Oceanic rings and jets as statistical equilibrium states

Antoine Venaille *
GFDL-AOS, Princeton, O8540 NJ, US
Freddy Bouchet †
CNRS, ENS-Lyon, 69007 Lyon, France

ABSTRACT

Equilibrium statistical mechanics of two-dimensional flows provides an explanation and a prediction for the self-organization of large scale coherent structures. This theory is applied in this paper to the description of oceanic rings and jets, in the framework of a 1.5 layer quasi-geostrophic model. The theory predicts the spontaneous formation of regions where the potential vorticity is homogenized, with strong and localized jets at their interface. Mesoscale rings are shown to be close to a statistical equilibrium: the theory accounts for their shape, their drift, and their ubiquity in the ocean, independently of the underlying generation mechanism. At basin scale, inertial states presenting mid basin eastward jets (and then different from the classical Fofonoff solution) are described as marginally unstable states. In that case, considering a purely inertial limit is a first step toward more comprehensive out of equilibrium studies that would take into account other essential aspects, such as wind forcing.

1. Introduction

Large scale coherent structures are ubiquitous in the ocean. Understanding the physical mechanism underlying their formation and persistence remains a major theoretical challenge.

At mesoscale, oceanic turbulence is mostly organized into westward propagating rings, as revealed by altimetry (Chelton et al. 2007), or comprehensive numerical ocean models, e.g. (Hallberg and Gnanadesikan 2006). Since typical eddy turnover times are much shorter than dissipation and forcing time scales, these rings can be studied in the inertial limit, for which forcing and dissipation are neglected.

At basin scale, the dynamics are strongly influenced by forcing and dissipation: wind forcing plays the leading role in setting the gyre structures through the Sverdrup balance, and the concomitant effect of planetary vorticity gradients and dissipation explains their westward intensification (Pedlosky 1998). Because none of these mechanisms are conservative processes, the inertial approach does not take these essential aspects into account. Conversely, existing theories give no clear explanation of the existence of strong and robust eastward jets in the inertial part of these currents. The classical wind driven ocean theory and the inertial approach both give an incomplete picture, and complement each other. A useful step towards a comprehensive non-equilibrium theory that would combine both approaches is to study mid-basin eastward jets in the inertial limit. Such is the focus of this paper.

On the one hand, the problem of the self-organization of a turbulent flow involves a huge number of degrees of freedom coupled together via complex non-linear interactions. This situation makes any deterministic approach illusory, if not impossible. On the other hand, there can be abrupt and drastic changes in the large scale flow structure when varying a single parameter such as the energy of the flow, or its circulation. It is then appealing to study this problem with a statistical mechanics approach, which reduces the problem of large-scale organization of the flow to the study of states depending on a few key parameters only.

There exists a theory, the Robert-Sommeria-Miller (RSM hereafter) equilibrium statistical mechanics, that explains the spontaneous organization of unforced and undissipated two-dimensional and geophysical flows (Robert 1990; Miller 1990; Robert 1991; Robert and Sommeria 1991). From the knowledge of the energy and the global distribution of potential vorticity levels provided by an initial condition, this theory predicts the large scale flow as the most probable outcome of turbulent mixing.

The aim of this paper is to apply this theory in the framework of an 1.5 layer quasi-geostrophic model. More precisely, the following question is addressed: can rings and jets be interpreted as statistical equilibria ?

The first attempt to use equilibrium statistical mechan-

ics ideas to explain the self-organization of 2D turbulence was performed by Onsager (1949) in the framework of the point vortex model. In order to treat flows with continuous vorticity fields, another approach has been proposed by Kraichnan in the framework of the truncated Euler equations (Kraichnan and Montgomery 1980). The truncation has a drastic consequence: only the energy and the enstrophy are conserved quantities, while any function of the vorticity is conserved for the Euler equation. This work has inspired a quadratic-invariant statistical theory for quasi-geostrophic flows over topography (Salmon et al. 1976), which is also equivalent to the phenomenological approach of Bretherton and Haidvogel (1976), relying on a minimum enstrophy principle. These theories all predict the emergence of a large scale flow characterized by a linear relationship between streamfunction and potential vorticity.

The existence of such a linear relation was assumed by Fofonoff (1954) for analytical convenience, in earlier work on inertial ocean circulation, independently of statistical mechanics approaches. Fofonoff was able to compute explicitly an inertial solution in the low energy limit. The emergence of such flows has then been observed in numerical simulations of freely evolving quasi-geostrophic flows (Zou and Holloway 1994; Wang and Vallis 1994) . Energy-enstrophy theories have been proven successful to interpret these Fofonoff flows as statistical equilibria, but they have not been able to account for most of the observed inertial features of oceanic flows, such as coherent rings and mid-basin eastward jets.

For the Euler or quasi-geostrophic dynamics, a major drawback of energy-enstrophy statistical theories is the loss of the additional invariants of the dynamics, which has no other justification except for being able to solve the mathematics. The generalization of Onsager's ideas to the Euler and quasi-geostrophic equations with continuous vorticity field, taking into account all invariants, has lead to the RSM theory (Robert 1990; Miller 1990; Robert 1991; Robert and Sommeria 1991).

The RSM theory has made possible an explanation of the shape of the Red Great Spot of Jupiter (Bouchet and Sommeria 2002), as well as of its position in the Southern Hemisphere (Turkington et al. 2001). The theory has also been applied by Prieto and Schubert (2001) in the context of atmospheric dynamics to predic equilibrium states of the stratospheric polar vortex. In the oceanic context, RSM theory has been applied to the description of self-organization phenomena following deep convection events (Dibattista and Majda 2000; Dibattista et al. 2002), in the framework of the Heton model of Hogg and Stommel (1985).

Several studies, among which are Abramov and Majda (2003); Dubinkina and Frank (2010), have specifically addressed the importance of higher potential vorticity moments for the equilibrium states. Taking into account these additional dynamical invariants provides a much richer variety of equilibrium states than previous energy-enstrophy theories. For instance, RSM theory predicts for some initial conditions a relationship between streamfunction and potential vorticity close to a step function. Most solutions fall in that case in the limit of small Rossby radius of deformation and large energies (Bouchet 2001). Such relations are associated with fronts between regions of homogenized potential vorticity. It will be explained in this paper that most interesting statistical equilibria in the oceanic context (i.e. rings and jets) fall into this class of solutions. Computing a class of equilibrium states associated with specified relations between potential vorticity and streamfunction allows for more general results than an initial value problem, where one would compute the statistical equilibrium state associated with a given, particular initial condition.

Most of oceanic coherent structures are surface intensified, with most of their kinetic energy located above the thermocline (around 1000 m depth). In addition, eastward jets and rings are characterized by a jet width of the order of the first baroclinic Rossby radius of deformation. In this paper, we consider the simplest ocean model that takes into account this vertical structure and this typical horizontal length scale, namely an equivalent barotropic, 1.5 layer quasi-geostrophic model.

So far most oceanic applications of equilibrium statistical mechanics have been adressed in the limit of infinite Rossby radius of deformation, i.e. in the framework of the barotropic quasi-geostrophic model, see e.g. Salmon (1998). The novelty of the present paper is to consider the other limit, when the Rossby radius of deformation is small, which allows analytical computations of statistical equilibria, following the work of Bouchet and Sommeria (2002). This assumption provides important insights for more general situations, even when such a scale separation does not exist. This is also a first step before considering the shallow water model, which is consistent with the scale separation between the Rossby radius of deformation and the domain scale.

In the limit of small Rossby radius of deformation, it has been shown that the computation of RSM statistical equilibria can be simplified into a Van-der-Walls Cahn-Hilliard variational problem (Bouchet 2008). These variational problems explain the formation and the shape of bubbles in thermodynamics. The existence of this formal analogy between the formation of bubbles and the self-organization of potential vorticity in geophysical flows has been very fruitful in the description of Jovian vortices.

This paper puts forward this approach in the oceanic context. It is organized as follows. Equilibrium statistical mechanics of the 1.5 layer quasi-geostrophic model is presented in the first section. The method to compute analytically statistical equilibrium states in the limit of small

Rossby radius of deformation is presented on the second section. It allows for a justification of the potential vorticity homogenization theory of Rhines and Young (1982) without invoking any dissipation mechanism. These ideas are applied in the third section to the case of a zonal channel on a beta plane. It allows to interpret mesoscale rings as statistical equilibrium states, and to account for their observed drift. The case of mid basin eastward jets in a closed domain is studied in a fourth section. It is shown that such jets are critical points of the RSM theory: they can be either global or local entropy maxima, minima or saddle, depending on the parameters. Notations and symbols are referenced in table 1.

