ON SAGITTARIUS A* THEORY AND OBSERVATIONS

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Abstract. Massive and supermassive "dust" spheres (with a zero internal pressure) collapse to "full globes" of finite volumes, whose surfaces have the properties of the event horizon around a mass-point. This fact explains the observational data concerning Sagittarius A* (SgrA*). By virtue of Hilbert's repulsive effect, both the event horizon of a mass-point and the event horizon of a "full globe" cannot "swallow" anything.

Summary – **1**, **2**. Some observational data about SgrA*. – **3**. The geodesics of Schwarzschild's manifold created by a point-mass and the gravitational repulsion. – **3bis**. Inadequacy of the proper time as evolution parameter of the geodesic motions in a Schwarzschild's manifold. – **4**. Kerr's manifold and gravitational repulsion. – **5**, **5bis**. A 'dust" sphere collapses to a 'full globe" of a finite volume; *et cetera*. – **6**. Explanation of the observational data about SgrA*. – *Appendix A*: On celestial objects endowed with a magnetic moment. – *Appendix B*: Attraction and repulsion in Hilbert's and Droste's treatments. –

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1. – In the paper "The event horizon of Sagittarius A*" by Broderick *et al.* [1], the authors affirm that recent millimeter and infrared observations of the supermassive centre SgrA* of the Milky Way require the existence of the ideal surface of an event horizon. In the present paper we show that the observational data of [1] are perfectly consistent with the existence of a *real*, peculiar surface, which has all the properties of the above event horizon. Remark that the horizons have *no* "swallowing" property – a fact which is commonly ignored.

2. – According to Broderick *et al.* [1], by virtue of its relative proximity SgrA* is the best candidate to the possession of the ideal surface of an event horizon.

Observations of massive stars in its vicinity have given the following values for its mass and its distance: mass $M = (4.5 \pm 0.4) \times 10^6$ solar masses;

distance $D = (8.4 \pm 0.4)$ kpc; SgrA* is confined within 40AU. The radiative emission (luminosity of 10^{36} erg s^{-1}) is strongly non-thermal, and is distributed from the radio to the γ-rays.

The substance of the arguments of Broderick *et al.* [1] is admirably summarized in the first paragraph of the final sect.**4**, which we report literally: "Recent infrared and mm-VLBI observations imply that if the matter accreting onto Sgr A* comes to rest in a region visible to distant observers, the luminosity associated with the surface emission from this region satisfies $L_{\mathrm{surf}}/L_{\mathrm{acc}} \lesssim 0.003$. Equivalently, these observations require that 99.6% of the gravitational binding energy liberated during infall is radiated in some form prior to finally settling. These numbers are inconsistent by orders of magnitude with our present understanding of the radiative properties of Sgr A*'s accretion flow specifically and relativistic accretion flows generally. Therefore, it is all but certain that no such surface can be present, i.e., *an event horizon must exist.*"

Now, if we take into account the decisive role of the Hilbertian gravitational *repulsion* [2], [3], which is neglected by our authors [1], the picture changes drastically, and the above inequality $L_{\mathrm{surf}}/L_{\mathrm{acc}} \lesssim 0.003$ becomes quite comprehensible, as we shall see in the sequel.

3. – For the computation of the geodesics of the Schwarzschild manifold created by a material point, Hilbert [2] starts from the standard (Hilbert-Droste-Weyl) form of the interval ds:

$$(1)\quad \mathrm{d}s^2 = \frac{r}{r-\alpha}\,\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\,\mathrm{d}\varphi^2 - \frac{r-\alpha}{r}\,\mathrm{d}t^2 \quad ; \quad (c=G=1) \quad ,$$

where $\alpha \equiv 2m$, and m is the mass of the gravitating point – if M is the mass in CGS units, we have $M = c^2m/G$. (The original Schwarzschild's form of ds^2 can be obtained from eq. (1) with the substitution $r \rightarrow (r^3+\alpha^3)^{1/3}$.)

