Acceleration of particles by black holes - general explanation

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We give simple and general explanation to the effect of unbound acceleration of particles by black holes. It is related to the fact that the scalar product of a timelike vector of the four-velocity of an ingoing particle and the lightlike horizon generator tends to zero in some special cases, so the condition of "motion forward in time" is marginally satisfied. In this sense, an ingoing particle with special relation between parameters imitates the property of infinite redshift typical of any outgoing particle near the future horizon of a black hole. We check this assertion using the Reissner-Nordström and rotating axially-symmetric metrics as examples.

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I. INTRODUCTION

Recently, an interesting observation was made in [1] about acceleration of particles near the horizon of a rotating black hole to unlimited energies $E_{c.m.}$ in the centre of mass frame. In this sense, a black hole can act a cosmic supercollider that is very promising from the viewpoint of new physics expected at the Planck scale. The series of papers followed where details of this process were studied [2] - [11] and its generalization [12] and extension to charged nonrotating black holes [13] were suggested. The goal of the present work is to give a general and comprehensive explanation to this interesting effect. Rather surprisingly, it turns out that such an explanation is very simple and relies not on the details of theory but on the mutual properties of particles and a light cone near the future horizon of a black hole. Thus, we generalize previous observations and elucidate the underlying reason for the

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manifestation of the effect under consideration diversity of the particular metrics or even their classes (Kerr, Kerr-Newman, Reissner-Nordström, stringy, Kaluza-Klein black holes, axially-symmetric rotating dirty black holes, etc.) considered in aforementioned papers.

II. BASIC FORMULAS

It would seem that the effect connected with acceleration of particles requires necessarily detailed analysis of equations of their motion. It is just the approach developed in previous works [1] - [13]. Instead, in the present work we focus attention on what happens to the four-velocity of a particle with respect to its local light cone in the immediate vicinity of the horizon. Let us consider the collision of two particles near the future horizon of a black hole. In doing so, one should clearly distinct two different cases: 1) particles move in the opposite directions (towards the horizon and away from it), 2) both particles move towards the horizon. Actually, the first case was discussed in [14] (although the corresponding condition was not explicitly pronounced there) a long time ago. The second case is discussed in the series of aforementioned papers [1] - [13].

We will use the following geometric construction. Let us introduce in the point P under consideration and its vicinity the tetrad with lightlike vectors l^{μ}, N^{μ} and spacelike vectors a^{μ}, b^{μ} orthogonal to them. Here, the vectors l^{μ}, N^{μ} are normalized, say, as $l^{\mu}N_{\mu}=-1$.

Then,

$$g_{\alpha\beta}=-l_{\alpha}N_{\beta}-l_{\beta}N_{\alpha}+\sigma_{\alpha\beta} \tag{1}$$

where $\sigma_{\alpha\beta}=a_{\alpha}b_{\beta}+a_{\beta}b_{\alpha}$, $l^{\alpha}\sigma_{\alpha\beta}=N^{\alpha}\sigma_{\alpha\beta}=0$, a_{α} and b_{α} are spacelike (see, for example, textbook [15]). We assume that it is the vector l^{μ} that becomes the generator of the future horizon. Let us also introduce the quantity $a(u)\equiv -u^{\mu}l_{\mu}\equiv -(ul)$ where $(uu)=-1$, u^{μ} is the timelike vector having the meaning of the four-velocity. As both vectors u^{μ}, l^{μ} are assumed to be future-directed, $\alpha>0$ (motion "forward in time"). In general, we can write down the vector u^{μ} of any particle as

$$u_{i}^{\mu}=\frac{l^{\mu}}{2\alpha_{i}}+\alpha_{i}N^{\mu}+s_{i}^{\mu},\ s_{i}^{\mu}=A_{i}a^{\mu}+B_{i}b^{\mu} \tag{2}$$

where $i=1,2$ labels the particles. Then,

$$-(u_{1}u_{2})=\frac{1}{2}(\frac{\alpha_{1}}{\alpha_{2}}+\frac{\alpha_{2}}{\alpha_{1}})-(s_{1}s_{2}). \tag{3}$$

The energy in the centre of mass frame [1] - [13] is equal to $E^2_{c.m.} = m_1^2 + m_2^2 - 2m_1m_2(uv)$ (m_i are rest masses of particles), so

$$E^2_{c.m.} = m_1^2 + m_2^2 + m_1m_2[\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1} - 2(s_1s_2)]. \tag{4}$$

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III. INGOING VERSUS OUTGOING PARTICLES IN THE VICINITY OF THE HORIZON

A. Case 1.

Let particle 1 be going from the immediate vicinity of the horizon in the outward direction. As we are dealing with the future horizon of a black hole, particle 1 moves almost in the direction of the horizon generator and it follows that the condition

$$\alpha_1 \to 0 \tag{5}$$

is satisfied for it. Meanwhile, α_2 is arbitrary positive quantity. Then, it is seen from (4) that $E^2_{c.m.} \to \infty$. One can say that this is just direct consequence of infinite redshift near the horizon.

B. Case 2

This case (both particles move towards the horizon) is much more interesting since the frame of the centre of mass falls down with both particles [1], so the possible effect of unbound acceleration is not direct manifestation of the redshift. In general, as it is seen from (4), $E^2_{c.m.}$ remains finite even in the vicinity of the horizon for any nonzero α_1, α_2. However, let us now *assume* that (5) holds now (in case 1 this was satisfied automatically). In other words, an ingoing particle imitates the property of infinite redshift (5) typical of an outgoing particle near the horizon. Then, again it follows from (4), (5) that $E^2_{c.m.} \to \infty$. This is just the effect discovered in [1] and studied in [2] - [13]. Thus, in case 2 the special condition (5) is needed. It relates the parameters of a particle like the energy and angular momentum or the energy and electric charge, etc. (see examples below).