2. Statistical mechanics of the 1.5 layer, equivalent barotropic quasi-geostrophic model

a. *The 1.5 layer quasi-geostrophic model, its dynamical invariants and its dynamical equilibria*

The simplest possible inertial mid-latitude ocean model taking into account the stratification of the oceans and the sphericity of the Earth is considered in this paper. This is the unforced, undissipated, 1.5 layer quasi-geostrophic model on a beta plane:

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0, \quad \text{with } \mathbf{v} = \mathbf{e}_z \times \nabla\psi \ , \tag{1}$$

$$\text{and } q = \nabla^2\psi - \frac{\psi}{R^2} + \beta y \ . \tag{2}$$

For the boundary conditions, two cases will be distinguished, depending on the domain geometry $\mathcal{D}$. In the case of a closed domain, there is an impermeability constraint (no flow across the boundary), which amounts to a constant streamfunction along the boundary. To simplify the presentation, the condition $\psi = 0$ at boundaries will be considered[1]. In the case of a zonal channel, the streamfunction ψ is periodic in the x direction, and the impermeability constraint applies on northern and southern boundaries. In the remaining, length scales are nondimensionalized such that the domain area $|\mathcal{D}|$ is equal to one.

According to Noether's Theorem, each symmetry of the system is associated with the existence of a dynamical invariant, see e.g. Salmon (1998). These invariants are crucial quantities, because they provide strong constraints for the flow evolution. Starting from (1), (2) and the aforementioned boundary conditions one can prove that quasi-geostrophic flows conserve the energy:

$$E = \frac{1}{2}\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \left[(\nabla\psi)^2 + \frac{\psi^2}{R^2}\right] = -\frac{1}{2}\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, (q-\beta y)\,\psi. \tag{3}$$

[1] The physically relevant boundary condition should be $\psi = \psi_{fr}$ where ψ_{fr} is determined by using the mass conservation constraint $\int \mathrm{d}\mathbf{r}\ \psi = 0$ (ψ is proportional to interface variations). Taking $\psi = 0$ does not change quantitatively the solutions in the domain bulk, but only the strength of boundary jets.

Additionally, the quasi-geostrophic dynamics (1) is a transport by an incompressible flow, so that the area $\gamma(\sigma)\, d\sigma$ occupied by a given vorticity level σ is a dynamical invariant. The quantity $\gamma(\sigma)$ will be referred to as the global distribution of potential vorticity. The conservation of the distribution $\gamma(\sigma)$ is equivalent to the conservation of any moment of the potential vorticity $\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, q^n$, and is related to particle relabelling symmetry Ripa (1981); Salmon (1998).

The stationary points of the quasi-geostrophic equations (1), referred to as *dynamical equilibrium states*, satisfy $\mathbf{v}\cdot\nabla q = \nabla\psi\times\nabla q = 0$. It means that dynamical equilibria are flows for which streamlines are isolines of potential vorticity. Then, any state characterized by a $q-\psi$ functional relationship is a dynamical equilibrium.

At this point, we need a theory i) to support the idea that the freely evolving flow dynamics will effectively self-organize into a dynamical equilibrium state ii) to determine the $q-\psi$ relationship associated with this dynamical equilibrium iii) to select the dynamical equilibria that are likely to be observed. This is the goal and the achievement of equilibrium statistical mechanics theory, presented in the next subsection.

b. *The equilibrium statistical mechanics of Robert-Sommeria-Miller (RSM)*

The RSM statistical theory (Robert 1990; Miller 1990; Robert 1991; Robert and Sommeria 1991) is introduced on a heuristic level in the following. There exist rigorous justifications of the theory (Michel and Robert 1994; Ellis et al. 2004; Bouchet and Corvellec 2010).

A microscopic state is defined by its potential vorticity field $q(\mathbf{r})$. If taken as an initial condition, such a fine grained field would evolve toward a state with filamentation at smaller and smaller scales, while keeping in general a well defined large scale organization. Then, among all the possible fine grained states, an overwhelming number are characterized by these complicated small scale filamentary structures. This phenomenology gives a strong incentive for a mean-field approach, in which the flow is described at a coarse-grained level. For that purpose, the probability $\rho(\sigma,\mathbf{r})\mathrm{d}\sigma$ is introduced to measure a potential vorticity level σ at a point $\mathbf{r} = (x,y)$. The probability density field ρ defines a macroscopic state of the system. The corresponding averaged potential vorticity field, also referred to as *coarse-grained*, or *mean-field*, is $\overline{q}(\mathbf{r}) = \int_\Sigma \mathrm{d}\sigma\ \sigma\rho(\sigma,\mathbf{r})$, with the average streamfunction $\bar{\psi}$ defined by $\overline{q} = \nabla^2\bar{\psi} - \bar{\psi}/R^2 + \beta y$, and where $\Sigma =]-\infty,\ +\infty[$. Many microscopic states can be associated with a given macroscopic state. The cornerstone of the RSM statistical theory is the computation of the most probable state ρ_{eq},

TABLE 1. Symbols and notations used in the text.

Symbol	definition
$\mathbf{e}_{x,y,z}$	unit vectors in the meridional (x), zonal (y) and vertical (z) direction
$\mathbf{r} = (x, y)$	coordinate of a point, with $\nabla = (\partial_x,\ \partial_y)$
t	time coordinate
R	Rossby radius of deformation
f_0	Coriolis parameter
β	planetary vorticity gradient
$\mathcal{D}$	domain where the flow takes place (with $\lvert\mathcal{D}\rvert = 1$)
$\mathbf{u}(\mathbf{r})$	velocity field
$q(\mathbf{r}, t)$	(fine grained) potential vorticity field
σ	level of potential vorticity, with $\sigma \in \Sigma =]-\infty,\ +\infty[$
$\overline{q}$, $\overline{\psi}$	(coarse grained) mean field potential vorticity and streamfunction
E a	Energy of the flow
$\gamma(\sigma)$	global distribution of potential vorticity levels
$\rho(\mathbf{r}, \sigma)$	probability distribution function
$\mathcal{S}[\rho]$	Mixing entropy
$\mathcal{N}[\rho] = N$	normalization constraint
$D_\sigma[\rho] = \gamma(\sigma)$	constraint on global potential vorticity distribution
$\mathcal{E}[\rho] = E$	constraint on the energy
$\zeta(\mathbf{r})$	Lagrange multiplier associated with the normalization $\mathcal{N}$
$\alpha(\sigma)$	Lagrange multiplier associated with the global vorticity distribution D_σ
λ	Lagrange multiplier associated with the energy $\mathcal{E}$
$g_\alpha(\lambda\psi)$	$q - \psi$ relation at equilibrium
$\phi = \psi/R^2$	rescaled streamfunction
$\mathcal{F}[\phi]$, F	Free energy functional and equilibrium free energy
$f(\phi)$	specific free energy
$C = -\lambda R^2$	rescaled Lagrange multiplier associated with the energy constraint
$M = \int_\mathcal{D} \mathrm{d}\mathbf{r}\phi$	constraint for ϕ
μ	Lagrange parameter associated with M
ϕ_1, ϕ_2	values of ϕ in a given phase
q_1, q_2	values of q in a given phase
A_1, A_2	domain (and area) occupied by a given phase
L	perimeter of the interface between phases
r	curvature radius of the interface
η	Lagrange multiplier associated with the constraint on A_2 when minimizing L
$\tau = R\widetilde{\tau}$	coordinate across the interface
$\phi_{jet}(\tau)$	jet profile
$U_{jet} = (\phi_2 - \phi_1)R$	velocity of the jet
$\mathcal{F}_{int} = cRL(\phi_2 - \phi_1)^2$	Free energy of the interface, with $c \sim 1$
$\mathcal{L}[q] = \mathcal{L}^i = \mathcal{L}^f$	constraint on the linear momentum (i : inital ; f : final)
y_f, y_{jet}	latitude of the ring center of of jet latitude
$\widetilde{\beta} = \beta/R^2$	rescaled beta coefficient
$\mathcal{F}_\beta$	contribution of the beta term to the free energy
$l(x)$	perturbation of the zonal interface
$\mathcal{F}_R = \mathcal{F}_{int} + \mathcal{F}_\beta$	First order corrections to the free energy
L_x, L_y	zonal and meridional extension of the closed domain
L_{ring}	diameter of the ring
U_d	constant zonal flow in the deep layer

that maximizes the Boltzmann-Gibbs (or mixing) entropy

$$\mathcal{S}[\rho] \equiv -\int_\mathcal{D} \mathrm{d}\mathbf{r} \int_\Sigma \mathrm{d}\sigma\, \rho \log \rho\ , \tag{4}$$

while satisfying the constraints associated with each dy-

namical invariant. The mixing entropy (4) is a quantification of the number of microscopic states q corresponding to a given macroscopic state ρ. The state ρ_{eq} is not only the most probable one: an overwhelming number of microstates are effectively concentrated close to it (Michel and Robert 1994). This gives the physical explanation and the prediction of the large scale organization of the flow.

In the remaining of this paper, the term *global entropy maximum* or *stable equilibrium state* will be used for any global maximizer ρ of the entropy (4) satisfying the constraints. The term *local entropy maximum* or *metastable equilibrium state* will be used for any state ρ that is a local maximizer of the entropy (4), satisfying the constraints.

To compute statistical equilibria, the constraints must be expressed in term of the macroscopic state ρ:

- The local normalization $N[\rho](\mathbf{r}) \equiv \int_\Sigma \mathrm{d}\sigma\ \rho(\sigma,\mathbf{r}) = 1$,
- The global potential vorticity distribution $D_\sigma[\rho] \equiv \int_\mathcal{D} d\mathbf{r}\,\rho(\sigma,\mathbf{r}) = \gamma(\sigma)$,
- The energy $\mathcal{E}[\rho] \equiv -\frac{1}{2}\int_\mathcal{D} \mathrm{d}\mathbf{r} \int_\Sigma \mathrm{d}\sigma\,\rho(\sigma-\beta y)\,\overline{\psi} = E$.

