Uniqueness of collinear solutions for the relativistic three-body problem

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Abstract

Continuing work initiated in an earlier publication [Yamada, Asada, Phys. Rev. D **82**, 104019 (2010)], we investigate collinear solutions to the general relativistic three-body problem. We prove the uniqueness of the configuration for given system parameters (the masses and the largest separation among them). First, we show that the equation determining the distance ratio among the three masses, which has been obtained as a seventh-order polynomial in the previous paper, has at most three positive roots, which apparently provide three cases of the distance ratio. It is found, however, that, even for such cases, there exists one physically reasonable root and only one, because the remaining two positive roots do not satisfy the slow motion assumption in the post-Newtonian approximation and are thus discarded. This means that, especially for the restricted three-body problem, exactly three positions of a third body are true even at the post-Newtonian order. They are relativistic counterparts of the Newtonian Lagrange points L_1, L_2 and L_3. We show also that, for given system parameters, the angular velocity of the post-Newtonian collinear configuration is smaller than that for the Newtonian case.

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I. INTRODUCTION

The three-body problem in the Newton gravity belongs among classical problems in astronomy and physics (e.g, [1, 2]). In 1765, Euler found a collinear solution for the restricted three-body problem, where one of three bodies is a test mass. Soon later, his solution was extended for a general three-body problem by Lagrange, who also found an equilateral triangle solution in 1772. Now, the solutions for the restricted three-body problem are called Lagrange points L_1, L_2, L_3, L_4 and L_5, which are described in textbooks of classical mechanics [2]. SOHO (Solar and Heliospheric Observatory) and WMAP (Wilkinson Microwave Anisotropy Probe) launched by NASA are in operation at the Sun-Earth L_1 and L_2, respectively. LISA (Laser Interferometer Space Antenna) pathfinder is planned to go to L_1. Lagrange points have recently attracted renewed interests for relativistic astrophysics [3, 4], where they have discussed the gravitational radiation reaction on L_4 and L_5 by numerical methods.

As a pioneering work, Nordtvedt pointed out that the location of the triangular points is very sensitive to the ratio of the gravitational mass to the inertial one [5]. Along this course, it is interesting as a gravity experiment to discuss the three-body coupling terms at the post-Newtonian order, because some of the terms are proportional to a product of three masses as $M_1 \times M_2 \times M_3$. Such a term appears only for relativistic three (or more) body systems: For a relativistic binary with two masses M_1 and M_2, there exist $M_1^2 M_2$ and $M_1 M_2^2$ without such a three-mass product. For a Newtonian three-body system, we have only the two-body coupling terms proportional to $M_1 M_2$, $M_2 M_3$ or $M_3 M_1$.

The relativistic periastron advance of the Mercury is detected only after much larger shifts due to Newtonian perturbations by other planets such as the Venus and Jupiter are taken into account in the astrometric data analysis. In this sense, effects by the three body coupling are worthy to investigate. Nevertheless, most of post-Newtonian works have focused on either compact binaries because of our interest in gravitational waves astronomy or N-body equation of motion (and coordinate systems) in the weak field such as the solar system (e.g. [6]). Actually, future space astrometric missions such as Gaia [7, 8] require a general relativistic modeling of the solar system within the accuracy of a micro arc-second [9]. Furthermore, a binary plus a third body have been discussed also for perturbations of gravitational waves induced by the third body [10–13].

After efforts to find a general solution, Poincare proved that it is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential. Namely, we cannot analytically obtain all the solutions. Nevertheless, the number of new solutions is increasing [14]. Therefore, the three-body problem still remains an open issue even for Newton gravity.

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time. Hence it is important to take account of general relativistic effects on three-body configurations. The figure-eight configuration that was found decades ago [15, 16] has been recently studied at the first post-Newtonian [17] and also the second post-Newtonian orders [18]. According to their numerical investigations, the solution remains true with a slight change in the figure-eight shape because of relativistic effects.

On the other hand, the post-Newtonian collinear configuration obtained in the previous paper [19] may offer a useful toy model for relativistic three-body interactions, because it is tractable by hand without numerical simulations. This solution is an relativistic extension of *Euler's collinear one*, where three bodies move around the common center of mass with the same orbital period and always line up.

In fact, their formulation leads to a seventh-order equation determining the distance ratio among masses [19]. Here, it should be noted that only positive roots are acceptable, because the distance ratio must be positive. Properties of the master equation has not been known yet. How many positive roots for it are there? The main purpose of this paper is to *analytically* investigate the number of the positive roots. In particular, we shall prove the uniqueness of the configuration for given system parameters (the masses and the largest separation among them).

This paper is organized as follows. In section II, we briefly summarize formulations for collinear solutions at the Newtonian and post-Newtonian orders. We discuss positive roots for the seventh-order equation for determining the distance ratio in section III. In section IV, we show the uniqueness of the configuration for given system parameters (the masses and the largest separation among them). We also compare the angular velocity of the post-Newtonian collinear configuration with that for the Newtonian one. Section V is devoted to the conclusion. We provide some detailed calculations regarding the angular velocity of collinear configurations in Appendix.

