Free Fock space and functional calculus approach to the n-point information about the "Universe"

Jerzy Hanćkowiak

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Abstract

Starting from a differential equation for the unique field $\varphi(\tilde{x})$, where the vector $\tilde{x}$ contains space-time and the discrete field characteristics, the equation for the generating vector |V> of the n-point information (correlation and smeared functions) in the free Fock space is derived. In derived equation, due to appropriate extension of the right invertible operators, the physical vacuum vector $|0>_{ph}$appears with a global characteristic of the field φ.

For so called resolvent regularization of the original systems, the closed equations for the n-point information are analysed with the help of functional calculus.

key words: the strong and a weak formulation, a right invertible operator, a regularized or modified theory, a generalized resolvent, functional calculus

Zielona Gora University, e-mail: hanckowiak@wp.pl

1 Introduction

In spite of a growing trend in physics to define the physical world as being made of information itself and thus information is defined in this way, I am using this term to express some knowledge about things and ideas. To gain this information - before you need to formulate a suitable question, see Titus Lucretius Carus *De Rerum Natura.* According to Britannica Concise Encyclopaedia - equations, in essence, - are questions. To get corresponding knowledge (information about things and ideas) we have to solve these equations which in many cases is not an easy task. The trouble is that the questions are too detailed and hence the idea of *weak formulation* of the original equations is used. In Internet we can find the following characterization of this idea:

"Weak formulations are an important tool for the analysis of mathematical equations that permit the transfer of concepts of linear algebra **to solve problems in other fields** such as partial differential equations. In a weak formulation, an equation is no longer required to hold absolutely (and this is not even well defined) and has instead *weak solutions* only with respect to certain "test vectors" or "test functions"". See Wikipedia <Weak formulation>; the page last modified on 27 January 2010 by unknown author?.

An example in which the idea of weak formulation is used is a celebrated Galerkin method. In this method the original equations are not changed but the original spaces in which solutions are searched are drastically changed, for example, when the original space is substituted by a finite dimensional usually a low dimensional subspace. Surprising is that in this way in many cases you can get quite correct results even for very complicated systems describing, for example, the fluid flows, see, e.g.,[1], [2], [3] and

Internet.

We have to remember, however, that the weak solutions of the weak equations, obtained in a frame of reduced-order philosophy, are not solutions of the original problem, see [5].

In the paper presented, in contrary to the canonical situation, we propose such use of the weak formulation of the original theory that the weak solutions have a clear physical interpretation and perhaps nice mathematical properies. So, instead of the original mostly partial differential equations (PDE) with the initial and boundary conditions (IBC) defined in a sharp way, we consider PDE in which IBC are trated as random or smooth quantities. As a consequence of that approach, the **original nonlinear equations** are substituted by **the linear equations** for correlation functions or their generalizations in both cases called the n-point functions (n-pfs) or as in the title - the n-point information. This step can be treated as a weak formulation of the original nonlinear system of equations because considered linear equations contains solutions of the original theory as well as the new solutions. The second step in the paper is quite opposite to what is done in any **canonical weak formulation**: instead of narrowing the space in which n-point information are considered, we move to a larger space - the free Fock space (in which we do not postulate the permutation symmetry of functions). However, in this enlarged (free) Fock space - like in a smaller space of weak formulation - the transfer of concepts of linear algebra is possible and even general solutions in many cases can be constructed.

Given the extraordinary ease of constructing unilaterally reverse operations to many operators which appear in the free Fock space and by introducing an additional parameter (minor coupling constant) can be derived new equations for n-pfs. In this study and others we were trying to better understand derived equations.

In Secs 2 and 3 we define the strong and weak formulations of the considered equations.

In Sec.4 the free (super, general) Fock space metodolodgy is described and the basic equation for the correlation functions, (11) is postulated.

Sec.5 is devoted to the canonical perturbation theory applied to the Eq.11 and a determination of the arbitrary terms which appear in the free Fock space, see also Sec.8.

It seems that very interesting is the idea of regularization of the true culprit of many problems of nonlinear theory:

$$\lambda_1 \hat{N} \to \lambda_1 (\hat{I} + \lambda_2 \hat{M})_R^{-1} \hat{N} \tag{1}$$

whereby in the derived equations in addition to the usual sum of operators related to the linear $((\hat{L}+\hat{G}))$ and nonlinear $(\hat{N})$ parts of the theory, see Eq.2 and Eq.11, the product $(\lambda_2(\hat{L}+\hat{G})\hat{M})$ appears, see Secs 7 and 8. In a particular case of regularization, $\hat{M} = \hat{N}$, - the closed equations for n-pfs - are obtained. In Sec.9 remarks about functional calculus are given. An example of such equations and the general solution for the 1-pf in the case of the φ^3- model is discussed in Sec.10 where some general remarks are also included.

This paper is a continuation of work [9] with , as we hope, better use of the operator-valued functions and with useful extension of considered operators. In comparison with work [6], where evolutionary type of equations were considered, in this paper - equations of “resovent type” are used, for comparison, see [21].

2 Strong (exact) formulation

We will assume that a theory is formulated by the following integro-differential equations

$$L[\tilde{x}; \varphi(\tilde{x})] + \lambda N[\tilde{x}; \varphi] + G(\tilde{x}) = 0 \qquad (2)$$

with a linear and nonlinear dependence on the unique field φ (first and second terms) and a free term G. In the case of homogeneous environment or materials, the operators L and N do not depend explicitly on the vector variable $\tilde{x} \in B$ where a set B describes the domain of the unique field φ . In order to improve the description of equations the components of the vector $\tilde{x}$ contain also discrete indexes which usually appear as subindices of the fields or functions. It turns out that riddance of the lower and upper indices is a big improvement of description. In this way only one *unique field* φ is considered. The coupling constant λ, later denoted by $\lambda_1$and called the major coupling constant, contains the memory of the nonlinearity of the original, strong formulation (2). This is usually an expansion parameter in the perturbation approach to the statistical and quantum fields.

We rewrite the Eq.2 as

$$(L_0\varphi)(\tilde{x}) + L_1[\tilde{x}; \varphi(\tilde{x})] + \lambda N_0[\tilde{x}; \varphi(\tilde{x})] + \lambda N_1[\tilde{x}; \varphi] + G(\tilde{x}) = 0 \quad (3)$$

This is an equation for the unique field $\varphi(\tilde{x})$ in which operators (functionals) with sub index "o" denote a local or self interaction of a cell or particle or field, but sub index "1" denotes an interaction among the constituents of the physical system. An additional restriction of particular terms in Eq.3 comes from a co-variant character of proposed equations with respect to symmetry transformations of the theory. A symmetry of equations can be used to introduce another averages (or smoothing) than the ensemble averages with a simple, geometrical interpretation and often direct measured. It is interesting that these two kind of smoothing procedure lead to identical equations for n-pfs, [?],[?]. This is a happy coincidence because in this way we can compare theory with experiment without resolving the ergodic problem, [?]. The problem of calculation of the time averages is related to solving equations upon n-pfs with appropriate additional conditions.

3 Weak formulation can be linear, multitime and noncommuting. Positivity conditions

These three features of presented here weak formulation are chosen not for a provocative purpose or to illustrate a philosophical doctrine that science is a matter of convention but to draw the attention of practically oriented reader that they together can also be applied in numerical methods. Linearity is associated with randomization or/and smoothing of description, multitime is also related to a more complete randomization description in which the time is not distinguish and is treated as other variables. In result, the Kraichnan-Lewis multitime correlation functions instead of the one time Reynolds' or Hopf's correlation functions are used:

$$\varphi(\tilde{x}) \Rightarrow < \varphi(\tilde{x}_1) \cdots \varphi(\tilde{x}_n) >; \; n = 1, 2, ..., \infty \tag{4}$$

Finally, the noncommuting variables are associated with an additional generalization of the arena in which systems are described: We introduce the free Fock space in which the correlation functions or smoothed n-point functions (n-pfs) need not be permutation symmetric. To stress this fact we will call n-pfs the *n-point information*. In this last step we do not project the original equations, in our case equations for n-point information, but we enlarge the space in which solutions are searched. This is exactly opposite to what it is done in the Galerkin methods. Nevertheless, like in the Galerkin methods, where corresponding space is diminished(!), this permits the transfer of concepts of linear algebra to solve considered equations. Among these concepts we take the generators for Cuntz algebra, right and left invertible operators and plenty projectors using of which allows us to construct varies final formulas for generating vectors generating the correlation functions or, in the general case, n-pfs, which we call the n-point information. In the case of the correlation functions we have important restrictions:

$$< \varphi(x)^{2n} > \geq 0 \tag{5}$$

for n=0,1,2...These restrictions we will call the *positivity conditions,* for correlation functions. The case

$$< \varphi(x)^{2n} > = 0 \Longleftrightarrow \varphi \equiv 0 \tag{6}$$

means a trivial theory.