Because of the overwhelming number of states with only small scale fluctuations around the mean field potential vorticity, and because energy is a large scale quantity, contributions of these fluctuations to the total energy are negligible with respect to the mean-field energy.

The first step toward computations of RSM equilibria is to find critical points ρ of the mixing entropy (4). In order to take into account the constraints, one needs to introduce the Lagrange multipliers $\zeta(\mathbf{r})$, $\alpha(\sigma)$, and λ associated respectively with the local normalization, the conservation of the global vorticity distribution and of the energy. Critical points are solutions of:

$$\forall\ \delta\rho \quad \delta\mathcal{S} - \lambda\delta\mathcal{E} - \int_\Sigma \mathrm{d}\sigma\ \alpha\delta D_\sigma - \int_\mathcal{D} \mathrm{d}\mathbf{r}\ \zeta\delta N = 0\ , \quad (5)$$

where first variations are taken with respect to ρ. This leads to $\rho = N\exp(\lambda\sigma\psi(\mathbf{r}) - \alpha(\sigma))$ where N is determined by the normalization constraint $\left(\int \mathrm{d}\sigma\ \rho = 1\right)$.

Statistical equilibria are dynamical equilibria characterized by a functional relation between potential vorticity and streamfunction:

$$\bar{q} = \frac{\int_\Sigma \mathrm{d}\sigma\ \sigma e^{\lambda\sigma\psi(\mathbf{r})-\alpha(\sigma)}}{\int_\Sigma \mathrm{d}\sigma\, e^{\sigma\lambda\psi(\mathbf{r})-\alpha(\sigma)}} = g_\alpha\left(\lambda\bar{\psi}\right) \quad (6)$$

It can be shown that g_α is a monotonic, increasing and bounded function of $\lambda\overline{\psi}$ for any global distribution $\gamma(\sigma)$ and energy E. These critical points can either be entropy minima, saddle or maxima. To find statistical equilibria, one needs then to select the entropy maxima.

At this point, two different approaches could be followed. The first one would be to consider a given small scale distribution $\gamma(\sigma)$ and energy E, and then to compute the statistical equilibria corresponding to these parameters. In practice, especially in the geophysical context, one does not have empirically access to the fine grained vorticity distribution, but rather to the $q-\psi$ relation (6) of the large scale flow. The second approach, followed in the remaining of this paper, is to study statistical equilibria corresponding to a given class of $q-\psi$ relations.

More precisely, we will consider the class of $q-\psi$ relations that admit an inflexion point, as for instance *tanh* functions (see figure 1 for a precise definition). It will be shown in the following that the existence of an inflexion point in the $q-\psi$ relation allows for statistical equilibria characterized by fronts of potential vorticity. In that respect, this is the relevant class of $q-\psi$ relations to describe either rings or zonal jets.

c. Simplification of the variational problem

In the general case, computing directly the entropy maxima is not an easy program. A major difficulty comes from the infinite number of constraints given by dynamical invariants. Rather than dealing with such complicated variational problem, a widely used method in statistical mechanics is to consider a dual variational problem, which has the same critical points as the initial one, but that is less constrained: any solution of the less constrained (easier to solve) problem is a solution of the more constrained problem, see e.g. Ellis et al. (2000).

Let us for instance consider the ensemble of rescaled streamfunction fields $\phi = \psi/R^2$, that satisfy the constraint $\int_\mathcal{D} \mathrm{d}\mathbf{r}\ \phi = M$. This constraint is equivalent to the conservation of the first moment of potential vorticity $\int_\mathcal{D} \mathrm{d}\mathbf{r}\ q$, at leading order in R. Let us then look for the minimizers of the free energy functional

$$\mathcal{F} = \int_\mathcal{D} \mathrm{d}\mathbf{r}\ \left[\frac{R^2}{2}(\nabla\phi)^2 + f(\phi) - \beta\phi y\right]\ , \quad (7)$$

where $f(\phi)$ is a specific free energy to be defined precisely in the next paragraph. This provides a variational problem

$$F = \min_\phi \left\{\mathcal{F}[\phi]\ \middle|\ \int_\mathcal{D} \mathrm{d}\mathbf{r}\ \phi = M\right\}\ , \quad (8)$$

which is much simpler than the one of the RSM statistical theory, since only one constraint is kept. Critical points of this problem are solutions of $\delta\mathcal{F} - \mu\int_\mathcal{D} \mathrm{d}\mathbf{r}\ \delta\phi = 0$, for any perturbation $\delta\phi$, where α is the Lagrange multiplier associated with the constraint. A part integration and the relation $q = R^2\nabla^2\phi - \phi + \beta y$ give $\delta\mathcal{F} = \int \mathrm{d}\mathbf{r}\ (f'(\phi) - \phi - q)\,\delta\phi$. Critical points satisfy therefore the relation $q = f'(\phi) - \phi - \mu$. These critical points are the same as the RSM critical points, given by equation (6), provided that

$$f'(\phi) = g_\alpha(\lambda R^2\phi) + \phi + \mu\ . \quad (9)$$

This relation defines the specific free energy f. One can see that the *tanh*-like shape of g_α with a sufficiently steep slope at its inflection point (which implies $\lambda < -C/R^2$, where C is a sufficiently large constant of order one) leads to a double-well shape for f, as illustrated on figure 1. This double-well shape will be an important ingredient for the computation of statistical equilibria in the following.

It has been proven by Bouchet (2008) that for each minimizer ϕ of the variational problem (8), there exists a set of constraints E, $\gamma(\sigma)$ such that ϕ is the mean-field streamfunction of the RMS statistical equilibrium state ρ associated with the constraints E, $\gamma(\sigma)$. In other words, any local (global) free energy minimizer ϕ can be interpreted as a local (global) entropy maximum state of the RMS theory. This result will be used in the remaining of this paper to compute RSM statistical equilibria by solving the variational problem (8).

3. Potential vorticity homogenization and jets as statistical equilibria

Assuming that there is no beta effect ($\beta = 0$), that $f(\phi)$ has a double-well shape, and considering the limit $R \ll 1$, the variational problem (8) becomes analogous to the Van der Walls – Cahn – Hilliard model that describes phase separation and phase coexistence in thermodynamics (Bouchet 2001, 2008). This formal analogy provides then an interesting physical interpretation of self-organization phenomena in geostrophic turbulence.

a. Solutions of the Van der Walls – Cahn – Hilliard variational problem

The remaining of this subsection is devoted to the resolution of this Van der Walls – Cahn – Hilliard variational problem, with heuristic considerations. There exist rigorous mathematical justifications of these physical arguments (Modica 1987; Bouchet 2001).

At zeroth order in R, the function $f(\phi)$ plays the dominant role in the free energy functional $\mathcal{F}$ given by (7). In order to minimize $\mathcal{F}$, the streamfunction ϕ must therefore be equal to one of the two minima of the specific free energy $f(\phi)$ (the points ϕ_1 and ϕ_2 on figure 1). Each of these minima corresponds to one phase. Without the constraint $\int_\mathcal{D} \mathrm{d}\mathbf{r}\phi = M$, one of the two uniform solutions $\phi = \phi_1$ or $\phi = \phi_2$ would minimize $\mathcal{F}$: the system would have only one phase. But in order to satisfy the constraint $\int_\mathcal{D} \mathrm{d}\mathbf{r}\ \phi = M$, the system has to split into sub-domains: part of it with phase $\phi = \phi_1$ and part of it with phase $\phi = \phi_2$. In terms of free energy minimization, the coexistence of these two phases is possible only if $f(\phi_1) = f(\phi_2)$. Using equation (9), one can always choose the Lagrange parameter μ to satisfy this condition, see figure 1 for a graphical interpretation. In physical space, the area occupied by each of the phases is denoted A_1 and A_2 respectively, see figure 2. These values are fixed by the constraint $\int_\mathcal{D} \mathrm{d}\mathbf{r}\ \phi = M$, which gives at leading order $\phi_1 A_1 + \phi_2 A_2 = M$ and by the geometrical constraint $A_1 + A_2 = 1$ (where 1 is the area of the domain).