It is easy to see that there are only *plane* trajectories, and therefore it suffices to consider only one value for ϑ, *e.g.* $\pi/2$. Eq. (1) has as an evident consequence the following first integrals of the geodesic motions, where A, B, C are constants with respect to the affine parameter p:

$$(2)\qquad \frac{r}{r-\alpha}\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 + r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}p}\right)^2 - \frac{r-\alpha}{r}\left(\frac{\mathrm{d}t}{\mathrm{d}p}\right)^2 = A \quad ;$$

$$(3)\qquad r^2\,\frac{\mathrm{d}\varphi}{\mathrm{d}p} = B \quad ;$$

$$(4)\qquad \frac{r-\alpha}{r}\,\frac{\mathrm{d}t}{\mathrm{d}p} = C \quad .$$

Clearly, A is negative for the test-particles and zero for the light-rays. With a suitable choice of p, we can put $C = 1$. Then, by eliminating t and p from eqs. (2)–(3)–(4), we obtain the *general formula* of *all* the geodesic lines:

(5) $$\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 = \frac{1+A}{B^2} - \frac{A\alpha}{B^2}\,\varrho - \varrho^2 + \alpha\varrho^3 \left[= \left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 \frac{1}{B^2}\right] \quad ,$$

where $\varrho \equiv 1/r$. Remark that the coordinate r in eq. (5) must satisfy the condition $r > \alpha$, because the progenitor eqs. (2) and (4) do not hold for $r \leq \alpha$. Remark that by substituting $\mathrm{d}p = [(r-\alpha)/r]\,\mathrm{d}t$ in eqs. (2)–(3) one obtains two first integrals with respect to the evolution parameter t.

Circular and radial orbits are evidently possible; however, for the circular motions it is necessary to use also the Lagrangean equation of motion for r:

(6) $$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{2r}{r-\alpha}\frac{\mathrm{d}r}{\mathrm{d}p}\right) + \frac{\alpha}{(r-\alpha)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 - 2r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}p}\right)^2 + \frac{\alpha}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}p}\right)^2 = 0 \quad ,$$

since, when $\mathrm{d}r/\mathrm{d}p = 0$, this equation is *not* an analytical consequence of eqs. (2)–(3)–(4).

One finds that the velocity $v = r\,\mathrm{d}\varphi/\mathrm{d}t$ on a *circular* orbit is given by

(7) $$v^2 = \left(\frac{r\,\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \frac{\alpha}{2r} \quad ;$$

for the test-particles we have that

(8) $$v < \frac{1}{\sqrt{3}} \quad ,$$

and

(9) $$r > \frac{3\alpha}{2} \quad ,$$

a clear example of the existence of a gravitational ***repulsion***; for the light-rays there is a unique circular trajectory, for which

(10) $$v = \frac{1}{\sqrt{3}} \quad ,$$

(11) $$r = \frac{3\alpha}{2} \quad ,$$

and we see that also the light "feels" the gravitational ***repulsion***. Of course, the inequalities (9) and (11) can be proved with the use of *both* the evolution parameters, t and p. (And also with the proper time s on the circular geodetics for relation (9)).

For the *radial* trajectories of the geodesic motions there is gravitational attraction where

(12) $$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| < \frac{1}{\sqrt{3}}\,\frac{r-\alpha}{r} \quad ,$$

and gravitational ***repulsion*** where

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| > \frac{1}{\sqrt{3}}\frac{r-\alpha}{r} \quad . \tag{13}$$

If r^* is the value of r for which $\mathrm{d}^2r/\mathrm{d}t^2 = 0$ – attraction and repulsion counterbalance each other –, the velocity $\mathrm{d}r/\mathrm{d}t$ has its maximal value at $r = r^*$:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{\max} = \frac{1}{\sqrt{3}}\frac{r^*-\alpha}{r^*} \quad . \tag{14}$$

For the light-rays we have from $\mathrm{d}s^2 = 0$:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| = \frac{r-\alpha}{r} \quad : \tag{15}$$

the light is ***repulsed everywhere*** by the gravitating point-mass m; its velocity increases from zero at $r = \alpha$ to 1 at $r = \infty$.