4 The free Fock space and n-points information

To deal with an infinite collection of correlation functions or n-pfs, $< \varphi(\tilde{x}_1) \cdots \varphi(\tilde{x}_n) >$, the one generating vector $|V >$can be introduced by means of which all these n-pfs can be reproduced. Using such a vector we describe the infinite system of branching equations for n-pfs in a compact form of one vector equation which can be transformed in varies equivalent and useful forms. From definition

$$|V> = \sum_{n=1} \int d\tilde{x}_{(n)} < \varphi(\tilde{x}_1) \cdots \varphi(\tilde{x}_n) > \hat{\eta}^\star(\tilde{x}_1) \cdots \hat{\eta}^\star(\tilde{x}_n)|0> + V_0|0> \tag{7}$$

where operators $\hat{\eta}^\star(\tilde{x})$, hermitian conjugat to the operator $\hat{\eta}(\tilde{x})$, satisfy the Cuntz relations

$$\hat{\eta}(\tilde{x})\hat{\eta}^\star(\tilde{y}) = \hat{I} \cdot \delta(\tilde{x} - \tilde{y}) \tag{8}$$

which mean that ranges of operators $\hat{\eta}^\star$ are pairwise orthogonal and in fact the expantion (7) is a generalization of the idea of expansion of a vector by means of a multiple orthogonal base. Here operator $\hat{I}$- unit operator, δ- is a product of Kronecker's delta (discrete case) and vector $|0>$ represents, using quantum field theory language, a "vacuum",

$$\hat{\eta}(\tilde{x})|0> = 0 \tag{9}$$

see[**?**],[**?**] and [9]. Set of vectors (7) form the free linear Fock space F. Lack of commutation between quantities $\hat{\eta}*$in the generating vectors |V> (free Fock space) does not exclude the possibility that n-pfs $< \varphi(\tilde{x}_1) \cdots \varphi(\tilde{x}_n) >$are permutation symmetric, see below.

Conditions (8) and (9) are enough to show that

$$< 0|\hat{\eta}(\tilde{y}_1) \cdots \hat{\eta}(\tilde{y}_n)|V> = < \varphi(\tilde{y}_1) \cdots \varphi(\tilde{y}_n) > \tag{10}$$

Since operators $\hat{\eta}$ do not commute, the above formula is able to retrieve from the generating vector7 also permutation non-symmetrical n-pfs like in the case of quantum fields $\hat{\varphi}$. But a true reason for introducing non-commuting field $\hat{\eta}$ is such that operators introduced below and constructed by means of the operators $\hat{\eta}$,$\hat{\eta}^{\star}$ can be right or left invertible, inverses to which can be easily constructed. This leads to a variety of useful formulas for n-pfs.

We postulate the following equations for the n-pfs $<\varphi(\tilde{x}_1)\cdots(\tilde{x}_n)>$ which by means of the generating vector (7) can be described in a compact way:

$$(\hat{L}+\lambda\hat{N}+\hat{G})|V>=\hat{P}_0|V>+\lambda\hat{P}_0\hat{N}|V>\equiv|0>_{ph} \tag{11}$$

with operators

$$\begin{aligned}\hat{L}= & \int\hat{\eta}*(\tilde{x})L[\tilde{x};\hat{\eta}]d\tilde{x}+|0><0|= \\ & \int\hat{\eta}*(\tilde{x})L(\tilde{x},\tilde{y})\hat{\eta}(\tilde{y})d\tilde{x}d\tilde{y}+\hat{P}_0\end{aligned} \tag{12}$$

$$\hat{N}=\int\hat{\eta}*(\tilde{z})N[\tilde{z};\hat{\eta}]d\tilde{z}+\hat{P}_0\hat{N} \tag{13}$$

and

$$\hat{G}=\int\hat{\eta}*(\tilde{x})G(\tilde{x}) \tag{14}$$

see[?],[?]. A small modification of the r.h.s. of Eq.11 is connected with a demand of right invertability of the operators $\hat{L}$ and $\hat{N}$ what force us to add terms $\hat{P}_0|V>$and $\lambda\hat{P}_0\hat{N}|V>$. For a concrete choice of that term, see (108). In fact, the $|0>_{ph}\neq|0>$ have to be used only for $\hat{G}\neq 0$ and $\hat{N}\neq\hat{0}$. It reminds us of a distant analogy with virtual particles of Quantum Field Theory and therefore is called the *physical vacuum.* *I*n fact the r.h.s. of Eq.11 comes from the fact that the original Eq.2 and the averaging process, $<...>$, does not say anything about zero component of the equation.

Eq.11 means that we have chosen averages with respect to used additional conditions (ensemble averages). As we said in Sec.3, in the case of homogeneous system, both types of averages lead to the same equations for the correlation functions.

By introducing projectors $\hat{P}_n$ projecting on the consecutive terms of the expansion (7), we can express the projection properties of operators (12-14) as follows:

$$\hat{P}_n\hat{L} = \hat{L}\hat{P}_n \tag{15}$$

(diagonal), where n=0,1,2,...,

$$\hat{P}_n\hat{N} = \sum_{n<m}\hat{P}_n\hat{N}\hat{P}_m \tag{16}$$

(upper triangular), where n=0,1,2,..., see (13) and (108).

$$\hat{P}_n\hat{G} = \hat{G}\hat{P}_{n-1} \tag{17}$$

(lower triangular), where n=1,2,.... The operator values function $N[\tilde{z};\hat{\eta}]$ can be a polynomial or other function depending on the vector variable $\tilde{z}$ and the operator variables $\hat{\eta}(\tilde{x})$ indexed by the vector variable $\tilde{x}$. The operator $\hat{N}$ is related to a nonlinear part of the strong formulation of theory (the original differential equations (2). The operator $\hat{G}$ describes a source term with a function $G(\tilde{x})$ correponding to the external forces, for example. It is symtomatic that diagonal and upper triangular operators describe an interaction or selfinteraction of the constituents of the system and that lower triangular operators describe an interaction with the external world or quantum properties of the system (microworld).

The simplest diagonal operator is the unit operator

$$\hat{I} = |0 >< 0| + \int \hat{\eta} * (\tilde{x})\hat{\eta}(\tilde{x})d\tilde{x} \tag{18}$$

Other diagonal operators are the projectors used in formulas (15-17) and constructed by means of the tensor product of vectors:

$$\hat{P}_n = \int \hat{\eta} * (\tilde{x}_1) \cdots \hat{\eta} * (\tilde{x}_n)|0 >< 0|\hat{\eta}(\tilde{x}_n) \cdots \hat{\eta}(\tilde{x}_1)d\tilde{x}_{(n)} \tag{19}$$

where $\hat{P}_0 = |0 >< 0|$. They form a complete set of orthogonal projectors:

$$\sum_{n=0} \hat{P}_n = \hat{I}, \quad and \; \hat{P}_m\hat{P}_n = \hat{P}_n\delta_{mn} \tag{20}$$

We can say that projections $\hat{P}_n|V >$, for n=1,2,..., provide n-points information about the local nature of the system but the projection $\hat{P}_0|V >$provides rather global, agregated information. In the case of system representing the Universe, the r.hs. of Eqs like (11), (59), (76), (80) can be interpreted as a vacuum, see [9]. Thus, in this interpretation the (classical) vacuum contains the global information about the Universe. Like in QFT a non-trivial structure of the vacuum arises only through the nonlinear theory and this is a positive element that might enable its stady. Identical equations as (11) take place in QFT, for vacuum expectation values

$$< \hat{\varphi}(\tilde{x}_1)...\hat{\varphi}(\tilde{x}_n) > \tag{21}$$

where $\hat{\varphi}$are now operators with appropriate equal time commutators. In this case however, n-pfs (10) are not permutationally symmetric.