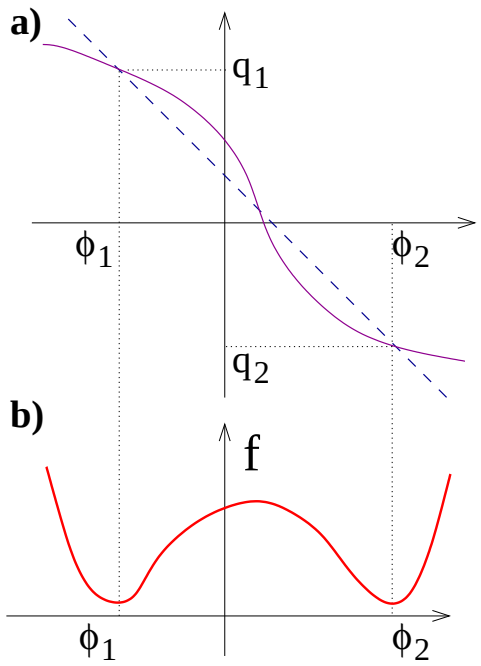

FIG. 1. **a)** In this paper, we consider the class of functions g_α, defined by equation (6), that have a *tanh*-like shape, namely i) increasing with $\lambda\psi$ ii) bounded for $\psi \to \pm\infty$ iii) with a single inflection point. In addition, we assume that $\lambda < -C/R^2$, where $Cg'_\alpha > 1$ at the inflexion point, so that the $q - \phi$ relation (with $\phi = \psi/R^2$), represented as a **plain line**, is decreasing and cross three times the **dashed line** $q = -\phi - \mu$. **b)** The double-well shape of the specific free energy $f(\phi)$ appearing in the expression of the free energy functional (7). This function is related to the $q-\psi$ relation through equation (9). The Lagrange parameter μ is chosen such that the specific free energy of the two minima are the same ($f(\phi_1) = f(\phi_2)$), which allows phase coexistence.

The interface between the sub-domains characterized by ϕ_1 and ϕ_2 corresponds to an abrupt variation of streamfunction. The term $R^2 (\nabla\phi)^2$ in the expression (7) of the free energy $\mathcal{F}$ is negligible everywhere except around this interface, on a typical width of order R. The interface is therefore associated with a strong and localized jet directed along this interface, with a typical velocity $U_{jet} = (\phi_1 - \phi_2) R$, and a typical width R.

The actual jet profile is computed in Appendix A, by minimizing the free energy associated with this profile. It is shown in this Appendix that the jet gives always a positive

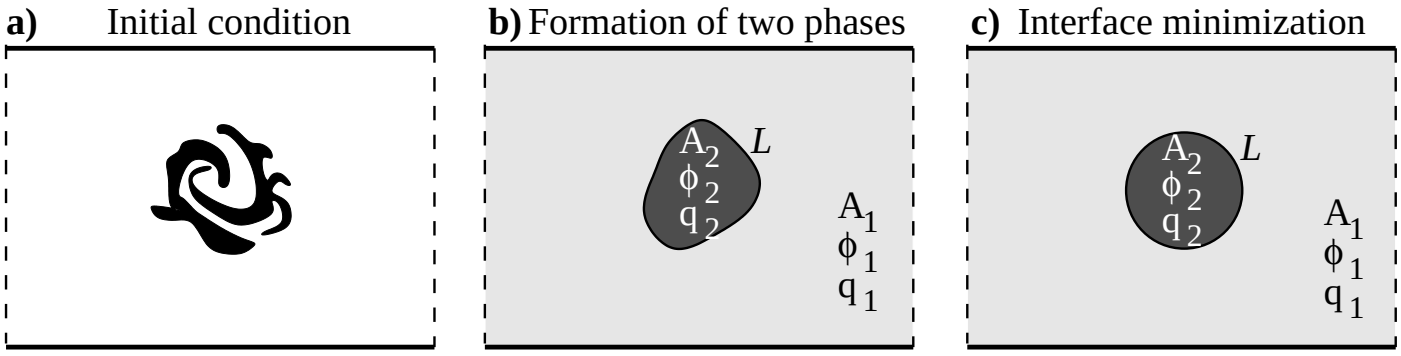

FIG. 2. Resolution of the Van der Walls variational problem (8). a) Example of an initial condition for the potential vorticity field. Note that this initial condition could as well have many levels of potential vorticity. b) At zeroth order in R, ϕ takes two values ϕ_1, ϕ_2 on two sub-domains A_1 and A_2 corresponding to the coexistence of two phases of homogenized potential vorticity q_1, q_2. These sub-domains are separated by strong jets of typical width $l_{jet} = R$ and velocity $U_{jet} = (\phi_2 - \phi_1)R$. c) The actual shape of the structure, or equivalently the position of the jets, is obtained by minimizing its perimeter L for a fixed value of A_2.

contribution to the free energy:

$$\mathcal{F}_{int} = c\,(\phi_2 - \phi_1)^2\, RL \ , \quad \text{with } c \sim 1 \ . \tag{10}$$

In order to minimize this interfacial free energy, the perimeter of the jet L must be minimal, taking into account the constraints given by the fixed areas A_1 and A_2. We thus look for the curve with the minimal length that bounds a given surface. The solution of this classical problem is that the interface is either a circle or a straight line.

To conclude, the computation of the statistical equilibria predicts i) the formation of two phases of constant streamfunction, with strong and localized jets at these interface ii) the structure of these jets iii) the shape of these jets, which is determined by an isoperimetrical problem: the minimization of the interface length for a fixed enclosed area.

b. Link with potential vorticity homogenization theories

The class of equilibria described above corresponds to those characterized by a $q - \psi$ relation with a single inflection point (a *tanh*-like shape), and with a sufficiently steep derivative at this inflexion point in order to ensure a double-well shape for the specific free energy $f(\phi)$. These $q-\psi$ relations imply that sub-domains of constant streamfunction are also sub-domains of constant coarse grained potential vorticity. It means that the potential vorticity is homogenized in these sub-domains.

Statistical mechanics provides therefore a physical explanation for the potential homogenization theory of Rhines and Young (1982), without invoking any dissipation mechanism. Potential vorticity homogenization is now a widely used concept in geophysical fluid dynamics, with, among others, application to thermocline theory (Rhines and Young 1982) or stratospheric polar vortex dynamics (McIntyre and Palmer 1983).

In the case of freely evolving 1.5 layer quasi-geostrophic dynamics, statistical mechanics predicts not only the spontaneous formation of regions where potential vorticity is homogenized at a coarse grained level, but also the shape of the interface between these regions, corresponding to jets, where vorticity gradients are confined. The formation of regions with different values of coarse grained potential vorticity is essential to insure conservation of the total energy. Importantly, such states could not be achieved with eddy parameterization based on local down-gradient diffusivities: this term would lead to the formation of a single phase of homogenized potential vorticity, which would in general not statisfy the energy constraint.

Statistical mechanics arguments also account for the irreversible nature of mixing: an overwhelming number of fine grained microscale are associated with a given coarse grained equilibrium state. The only effect of a weak small scale dissipation process would be to smooth out locally fine-grained fluctuations of potential vorticity, leaving unchanged its coarse-grained structure.

These results are complementary to previous work focusing on the dynamics of potential vorticity mixing, using chaotic advection theory (Pierrehumbert 1991). Chaotic advection theory has been proven successful to account for many observed feature of potential vorticity mixing, but unlike statistical mechanics, it provides in general no prediction for the final state of the large scale flow, especially when there is no scale separation between mean and eddies in the initial condition.

We discuss in the following how this tendency toward formation of subdomains with homogenized potential vorticity generalizes in presence of β effect, and we apply these results to the description of some features of geostrophic

turbulence in the ocean.

4. Application to oceanic rings

Observations show that mesoscale oceanic rings exist everywhere in the ocean, and particularly near western boundary currents, where high levels of eddy kinetic energy levels are largely associated with their presence, see e.g. Olson (1991) for a review and Morrow et al. (2004); Chelton et al. (2007) for recent altimetry measurements. Both cyclonic and anticyclonic rings propagate westward at speed βR^2. They also present an additional small meridional drift, poleward for cyclonic rings and equatorward for anti-cyclonic rings.

Among others, contributions of Nof (1981), of Flierl (1987) and of Cushman-Roisin et al. (1990) have provided insights on the dynamics and on the mechanisms responsible for the generation of the rings, in a large class of models and configurations. Various dynamical mechanisms accounting for the formation of the rings have been pointed out: vortex sheding above topography, pinch off process during the non-linear evolution of a meandering jet, boundary layer separation, or large scale organization of initially small turbulent disturbances. Despite these very different generation mechanisms, the striking ressemblance between observed oceanic rings in very different regions of the ocean, which has long been recognized as a *surprising result* (Olson 1991), suggests that at least some aspect of these coherent structures can be studied independently of their generation mechanism.

This is precisely the interest of statistical mechanics, which accounts for spontaneous formation of circular structures surrounded by a jet, i.e. the self-organization of mesoscale turbulence into rings. Many studies have been devoted to the study of the dynamics of an isolated mesoscale vortex with a prescribed shape. Statistical mechanics gives a complelentary point of view by predicting the shape of the vortex itself.

a. The westward drift of the rings

To show that westward propagating circular rings can be interpreted as equilibrium states, two important ingredients must be taken into account: i) the beta effect βy ii) a domain invariant by translation in the zonal direction. A drifting ring could not exist as statistical equilibria in a closed domain, since it would be destroyed when arriving on the western boundary. The zonal translational invariance of the problem has important consequences. It is shown in Appendix B that a change of Galilean reference frame in the zonal direction translates as a beta effect in the expression of potential vorticity. Moreover, in a reference frame moving at velocity $-\beta R^2 \mathbf{e}_x$, the beta effect is exactly canceled out.