Throughout this paper, we take the units of $G = c = 1$.

II.   EQUATION FOR THE DISTANCE RATIO AMONG THREE MASSES

Let us begin by summarizing the derivation of the Euler's collinear solution for the circular three-body problem in Newton gravity. We consider Euler's solution, for which each mass moves around their common center of mass denoted as $\boldsymbol{X}_G$ with a constant angular velocity ω. Hence, it is convenient to use the corotating frame with the same angular velocity ω. We choose an orbital plane normal to the total angular momentum as the $x - y$ plane in such a corotating frame. We locate all the three bodies on a single line, along which we take the x-coordinate. The location of each mass M_I ($I = 1, 2, 3$) is written as $\boldsymbol{X}_I \equiv (x_I, 0)$. Without loss of generality, we assume $x_3 < x_2 < x_1$. Let R_I define the relative position of each mass with respective to the center of mass $\boldsymbol{X}_G \equiv (x_G, 0)$, namely $R_I \equiv x_I - x_G$ ($R_I \neq |\boldsymbol{X}_I|$ unless $x_G = 0$). We choose $x = 0$ between M_1 and M_3. We thus have $R_3 < R_2 < R_1$, $R_3 < 0$ and $R_1 > 0$.

It is convenient to define a ratio as $R_{23}/R_{12} = z$, which is an important variable in the following formulation. Then we have $R_{13} = (1 + z)R_{12}$. The equation of motion becomes

$$R_1 \omega^2 = \frac{M_2}{R_{12}^2} + \frac{M_3}{R_{13}^2}, \tag{1}$$

$$R_2 \omega^2 = -\frac{M_1}{R_{12}^2} + \frac{M_3}{R_{23}^2}, \tag{2}$$

$$R_3 \omega^2 = -\frac{M_1}{R_{13}^2} - \frac{M_2}{R_{23}^2}, \tag{3}$$

where we define

$$\boldsymbol{R}_{IJ} \equiv \boldsymbol{X}_I - \boldsymbol{X}_J, \tag{4}$$

$$R_{IJ} \equiv |\boldsymbol{R}_{IJ}|. \tag{5}$$

First, we subtract Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2) and use $R_{12} \equiv |\boldsymbol{X}_1 - \boldsymbol{X}_2|$ and $R_{23} \equiv |\boldsymbol{X}_2 - \boldsymbol{X}_3|$. Such a subtraction procedure will be useful also at the post-Newtonian order, because we can avoid directly using the post-Newtonian center of mass [21, 22]. Next, we compute a ratio between them to delete ω^2. Hence a fifth-order equation is obtained as

$$(M_1 + M_2)z^5 + (3M_1 + 2M_2)z^4 + (3M_1 + M_2)z^3 - (M_2 + 3M_3)z^2 - (2M_2 + 3M_3)z - (M_2 + M_3) = 0. \tag{6}$$

Now we have a condition as $z > 0$. Descartes' rule of signs (e.g., [20]) states that the number of positive roots either equals to that of sign changes in coefficients of a polynomial or less

than it by a multiple of two. According to this rule, Eq. (6) has the only positive root $z > 0$, though such a fifth-order equation cannot be solved in algebraic manners as shown by Galois (e.g., [20]). After obtaining z, one can substitute it into a difference, for instance between Eqs. (1) and (3). Hence we get ω.

In order to include the dominant part of general relativistic effects, we take account of the terms at the first post-Newtonian order. Namely, the massive bodies obey the Einstein-Infeld-Hoffman (EIH) equation of motion as [21, 22]

$$
\begin{aligned}
\frac{d\boldsymbol{v}_K}{dt} = & \sum_{A \neq K} \boldsymbol{R}_{AK} \frac{M_A}{R_{AK}^3} \left[1 - 4 \sum_{B \neq K} \frac{M_B}{R_{BK}} - \sum_{C \neq A} \frac{M_C}{R_{CA}} \left(1 - \frac{\boldsymbol{R}_{AK} \cdot \boldsymbol{R}_{CA}}{2R_{CA}^2} \right) \right. \\
& \left. + v_K^2 + 2v_A^2 - 4\boldsymbol{v}_A \cdot \boldsymbol{v}_K - \frac{3}{2} \left(\boldsymbol{v}_A \cdot \boldsymbol{n}_{AK} \right)^2 \right] \\
& - \sum_{A \neq K} (\boldsymbol{v}_A - \boldsymbol{v}_K) \frac{M_A \boldsymbol{n}_{AK} \cdot (3\boldsymbol{v}_A - 4\boldsymbol{v}_K)}{R_{AK}^2} \\
& + \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \boldsymbol{R}_{CA} \frac{M_A M_C}{R_{AK} R_{CA}^3},
\end{aligned}
\tag{7}
$$

where $\boldsymbol{v}_I$ denotes the velocity of each mass in an inertial frame and we define

$$
\boldsymbol{n}_{IJ} \equiv \frac{\boldsymbol{R}_{IJ}}{R_{IJ}}.
\tag{8}
$$

We obtain a lengthy form of the equation of motion for each body. By subtracting the post-Newtonian equation of motion for M_3 from that for M_1 for instance, we obtain the equation as [19]