5 Right invertible linear part of the theory and approximated solutions

If the kernel L of a diagonal operator $\hat{L}$ is a right invertible:

$$\int L(\tilde{x},\tilde{y})L_R^{-1}(\tilde{y},\tilde{z})d\tilde{z} = \delta(\tilde{x}-\tilde{z}) \tag{22}$$

then the operator $\hat{L}$ is a right invertible in the Fock space F, see 7-9. A right inverse to $\hat{L}$, denoted by $\hat{L}_R^{-1}$can be constructed as

$$\hat{L}_R^{-1} = \int \hat{\eta} * (\tilde{z})L_R^{-1}(\tilde{z},\tilde{w})\hat{\eta}(\tilde{w})d\tilde{z}d\tilde{w} + \hat{P}_0 \tag{23}$$

where, as before, the symbol $\int$ means the sumation or integration with respect to components of vectors $\tilde{z}$,$\tilde{w}$. In fact, the operator $\hat{L}_R^{-1}$satisfies the weaker equation:

$$\hat{L}\hat{L}_R^{-1} = \hat{I} \tag{24}$$

Having constructed a right inverse to a given operator $\hat{L}$, we can construct the projector on the null space of $\hat{L}$:

$$\Pi_L = \hat{I} - \hat{L}_R^{-1}\hat{L} \tag{25}$$

Multiplying Eq.11 by a right inverse operator $(\hat{L}+\hat{G})$, we can describe this equation in an equivalent way as follows

$$\begin{gathered}[\hat{I} + \lambda(\hat{L}+\hat{G})_R^{-1}\hat{N}]|V>= \hat{\Pi}_{L+G}|V> + \\ (\hat{L}+\hat{G})_R^{-1}\left(\hat{P}_0|V> + \lambda\hat{P}_0\hat{N}|V>\right)\end{gathered} \tag{26}$$

where the projector on the null space of the operator $(\hat{L}+\hat{G})$ is

$$\hat{\Pi}_{L+G} = \hat{I} - (\hat{L} + \hat{G})_R^{-1}(\hat{L} + \hat{G}) \tag{27}$$

From point of view of Eq.11, considered in the free (full, super) Fock space F, the projection $\hat{\Pi}_{L+G}|V>$can be any vector from space $\hat{\Pi}_{L+G}F$. With different right inverse operators$\hat{L}_R^{-1}$, different projectors $\hat{\Pi}_{L+G}$on the null space of the operator $(\hat{L} + \hat{G})$ are constructed. In result, a general solution to the generating vector |V> obtained by means Eq.26 has different parametrizations.

From definition of smoothing operations given in Sec.3, we see that obtained n-pfs are permutation symmetric. Denoting by $\hat{S}$ a projector on vectors generating permutation symmetric n-pfs , we should have, for the *physical solutions*:

$$|V >= \hat{S}|V > \tag{28}$$

Hence and from Eq.26 , we get

$$\begin{gathered}[\hat{I} + \lambda\hat{S}(\hat{L} + \hat{G})_R^{-1}\hat{N}]|V >= \\ \hat{S}\hat{\Pi}_{L+G}|V > +\hat{S}(\hat{L} + \hat{G})_R^{-1}\left(\hat{P}_0|V > +\lambda\hat{P}_0\hat{N}|V >\right)\end{gathered} \tag{29}$$

Eq.29 was derived by using first Eq.28 and next acting on Eq.26 with projector $\hat{S}$. Similar procedure is used in the Galerkin method, however, there is an important difference with Galerkin approach, namely - Eq.28 - is an exact equation and hence the solutions to Eq.29 can also be exact solutions of the original problem.

Taking into account the projection properties of operators $\hat{S}$, $\hat{L}_R^{-1}$ and $\hat{G}$, Eq.11 can be also described as:

$$\{\hat{I} + \lambda(\hat{I} + \hat{S}\hat{L}_R^{-1}\hat{G})^{-1}\hat{S}\hat{L}_R^{-1}\hat{N}\}|V >=$$

$$\hat{S}\hat{\Pi}_L|V> +\hat{S}(\hat{L}+\hat{G})_R^{-1}\left(\hat{P}_0|V> +\lambda\hat{P}_0\hat{N}|V>\right) \tag{30}$$

This or Eq.29 are the output equatios for calculating successive approximations to the generating vector |V> expanded in the positive powers of the coupling λ standing at the nonlinear part of original theory (2):

$$|V>=\sum_{j=0}^{\propto}\lambda^j|V>^{(j)} \tag{31}$$

The arbitary element of Eq.26, $\hat{\Pi}_L|V>\in\hat{\Pi}_L F$, can also be expanded in this way

$$\hat{\Pi}_L|V>=\sum_{j=0}^{\propto}\lambda^j(\hat{\Pi}_L|V>)^{(j)} \tag{32}$$

The terms of the above series can be restricted by the permutation symmetry of n-pfs. In the case of symmetrical solutions (28) to Eq.29 we can assume that the symmetry part of the arbitrary element $\hat{\Pi}_L|V>$is given by a linear theory:

$$\hat{S}\hat{\Pi}_{L+G}|V>=\hat{S}|V>^{(0)}=|V>^{(0)} \tag{33}$$

It is a common situation accompaning to almost every approximated and exact theory, see also Sec.8. It means that only due to terms with $\lambda\neq 0$ higher approximations appear. So the zeroth order approximation

$$|V>^{(0)}=\hat{S}\hat{\Pi}_L|V>^{(0)} \tag{34}$$

The first order approximation

$$|V>^{(1)}=-\{(\hat{I}+\hat{S}\hat{L}_R^{-1}\hat{G})^{-1}\hat{S}\hat{L}_R^{-1}\hat{N}\}|V>^{(0)} \tag{35}$$

The second order approximation

$$|V>^{(2)}=-\{(\hat{I}+\hat{S}\hat{L}_R^{-1}\hat{G})^{-1}\hat{S}\hat{L}_R^{-1}\hat{N}\}|V>^{(1)} \tag{36}$$

and so on. All these approximations can be obtained from a single vector formula:

$$|V>=$$
$$\{\hat{I}+\lambda(\hat{I}+\hat{S}\hat{L}_R^{-1}\hat{G})^{-1}\hat{S}\hat{L}_R^{-1}\hat{N}\}^{-1}(\hat{I}+\hat{S}\hat{L}_R^{-1}\hat{G})^{-1}\hat{S}\hat{\Pi}_L|V> \tag{37}$$

and it is not inconceivable that these compact formula provides a new look at old divergent problems of unrenormalizable theories.

Let us focuse on the zeroth order approximation, (34). It is a symmetrical solution to the Eq.11, for $\lambda=0$. We will consider this equation with additional simplification that exterial forces acting on the system and represented by the operator $\hat{G}=0$. So we get the following vector equation

$$\hat{L}|V>^{(0)}=0 \tag{38}$$

describing a linear theory. In the "components" form it looks as follows:

$$\int d\tilde{x}_1 L(\tilde{x},\tilde{x}_1)<\varphi(\tilde{x}_1)\cdots\varphi(\tilde{x}_n)>^{(0)}=0 \tag{39}$$

for n=1,2,... Let us assume that we know a function $\triangle$ which satisfies equation

$$\int d\tilde{x}_1 L(\tilde{x}, \tilde{x}_1) \triangle(\tilde{x}_1, \tilde{y}) = 0 \tag{40}$$

for all $\tilde{x}, \tilde{y}$. With functions $\triangle$, we can construct, for even n, symmetrical solutions to Eqs 39 as follows:

$$\begin{gathered} < \varphi(\tilde{x}_1) \cdots \varphi(\tilde{x}_n) >^{(0)} \equiv V(\tilde{x}_1, ..., \tilde{x}_n)^{(0)} = \\ \frac{1}{n!} \sum_{perm} \triangle(\tilde{x}_1, \tilde{x}_2) \triangle(\tilde{x}_3, \tilde{x}_4) \cdots \triangle(\tilde{x}_{n-1}, \tilde{x}_n) \end{gathered} \tag{41}$$

and zero, for odd n. Because of symmetry, the permutation operation is taken in the sum. In a graph representation, with vertexes denoted by $\tilde{x}_1, ..., \tilde{x}_n$, a particular terms of sum 41are represented by graphs with one edge vertexes. It is easy to see that, for such solutions, conditions (5) are satisfied. In addition, we also expect that these conditions are also fulfilled for the higher order approximations for the correlation functions when the perturbation parameter, the coupling constant λ, is a small quantity in Eq.30.

As an example of 41,

$$< \varphi(\tilde{x}) >^{(0)} = 0 \tag{42}$$

$$< \varphi(\tilde{x}) \varphi(\tilde{y}) >^{(0)} = \frac{1}{2!} (\triangle(\tilde{x}, \tilde{y}) + \triangle(\tilde{y}, \tilde{x})) = \triangle(\tilde{x}, \tilde{y}) \tag{43}$$

where the permutation symmetry of function $\triangle$ is assumed. Hence we see that in zeroth order approximation the correlation functions (41) are sums of products of the 2-point function43.