We conclude that in a domain invariant by translation in the zonal direction, statistical equilibria obtained by the minimization of the Van-Der-Walls Cahn Hilliard variational problem (8) without beta effect are also statistical equilibria with beta effect, but drifting westward at speed $V = -\beta R^2$. A flow configuration with drifting disks (the rings) of homogenized potential vorticity evolving in a background flow made of homogenized potential vorticity (with a different value than for the rings) is a statistical equilibrium.

b. The poleward drift of cyclones and the equatorward drift of anticyclones

If the flow actually reaches a local statistical equilibrium, then not only the ring is composed of an homogenized region of potential vorticity, but also the background flow. In figure 3 is represented the case of an isolated patch of potential vorticity ($q = q_i$ within the ring of area A_i centered on $y = 0$) on a beta plane ($q = \beta y$ elsewhere). This situation is common in the ocean, for instance when Agulhas rings arrive in quiescent regions of the Atlantic ocean. This state is not a statistical equilibrium state, the background potential vorticity is not homogenized.

It is shown in the following that the observed asymmetric small meridional drift of cyclonic and anticyclonic rings can be understood as a tendency for the system to reach the statistical equilibrium. One needs for that purpose to consider the conservation of the linear momentum:

$$\mathcal{L} = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\ qy\ ,$$

which is the dynamical invariant associated with the zonal translational symmetry. The linear momentum of the initial condition is, at lowest order in R:

$$\mathcal{L}^i \approx \beta \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\ y^2 - \beta \int_{A_i} \mathrm{d}\mathbf{r}\ y^2\ ,$$

where A_i is the initial area of the ring. Considering the limit of small rings compared to the domain size, the first term of the right hand side dominates the second one, and $\mathcal{L}^i \approx \beta L_x L_y^3/24$. Assuming the statistical equilibrium is reached in the final state, the flow is made of two phases of homogenized potential vorticity: the background phase, with value $q = q_b$, and the ring's phase of area A_f, centered at latitude $y = y_f$, with value $q = q_f$ (same sign as q_i) and with $|q_f| > |q_b|$. The linear momentum of this final state is

$$\mathcal{L}^f \approx q_b \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\ y - (q_f - q_b) \int_{A_f} \mathrm{d}\mathbf{r}\ y\ .$$

Finally, the linear momentum conservation $\mathcal{L}^i = \mathcal{L}^f$ gives

$$\frac{\beta}{24} L_x L_y^3 \approx -\left(q_f - q_b\right) y_f A_f\ .$$

Physically, this means that for statistical mechanics reason, the background potential vorticity has to be homogenized.

This leads to a loss of linear momentum that must be compensated by a latitude shift of the ring center. Rings with negative potential vorticity ($(q_f - q_b) < 0$) decrease their latitude ($y_f < 0$), while rings with positive potential vorticity ($q_f - q_b) > 0$ increase their latitude ($y_f > 0$). Then, in order to reach the statistical equilibrium, the ring has to drift northward if it is made of an initial positive potential vorticity patch, and southward if it is made of an initial negative potential vorticity patch. This corresponds always to a poleward drift for cyclonic structures and an equatorward drift for for anticyclonic structures, just as what is reported from altimetry measurements (Morrow et al. 2004; Chelton et al. 2007).

We emphasize that these statistical mechanics arguments are complementary to previous work: the dynamics of the beta drift problem has for instance been widely studied numerically, see e.g. McWilliams and Flierl (1979); Sai-Lap Lam and Dritschel (2001) and references therein. It has for instance been shown that the ring obeys a complicated dynamics during its earlier evolution. The ring must indeed cedes part of its energy in order to homogenize the background potential vorticity. An efficient way to do so is by pulsations of the rings that generate Rossby waves. Such Rossby waves then propagate away, interact in a complicated way with the ring and eventually break and mix the background potential vorticity.

In the previous analysis, it has been implicitly assumed that the energy of the initial ring was greater than the energy required to mix the background potential vorticity. This minimal energy required to mix the background flow is given by the potential energy associated with a zonal current compensating the beta effect. This assumption amounts to say that the energy of the system is high enough to have a double-well shape for the specific free energy $f(\phi)$, and then to allow for the formation of an equilibrium state with phase separation between two regions of homogenized potential vorticity. If this would not be the case, the initial ring would lose its cohesion, and its potential vorticity would spread on the whole domain.

c. Conclusion: oceanic rings are local statistical equilibria

In the ocean, the scale separation between the size of the rings and the Rossby radius of deformation is satisfied only to a limited extent. This scale separation has been assumed for technical reasons only: it allows for explicit analytical computations of the equilibria. The results obtained in this limit actually apply for far more general situations. This is confirmed by numerical computations of RSM equilibria, in which rings are obtained as local equilibria even when the scale separation is not satisfied, see e.g. figure 4.

To conclude, the quasi-circular shape of oceanic rings and their westward propagation suggest that these coherent structures can be interpreted as local statistical equilibria. The existence of a meridional drift shows a departure from the prediction of the equilibrium theory. However, the fact that this drift can be interpreted as a tendency to reach to equilibrium state shows that these structures remain close to an equilibrium state.

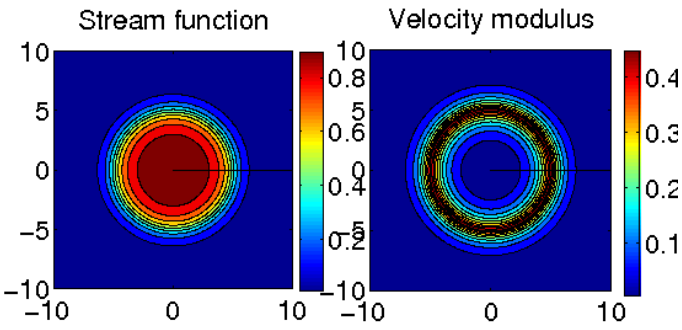

FIG. 4. Circular vortex as a statistical equilibrium of the quasi-geostrophic model, with $R < L_{ring}$. Although analytical computations are carried in the limit $R \ll L_{ring}$, the results are expected to hold when this scale separation does not exit. It is a circular patch of (homogenized) potential vorticity in a background of homogenized potential vorticity, with two different values. The velocity field (right panel) has a ring structure. The width of the jet surrounding the ring has the order of magnitude of the Rossby radius of deformation R.

Rings are local (metastable), and not global statistical equilibria of the equivalent barotropic model: a global equilibria would imply the coalescence of all existing rings into a single large scale vortex, in order to minimize the total interface between the different regions of homogenized potential vorticity.

5. Application to mid-basin eastward jets

Another region of the ocean where strong jets of typical width given by the Rossby radius of deformation R are localized along an interface separating two regions of homogenized potential vorticity is the inertial part of mid-latitude eastward jets, as the Kuroshio or the Gulf stream currents. The inertial part of these currents is located in the regions where the western boundary currents separate from the coastline and self-organize downstream into a strong eastward jet. Because mid-basin eastward jets fill a large part of oceanic basins, and because the existence of a western boundary is an essential ingredient for their formation, one must look for statistical equilibria in a close domain in that case.

We ask in this section whether mid-basin eastward jets in a closed domain are statistical equilibria.

