ondes Élastiques dans les Solides* (Masson et C^{ie}, 1974)

⁴ R. Truell, C. Elbaum and B. B. Chick, *Ultrasonic Methods in Solid State Physics* (Academic Press, New York, 1969)

⁵ J. P. Musgrave, *Crystal acoustics* (San Francisco: Holden-Day, 1970)

⁶ F. I. Fedorov, *Theory of elastic waves in crystals* (New York: Plenum Press, 1968)

⁷ Vladimir K. Ignatovich, Loan T. N. Phan, "Those wonderful elastic waves," Am. J. Phys. **77**, 1162-72 (2009).

⁸ V. K. Ignatovich, "The physics of Ultracold neutrons" Plenum press, Oxford, 1990.

⁹ Lev D. Landau and Eugin M. Lifshitz, *Theoretical Physics, Vol 7 Theory of Elasticity*. (Oxford: Reed educational and Professional Publishing ltd, 1999)

- ¹⁰ A. P. Kiselev, “Energy flux of elastic waves,” *Journal of Soviet Mathematics*, **19**, 1372-1375. (1982), <http://www.springerlink.com/content/rq11hv616ju7q464/fulltext.pdf>
- ¹¹ I. A. Victorov, *Sound surface waves in solids* (M.: Nauka, 1981) (in Russian).
- ¹² In⁷ in this formula one factor $\cos^2 \varphi$ in denominator was missed