$$
R_{13}\omega^2 = F_N + F_M + F_V \omega^2,
\tag{9}
$$

where we denote $a \equiv R_{13}$ and the Newtonian term F_N and the post-Newtonian parts F_M (dependent on the masses only) and F_V (velocity-dependent part divided by ω^2) are defined

as

$$F_N = \frac{M}{a^2 z^2}\left[(\nu_1 + \nu_3)z^2 + (1 - \nu_1 - \nu_3)(1 + z^2)(1 + z)^2\right], \tag{10}$$

$$\begin{aligned}
F_M = -\frac{M^2}{a^3 z^3}\Big[& (4 - 4\nu_1 + \nu_3)(1 - \nu_1 - \nu_3) \\
& + (12 - 7\nu_1 + 3\nu_3)(1 - \nu_1 - \nu_3)z \\
& + (12 - \nu_1 + \nu_3)(1 - \nu_1 - \nu_3)z^2 \\
& + (8 - 7\nu_1 - 7\nu_3 + 8\nu_1\nu_3 + 3\nu_1^2 + 3\nu_3^2)z^3 \\
& + (12 + \nu_1 - \nu_3)(1 - \nu_1 - \nu_3)z^4 \\
& + (12 + 3\nu_1 - 7\nu_3)(1 - \nu_1 - \nu_3)z^5 \\
& + (4 + \nu_1 - 4\nu_3)(1 - \nu_1 - \nu_3)z^6\Big], \tag{11}
\end{aligned}$$

$$\begin{aligned}
F_V = \frac{M}{(1 + z)^2 z^2}\Big[& -\nu_1^2(1 - \nu_1 - \nu_3) \\
& - 2\nu_1(1 + \nu_1 - \nu_3)(1 - \nu_1 - \nu_3)z \\
& + (2 - 2\nu_1 + \nu_3 + 6\nu_1\nu_3 - 3\nu_3^2 + \nu_1^3 - 3\nu_1^2\nu_3 - 3\nu_1\nu_3^2 + \nu_3^3)z^2 \\
& + 2(2 - \nu_1 - \nu_3)(1 + \nu_1 + \nu_3 - \nu_1^2 + \nu_1\nu_3 - \nu_3^2)z^3 \\
& + (2 + \nu_1 - 2\nu_3 - 3\nu_1^2 + 6\nu_1\nu_3 + \nu_1^3 - 3\nu_1^2\nu_3 - 3\nu_1\nu_3^2 + \nu_3^3)z^4 \\
& - 2\nu_3(1 - \nu_1 + \nu_3)(1 - \nu_1 - \nu_3)z^5 \\
& - \nu_3^2(1 - \nu_1 - \nu_3)z^6\Big], \tag{12}
\end{aligned}$$

respectively. Here, we define the mass ratio as $\nu_I \equiv M_I/M$ for the total mass $M \equiv \sum_I M_I$ and make a frequent use of $\nu_2 = 1 - \nu_1 - \nu_3$. It should be noted that in this truncated calculation we ignore the second post-Newtonian (or higher order) contributions so that we can replace, for instance, v_1 by $R_1\omega$ (with using the Newtonian R_1) in post-Newtonian velocity-dependent terms such as v_1^2.

In the similar manner to the above Newtonian formulation, straightforward but lengthy calculations lead to a seventh-order equation as [19]

$$F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0, \tag{13}$$

6

where we define

$$A_7 = \frac{M}{a}\left[-4 - 2(\nu_1 - 4\nu_3) + 2(\nu_1^2 + 2\nu_1\nu_3 - 2\nu_3^2) - 2\nu_1\nu_3(\nu_1 + \nu_3)\right], \tag{14}$$

$$A_6 = 1 - \nu_3 + \frac{M}{a}\left[-13 - (10\nu_1 - 17\nu_3) + 2(2\nu_1^2 + 8\nu_1\nu_3 - \nu_3^2)\right.$$
$$\left. + 2(\nu_1^3 - 2\nu_1^2\nu_3 - 3\nu_1\nu_3^2 - \nu_3^3)\right], \tag{15}$$

$$A_5 = 2 + \nu_1 - 2\nu_3 + \frac{M}{a}\left[-15 - (18\nu_1 - 5\nu_3) + 4(5\nu_1\nu_3 + 4\nu_3^2)\right.$$
$$\left. + 6(\nu_1^3 - \nu_1\nu_3^2 - \nu_3^3)\right], \tag{16}$$

$$A_4 = 1 + 2\nu_1 - \nu_3 + \frac{M}{a}\left[-6 - 2(5\nu_1 + 2\nu_3) - 4(2\nu_1^2 - \nu_1\nu_3 - 4\nu_3^2)\right.$$
$$\left. + 2(3\nu_1^3 + \nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 3\nu_3^3)\right], \tag{17}$$