To find a connection of the zeroth order 2-pf Δ with the general solution to Eq.2, in the case of $\lambda = 0$, $G = 0$, we represent this solution in the form

$$\varphi^{(0)}[\tilde{x},\alpha] = \int d\tilde{u}\Gamma(\tilde{x},\tilde{u})\alpha(\tilde{u}) \tag{44}$$

in which the *kernel* Γ of the general solution (2) $(\lambda = 0, G = 0)$ represents possible elementary solutions to Eqs (2) and40 - labeled by the vector variable $\tilde{u}$ which are summed with $\tilde{u}$- dependent factor α. We do not define here a set to which the vector parameteru $\tilde{u}$ belongs. From 43) and 44 we get

$$\begin{gathered} < \varphi(\tilde{x})\varphi(\tilde{y}) >^{(0)} \equiv \triangle(\tilde{x},\tilde{y}) = \\ \int d\tilde{u}d\tilde{w}\delta\alpha\Gamma(\tilde{x},\tilde{u})\Gamma(\tilde{y},\tilde{w})\alpha(\tilde{u})\alpha(\tilde{w})P[\alpha] \end{gathered} \tag{45}$$

Denoting the functional integral occurring in (45) as

$$\int \delta\alpha\alpha(\tilde{u})\alpha(\tilde{w})P[\alpha] = P(\tilde{u},\tilde{w}) \tag{46}$$

we get for the function $\triangle$, the following expression:

$$\triangle(\tilde{x},\tilde{y}) \ = < \varphi(\tilde{x})\varphi(\tilde{y}) >^{(0)} = \int d\tilde{u}d\tilde{w}\Gamma(\tilde{x},\tilde{u})\Gamma(\tilde{y},\tilde{w})P(\tilde{u},\tilde{w}) \tag{47}$$

which relates the correlation function $\triangle$ between cells with the *kernel* Γ of the general solution to (2), at $\lambda = 0$,G=0, and the *second order moments* $P(\tilde{u},\tilde{w})$ of the *probability density* $P[\alpha]$.

The kernel Γ of the general solution to Eq.2, in case $(\lambda = 0, G = 0)$, can be constructed by means of a right inverse L_R^{-1}, see 22:

$$\Gamma(\tilde{x},\tilde{u}) \equiv \Pi_L(\tilde{x},\tilde{u}) = \delta(\tilde{x}-\tilde{u}) - \int d\tilde{z}L_R^{-1}(\tilde{x},\tilde{z})L(\tilde{z},\tilde{u}) \tag{48}$$

In such case this is a projector (idempotent):

$$\Gamma(\tilde{x},\tilde{u}) = \int d\tilde{z}\Gamma(\tilde{x},\tilde{z})\Gamma(\tilde{z},\tilde{u}) \tag{49}$$

and the subspace described by the projector Γ can be identified with all solutions to Eq.2 . In many cases, with every subspce, one can define a linear differential equation, see [13]. In fact a differential equation can be *represented geometrically* by the set of all solutions, and a *symmetry of the differential equation* is defined as a map which transforms this set into itself,[14].

For a symmetrical Γ

$$\Gamma(\tilde{x},\tilde{y}) = \Gamma(\tilde{y},\tilde{x}) \tag{50}$$

and the Gaussian

$$P(\tilde{u},\tilde{w}) = \delta(\tilde{u}-\tilde{w}) \tag{51}$$

we get from (45)

$$\triangle(\tilde{x},\tilde{y}) \equiv < \varphi(\tilde{x})\varphi(\tilde{y}) >^{(0)} = \Gamma(\tilde{x},\tilde{y}) \tag{52}$$

In other words, for the linear systems (2) described by the symmetrical, idempotent kernel$\Gamma$of the general solution and the Gaussian type moments (51), the zeroth order 2-pf is identical with this kernel (sic!). It is astonishing case which shows that, at least at the zeroth order level, information lost by averaging or smoothing procedures ($< \varphi(\tilde{x}) >^{(0)} = 0 \, for \, all \, \tilde{x}$) can be recovered with the help of 2-point correlation function (52). It is not excluded that the following hypothesis is true: - in the case of nonlinear equations a similar phenomen takes place with the help of the higher order correlation functions and the Gauss smoothing.

For φ^3nonlinear, local theory, the above properties are particularly important because they allow to substitute the bilinear products $\Delta \cdot \Delta$occurring in such a theory by the one Δ, see Sec.5.

It is interesting to notice that condition (50) called sometimes the *reverse normalized condition* is in fact one additional condition imposed on a right inverse operator to get a solution to (2) with smallest Euclidean norm (generalized inverse or pseudo-inverse called also Moore-Penrose inverse). This illustrates the kind of processes described by Eq.52. It turns out that every bounded operator $L:\ K \to H$ (Hilbert spaces) with closed range has a generalized inverse. To see properties of such operators, see[15].

6 Right invertible "nonlinear" part of the theory

In this case we will assume and it actually takes place in the free Fock space, [6]-[9] that a right inverse operator $\hat{N}_R^{-1}$exists to the operator $\hat{N}$:

$$\hat{N}\hat{N}_R^{-1} = \hat{I} \tag{53}$$

and that the operator $\hat{N}_R^{-1}$can be explicitly constructed in the most general form, [6]-[9]. Introducing a projector on the null space of the operator $\hat{N}$:

$$\hat{P} = \hat{I} - \hat{N}_R^{-1}\hat{N} \equiv \hat{I} - \hat{Q}_N; \quad < 0|\hat{N} = 0 \tag{54}$$

we can write equivalently Eq.11 as follows:

$$\begin{gathered}[\hat{I} + \lambda^{-1}\hat{N}_R^{-1}(\hat{L} + \hat{G})]|V >= \hat{P}|V > + \\ \lambda^{-1}\hat{N}_R^{-1}\left(\hat{P}_0|V > +\lambda\hat{P}_0\hat{N}|V >\right)\end{gathered} \tag{55}$$

With symmetry (28) we get

$$[\hat{I} + \lambda^{-1}\hat{S}\hat{N}_R^{-1}(\hat{L} + \hat{G})]|V >= \hat{S}\hat{P}|V > + \lambda^{-1}\hat{S}\hat{N}_R^{-1}\left(\hat{P}_0|V > + \lambda\hat{P}_0\hat{N}|V >\right) \tag{56}$$

Assuming that with limes $\lambda \to 0$ we get finite results for the generating vector |V>, we have

$$|V(\lambda = \infty) >= \hat{S}\hat{P}|V(\infty) > + \hat{P}_0\hat{N}|V(\infty) > \tag{57}$$

In a spirit of perturbation theory we assume that the arbitrary term of solutions to Eq.56

$$\hat{S}\hat{P}|V >= \hat{S}\hat{P}|V(\infty) >= |V(\lambda = \infty) > -\hat{P}_0\hat{N}|V(\infty) >=? \tag{58}$$

and a possible answer to that is a real issue.

7 Modified or regularized theory

We mean by this an introducing to the term representing the nonlinear part of the original theory - one extra parameter: $\hat{N} \to \hat{N}(\lambda_2)$. This parameter is called the *minor coupling constant* and is denoted by λ_2 where the original parameter λ, now denoted by λ_1, will be called the *major coupling constant.* If at the end of calculations we go with parameter λ_2 to zero and with $\lambda_2 \to 0$, $\hat{N}(\lambda_2) \to \hat{N}$ we will speak about a regularization of the theory in other case about its modification. In certain cases such theories with two coupling constants $\lambda_1, \lambda_2 \neq 0$ satisfies the Dyson criterion of convergency,[16], and hence the name - *regularization*. In other words, divergences of the usual perturbation theory can be interpreted as residues of the deformed, nonlocal and more basic theory which is currently still unknown. For other interesting ideas concerning renormalizability of a theory see [17].

So, instead of Eq.11 we consider the *resolvent regularized or modified equation*

$$\left(\hat{L} + \lambda_1(\hat{I} + \lambda_2\hat{M})_R^{-1}\hat{N} + \hat{G}\right)|V>=$$
$$\hat{P}_0|V> +\lambda_1\hat{P}_0(\hat{I} + \lambda_2\hat{M})_R^{-1}\hat{N}|V>\equiv |0>_{reg} \qquad (59)$$

where the operator $(\hat{I} + \lambda_2\hat{M})_R^{-1}$is a right inverse operator to the operator $(\hat{I} + \lambda\hat{M})$. In the mathematical literature, this operator is called a *generalized resolvent of the operator $\hat{M}$,* [11]. See [21], for other use of expression - the resolvent equations.

The r.h.s. of Eq.59 comes from the fact that the original Eq.2 and the averaging process, $<...>$, does not say anything about zero component of the equation. Therefore, acting on this equation with projector $\hat{P}_0$we should get identity.