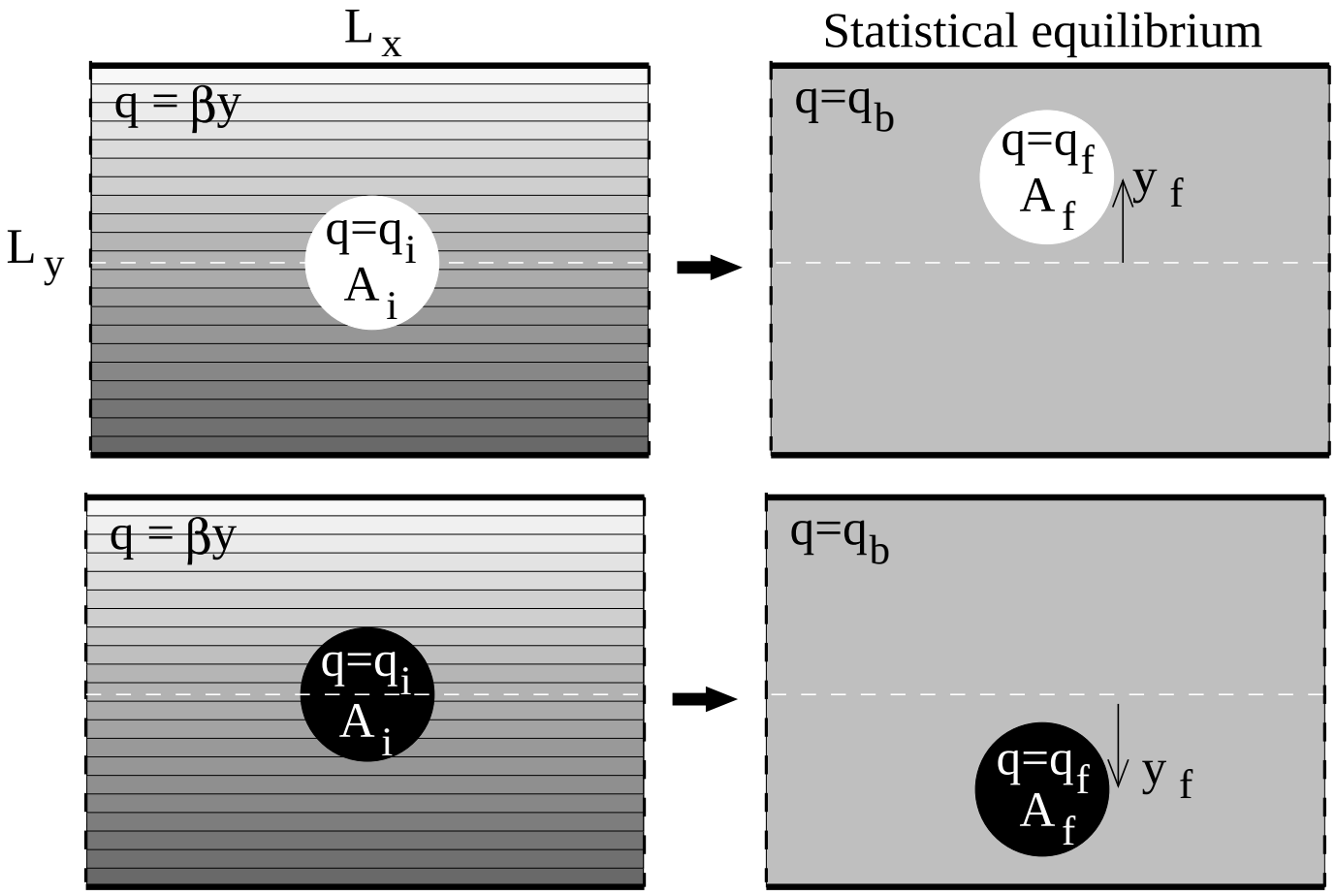

FIG. 3. Explanation of the meridional drift of rings, as a tendency to reach the statistical equilibria. On the left, initial conditions, with a white disk of positive potential vorticity in the upper panel, and a black disk of negative potential vorticity in the lower panel. In both cases the disk evolves on an initial beta plane with no background flow. On the right, the corresponding statistical equilibrium.

In view of the applications to mid-basin ocean jets, we assume a situation in which the global distribution of potential vorticity is symmetric: two phases characterized by symmetric values of streamfunction ($\phi_1 = -\phi_2 = U_{jet}/(2R)$), each of them filling half the domain area ($A_1 = A_2 = 1/2$) We do not report the computations for the non symmetric case, but the main results obtained in this simpler case remain unchanged.

a. Without beta effect, mid basin eastward jets are statistical equilibria of the quasi-geostrophic model

The value $\phi = \phi_{1,2}$ for the two coexisting phases is not compatible with the boundary condition $\phi = 0$. As a consequence, there exists a boundary jet in order to match a uniform phase $\phi = \phi_{1,2}$ to the boundary conditions. Just like interior jets, treated in section 3, these jets contribute to the first order free energy, which gives the boundary jet structure and shape. The symmetry of the problem ($\phi_1 = -\phi_2$) implies that boundary jets of each phase give the same contribution to the free energy. Because the boundary length is a fixed quantity, the free energy minimization amounts to the minimization of the interior jet length only, just as in previous subsections. The interior jet position and shape is thus given by the minimization of the interior jet length with fixed area $A_2 = 1/2$.

The jet has to be straight or circular. There are three possible interface configurations with straight or circular jets: i) the zonal jet configuration (jet along the x axis) with $L = L_x$, ii) the meridional jet configuration (jet along the y axis) with $L = L_y$, iii) and an interior circular vortex, with $L = \sqrt{2\pi}$. The case i) of a zonal jet is a global interface minimum (and then a global equilibrium state) if and only if the aspect ratio satisfies $L_x/L_y < 1$. For $L_x/L_y > 1$, these solutions become metastable states (local

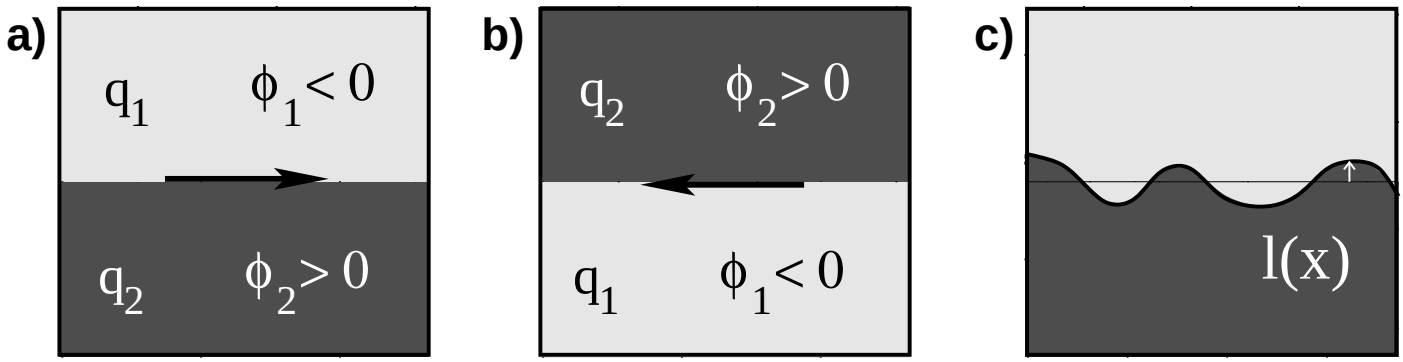

FIG. 5. a) Eastward jet configuration b) Westward jet configuration c) Perturbation of the interface for the eastward jet configuration, to determine when this solution is a local equilibrium. Without beta effect, both the eastward and the westward configurations are entropy maxima. With positive beta effect the westward jet becomes the the global entropy maximum, and the eastward jet becomes metastable provided that beta is small enough.

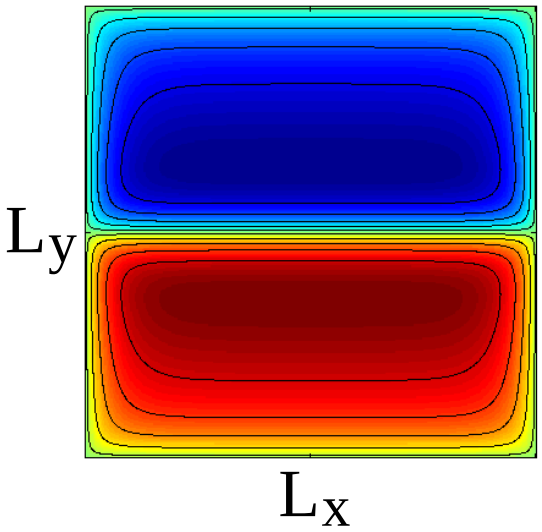

FIG. 6. Streamfunction of the solution presenting an eastward jet with beta effect (red: positive values, blue: negative values), associated with case (a) of figure 5. The jet width is of order R. This solution is a statistical equilibrium for $L_x < \pi(U_{jet}/\beta)^{1/2}$.

entropy maximum).

We conclude that without beta effect, mid-latitude eastward jets are statistical equilibria. Because of the symmetry $\phi_1 = -\phi_2$, solutions presenting eastward and westward jets are equivalent: westward jets are also statistical equilibria.

b. With beta effect, eastward jets becomes metastable or unstable

Contrary to the case of the zonal channel, the beta effect can not be cancelled out by a change of Galilean reference frame in the case of a closed domain. One can therefore not avoid taking into account this term in the computation of the equilibrium free energy.

One can readily see on the expression (7) of the free energy, that when $\beta \neq 0$, the additional term

$$\mathcal{F}_\beta = -\beta \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\ \phi y$$

breaks the symmetry $\pm q$. The westward jet case (with $\phi < 0$ on the southern part of the domain and $\phi > 0$ on the northern part) is more favorable in terms of free energy minimization than the eastward jet case (with $\phi > 0$ in the southern part of the domain and $\phi < 0$ in the northern part): westward jets become the only global equilibria for $\beta > 0$ and aspect ratio $L_x/L_y > 1$.

Let us be more precise by considering the limit of small beta effect, with the scaling $\beta \sim R\widetilde{\beta}$. With that scaling, the equivalent topography does not play any role at zeroth order in the variational problem (8). We thus still conclude that phase separation occurs, with sub-domains of fixed areas A_1 and A_2, separated by jets whose transverse structure is described in Appendix A.

It is shown in this same Appendix that the interface gives a contribution $\mathcal{F}_{int} = cRL\,(\phi_2 - \phi_1)^2$. Let us now compute the first order contribution of the additional term $\mathcal{F}_\beta$. Using the zeroth order result $\phi = \phi_1$ on sub-domain A_1 and $\phi = \phi_2$ on sub-domain A_2, one obtains

$$\mathcal{F}_\beta = -\,(\phi_2 - \phi_1)\,\beta \int_{A_2} \mathrm{d}\mathbf{r}\ y\ ,$$

plus an unimportant constant. Finally, the total first order contribution to the free energy is

$$\mathcal{F}_R = cR\,(\phi_2 - \phi_1)^2\,L - (\phi_2 - \phi_1)\,\beta \int_{A_2} \mathrm{d}\mathbf{r}\ y\ . \qquad (11)$$

Recalling that first variations of the length are proportional to the inverse of the curvature radius r of the interface (Gelfand and S.V. 1967), the minimization of (11), with fixed area A_2 gives

$$(\phi_2 - \phi_1) R\widetilde{\beta} y + \eta = \frac{cR\left(\phi_2 - \phi_1\right)^2}{r} , \qquad (12)$$

where η is a Lagrange parameter associated with the conservation of the area A_2.