$$A_3 = -(1 - \nu_1 + 2\nu_3) + \frac{M}{a}\left[6 + 2(2\nu_1 + 5\nu_3) - 4(4\nu_1^2 + \nu_1\nu_3 - 2\nu_3^2)\right.$$
$$\left. + 2(3\nu_1^3 + 2\nu_1^2\nu_3 - \nu_1\nu_3^2 - 3\nu_3^3)\right], \tag{18}$$

$$A_2 = -(2 - 2\nu_1 + \nu_3) + \frac{M}{a}\left[15 - (5\nu_1 - 18\nu_3) - 4(4\nu_1^2 + 5\nu_1\nu_3)\right.$$
$$\left. + 6(\nu_1^3 + \nu_1^2\nu_3 - \nu_3^3)\right], \tag{19}$$

$$A_1 = -(1 - \nu_1) + \frac{M}{a}\left[13 - (17\nu_1 - 10\nu_3) + 2(\nu_1^2 - 8\nu_1\nu_3 - 2\nu_3^2)\right.$$
$$\left. + 2(\nu_1^3 + 3\nu_1^2\nu_3 + 2\nu_1\nu_3^2 - \nu_3^3)\right], \tag{20}$$

$$A_0 = \frac{M}{a}\left[4 - 2(4\nu_1 - \nu_3) + 2(2\nu_1^2 - 2\nu_1\nu_3 - \nu_3^2) + 2\nu_1\nu_3(\nu_1 + \nu_3)\right]. \tag{21}$$

Here, the sign of Eq. (21) is chosen so that it can agree with the fifth-order equation Eq. (6) in the Newtonian limit of $M/a \to 0$. This seventh-order equation is antisymmetric for exchanges between ν_1 and ν_3, only if one makes a change as $z \to 1/z$. This antisymmetry may validate the complicated form of each coefficient.

Once a positive root for Eq. (13) is found, the root z can be substituted into Eq. (9) in order to obtain the angular velocity ω.

The angular velocity including the post-Newtonian effects is obtained from Eq. (9) as [19]

$$\omega = \omega_N \left(1 + \frac{F_M}{2F_N} + \frac{F_V}{2R_{13}}\right), \tag{22}$$

where $\omega_N \equiv (F_N/R_{13})^{1/2}$ denotes the angular velocity of the Newtonian collinear orbit.

III. EXISTENCE OF POSITIVE ROOTS

In this section, we show that there always exist positive roots for the seventh-order equation that has been derived as Eq. (13). This is nothing but the existence of the post-Newtonian collinear solution.

For the later convenience, we recover ν_2 and thus rewrite a coefficient A_0 as

$$A_0 = 2\frac{M}{a}(\nu_2 + \nu_3)(2\nu_2 + 2\nu_3 + \nu_2\nu_3), \tag{23}$$

which immediately leads to $A_0 > 0$.

In the similar manner, one can show $A_7 < 0$. An alternative but powerful way to see this is using the antisymmetry of the seventh-order equation for transformations between masses M_1 and M_3 as $\nu_1 \leftrightarrow \nu_3$ and $z \leftrightarrow 1/z$. This transformation makes a change as $A_0 \to -A_7$. By using $A_0 > 0$, therefore, we have always $A_7 < 0$.

Bringing the above results together, we have $F(0) = A_0 > 0$ and $F(\infty) = A_7 z^7|_{z\to\infty} < 0$. Therefore, the number of positive roots for $F(z) = 0$ either equals to one or more than it by a multiple of two.

Let us investigate the seventh-order equation in order to more precisely determine the number of positive roots. In the Newtonian fifth-order equation by Eq. (6), we have $A_{N5} > 0$, $A_{N4} > 0$, $A_{N3} > 0$, $A_{N2} < 0$, $A_{N1} < 0$, $A_{N0} < 0$. In the post-Newtonian approximation, the post-Newtonian parts must be much smaller than the Newtonian ones ($|A_{PNk}| \ll |A_{Nk}|$ for each k), so that the post-Newtonian correction cannot change the sign of each coefficient A_k. We thus have $A_6 > 0$, $A_5 > 0$, $A_4 > 0$, $A_3 < 0$, $A_2 < 0$, $A_1 < 0$. By combining them with $A_7 < 0$ and $A_0 > 0$, the number of sign changes of the coefficients in Eq. (13) is necessarily three. Therefore, Descartes' rule of signs indicates that Eq. (13) has either one or three

TABLE I: Values of z, ω and $a\omega$ for Figure 1. Here are three positive roots, where we assume $z_1 < z_2 < z_3$.

	z_1	z_2	z_3
z	3.635×10^{-4}	1.000	2751
ω	8.723×10^{-5}	2.449×10^{-6}	8.723×10^{-5}
$a\omega$	0.8723	0.02449	0.8723

roots. We can easily understand that one of them is a correction to the Newtonian orbit. What are the other two roots at all? We shall investigate them in next section.

IV. UNIQUENESS OF THE POST-NEWTONIAN COLLINEAR SOLUTION

Figure 1 shows that the equation has three positive roots, where we assume $\nu_1 = 1/7$, $\nu_2 = 5/7$, $\nu_3 = 1/7$, $a/M = 10^4$ ($v \sim 10^{-2}$). Table I shows numerical values of z, ω and $a\omega$ for Figure 1. Two out of the three roots do not satisfy a slow-motion condition for the post-Newtonian approximation as shown below.