Multiplying the last equation by the operator $(\hat{I} + \lambda\hat{M})$, we remove, at least from the l.h.s. of Eq.59, ambiguities contained into the right inverse operator and get

$$\left[(\hat{I} + \lambda_2\hat{M})(\hat{L} + \hat{G}) + \lambda_1\hat{N}\right]|V>=$$
$$(\hat{I} + \lambda_2\hat{M})\left(\hat{P}_0|V> +\lambda_1\hat{P}_0(\hat{I} + \lambda_2\hat{M})_R^{-1}\hat{N}|V>\right) \qquad (60)$$

This equation can also be treated as a *regularization* of the original Eq.11 in a sense that, for $\lambda_2 \to 0$, (60) tends to Eq.11.

Eq.60 is not equivalent to (59} because it was derived by multiplying by the right invertible operator $(\hat{I} - \lambda_2\hat{M})$. This operation is equivalent to the operation of multiplication by the projector

$$\hat{Q}_{(I+\lambda_2 M)} = (\hat{I} + \lambda_2 \hat{M})_R^{-1}(\hat{I} + \lambda_2 \hat{M}) \tag{61}$$

We have here an interesting situation in which the projected Eq.60 tends in the limes, $\lambda_2 \to 0$, to the original Eq.11. Is it possible from the point of view of logic? The answer to this question is positive, if we take into account that equation (4.5) considered in the Fock space are *overdetermined*. Why? Because Eqs11 or its regularized version (59) or (60) are exactly the same in the free Fock space as well as in the physical space (28).

Now we transform the Eq.60 in an equivalent way taking into account the right invertibility of the operators $\hat{M}$ and $\hat{L}$. We get

$$\left\{\hat{I} + [\lambda_2 \hat{M}(\hat{L} + \hat{G})]_R^{-1}[\hat{L} + \hat{G} + \lambda_1 \hat{N}]\right\}|V>= \hat{P}_M|V> + \\ [\lambda_2 \hat{M}(\hat{L} + \hat{G})]_R^{-1}\left(\hat{P}_0|V> + \lambda_1 \hat{P}_0(\hat{I} + \lambda_2 \hat{M})_R^{-1}\hat{N}|V>\right) \tag{62}$$

where

$$\hat{P}_M = \hat{I} - [\hat{M}(\hat{L} + \hat{G})]_R^{-1}\hat{M}(\hat{L} + \hat{G}) \tag{63}$$

is a projector on the null space of the operator $\hat{M}(\hat{L} + \hat{G})$. This projector has the following structure:

$$\hat{P}_M = \hat{I} - (\hat{L} + \hat{G})_R^{-1}\hat{Q}_M(\hat{L} + \hat{G}) \tag{64}$$

with projector $\hat{Q}_M = \hat{M}_R^{-1}\hat{M}$ and a right inverse

$$(\hat{L} + \hat{G})_R^{-1} = \hat{L}_R^{-1}(\hat{I} + \hat{G}\hat{L}_R^{-1})_R^{-1} \tag{65}$$

The projected vector of |V>, $\hat{P}_M|V>$, is an arbitrary vector of the null space of the operator $\hat{M}(\hat{L}+\hat{G})$. It describes an additional freedom which a deformed theory brings to the original theory given by Eq.11. From (27) we have

$$\hat{P}_M \hat{\Pi}_{L+G} = \hat{\Pi}_{L+G} \tag{66}$$

It is also possible to have a different perspective on the Eq.59. This time we will not weaken this equation but we represent the operator $\hat{R}(\lambda_2) \equiv (\hat{I} + \lambda_2 \hat{M})^{-1}_R$ in a more explicit way assuming that $\hat{M}$ is a right invertible operator. Then we get

$$\begin{gathered} \hat{R}(\lambda_2) \equiv (\hat{I} + \lambda_2 \hat{M})^{-1}_R = \\ [\hat{I} + (\lambda_2 \hat{M})^{-1}_R]^{-1}[(\lambda_2 \hat{M})^{-1}_R + \hat{\Gamma}_M \hat{R}(\lambda_2)] \end{gathered} \tag{67}$$

with an arbitrary element $\hat{\Gamma}_M \hat{R}(\lambda_2)$ where the projector

$$\hat{\Gamma}_M = \hat{I} - \hat{M}^{-1}_R \hat{M} \equiv \hat{I} - \hat{Q}_M \tag{68}$$

It is interesting that a choice

$$\hat{\Gamma}_M \hat{R}(\lambda_2) = 0 \tag{69}$$

leads to a generalized resolvent $\hat{R}(\lambda_2)$ satisfying the resolvent equation as

$$\frac{d}{d\lambda_2} \hat{R} = -\hat{M} \hat{R}^2 \tag{70}$$

which is also satisfied for the operators $\hat{M}$ with the usual resolvents. This equation, in turn, ensures that, for at least right invertible operators $\hat{M}$, the choice $\hat{R} = [\hat{I} + (\lambda_2 \hat{M})^{-1}_R]^{-1} (\lambda_2 \hat{M})^{-1}_R$ satisfies the same condition as for the non singular $\hat{M}$, namely: $\hat{R} \to 0$, for $\lambda_2 \to \infty$. In practical considerations, the term $\hat{\Gamma}_M \hat{R}(\lambda_2) \neq 0$ can be used when there are problems with limes $\lambda_2 \to 0$.

Let us also notice that in the zero-$\lambda_2 limit$ the vector

$$|\Psi(\lambda_2)>\equiv [\hat{I}+(\lambda_2\hat{M})_R^{-1}]^{-1}(\lambda_2\hat{M})_R^{-1}|\Phi>\to|\Phi> \tag{71}$$

at the assumption

$$\lambda_2\hat{M}|\Psi(\lambda_2)>\to 0, \quad for\ \lambda_2\to 0 \tag{72}$$

because from (71) results that

$$[\lambda_2\hat{I}+\hat{M}_R^{-1}]|\Psi(\lambda_2)>=\hat{M}_R^{-1}|\Phi> \tag{73}$$

and finally

$$[\lambda_2\hat{M}+\hat{I}]|\Psi(\lambda_2)>=|\Phi> \tag{74}$$

In other words the operator

$$[\hat{I}+(\lambda_2\hat{M})_R^{-1}]^{-1}(\lambda_2\hat{M})_R^{-1}\to\hat{I} \tag{75}$$

behaves as the unit operator, with $\lambda_2\to 0$, if the vector $\hat{M}|\Psi(\lambda_2)>$has oscillations suppressed by $\lambda_2\to 0$. In fact, the paper presented is about the indeterminate expression $0\cdot\infty$ or rather -$\hat{0}\cdot\hat{\infty}$- because we are dealing here with operators.

Eq.59 with representation (67) is following:

$$\begin{aligned}&\left(\hat{L}+\hat{G}\right)|V>+\\ &\lambda_1\left([\hat{I}+(\lambda_2\hat{M})_R^{-1}]^{-1}[(\lambda_2\hat{M})_R^{-1}+\hat{\Gamma}_M\hat{R}(\lambda_2)]\right)\hat{N}|V>\\ &=\hat{P}_0|V>+\lambda_1\hat{P}_0(\hat{I}+\lambda_2\hat{M})_R^{-1}\hat{N}|V>\end{aligned} \tag{76}$$

With a right inverse operator, $(\hat{L}+\hat{G})_R^{-1}$, we can transform this equation in an equivalent way as

$$|V> +\lambda_1(\hat{L}+\hat{G})_R^{-1}[\hat{I}+(\lambda_2\hat{M})_R^{-1}]^{-1}\cdot$$
$$\left([(\lambda_2\hat{M})_R^{-1}+\hat{\Gamma}_M\hat{R}(\lambda_2)]\right)\hat{N}|V>$$
$$=\hat{\Pi}_{L+G}|V> +$$
$$(\hat{L}+\hat{G})_R^{-1}\left(\hat{P}_0|V> +\lambda_1\hat{P}_0(\hat{I}+\lambda_2\hat{M})_R^{-1}\hat{N}|V>\right) \qquad (77)$$

where a projector on the null space of the operator $(\hat{L}+\hat{G})$ is given by

$$\hat{\Pi}_{L+G}=\hat{I}-(\hat{L}+\hat{G})_R^{-1}(\hat{L}+\hat{G}) \qquad (78)$$

With condition (28), Eq.77 leads to an equation in which the permutation symmetry of solutions is explicitly taken into account:

$$|V> +\lambda_1\hat{S}(\hat{L}+\hat{G})_R^{-1}\cdot$$
$$\left([\hat{I}+(\lambda_2\hat{M})_R^{-1}]^{-1}[(\lambda_2\hat{M})_R^{-1}+\hat{\Gamma}_M\hat{R}(\lambda_2)]\right)\hat{N}|V>$$
$$=\hat{S}\hat{\Pi}_{L+G}|V> +$$
$$\hat{S}(\hat{L}+\hat{G})_R^{-1}\left(\hat{P}_0|V> +\lambda_1\hat{P}_0(\hat{I}+\lambda_2\hat{M})_R^{-1}\hat{N}|V>\right) \qquad (79)$$

For a choice of (95), this equation in which the arbitrary element - the projection $\hat{\Pi}_{L+G}|V>$- is the same as in the theory with major coupling constant $\lambda_1=0$, see Sec.5, can be closed, see the next section. It is noteworthy that perhaps equations: (62), (79) are a new type of equations which offer a new perspective on the calculation of n-pfs, see Sec.10. It is encouraging that in both formulations unspecified elements can be chosen in such a way that similar results can be obtained.