Test-particles and light-rays arrive at $r = \alpha$ with $\mathrm{d}r/\mathrm{d}t = 0$ and $\mathrm{d}^2r/\mathrm{d}t^2 = 0$: *the spatial surface $r = \alpha$ represents for them an insuperable barrier.*

This basic fact can be also illustrated starting from eq. (5). We consider the instance of the light-rays ($A = 0$); for the test-particles ($A < 0$) the results are qualitatively the same. The general orbit of the light-rays is given by

$$\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 = \frac{1}{B^2} - \varrho^2 + \alpha\varrho^3 \quad , \quad (\varrho \equiv 1/r) \quad ; \tag{5'}$$

this equation has for $B = 3\sqrt{3}\alpha/2$ the circle $r = 3\alpha/2$ as Poincaré's "cycle" [2]. Let us characterize a ray with the segment B – see Fig. 1 –, which gives at infinity its distance from the vertical radial line. Fig. 1 represents intuitively some integral curves of eq. (5′) obtained with Poincaré's cycle theory [3]. When $B < 3\sqrt{3}\alpha/2$, the light-ray arrives at $r = \alpha$ *and ends there.* When $B = 3\sqrt{3}\alpha/2$, it comes near asymptotically by spiralling to the circle $r = 3\alpha/2$. When $B > 3\sqrt{3}\alpha/2$, the ray performs, in general, several revolutions round this circle, and then goes to infinity. Fig. 1 shows three rays of the last kind, one of them performs a revolution. Clearly, the vertical line is characterized by the limit $B \to 0$ – and therefore it *ends at $r = \alpha$.*

Droste [4] makes a detailed investigation of eq. (5), whose solution is given by Weierstraß' elliptic function. He emphasizes that eq. (1), and all the formulae that are a consequence of this $\mathrm{d}s^2$, are valid only for $r > \alpha$. In reality, the modification $(- + ++)$ for $r < \alpha$ of the signature $(+ + +-)$ of eq. (1) – with interchanged roles of r and t – is an absurdity. Not always does geometry coincide with physics.

In sect.**7** of [4] we find a study of the conditions under which there is gravitational attraction or gravitational repulsion in the radial geodetic motions. Droste's treatment is a little different from that of Hilbert [2], because he uses a velocity $\mathrm{d}\delta/\mathrm{d}t = (\mathrm{d}r/\mathrm{d}t)(1-\alpha r^{-1})^{-1/2}$, which is the time derivative

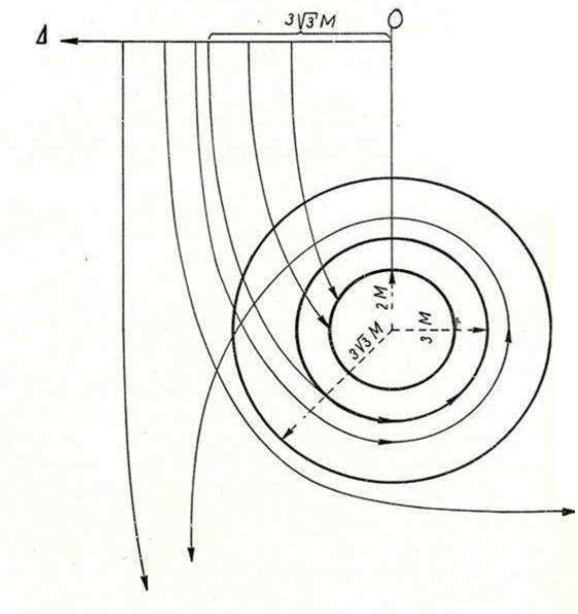

FIGURE 1. See von Laue [3] – This Author writes M in lieu of our m, and Δ in lieu of our B.

of the metric distance δ of the generic point (r, ϑ, φ) from $r = \alpha$. In *App. B* we give a comparison between Hilbert's and Droste's treatments.