Here we show that the remaining two positive roots must be discarded. Because of the antisymmetry of Eq. (13) for the transformation as $z \leftrightarrow 1/z$, the two roots must be a pair through this transformation associated with exchanges between M_1 and M_3. Let the smaller root and larger one denoted as z_S and z_L, respectively.

First, we consider the smallest positive root z_S, where we assume $z_S \ll 1$. Then, Eq.(13) is approximated as

$$A_1 z_S + A_0 = 0, \tag{24}$$

where A_0 starts at the post-Newtonian order without Newtonian terms and $A_1 = A_{N1}+A_{PN1}$ has both the Newtonian terms and post-Newtonian corrections ($|A_{N1}| \gg |A_{PN1}|$). We thus obtain an approximate form of the smallest root as

$$
\begin{aligned}
z_S &= -\frac{A_0}{A_{N1}} \\
&= O\left(\frac{M}{a}\right), \tag{25}
\end{aligned}
$$

where we used Eqs. (20) and (21). This implies that z_S is indeed of the post-Newtonian

9

order, in consistent with $z_S \ll 1$. At this point, however, we cannot discard this smallest root z_S.

As a next step, let us make an order-of-magnitude estimation for the angular velocity ω_S that satisfies Eq. (9) for z_S, where ω_S denotes the angular velocity corresponding to z_S. We obtain from Eqs. (10), (11) and (12)

$$
\begin{aligned}
F_N &= O\left(\frac{M}{a^2 z_S^2}\right) \\
&= O\left(\frac{1}{M}\right),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
F_M &= O\left(\frac{M^2}{a^3 z_S^3}\right) \\
&= O\left(\frac{1}{M}\right),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
F_V &= O\left(\frac{M}{z_S^2}\right) \\
&= O\left(\frac{a^2}{M}\right).
\end{aligned}
\tag{28}
$$

We thus find $R_{13} = a \ll F_V$ because $M \ll a$. Therefore, we find $F_N \sim F_M \sim F_V \omega_S^2 \gg R_{13}\omega_S^2$ in Eq. (9). This leads to

$$
\omega_S = O\left(\frac{1}{a}\right),
\tag{29}
$$

though $\omega_N^2 = O(M/a^3)$ for the Newtonian case. Eq. (29) implies an extremely fast rotation, since the rotational velocity becomes $v_S \approx a\omega_S = O(1)$, namely, comparable to the speed of light. This unacceptable branch of such an extremely fast motion contradicts with the post-Newtonian approximation. Hence, z_S must be abandoned.

Next, we consider the largest positive root z_L, where we assume $z_L \gg 1$. Then, Eq.(13) is approximated as

$$
A_7 z_L + A_6 = 0,
\tag{30}
$$

where A_7 starts at the post-Newtonian order without Newtonian terms and $A_6 = A_{N6} + A_{PN6}$ has both the Newtonian terms and post-Newtonian corrections ($|A_{N6}| \gg |A_{PN6}|$). We thus obtain an approximate form of the largest root as

$$
\begin{aligned}
z_L &= -\frac{A_7}{A_{N6}} \\
&= O\left(\frac{a}{M}\right),
\end{aligned}
\tag{31}
$$

10

where we used Eqs. (14) and (15). This implies that z_L^{-1} is indeed of the post-Newtonian order, in consistent with $z_L \gg 1$. At this point, however, we cannot discard this largest root z_L.

As a next step, let us make an order-of-magnitude estimation for the angular velocity ω_L that satisfies Eq. (9) for z_L, where ω_L denotes the angular velocity corresponding to z_L. We obtain from Eqs. (10), (11) and (12)

$$
\begin{aligned}
F_N &= O\left(\frac{M z_L^2}{a^2}\right) \\
&= O\left(\frac{1}{M}\right),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
F_M &= O\left(\frac{M^2 z_L^3}{a^3}\right) \\
&= O\left(\frac{1}{M}\right),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
F_V &= O\left(M z_L^2\right) \\
&= O\left(\frac{a^2}{M}\right).
\end{aligned}
\tag{34}
$$

We thus find $R_{13} = a \ll F_V$ because $M \ll a$. Therefore, we find $F_N \sim F_M \sim F_V \omega_L^2 \gg R_{13}\omega_L^2$ in Eq. (9). This leads to

$$
\omega_L = O\left(\frac{1}{a}\right),
\tag{35}
$$

though $\omega_N^2 = O(M/a^3)$ for the Newtonian case. Eq. (35) implies an extremely fast rotation, since the rotational velocity becomes $v_L \approx a\omega_L = O(1)$, namely, comparable to the speed of light. This unacceptable branch of such an extremely fast motion contradicts with the post-Newtonian approximation. Hence, also z_L must be abandoned.

We should remember the transformation as $z \leftrightarrow 1/z$, namely $1 \leftrightarrow 3$. Hence, z_S and z_L correspond to each other as $z_S = 1/z_L$. In this sense, it seems natural that the above argument for discarding z_L is very similar to that of z_S.