There is possibility of a different regularization of Eq.11:

$$\left(\hat{L} + \lambda_1 \hat{N}(\hat{I} + \lambda_2 \hat{M})_R^{-1} + \hat{G}\right)|V>=$$
$$\hat{P}_0|V> + \lambda_1 \hat{P}_0 \hat{N}(\hat{I} + \lambda_2 \hat{M})_R^{-1}|V> \tag{80}$$

Hence and from (67) we get

$$\left\{\hat{L} + +\hat{G}\right\}|V> +$$
$$\lambda_1 \hat{N}\left([\hat{I} + (\lambda_2 \hat{M})_R^{-1}]^{-1}[(\lambda_2 \hat{M})_R^{-1} + \hat{\Gamma}_M \hat{R}(\lambda_2)]\right)|V>$$
$$= \hat{P}_0|V> + \lambda_1 \hat{P}_0 \hat{N}(\hat{I} + \lambda_2 \hat{M})_R^{-1}|V> \tag{81}$$

With choice (95)

$$|V> + (\hat{L} + \hat{G})_R^{-1} \cdot$$
$$\left(\frac{\lambda_1}{\lambda_2}[\hat{I} + (\lambda_2 \hat{N})_R^{-1}]^{-1} + \lambda_1 \hat{N}\left([\hat{I} + (\lambda_2 \hat{N})_R^{-1}]^{-1} \hat{\Gamma}_N \hat{R}(\lambda_2)]\right)\right) \cdot$$
$$\cdot|V>= \hat{\Pi}_{L+G}|V> +$$
$$(\hat{L} + \hat{G})_R^{-1}\left(\hat{P}_0|V> + \lambda_1 \hat{P}_0 \hat{N}(\hat{I} + \lambda_2 \hat{N})_R^{-1}|V>\right) \tag{82}$$

With assumption (28) the above equation leads to its symmetrized version

$$|V> + \hat{S}(\hat{L} + \hat{G})_R^{-1} \cdot$$
$$\left(\frac{\lambda_1}{\lambda_2}[\hat{I} + (\lambda_2 \hat{N})_R^{-1}]^{-1} + \lambda_1 \hat{N}\left([\hat{I} + (\lambda_2 \hat{N})_R^{-1}]^{-1} \hat{\Gamma}_N \hat{R}(\lambda_2)]\right)\right) \cdot$$
$$\cdot|V>= \hat{S}\hat{\Pi}_{L+G}|V> +$$
$$\hat{S}(\hat{L} + \hat{G})_R^{-1}\left(\hat{P}_0|V> + \lambda_1 \hat{P}_0 \hat{N}(\hat{I} + \lambda_2 \hat{N})_R^{-1}|V>\right) \tag{83}$$

This equation as other previous equations, for the choice (95), leads to closed equations for n-pfs. A positive fact is that, as in perturbation theory, the undetermined vector $\hat{S}\hat{\Pi}_{L+G}|V>$can be identified with the original linear theory ($\hat{N}=0$) and that for derivation of the last equation we did not project, besides their symmetrization, the previous equations. This last fact makes that weakening of the original Eqs (2) still can be interpreted as the appropriate averaging.

The arbitrary term, $\hat{\Gamma}_N\hat{R}(\lambda_2)$, can be chosen in such a way to remove the “secular” terms which arise when the $\lambda_2 \to 0$.

Finally, we can connect two kind of regularizations and consider the following regularization:

$$\hat{N} \to \frac{1}{2}\left(\hat{N}(\hat{I}+\lambda_2\hat{N})_R^{-1} + (\hat{I}+\lambda_2\hat{N})_R^{-1}\hat{N}\right) \tag{84}$$

to mimic in some degree a commutation of $\hat{N}$ with operator $(\hat{I}+\lambda_2\hat{N})^{-1}$if the last operator would exist.

For such regularization and choice (69) chosen only for simplicity, we get the following equation:

$$\begin{gathered}\left\{\hat{I}+\tfrac{\lambda_1}{2\lambda_2}\hat{S}(\hat{L}+\hat{G})_R^{-1}\left([\hat{I}+(\lambda_2\hat{N})_R^{-1}]^{-1}[\hat{Q}_N+\hat{I}]\right)\right\}\cdot \\ \cdot|V>=\hat{S}\hat{\Pi}_{L+G}|V>+\hat{S}(\hat{L}+\hat{G})_R^{-1}\cdot \\ \left(\hat{P}_0|V>+\tfrac{\lambda_1}{2}\hat{P}_0\left(\hat{N}(\hat{I}+\lambda_2\hat{N})_R^{-1}+(\hat{I}+\lambda_2\hat{N})_R^{-1}\hat{N}\right)|V>\right)\end{gathered} \tag{85}$$

We see that at the diagonal terms the rate of coupling constants $\lambda_{1,2}$appears and particularly, the 1-pf V only has such a dependence. This means that, for monomial theories at least and used assumptions, the 1-pfs V, in the $lim\lambda_2 \to 0$, stop to depend on the major coupling constant λ_1. If we agree that exact closure of equations for n-pfs V is equivalent to a summation of the infinite terms of the canonical perturbation theory, we can compare this result with disappearing of the effective coupling constant of the perturbation theory. This rather negative result can be treated as a hint that $\lambda_2 \neq 0$, and that non polynomial theories should be taken into consideration, or, that $\lambda_1$depends on $\lambda_2$in such a way that $\frac{\lambda_1}{\lambda_2} \to const$ when $\lambda_2 \to 0$ (some kind of renormalization of the original theory). We can imagine the following situation in which the developed type of theories can be used: For a certain class of initial and/or boundary conditions ("laminar" conditions), phenomena are well described by the polynomial theory (2) with $\lambda_2 = 0$. For others ("turbulent" conditions, we need to use averages or smearing and then non polynomial terms ($\lambda_2 \neq 0$) have to be taken into account because, as we see in averaged version of Eqs (2), Eqs (59) and (60), $\lambda_2$enters these equations in the inverse powers. In other words, the non polynomial terms become apparent only when considered situations (boundary and initial conditions) are becoming increasingly complex and averaged solutions and correlation functions are needed.

Let us show that at least for a choice (69) the right invertible operator $\hat{M}$ dose not commute with its general resolvent $\hat{R} = [\hat{I} + (\lambda\hat{M})_R^{-1}]^{-1}(\lambda\hat{M})_R^{-1}$: We get

$$\begin{aligned}\left[\hat{M},\hat{R}\right] = \hat{M}\tfrac{(\lambda\hat{M})_R^{-1}}{\hat{I}+(\lambda\hat{M})_R^{-1}} - \tfrac{(\lambda\hat{M})_R^{-1}}{\hat{I}+(\lambda\hat{M})_R^{-1}}\hat{M} = \\ \tfrac{\lambda^{-1}\hat{I}}{\hat{I}+(\lambda\hat{M})_R^{-1}} - \tfrac{\lambda^{-1}\hat{Q}_M}{\hat{I}+(\lambda\hat{M})_R^{-1}} = \\ \tfrac{\lambda^{-1}\hat{\Gamma}_M}{\hat{I}+(\lambda\hat{M})_R^{-1}} = \lambda^{-1}\hat{\Gamma}_M\end{aligned} \tag{86}$$

see (64) and (68), for definitions of appropriate projectors.