3bis. – In sect.**3** we have emphasized that the geodesic lines for which $B < 3\sqrt{3}\alpha/2$ arrive at $r = \alpha$ and end there. A result which generalizes the previous conclusion that for the radial geodesics we have $\mathrm{d}r/\mathrm{d}t = 0 = \mathrm{d}^2r/\mathrm{d}t^2$ at $r = \alpha$.

Some theoreticians do not like the use of the "Systemzeit" t; they prefer the proper time of the test-particles (and an affine parameter for the light-rays). However, in the present context the use of the proper time is not reasonable. Indeed, let us consider the analogues of the first integrals (2)–(3)–(4) with the proper time s in lieu of the affine parameter p; we have:

$$\frac{r}{r-\alpha}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 - \frac{r-\alpha}{r}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = -1 \quad ; \tag{16}$$

$$r^2\,\frac{\mathrm{d}\varphi}{\mathrm{d}s} = L \quad ; \tag{17}$$

$$\frac{r-\alpha}{r}\,\frac{\mathrm{d}t}{\mathrm{d}s} = E \quad , \tag{18}$$

where L and E are two constants. Formal deductions from (16)–(17)–(18) yield:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = E^2 - \frac{r-\alpha}{r^3}\,L^2 - \frac{r-\alpha}{r} \quad ; \tag{19}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = \frac{L^2}{2}\left(\frac{2}{r^3} - \frac{3\alpha}{r^4}\right) - \frac{\alpha}{2r^2} \quad ; \tag{20}$$

from which, when $r = \alpha$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2_{r=\alpha} = E^2 \quad ; \tag{21}$$

$$\left(\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}\right)_{r=\alpha} = \frac{1}{2}\left(\frac{L^2}{\alpha^3} - \frac{1}{\alpha}\right) \quad ; \tag{22}$$

However, eqs. (21) and (22) are actually meaningless: when $r = \alpha$, $\mathrm{d}s^2 = \infty$. (In his memoir [2] Hilbert did not use s as an evolution parameter!) – Obviously, computations with an affine parameter p as evolution parameter would give finite, *but p-dependent* values for $(\mathrm{d}r/\mathrm{d}p)^2_{r=\alpha}$ and $(\mathrm{d}^2 r/\mathrm{d}p^2)_{r=\alpha}$.

All the geodesic lines of the light-rays satisfy the equation $\mathrm{d}s^2 = 0$, from which it is very natural to infer that

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r\,(r-\alpha)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \left(\frac{r-\alpha}{r}\right)^2 \quad , \tag{23}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2_{r=\alpha} = 0 \quad ; \tag{24}$$

$$\left(\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\right)_{r=\alpha} = 0 \quad . \tag{25}$$

But also for the test-particles the "Systemzeit" t is the *unique* evolution parameter which gives always *real physical* results.

A last remark. In the Schwarzschildian *original* form of $\mathrm{d}s^2$ ($(r^3+\alpha^3)^{1/3}$ in lieu of r in eq. (1)), or in Brillouin's form ($r+\alpha$ in lieu of r in eq. (1)), the spatial region $0 \le r < \alpha$ is *absent*. The manifold is *maximally extended*. Of course, *all* the physical results which can be derived from eq. (1) can be obtained also with Schwarzschild's and Brillouin's metric forms. In particular, *the gravitational repulsion is a phenomenon of invariant character*, *i.e.* independent of the space-time reference frame.

4. – "It si widely believed that the gravitational field of any electrically neutral collapsing body will eventually approach [by virtue of an assumed rotation] the Kerr form." (Weinberg [5]).

Now, we have proved that a test-particle, or a light-ray, moving through the Kerr manifold along a radial geodesic in the negative direction of the radial coordinate arrive at the "stationary-limit"surface with a zero three-velocity and a positive, or zero, three-acceleration: a clear instance of a Hilbert's repulsive effect, whose action we have computed for a generic value of the ϑ-coordinate [6].