As a result, two of the three positive roots are discarded as unphysical ones. Hence, we complete the proof of the uniqueness.

We mention an application of the uniqueness theorem for the restricted three-body problem. We have three possibilities for choosing a test mass as $M_1 = 0$, $M_2 = 0$ or $M_3 = 0$. For

each case, we have only the single collinear solution. Therefore, the three equilibrium points exist along the symmetry axis of the system, and they are a generalization of Lagrange points L_1, L_2 and L_3.

Before closing this section, we mention an interesting property of the angular velocity of the collinear configurations. We have always an inequality as

$$\omega < \omega_N, \tag{36}$$

which means that the post-Newtonian orbital period measured in the coordinate time is longer than the Newtonian one. Detailed calculations are given in Appendix.

V. CONCLUSION

We proved the uniqueness of the collinear configuration for given system parameters (the masses and the largest separation among them). It was shown that the equation determining the distance ratio among the three masses, which has been obtained as a seventh-order polynomial in the previous paper, has at most three positive roots, which apparently provide three cases of the distance ratio. It was found, however, that there exists one physically acceptable root and only one. The remaining two positive roots are discarded in the sense that they do not satisfy the slow motion ansatz in the post-Newtonian approximation.

Especially for the restricted three-body problem, exactly three positions of a third body are true even at the post-Newtonian order. They are relativistic counterparts of the Newtonian Lagrange points L_1, L_2 and L_3.

It was shown also that, for given system parameters, the angular velocity of the post-Newtonian collinear configuration is smaller than that for the Newtonian case.

Our way of discussion seems to work at the second (and higher) post-Newtonian orders, because the slow motion approximation is a key in the above proof. Therefore, the uniqueness of collinear configurations for a three-body system may be true even at higher orders, precisely speaking, if the configuration has the Newtonian limit. It is an open question whether *fully* general relativistic systems admit a particular solution that can appear only for a fast motion case and thus has no Newtonian limit.

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Appendix A: Detailed calculations on the angular velocity

Let us prove $\omega < \omega_N$. Eq. (22) is rewritten as

$$\frac{\omega - \omega_N}{\omega_N} = \frac{F_M R_{13} + F_V F_N}{2 F_N R_{13}}. \tag{A1}$$

Here, F_N is positive and thus the denominator of the R.H.S. of Eq. (A1) is always positive. What we have to do is to investigate the sign of the numerator for the R.H.S. of Eq. (A1). The numerator is factorized as

$$\frac{M^2}{a^2 z^4 (1+z)^2} \times \sum_{k=0}^{10} a_k z^k, \tag{A2}$$

where we define

$$a_{10} = -(1 - \nu_1 - \nu_3)^2 \nu_3^2, \tag{A3}$$

$$a_9 = -(1 - \nu_1 - \nu_3)(4 + \nu_1 - 2\nu_3 - 4\nu_1\nu_3 + 2\nu_3^2 + 2\nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 4\nu_3^3), \tag{A4}$$

$$\begin{aligned} a_8 = &-(1 - \nu_1 - \nu_3)(18 + 4\nu_1 - 9\nu_3 + 3\nu_1^2 - 14\nu_1\nu_3 \\ &+ 2\nu_3^2 - \nu_1^3 + 7\nu_1^2\nu_3 + 2\nu_1\nu_3^2 - 6\nu_3^3), \end{aligned} \tag{A5}$$

$$\begin{aligned} a_7 = &-(1 - \nu_1 - \nu_3)(32 + 4\nu_1 - 13\nu_3 + 12\nu_1^2 - 18\nu_1\nu_3 \\ &+ 10\nu_3^2 - 4\nu_1^3 + 8\nu_1^2\nu_3 + 4\nu_1\nu_3^2 - 8\nu_3^3), \end{aligned} \tag{A6}$$

$$\begin{aligned} a_6 = &-(30 - 30\nu_1 - 37\nu_3 + 19\nu_1^2 - 12\nu_1\nu_3 + 27\nu_3^2 - 22\nu_1^3 + 18\nu_1^2\nu_3 \\ &+ 12\nu_1\nu_3^2 - 28\nu_3^3 + 6\nu_1^4 - 4\nu_1^3\nu_3 - 15\nu_1^2\nu_3^2 + 6\nu_1\nu_3^3 + 11\nu_3^4), \end{aligned} \tag{A7}$$

$$\begin{aligned} a_5 = &-2(12 - 13\nu_1 - 13\nu_3 + 11\nu_1^2 - 10\nu_1\nu_3 + 11\nu_3^2 - 11\nu_1^3 + 17\nu_1^2\nu_3 \\ &+ 17\nu_1\nu_3^2 - 11\nu_3^3 + 4\nu_1^4 - 3\nu_1^3\nu_3 - 14\nu_1^2\nu_3^2 - 3\nu_1\nu_3^3 + 4\nu_3^4), \end{aligned} \tag{A8}$$