8 Undetermined terms of the general solution.

If we confine ourselves to the solutions satisfying the condition (28), then the projected by the projector $\hat{S}$, Eq.62, can be written as follows:

$$\left\{\hat{I}+\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}[\hat{L}+\hat{G}+\lambda_1\hat{N}]\right\}|V>=\hat{S}\hat{P}_M|V>+$$
$$\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\left(\hat{P}_0|V>+\lambda_1\hat{P}_0(\hat{I}+\lambda_2\hat{M})_R^{-1}\hat{N}|V>\right) \tag{87}$$

Introducing the operator

$$\hat{A}=\hat{I}+\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}[\hat{L}+\hat{G}] \tag{88}$$

we rewrite Eq.87 as follows:

$$\{\hat{A}+\lambda_1\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{N}\}|V>=\hat{S}\hat{P}_M|V>+$$
$$\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\left(\hat{P}_0|V>+\lambda_1\hat{P}_0(\hat{I}+\lambda_2\hat{M})_R^{-1}\hat{N}|V>\right) \tag{89}$$

This equation may serve as a starting point for a perturbation theory of the modified Eq59with the major coupling constant $\lambda_1$as an expansion parameter and the operator $\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{N}$ as the perturbation operator. With assumption about non singularity of the operator $\hat{A}$, we can rewrite Eq.**??** as

$$\left\{\hat{I}+\lambda_1\hat{A}^{-1}\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{N}\right\}|V>=\hat{A}^{-1}\hat{S}\hat{P}_M|V>+$$
$$\hat{A}^{-1}\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\left(\hat{P}_0|V>+\lambda_1\hat{P}_0(\hat{I}+\lambda_2\hat{M})_R^{-1}\hat{N}|V>\right) \tag{90}$$

We will assume that, for $\lambda_1 = 0$, the theory is continuous and hence, the unknown term

$$\hat{A}^{-1}\hat{S}\hat{P}_M|V>^{(0)} + \hat{A}^{-1}\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{P}_0|V>^{(0)} = |V;\lambda_1 = 0> \equiv |V>^{(0)} \tag{91}$$

where the generating vector $|V>^{(0)}$generates correlation functions of the originally linear theory characterized by the zero coupling constant λ_1. In a general case , for $\lambda_1 \not\approxeq 0$, the undetermined element of Eq.90 is chosen in a such way that the IBC are satisfied. Additionally, if the expansion (31) is used and we get

$$\hat{A}^{-1}\hat{S}\hat{P}_M|V;\lambda_1,\lambda_2> + \hat{A}^{-1}\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{P}_0|V> = |V>^{(0)} + \sum_{j=1}^{\propto}\lambda_1^j(\hat{A}^{-1}\hat{S}\hat{P}_M|V;\lambda_2>)^{(j)} \tag{92}$$

then we try to choose the undetermined projections $\hat{S}\hat{P}_M|V;\lambda_2>^{(j)}$ in such a way that no divergent terms do not appear in the series (31), when $\lambda_2 \to 0$. We have assumed here that the original theory ($\lambda_2 = 0$) is well defined.

The simplest case is then again when

$$\begin{gathered}\hat{A}^{-1}\hat{S}\hat{P}_M|V> + \hat{A}^{-1}\hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{P}_0|V> \\ = |V>^{(0)} \Leftrightarrow \\ \hat{S}\hat{P}_M|V> + \hat{S}[\lambda_2\hat{M}(\hat{L}+\hat{G})]_R^{-1}\hat{P}_0|V> = \hat{A}(\lambda_2)|V>^{(0)}\end{gathered} \tag{93}$$

This assumption is in agreement with (91) and also corresponds to the case when the IBC are satisfied by the zero-order approximation and no divergences appear in the expansion with respect to the major coupling constant λ_1. The assumption that arbitrary term of Eq.90, $\hat{A}^{-1}\hat{S}\hat{P}_M|V>$, is identified with the original linear theory, see Eq.2, indicates that the further consequences of this theory are related to the nonlinear effects (perturbation of the original, linear theory, (Eq.2)) occurring in the system. Quite natural assumption which is a started point of almost every perturbation theory. Again we must recall that assumption (93) does not exclude that the part of arbitrary element $\hat{P}_M|V>\in \hat{P}_M F$

$$(\hat{I}-\hat{S})\hat{P}_M|V>=f(\lambda_1,\lambda_2) \tag{94}$$

responsible for the elimination of asymmetric parts in Eq.62 can be a complicated vector function allowing to satisfy the condition (28).

If at the choice (93) some bad, “secular” terms appear in the power series expansion with respect to the major coupling constant λ_1, see [18] then to remove such terms we can use the formula (92). It appears an open question: Is it possible using this type of equations to gain a new perspective on the renormalization in of quantum and statistical field theories?

Next we assume

$$\hat{M}=\hat{N} \tag{95}$$

because such a choice of the modified operator often leads to closed or partly closed equations for n-point correlation functions. It is natural that a candidate for modified theory should be looked among closed theories of the type (95). It is worth noting that a more general choice of the modifying operator

$$\hat{M}=\hat{\Lambda}\hat{N} \tag{96}$$

with a diagonal operator $\hat{\Lambda}$ can also be used to close the considered equations. We assume that operator $\hat{\Lambda}$is a non singular operator not to increase freedom of the theory contained in the projection $\hat{\Gamma}_M \hat{R}(\lambda_2)$, see (67). Nevertheless, the choice of (95) exhibits a kind of self-similarities of fractal theories.

9 Functional calculus and considered equations

The *functional calculus* is define sometimes as a branch of mathematics about inserting operators into functions to get in result meaningful new operators, see, e.g., [19], [20]. Such operator-valued functions often appear in solutions to many linear equations of physics and engineering problems and in our opinion are an important generalization of the concept of function.

From the functional calculus point of view we can say that our paper needs a definition of the function $(\hat{I} + \lambda_2 \hat{M})^{-1}$in the case of upper triangular operators $\hat{M}$ like (95). The natural and often used definition of this function as the power series

$$(\hat{I} + \lambda_2 \hat{M})^{-1} := \hat{I} - \lambda_2 \hat{M} + (\lambda_2 \hat{M})^2 - ... \tag{97}$$

for unbounded operators, in every its term can be incorrect and, moreover, in the case of upper triangular operators $\hat{M}$ very inconvenient because it introduces to every correlation function of basic Eq59 an infinite number of modification or regularization terms. Therefore, we used a different definition

$$(\hat{I} + \lambda_2 \hat{M})^{-1} := (\hat{I} + \lambda_2 \hat{M})_R^{-1} \tag{98}$$

where subscript "R" means a right inverse operator to the operator $\hat{I} + \lambda_2 \hat{M}$. So, instead of a senseless expression (97) we use well defined and explicitly constructed expression (98), see (67),

(95) and (107)-(109). The operator (98) is called the *generalized resolvent* of the operator $\hat{M}$. Interesting thing is that regularization or modification of equations (11), given by the generalized resolvent of the operator $\hat{M} = \hat{N}$, leads to closed equations for the correlation functions. Moreover, many operator-valued functions in the free Fock space can be defined by means of the type of Dunford-Cauchy integral with generalized resolvent (98):

$$f_{\lambda_2}(\hat{M}) \equiv \int\limits_{\Gamma} f_{\lambda_2}(\lambda)(\hat{I} + \lambda\hat{M})_R^{-1} d\lambda \tag{99}$$

It is not unlikely that this paper is the first step to understanding these operator valued functions in the context of equations for the correlation functions. We can say that we analyzed such a problem: to which conclusions leads a regularization or modification of Eq.11 which consists in replacing an upper triangular operator $\hat{N}$ by the diagonal + lower triangular operator, e.g.,

$$\hat{N} \to f_{\lambda_2}(\hat{N})\hat{N} = \left(\int\limits_{\Gamma} f_{\lambda_2}(\lambda)[\hat{I} + (\lambda\hat{N})_R^{-1}]^{-1}(\lambda N)_R^{-1} d\lambda \right) \hat{N} \tag{100}$$

or

$$\hat{N} \to \frac{1}{2}\left(f_{\lambda_2}(\hat{N})\hat{N} + \hat{N} f_{\lambda_2}(\hat{N}) \right) \tag{101}$$

see (67), (69) and (95). In the last formula like in (84) we took into account that different right inverse operators do not commute with each other: Denoting two different right inverses by $\hat{N}_R^{-1}$and $\hat{N}_R^{-1'}$if we assume their commutativity we come to the absurd:

$$\hat{N}([\hat{N}_R^{-1}, \hat{N}_R^{-1'}])|\Phi >= \hat{N}_R^{-1'}|\Phi > -\hat{N}_R^{-1}|\Phi >= 0 \quad (102)$$

for any vector $|\Phi >$. Also the operator valued functions constructed by different right inverses do not commute with each other and the original operator $\hat{N}$. For example, we expect that, at least formally, $\hat{N}$commute with operator valued functions like $(\hat{I} + \lambda\hat{N})^{-1}$or $exp(-\hat{N})$ and so on. Nevertheless, we try to mimic commutativity of their formal predecessors (formal inverse operators, see (97)). We can support this idea in the following way: in the case in which the problem in consideration is ill-posed and we use averages, then the original Eq.2 looses its meaning and a more free connection between averages and Eq.2 is recommended. In this way we substitute traditional theories with very complicated perturbation series and related to the open equations for correlation functions by a new, more flexible theories with closed equations like (**??**).