5. – In a recent paper [7] we have proved that if one takes into account the Hilbertian gravitational repulsion, even a "dust" sphere with an internal zero pressure collapses to a body of a *finite* volume. Precisely, the collapse ends when the mass m of the sphere has filled up the spatial region $0 \le r \le \alpha(\equiv 2m)$. It follows that the gravitational field for $r > \alpha$ of this (relatively small) spherical body coincides with the field of a point-mass m. Both these objects have the physical property that – by virtue of the Hilbertian repulsion – *they are incapable of "swallowing" anything*, light or material corpuscles. (*N.B. – For the adjustment of the internal to the external solution we assume for simplicity that the radius of the "full globe" is equal to* $2m + \varepsilon$, *with an arbitrary small* $\varepsilon > 0$).

Of course, the Euclidean formula for a spherical volume $V = (4/3)\pi r^3$ does not hold for the volume U of the "full globe"; the difference between V and U becomes larger and larger with the increase of the mass m.

It is instructive to give a generalization for $B \neq 0$ of the equations $(\mathrm{d}r/\mathrm{d}t)_{r=\alpha} = 0 = (\mathrm{d}^2r/\mathrm{d}t^2)_{r=\alpha}$, which we have previously recalled for the radial geodesics ($B = 0$).

We get from eqs. (5) and (3) (with $C = 1$):

$$(26)\qquad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = B^2\left(\frac{r-\alpha}{r}\right)^2\left(\frac{1+A}{B^2} - \frac{A\alpha}{B^2}\frac{1}{r} - \frac{1}{r^2} + \frac{\alpha}{r^3}\right) \quad ,$$

from which, when $B = 0$:

$$(26')\qquad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2_{B=0} = \left(\frac{r-\alpha}{r}\right)^2\left(1 + A\,\frac{r-\alpha}{r}\right) \quad .$$

For $r = \alpha$ eqs. (26) and (26′) give:

$$(27)\qquad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{r=\alpha} = \left(\frac{\mathrm{d}^2r}{\mathrm{d}t^2}\right)_{r=\alpha} = 0 \quad .$$

And from eq. (26):

$$(28)\qquad \pm\, B\,\mathrm{d}t = \frac{r}{r-\alpha}\,\frac{\mathrm{d}r}{\{[(1+A)/B^2] - (A\alpha/B^2 r) - 1/r^2 + \alpha/r^3\}^{1/2}} \quad ,$$

which tells us that test-particles and light-rays reach $r = \alpha$ after an infinitely long time: a result that is universally known, but whose real meaning is usually neglected, because of the preference given to the proper time s. However, this preference is here mathematically *baseless*, as we have proved in sect. **3bis**.

Remark that if you put in eq. (28) $r = (9/8)\alpha$, *e.g.*, instead of $r = \alpha$, you get a reasonable time interval Δt.

5bis. – Under given conditions, the Hilbertian gravitational repulsion in Schwarzschild's and Kerr's manifolds manifests itself in *all* the geodesic paths, in particular in the circular orbits and in the trajectories which arrive on the spatial surface $r = 2m$. Due to an undue preference for the proper time with respect to the coordinate-time (*i.e.*, the *time of the system*), only the circular orbits are commonly believed to be subjected to Hilbert's repulsive action. This limitation is quite illogical: indeed, it is clear that the gravitational repulsion does not suspend its action for the non-circular geodesics, *in primis* for the geodesics which "strike" the surface $r = 2m$.

6. – Back to the paper by Broderick *et al.* [1]. The spherically-symmetric point-mass m (sect.**3**) and the "full globe" $0 \leq r \leq 2m$ (sect.**5**) create an identical gravitational field in the external region $r > 2m$ – and both these gravitating objects give a gravitational repulsion under the illustrated conditions. For a diagram of some geodesics of light-rays, see Fig.1, that represents a correct mathematical counterpart (in a generic plane) of Fig.1-[1], which is reproduced with its legend in the following Fig.2. We emphasize that the authors restrict the gravitational repulsion to the unique circular orbit of the light-rays.