We conclude that zonal jets (i.e. an interface at latitude $y = y_{jet}$, with infinite curvature radius r) are solutions to this equation for $\eta = -R\left(\phi_2 - \phi_1\right)\widetilde{\beta} y_{jet}$. This shows that eastward and westward jets described in the previous section are therefore still critical points of entropy maximization.

The eastward jet configuration is the one with the region A_2 below the line $y = 0$ at the center of the domain. To determine if this configuration is a local statistical equilibria, let us consider perturbations of this interface (given by the line $y = l(x)$), while keeping constant the area occupied by both phases, see figure 5. There are two contributions competing with each other in the expression (11) of the free energy $\mathcal{F}_R$. Any perturbation increases the jet length $L = \int \mathrm{d}x \ \sqrt{1 + \left(l'\right)^2}$, where $l' = dl/dx$, and then increases the first term of the free energy (11) by $\delta\mathcal{F}_{int} = cR\left(\phi_2 - \phi_1\right)^2 \int dx \ \left(l'\right)^2$. Any perturbation decreases the second term of the free energy (11) by $\delta\mathcal{F}_\beta = -R\left(\phi_2 - \phi_1\right)\widetilde{\beta} \int dx \ l^2$. If the eastward jet solution is not a free energy minimum, it exists a perturbation of the interface leading to negative variations of the free energy $\delta\mathcal{F}_R = \delta\mathcal{F}_{int} + \delta\mathcal{F}_\beta$. Let us consider the particular case $l = l_k \sin\left(k\pi x/L_x\right)$ where $k \geq 1$ is an integer. Using $U_{jet} = \left(\phi_2 - \phi_1\right) R$ and $\beta = R\widetilde{\beta}$, the condition $\delta\mathcal{F}_R < 0$ gives $\beta > cU_{jet}\left(k\pi/L_x\right)^2$. The most unfavorable case is for the smallest value of k^2, i.e. $k^2 = 1$.

It leads to the necessary condition $\beta > cU_{jet}\pi^2/L_x^2$ for the eastward jet solution to be unstable (in term of statistical mechanics). It can actually be shown, using less straightforward considerations, that it is also a sufficient condition for instability. The *destabilizing* effect of increasing values of β contrasts with its stabilizing effect in classical criteria for barotropic instability, see e.g. Vallis (2006). It has actually been shown that such eastward jet solutions can simultaneously be unstable for statistical mechanics, and stable for non-linear perturbations (Venaille 2008; Venaille et al. 2010).

For a fixed value of β, eastward jets are local free energy maxima if the domain zonal extension is smaller than a critical value: $L_x < \pi\left(cU_{jet}/\beta\right)^{1/2}$. The streamfunction of such a state is presented in figure 6. For jets like the Gulf Stream, $U_{jet} \approx 1 \ m.s^{-1}$ and $\beta \approx 10^{-11} \ m^{-1}.s^{-1}$. Using $c \sim 1$, the critical domain length scale upon which eastward jet become unstable is $L_x \approx 300 \ km$. This length is smaller than the typical zonal extension of the inertial part of the Kuroshio or Gulf Stream currents (around 1000 km), but not by an order of magnitude, which suggests that these structures are marginally unstable. The instability is consistent with the fact that strong meanders and pinch off process occurs downstream of oceanic eastward jets. But the marginal nature of this instability is also consistent with the overall robustness of the global structure of the flow, which becomes a statistical equilibrium when the extension of the jet is small enough.

Note also that the beta effect can be compensated by assuming the existence of a prescribed deep zonal flow U_d. This would add a term $-U_d/R^2 y$ to the potential vorticity, i.e. on the right hand side of equation (2), that could compensate or even overcome the beta term. More precisely, for an eastward deep flow satisfying $U_d > \beta R^2$, the solution presenting an eastward jet becomes the global equilibrium state.

6. Conclusion and prospects

The aim of this paper was to present a point of view complementary to existing approaches that deal with coherent structures in the ocean. It was shown that the RSM statistical mechanics provides a unified framework that may be useful to study mesoscale and basin scale inertial flows.

The theory gives a physical explanation and a prediction for the self-organization of large scale oceanic coherent structure, independently of the underlying generation mechanism. It predicts the formation of subdomains of homogenized potential vorticity, with intense jets at the interface. Mesoscale rings can be interpreted as local equilibrium states of the RSM theory. Their shape and their drift can be understood in this framework. Mid-basin eastward jets are found marginally unstable states of the RSM theory, consistently with observations of these jets.

The interest of this approach relies on its generality (it does not depends on a particular flow configuration) and on its ability to describe qualitatively different observed regimes of self-organization, such as rings and zonal jets. The present study was achieved in the framework of a 1.5 layer quasi-geostrophic model, which is too simplistic to describe oceanic eddies quantitatively; however, generalizations and further investigations in the framework of more complex models can be built upon these results.

For instance, the flow was supposed to be confined in an upper active layer, which does not allow for vertical energy transfers. The next step will be to study predictions of RMS theory for the vertical structure itself.

A caveat of the present approach is that forcing and dissipation are not taken into account in the framework of the equilibrium theory: the input of the RSM theory is given

by the dynamical invariants. In the case of mesoscale rings, even if the dynamics can be considered close to an equilibrium state, forcing and dissipation play an important role in setting these dynamical invariants. In the case of basin scale jets, their marginal instability suggest that one can not avoid taking into account forcing and dissipation mechanisms to explain these structures. So far the inertial part of wind driven circulation has been mostly studied from the point of view of bifurcation theory, starting from a highly dissipated ocean and decreasing progressively frictional parameters, see e.g. Dijkstra and Ghil (2005). We argue that this problem can be tackled with another point of view, starting from the purely inertial limit (this paper), and adding small forcing and dissipation (future work built upon Bouchet and Simonnet (2009)). These two approaches are complementary and may be combined in the future in a more comprehensive non-equilibrium theory.

APPENDIX A

Computation of the jet profile

At leading order, minimization of the free energy leads to the formation of subdomains of constant streamfunction $\phi = \phi_1$ and $\phi = \phi_2$. The interface between these subdomains are associated with strong and localized jets. Let us assume that the curvature radius of the interface is much larger than R, which allows us to neglect what happens along the interface at leading order. Calling $\tau = R\widetilde{\tau}$ the coordinate in a direction along the normal to the interface, the jet profile $\phi_{jet}(\tau)$ across the interface must be such that it minimizes its contribution to the total free energy (7). The jet profile is therefore determined by solving a one dimensional variational problem:

$$\mathcal{F}_{int} = LR \min_{\phi_{jet}} \left\{ \int_{-\infty}^{+\infty} \mathrm{d}\widetilde{\tau} \left[\frac{1}{2} \left(\frac{d\phi_{jet}}{d\widetilde{\tau}} \right)^2 + f(\phi_{jet}) \right] \right\}, \tag{A1}$$

where L is the perimeter of the jet and $\mathcal{F}_{int}$ the free energy associated with the existence of this interfacial jet. Critical points of this variational problem are states that cancel the first variations of the free energy with respect to ϕ_{int}. They are solutions of

$$\frac{d^2\phi_{jet}}{d\widetilde{\tau}^2} = \frac{df}{d\phi_{jet}} \ . \tag{A2}$$

Making an analogy with mechanics, if ϕ_{jet} would be a particle position, τ would be the time, equation (A2) would describe the conservative motion of the particle in a potential $-f$. In order to connect the two different phases in the bulk, on each side of the interface, one has to consider solutions with boundary conditions $\phi \to \phi_1$ for $\tau \to -\infty$ and $\phi \to \phi_2$ for $\tau \to +\infty$. It exists a unique trajectory with such limit conditions. In the particle analogy, it is the trajectory connecting the two unstable fixed points ϕ_1 and ϕ_2, corresponding to the two bumps of the potential $-f$ (see figure 1).

The energy $\left(d\phi_{jet}/d\tau\right)^2/2 - f(\phi_{jet})$ is conserved during the evolution of ϕ with time τ. Using this conservation property and the boundary condition $\phi \to \phi_2$ for $\tau \to +\infty$, one obtains $f(\phi_{jet}) = \left(d\phi_{jet}/d\tau\right)^2/2$, plus an unimportant constant. Injecting this expression into the variational problem (A1), one obtains

$$\mathcal{F}_{int} = LR \int_{-\infty}^{+\infty} \left(d\phi_{jet}/d\tau\right)^2 d\tau = cRL \left(\phi_2 - \phi_1\right)^2 ,$$

with $c \sim 1$. An important physical consequence is that the jet at the interface always gives a positive contribution to the free energy of the equilibrium state, which is of order R and proportional to the interface length L.