$$\begin{aligned} a_4 = &-(30 - 37\nu_1 - 30\nu_3 + 27\nu_1^2 - 12\nu_1\nu_3 + 19\nu_3^2 - 28\nu_1^3 + 12\nu_1^2\nu_3 \\ &+ 18\nu_1\nu_3^2 - 22\nu_3^3 + 11\nu_1^4 + 6\nu_1^3\nu_3 - 15\nu_1^2\nu_3^2 - 4\nu_1\nu_3^3 + 6\nu_3^4), \end{aligned} \tag{A9}$$

$$\begin{aligned} a_3 = &-(1 - \nu_1 - \nu_3)(32 - 13\nu_1 + 4\nu_3 + 10\nu_1^2 - 18\nu_1\nu_3 \\ &+ 12\nu_3^2 - 8\nu_1^3 + 4\nu_1^2\nu_3 + 8\nu_1\nu_3^2 - 4\nu_3^3), \end{aligned} \tag{A10}$$

$$\begin{aligned} a_2 = &-(1 - \nu_1 - \nu_3)(18 - 9\nu_1 + 4\nu_3 + 2\nu_1^2 - 14\nu_1\nu_3 \\ &+ 3\nu_3^2 - 6\nu_1^3 + 2\nu_1^2\nu_3 + 7\nu_1\nu_3^2 - \nu_3^3), \end{aligned} \tag{A11}$$

$$a_1 = -(1 - \nu_1 - \nu_3)(4 - 2\nu_1 + \nu_3 + 2\nu_1^2 - 4\nu_1\nu_3 - 4\nu_1^3 - 2\nu_1^2\nu_3 + 2\nu_1\nu_3^2), \tag{A12}$$

$$a_0 = -(1 - \nu_1 - \nu_3)^2 \nu_1^2. \tag{A13}$$

We show $a_k < 0$ for each k. It is trivial that $a_0 < 0$ and $a_{10} < 0$. For $\nu_1 \leftrightarrow \nu_3$, we have a symmetry between $a_9 \leftrightarrow a_1$, $a_8 \leftrightarrow a_2$, $a_7 \leftrightarrow a_3$ and $a_6 \leftrightarrow a_4$. Therefore, it is sufficient to examine a_9, a_8, a_7, a_6 and a_5.

First, let us show $a_9 < 0$. The nontrivial factor in Eq. (A4) turns out to be positive by noting the following relation as

$$
\begin{aligned}
& 4 - 2\nu_3 - 4\nu_1\nu_3 + 2\nu_3^2 - 2\nu_1\nu_3^2 - 4\nu_3^3 \\
&= 4(1 - \nu_3)^3 + 10\nu_3(1 - \frac{4}{10}\nu_1 - \nu_3 - \frac{2}{10}\nu_1\nu_3) \\
&> 4(1 - \nu_3)^3 + 10\nu_3(1 - \frac{6}{10}\nu_1 - \nu_3) \\
&> 4(1 - \nu_3)^3 + 10\nu_3(1 - \nu_1 - \nu_3) \\
&> 0,
\end{aligned}
\tag{A14}
$$

where we used $\nu_1 \geqq 0$, $\nu_3 \leqq 1$, $\nu_1\nu_3 \leqq \nu_1$. Hence we find $a_9 < 0$.

We discuss the sign of a_8. By using $\nu_1 = 1 - \nu_2 - \nu_3$ to delete ν_1 and recover ν_2, the R.H.S. of Eq. (A5) is factorized as

$$
-\nu_2(24 - 7\nu_2 - 23\nu_3 + 4\nu_3^2 + \nu_2^3 + 10\nu_2^2\nu_3 + 15\nu_2\nu_3^2).
\tag{A15}
$$

One can show that the latter three terms are positive, since

$$
\begin{aligned}
24 - 7\nu_2 - 23\nu_3 &= 1 + 7(1 - \nu_2 - \nu_3) + 16(1 - \nu_3) \\
&> 0.
\end{aligned}
\tag{A16}
$$

Hence, the second factor in Eq. (A15) is always positive, which leads to $a_8 < 0$, and also $a_2 < 0$.

Next, we examine a_7. By recovering ν_2 to delete ν_1, the R.H.S. of Eq. (A6) is factorized as

$$
-\nu_2(44 - 16\nu_2 - 39\nu_3 + 2\nu_2\nu_3 + 16\nu_3^2 + 4\nu_2^3 + 20\nu_2^2\nu_3 + 24\nu_2\nu_3^2).
\tag{A17}
$$

A key thing is a positivity as

$$
\begin{aligned}
44 - 16\nu_2 - 39\nu_3 &= 5 + 16(1 - \nu_2 - \nu_3) + 23(1 - \nu_3) \\
&> 0,
\end{aligned}
\tag{A18}
$$

which immediately leads to $a_7 < 0$, and also $a_3 < 0$.