By virtue of the Hilbertian repulsive effect, the materials of the accretion flow arrive at $r = 2m$ with a zero velocity and a zero acceleration. Accordingly, the "hollow globe" $r < 2m$ around the point-mass m cannot "swallow" anything. The accretion materials (matter fragments and light) perform a very soft landing on the ideal surface $\Im$ of the "hollow globe", or on the physical surface $\sum$ of the "full globe" $0 \leq r \leq 2m$. It seems to us that these facts explain the observational data of [1]; in particular, it is obvious that the surfaces $\Im$ and $\sum$ have a *low* luminosity.

Quite generally, the effects of the Hilbertian gravitational repulsion suffice to cancel all the widespread (and less widespread) convictions about the exotic properties of the ideal surface $r = 2m$.

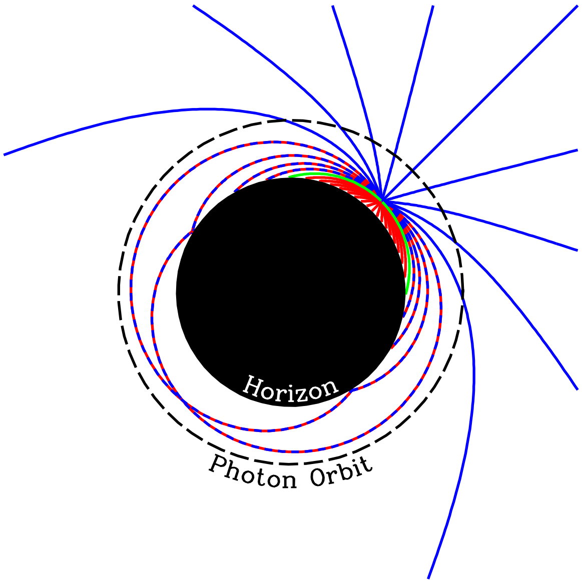

FIGURE 2. (From Broderick *et al.* [1]). Rays launched isotropically (every 10°) in the locally flat, stationary frame are lensed in a Schwarzschild spacetime. Those rays that are initially moving inwards, tangentially and outwards are shown in red, green and blue, respectively. Additionally, those that are launched initially moving outwards and are subsequently captured are red-blue dashed. For reference the horizon and photon orbit are shown. Generically, the fraction of rays that escape to infinity decreases as the emission point is moved towards the black hole, dropping below 50% at the photon orbit and dropping all the way to 0% at the horizon. As a consequence of this strong lensing, emitting objects that are contained within the photon orbit approximate the canonical pin-hole cavity example of a blackbody, becoming a perfect blackbody in the limit that the surface redshift goes to ∞.

APPENDIX A

In recent years, some authors have emphasized that there are observational proofs of the existence of *intrinsic magnetic moments* in BH-candidates – both of stellar masses and AGN-masses –, for instance in SgrA*. Now, the existence of a magnetic moment (of an appreciable magnitude) *forbids* the existence of the event horizon around a point-mass. The above authors have developed a sophisticated model of the collapse of massive and supermassive magnetic bodies with the purpose to offer an alternative explanation of the data by Broderick *et al.*; in particular, they give a special prominence to the action of a magnetic propeller driven outflow for explaining the low bolometric luminosity of SgrA*.

We think, however, that the reliability of this model (a heterogeneous offspring of GR and QED) is not evident.

APPENDIX B

It is useful to compare Droste's [4] and Hilbert's [2] treatments about attraction and repulsion in the radial geodesic motions through Schwarzschild manifold.