In the case of regularization of the original theory with nonlinearity described by the operator $\hat{N}$ we have a demand

$$lim_{\lambda_2 \to 0} f_{\lambda_2}(\hat{N}) = \hat{I} \quad (103)$$

or its weaker form, e.g.,

$$lim_{\lambda_2 \to 0} f_{\lambda_2}(\hat{N}) = \hat{I} - \hat{P}_0 \quad (104)$$

It is not clear whether the condition (103) or (104), although weaker than the condition (71) and (72), can be fulfilled for the operator valued functions occurring in formulas (100) and (101).

10 An example of the theory with a local nonlinear term, divergences and final remarks

Let us consider the equation:

$$\int L(\tilde{x},\tilde{y})\varphi(\tilde{y})d\tilde{y} + \lambda\varphi^3(\tilde{x}) = 0 \tag{105}$$

which is called the $\varphi^{(3)}$- or Hurst model. In this model the function $G = 0$ in Eq.2. Usually, the kernel L is a sum of the Dirac's deltas and its derivatives or their discrete version. For this model, the operator

$$\hat{L} = \int \hat{\eta} * (\tilde{x})L(\tilde{x},\tilde{y})\hat{\eta}(\tilde{y})d\tilde{x}d\tilde{y} + \hat{P}_0 \tag{106}$$

the operator

$$\hat{N} = \int \hat{\eta} * (\tilde{z})\hat{\eta}(\tilde{z})^2 d\tilde{z} + \hat{P}_0\hat{N} \tag{107}$$

where, to guarantee right invertibility of the operator $\hat{N}$, we have chosen

$$\hat{P}_0\hat{N} = \hat{P}_0\hat{N}\hat{P}_1 = \hat{P}_0 \int d\tilde{z}\hat{\eta}(\tilde{z})d\tilde{z}\hat{P}_1 \neq 0 \tag{108}$$

In this case a right inverse to the operator $\hat{N}$ can be chosen as

$$\hat{N}_R^{-1} = \int \{\hat{\eta} * (\tilde{y}))^2\hat{\eta}(\tilde{y})d\tilde{y} + \hat{N}_R^{-1}\hat{P}_0 \tag{109}$$

with

$$\hat{N}_R^{-1}\hat{P}_0 = \hat{P}_1\hat{N}_R^{-1}\hat{P}_0 = \hat{P}_1\hat{\eta}^\star(\tilde{y})\hat{P}_0 \tag{110}$$

We will use the regularized theory leading to closed equations, Secs 7-8. For a sake of simplicity, we use Eq.79 in the case of (95) and the operator $\hat{G} = 0$. We also choose (69), for the arbitrary element of the generalized resolvent (67). In result, we get equation

$$\left\{\hat{I} + \lambda_1 \hat{S}(\hat{L})_R^{-1}\left([\hat{I} + (\lambda_2\hat{N})_R^{-1}]^{-1}(\lambda_2\hat{N})_R^{-1}\right)\hat{N}\right\}|V> =$$
$$\hat{S}\hat{\Pi}_L|V> + \hat{S}(\hat{L})_R^{-1}\left(\hat{P}_0|V> + \lambda_1\hat{P}_0(\hat{I} + \lambda_2\hat{N})_R^{-1}\hat{N}|V>\right) \tag{111}$$

and further

$$\left\{\hat{I} + \tfrac{\lambda_1}{\lambda_2}\hat{S}(\hat{L})_R^{-1}\left([\hat{I} + (\lambda_2\hat{N})_R^{-1}]^{-1}\hat{Q}_N\right)\right\}|V> = \hat{S}\hat{\Pi}_L|V>$$
$$+\hat{S}(\hat{L})_R^{-1}\left(\hat{P}_0|V> + \lambda_1\hat{P}_0(\hat{I} + \lambda_2\hat{N})_R^{-1}\hat{N}|V>\right) \tag{112}$$

where projector

$$\hat{Q}_N = \hat{N}_R^{-1}\hat{N} \tag{113}$$

If projection $\hat{S}\hat{\Pi}_L|V\ >$can be calculated from Eq.112 in which $\lambda_1 = 0$ we would get

$$\hat{S}\hat{\Pi}_L|V\ >\ = \{\hat{I} - \hat{S}\hat{L}_R^{-1}\hat{P}_0\}|V>^{(0)} = \{\hat{I} - \hat{P}_0\}|V>^{(0)} \tag{114}$$

From Eq.112 we get then

$$\hat{P}_2\{\hat{I}+\tfrac{\lambda_1}{\lambda_2}\hat{S}\hat{L}_R^{-1}\left(\hat{I}-\lambda_2^{-1}\hat{N}_R^{-1}\right)\hat{Q}_N\}|V>= \\ \hat{P}_2\hat{S}\hat{\Pi}_L|V>=\hat{P}_2|V>^{(0)} \tag{115}$$

and

$$\hat{P}_1\{\hat{I}+\frac{\lambda_1}{\lambda_2}\hat{S}\hat{L}_R^{-1}\hat{Q}_N\}|V\ \ >\ =\hat{P}_1\hat{S}\hat{\Pi}_L|V>=\hat{P}_1|V>^{(0)} \tag{116}$$

where we took into account that the operator $\hat{N}_R^{-1}$is lower triangular and

$$\hat{P}_0\hat{Q}_N=0 \tag{117}$$

These are closed equations for the lowest - 1 and 2-pfs. We can't get an equation for the 0-pf, V_0, see (7). With Eq.116 we can express the 1-point solution as:

$$\hat{P}_1|V>=\hat{P}_1\{\hat{I}+\frac{\lambda_1}{\lambda_2}\hat{S}\hat{L}_R^{-1}\hat{Q}_N\}^{-1}|V>^{(0)} \tag{118}$$

To describe explicitly these equations we have to take into account that

$$\hat{Q}_N=\hat{N}_R^{-1}\hat{N}= \\ \int\hat{\eta}^\star(\tilde{x})^2\hat{\eta}(\tilde{x})^2d\tilde{x}+\hat{P}_1\int\hat{\eta}^\star(\tilde{z})\hat{P}_0\hat{\eta}(\tilde{y})d\tilde{y}\hat{P}_1 \tag{119}$$

and

$$\hat{L}_R^{-1}\hat{Q}_N=\int\hat{\eta}^\star(\tilde{x})L_R^{-1}(\tilde{x},\tilde{y})\hat{\eta}^\star(\tilde{y})\hat{\eta}(\tilde{y})^2d\tilde{x}d\tilde{y}+ \\ \hat{P}_1\hat{L}_R^{-1}\hat{P}_1\int\hat{\eta}^\star(\tilde{z})\hat{P}_0\hat{\eta}(\tilde{y})d\tilde{y}\hat{P}_1 \tag{120}$$

with arbitrary value for the variable $\tilde{z}$. From (116) we get the following equation for the 1-pf:

$$V(\tilde{x}) + \frac{\lambda_1}{\lambda_2} \int L_R^{-1}(\tilde{x}, \tilde{y}) d\tilde{y}\, V(\tilde{z}) = V^{(0)}(\tilde{x}) \tag{121}$$

with arbitrary fixed value for the variable $\tilde{z}$. Denoting the integral

$$\int L_R^{-1}(\tilde{x}, \tilde{y}) d\tilde{y} = a(\tilde{x}) \tag{122}$$

we rewrite Eq.121 as

$$V(\tilde{x}) + \frac{\lambda_1}{\lambda_2} a(\tilde{x}) V(\tilde{z}) = V^{(0)}(\tilde{x}) \tag{123}$$

Its solution is

$$V(\tilde{x}) \equiv V(\tilde{x}; \tilde{z}) = V^{(0)}(\tilde{x}) - \frac{\lambda_1}{\lambda_2} a(\tilde{x}) \frac{V^{(0)}(\tilde{z})}{1 + \frac{\lambda_1}{\lambda_2} a(\tilde{z})} \tag{124}$$

The result is rather frustrating because, for $\lambda_2 \to 0$, dependence of the 1-pf V on λ_1 disappears completely. Similar results we get for the 2-pf with trivial condition (6), for $\lambda_2 \to 0$. This could mean that perhaps we should keep $\lambda_2 \neq 0$, or, we should try to use other regularization also proposed in Sec.7.

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