We investigate a_6. Similarly to a_7, it is factorized as

$$-(3 + 34\nu_2 - \nu_3 - 11\nu_2^2 - 34\nu_2\nu_3 + \nu_3^2 - 2\nu_2^3 + 24\nu_2\nu_3^2 + 6\nu_2^4 + 28\nu_2^3\nu_3 + 33\nu_2^2\nu_3^2). \quad \text{(A19)}$$

Here, we see that the following two combinations both are positive,

$$
\begin{aligned}
34\nu_2 - 11\nu_2^2 - 34\nu_2\nu_3 &= 11\nu_2(1 - \nu_2 - \nu_3) + 23\nu_2(1 - \nu_3) \\
&> 0, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(A20)} \\
3 - \nu_3 - 2\nu_2^3 &> 2 - 2\nu_2^3 \\
&> 0, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(A21)}
\end{aligned}
$$

which show that Eq. (A19) is always negative. Hence, we show $a_6 < 0$, and also $a_4 < 0$.

Also for a_5, it is factorized as

$$
\begin{aligned}
-2(3 + 8\nu_2 - \nu_3 + 2\nu_2^2 - 11\nu_2\nu_3 + \nu_3^2 - 5\nu_2^3 \\
-7\nu_2^2\nu_3 + 12\nu_2\nu_3^2 + 4\nu_2^4 + 19\nu_2^3\nu_3 + 19\nu_2^2\nu_3^2).
\end{aligned}
\quad \text{(A22)}
$$

One can find the following rather tricky manipulation as

$$
\begin{aligned}
3 + 8\nu_2 - \nu_3 + 2\nu_2^2 - 11\nu_2\nu_3 - 5\nu_2^3 - 7\nu_2^2\nu_3 \\
= 3 + 5\nu_2[1 - \nu_2(\nu_2 + \nu_3) - \nu_3] - \nu_3 - 6\nu_2\nu_3 \\
> 3 + 5\nu_2(1 - \nu_2 - \nu_3) - \nu_3 - 6\nu_2\nu_3 \\
> \frac{1}{2}, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(A23)}
\end{aligned}
$$

where we used $0 < \nu_2 + \nu_3 < 1$ and $\nu_2\nu_3 \leqq 1/4$. Hence, we find $a_5 < 0$.

As a consequence, all the coefficients are always negative, which shows $\omega < \omega_N$ for any mass ratio.

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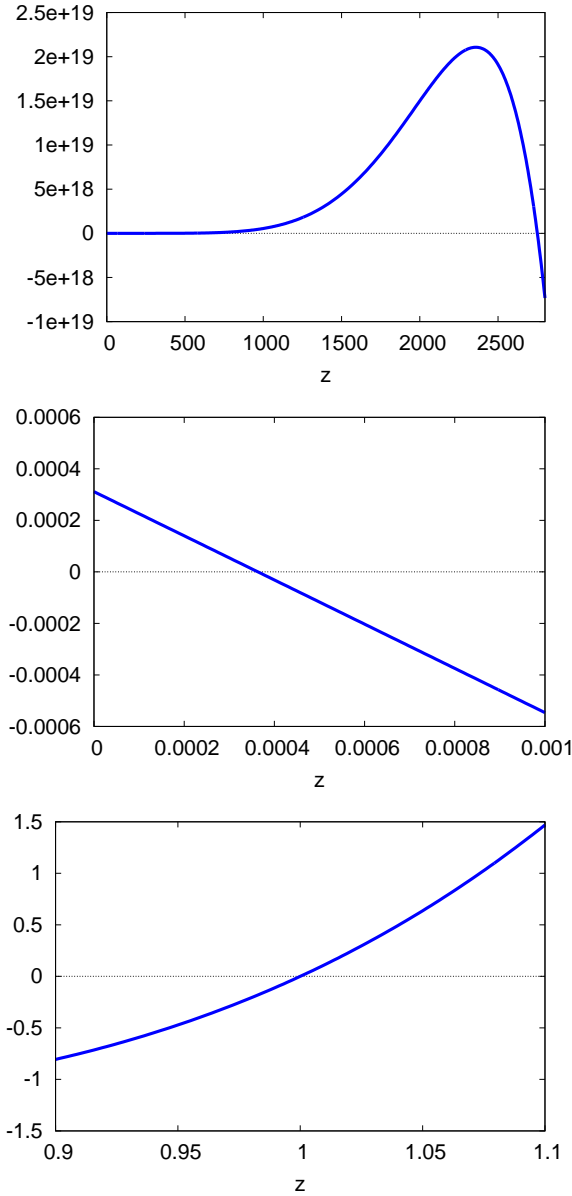[22] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Oxford, Pergamon 1962).

FIG. 1: Top panel: The seventh-order polynomial in the L.H.S. of Eq. (13). The horizontal axis is chosen as z. We take $\nu_1 = 1/7$, $\nu_2 = 5/7$, $\nu_3 = 1/7$, $a/M = 10^4$ ($v \sim 10^{-2}$) in order to exaggerate small effects in these figures. Clearly such a symmetric choice of the mass ratios produces a trivial root as $z = 1$, which makes it easy to check numerical calculations. M_2 is relatively large so that the centrifugal force can be large.

Middle panel: The seventh-order polynomial around the smallest positive root z_S.

Bottom panel: The polynomial around the moderate positive root.