Droste starts from this definition of radial geodesic velocity:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} := \frac{\mathrm{d}r}{\mathrm{d}t}\left(1-\frac{\alpha}{r}\right)^{-1/2} \quad , \quad (\alpha \equiv 2m) \quad , \tag{B1}$$

which is suggested by the metric radial interval $\mathrm{d}\delta := \mathrm{d}r(1-\alpha r^{-1})^{-1/2}$. Eq. (2) of sect.**3** gives the first integral

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(1-\frac{\alpha}{r}\right)^2\left[1+A\left(1-\frac{\alpha}{r}\right)\right] \quad , \tag{B2}$$

from which

$$\left(\frac{\mathrm{d}\delta}{\mathrm{d}t}\right)^2 = \left(1-\frac{\alpha}{r}\right)\left[1+A\left(1-\frac{\alpha}{r}\right)\right] \quad . \tag{B3}$$

As a consequence of

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{3\alpha}{2r\,(r-\alpha)}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{\alpha\,(r-\alpha)}{2r^3} \quad , \tag{B4}$$

which can be easily derived from eqs. (4) and (6) of sect.**3**, Droste arrives at the following expression for his acceleration $\mathrm{d}^2\delta/\mathrm{d}t^2$:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = -\frac{\alpha}{2r^2}\left[\left(1-\frac{\alpha}{r}\right)^{1/2} - \frac{2\,(\mathrm{d}\delta/\mathrm{d}t)^2}{\left(1-\frac{\alpha}{r}\right)^{1/2}}\right] \quad . \tag{B5}$$

Let us call r_* the value of r for which $\mathrm{d}^2\delta/\mathrm{d}t^2 = 0$ (attraction counterbalances repulsion); we have:

$$\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|_{r=r_*} = \frac{1}{\sqrt{2}}\left(1-\frac{\alpha}{r_*}\right)^{1/2} \quad , \tag{B6}$$

from which:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=r_*} = \frac{1}{\sqrt{2}}\left(1-\frac{\alpha}{r_*}\right) \quad ; \tag{B7}$$

but if r^* is the value of r for which $\mathrm{d}^2r/\mathrm{d}t^2 = 0$, we have with Hilbert [2]:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=r^*} = \frac{1}{\sqrt{3}}\left(1-\frac{\alpha}{r^*}\right) \quad ; \tag{B8}$$

we see that there is a non-negligible difference between Droste's B.(6)–(B.7) and Hilbert's (B.8) maximal velocities.

We think that *Hilbert's treatment must be preferred*, for the following reason.

The metric distance δ of the point (r, ϑ, φ) from $r = \alpha$ is:

(B9)

$$\delta = \int_\alpha^r \left(1 - \frac{\alpha}{r'}\right)^{-1/2} \mathrm{d}r' = r\,\left(1 - \frac{\alpha}{r}\right)^{1/2} + \alpha \ln\left|\left(\frac{r}{\alpha} - 1\right)^{1/2} + \left(\frac{r}{\alpha}\right)^{1/2}\right| \quad ;$$

by virtue of the arbitrariness in the choice of the coordinates, in particular of the radial coordinate, if we put $f(r) := \delta(r) + \alpha$, and perform in eq. (1) of sect.**3** the substitution $r \to f(r)$, we get a $\mathrm{d}s^2$ which is physically equivalent to the one of eq. (1). From the mathematical standpoint, this new expression of the $\mathrm{d}s^2$ is *diffeomorphic* to that of eq. (1) *for* $r > \alpha$. Now, let us choose a definition *à la* Hilbert for the radial velocity: we have $\mathrm{d}f(r)/\mathrm{d}t = \mathrm{d}\delta(r)/\mathrm{d}t$, *i.e.* Droste's definition (B.1).

However, all the consequences coincide now with those of Hilbert's treatment; in particular, $\mathrm{d}^2\delta(r)/\mathrm{d}t^2$ is now equal to zero if and only if $\mathrm{d}^2 r/\mathrm{d}t^2 = 0$. We see that a $\mathrm{d}s^2$ expressed *in toto* with the metric coordinate $f(r) = \delta(r) + \alpha$ gives the same results of the $\mathrm{d}s^2$ of eq. (1) in Hilbert's procedure.

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