APPENDIX B

Galilean invariance and beta effect

In the case of a zonal channel, the quasi-geostrophic equations (1) are invariant over a Galilean transformation in the zonal direction

$$x \to x' = x - Vt \ , \quad y \to y' = y \ , \quad t \to t' = t \ .$$

The velocity is transformed as $\mathbf{v} \to \mathbf{v}' = \mathbf{v} - V\mathbf{e}_x$, which, using the relation $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$, gives the transformation law for the streamfunction $\psi \to \psi' = \psi - Vy$. From the expression $q = \Delta\psi - \psi/R^2 + \beta y$, one obtains finally the transformation law for the potential vorticity $q \to q' = q + Vy/R^2$. Thus the expression for the dynamics in the new reference frame is

$$\frac{\partial q'}{\partial t'} + \mathbf{v}' \cdot \nabla' q' = 0, \quad \text{with } \mathbf{v}' = \mathbf{e}_z \times \nabla'\psi' \ ,$$

$$\text{and} \quad q' = \nabla'^2\psi' - \frac{\psi'}{R^2} + \left(\beta + \frac{V}{R^2}\right) y' \ .$$

REFERENCES

Abramov, R. V. and A. J. Majda, 2003: Statistically relevant conserved quantities for truncated quasigeostrophic flow. *Proceedings of the National Academy of Science*, **100**, 3841–3846.

Bouchet, F., 2001: *Mécanique statistique des écoulements géophysiques*. PHD, Université Joseph Fourier-Grenoble.

Bouchet, F., 2008: Simpler variational problems for statistical equilibria of the 2d euler equation and other systems with long range interactions. *Physica D Nonlinear Phenomena*, **237**, 1976–1981, arXiv:0710.5094.

Bouchet, F. and M. Corvellec, 2010: Invariant measures of the 2D Euler and Vlasov equations. *ArXiv e-prints*, 1003.5075.

Bouchet, F. and E. Simonnet, 2009: Random Changes of Flow Topology in Two-Dimensional and Geophysical Turbulence. *Physical Review Letters*, **102 (9)**, 094 504, 0804.2231.

Bouchet, F. and J. Sommeria, 2002: Emergence of intense jets and Jupiter's Great Red Spot as maximum-entropy structures. *Journal of Fluid Mechanics*, **464**, 165–207.

Bretherton, F. P. and D. B. Haidvogel, 1976: Two-dimensional turbulence above topography. *Journal of Fluid Mechanics*, **78**, 129–154.

Chelton, D. B., M. G. Schlax, R. M. Samelson, and R. A. de Szoeke, 2007: Global observations of large oceanic eddies. *Geo. Res. Lett.*, **34**, 15 606.

Cushman-Roisin, B., B. Tang, and E. P. Chassignet, 1990: Westward Motion of Mesoscale Eddies. *Journal of Physical Oceanography*, **20**, 758–768.

Dibattista, M. T. and A. J. Majda, 2000: An Equilibrium Statistical Theory for Large-Scale Features of Open-Ocean Convection. *Journal of Physical Oceanography*, **30**, 1325–1353.

Dibattista, M. T., A. J. Majda, and J. Marshall, 2002: A Statistical Theory for the "Patchiness" of Open-Ocean Deep Convection: The Effect of Preconditioning. *Journal of Physical Oceanography*, **32**, 599–626.

Dijkstra, H. A. and M. Ghil, 2005: Low-frequency variability of the large-scale ocean circulation: A dynamical systems approach. *Reviews of Geophysics*, **43**.

Dubinkina, S. and J. Frank, 2010: Statistical relevance of vorticity conservation in the Hamiltonian particle-mesh method. *Journal of Computational Physics*, **229**, 2634–2648.

Ellis, R. S., K. Haven, and B. Turkington, 2000: Large Deviation Principles and Complete Equivalence and Nonequivalence Results for Pure and Mixed Ensembles. *ArXiv Mathematics e-prints*, arXiv:math/0012081.

Ellis, R. S., R. Jordan, P. Otto, and B. Turkington, 2004: A Statistical Approach to the Asymptotic Behavior of a Class of Generalized Nonlinear Schrödinger Equations. *Communications in Mathematical Physics*, **244**, 187–208.

Flierl, G. R., 1987: Isolated eddy models in geophysics. *Annual Review of Fluid Mechanics*, **19**, 493–530.

Fofonoff, N. P., 1954: Steady flow in a frictionless homogeneous ocean. *J. Mar. Res.*, **13**, 254–262.

Gelfand, I. and F. S.V., 1967: *Calculus of variations*. Prentice Hall.

Hallberg, R. and A. Gnanadesikan, 2006: The Role of Eddies in Determining the Structure and Response of the Wind-Driven Southern Hemisphere Overturning: Results from the Modeling Eddies in the Southern Ocean (MESO) Project. *Journal of Physical Oceanography*, **36**, 2232.

Hogg, N. G. and H. M. Stommel, 1985: The Heton, an Elementary Interaction Between Discrete Baroclinic Geostrophic Vortices, and Its Implications Concerning Eddy Heat-Flow. *Royal Society of London Proceedings Series A*, **397**.

Kraichnan, R. H. and D. Montgomery, 1980: Two-dimensional turbulence. *Reports on Progress in Physics*, **43**, 547–619.

McIntyre, M. E. and T. N. Palmer, 1983: Breaking planetary waves in the stratosphere. *nature*, **305**, 593–600.

McWilliams, J. C. and G. R. Flierl, 1979: On the evolution of isolated, nonlinear vortices. *Journal of Physical Oceanography*, **9**, 1155–1182.

Michel, J. and R. Robert, 1994: Large deviations for young measures and statistical mechanics of infinite dimensional dynamical systems with conservation law. *Communications in Mathematical Physics*, **159**, 195–215.

Miller, J., 1990: Statistical mechanics of euler equations in two dimensions. *Phys. Rev. Lett.*, **65 (17)**, 2137–2140, doi:10.1103/PhysRevLett.65.2137.

Modica, L., 1987: The gradient theory of phase transitions and the minimal interface criterion. *Archive for Rational Mechanics and Analysis*, **98**, 123–142.

Morrow, R., F. Birol, D. Griffin, and J. Sudre, 2004: Divergent pathways of cyclonic and anti-cyclonic ocean eddies. *Geo. Res. Lett.*, **31**.

Nof, D., 1981: On the β-Induced Movement of Isolated Baroclinic Eddies. *Journal of Physical Oceanography*, **11**, 1662–1672.

Olson, D. B., 1991: Rings in the Ocean. *Annual Review of Earth and Planetary Sciences*, **19**, 283.

Onsager, L., 1949: Statistical hydrodynamics. *Nuovo Cimento*, **6 (No. 2 (Suppl.))**, 249–286.

Pedlosky, J., 1998: *Ocean Circulation Theory*. New York and Berlin, Springer-Verlag.

Pierrehumbert, R. T., 1991: Chaotic mixing of tracer and vorticity by modulated travelling Rossby waves. *Geophysical and Astrophysical Fluid Dynamics*, **58**, 285–319.

Prieto, R. and W. H. Schubert, 2001: Analytical Predictions for Zonally Symmetric Equilibrium States of the Stratospheric Polar Vortex. *Journal of Atmospheric Sciences*, **58**, 2709–2728.

Rhines, P. B. and W. R. Young, 1982: Homogenization of potential vorticity in planetary gyres. *Journal of Fluid Mechanics*, **122**, 347–367.

Ripa, P., 1981: Symmetries and conservation laws for internal gravity waves. *American Institute of Physics Conference Series*, American Institute of Physics Conference Series, Vol. 76, 281–306.

Robert, R., 1990: Etats d'equilibre statistique pour l'ecoulement bidimensionnel d'un fluide parfait. *C. R. Acad. Sci.*, **1**, 575–578.

Robert, R., 1991: A maximum-entropy principle for two-dimensional perfect fluid dynamics. *J. Stat. Phys.*, **65**, 531–553.

Robert, R. and J. Sommeria, 1991: Statistical equilibrium states for two-dimensional flows. *J. Fluid Mech.*, **229**, 291–310.

Sai-Lap Lam, J. and D. G. Dritschel, 2001: On the beta-drift of an initially circular vortex patch. *Journal of Fluid Mechanics*, **436**, 107–129.

Salmon, R., 1998: *Lectures on Geophysical Fluid Dynamics*. Oxford University Press.

Salmon, R., G. Holloway, and M. C. Hendershott, 1976: The equilibrium statistical mechanics of simple quasi-geostrophic models. *Journal of Fluid Mechanics*, **75**, 691–703.

Turkington, B., A. Majda, K. Haven, and M. Dibattista, 2001: Statistical equilibrium predictions of jets and spots on Jupiter. *PNAS*, **98**, 12 346–12 350.

Vallis, G. K., 2006: *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press, doi:10.2277/0521849691.

Venaille, A., 2008: *Mélange et circulation océanique: une approche par la physique statistique.* PHD thesis, Université Joseph Fourier-Grenoble.

Venaille, A., F. Bouchet, and E. Simonnet, 2010: Stability of intense eastward jets in Quasi-Geostrophic models of oceans. *Redaction, to be submitted to J. Fluid Mech.*

Wang, J. and G. K. Vallis, 1994: Emergence of Fofonoff states in inviscid and viscous ocean circulation models. *Journal of Marine Research*, **52**, 83–127.

Zou, J. and G. Holloway, 1994: Entropy maximization tendency in topographic turbulence. *Journal of Fluid Mechanics*, **263**, 361–374.