The above observations can be also reformulated as follows. Consider the vector ξ^{μ} which is timelike in the region where particles approach the horizon, $N^2 = -(\xi\xi) > 0$:

$$\xi^{\mu} = \frac{1}{2} l^{\mu} + N^2 N^{\mu}. \tag{6}$$

We can easily deduce two additional properties.

1) Let (5) be satisfied and (ξu) be finite. Then, the vector ξ^{μ} becomes lightlike in this limit.

Proof. It follows from (2) (6) that in this limit $(\xi u) \approx N^2(Nu) = -\frac{N^2}{2\alpha}$. As this quantity is finite, it follows from (5) that also $N \to 0$.

2) Let us, instead of (5), assume that $(\xi u) \to 0$. Then, (5) is satisfied and the vector ξ^{μ} becomes lightlike in this limit.

Proof. Multiplying (6) by u_{μ}, we observe that both terms are negative. Therefore, each of them vanishes separately in this limit, so $\alpha \to 0$, $N^2 \to 0$. As a consequence, $E^2_{c.m.} \to \infty$.

The situation where the vector ξ^{μ} is timelike in some region but becomes the lightlike on some hypersurface is typical of Killing horizons. However, we would like to emphasize that nowhere we used Killing equations.

The results under discussion can be reexpressed in another way with the help of Krusckal-like coordinates. Let, for simplicity, the metric can be written in the form

$$ds^2 = -CdXdY + \gamma_{ab} dx^a dx^b \tag{7}$$

where $a = 1, 2$ and the metric coefficient are regular functions of the coordinates X and Y (this is certainly possible for the nonrotating black holes). On the horizon $X = 0$ or $Y = 0$. Then, repeating the above arguments, we see that it follows from (5) that, say, near the horizon $X = 0$ the component of the four-velocity $u^X \sim \alpha \to 0$. Taking into account the regularity of the metric, we can write that $\alpha \sim X$, whence we have

$$\frac{dX}{d\tau} \sim X, \tag{8}$$

so

$$\tau \sim -\ln X \to \infty \tag{9}$$

in accordance with previous results for the Kerr [2], [10] or Reissner-Nordström [13] black holes. Such a feature is typical of collision with infinite energy near the extremal horizons [2], [12], [13]. For the nonextremal case, the analysis in the aforementioned articles showed that the energy is finite but, if parameters of the particle slightly differ from the critical relation, this energy can be made as large as one likes.

Let us now illustrate these general properties by two examples.

IV. EXAMPLES

A. Reissner-Nordström black hole

$$ds^2 = -dt^2N^2 + \frac{dr^2}{N^2} + r^2d\omega^2. \tag{10}$$

Here $d\omega^2 = \sin^2\theta d\phi^2 + d\theta^2$, $N^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ where M is the black hole mass, Q is its charge. The event horizon lies at $r = r_H = M + \sqrt{M^2 - Q^2}$. Consider a radial motion of the particle having the charge q and rest mass m. From the equations of motion one finds the components of the four-velocity for a pure radial motion in the direction towards the horizon:

$$u^0 = \frac{X}{N^2m},\ u^1 = -\frac{Z}{mN} \tag{11}$$

where $X = E - \frac{qQ}{r}$, $Z = \sqrt{X^2 - m^2N^2}$, the coordinates are $x^0 = t, x^1 = n$ (the proper distance $n = \int \frac{dr}{N}$), $x^2 = \theta$, $x^3 = \phi$.

Here, E is the conserved energy, dot denotes differentiation with respect to the proper time τ, u^μ is the four-velocity. The quantity $X_H = E - \frac{qQ}{r_H} \geq 0$, so it is positive for all $r > r_H$ (motion "forward in time"). Then, the vector (6) has the components $\xi^\mu = (1, 0, 0, 0)$ and coincides with the Killing vector. Let us also introduce two lightlike vectors $l^\mu = (1, N, 0, 0)$ and $N^\mu = \frac{1}{2}(\frac{1}{N^2}, -\frac{1}{N}, 0, 0)$, $(Nl) = -1$. The vectors a_μ and b_μ have nonzero components $a_\theta = r$, $b_\phi = r\sin\theta$. One can check that the equality (1) is satisfied.

Then,

$$-(\xi u) = \frac{X}{m}, \tag{12}$$

$$-(ul) = \frac{Z + X}{m} > 0,\ -(uN)N^2 = \frac{1}{2}\frac{X - Z}{m} > 0. \tag{13}$$

On the horizon $Z = X_H$ (hereafter we use subscript "H" for the values calculated on the horizon), $-(ul) = \frac{2X_H}{m} > 0$, $-(\xi u) = \frac{X_H}{m} > 0$, $-(uN) = \frac{m}{4X_H}$ is finite for any particles, except from those with $X_H = 0$, $qQ = Er_H$. For them, $-(ul) \to 0$, $-(\xi u) \to 0$, $-(uN) \to \infty$.

Then, the above consideration applies which leads to the result $E^2_{c.m.} \to \infty$. that agrees with the one obtained earlier [13].

From another hand, if we took the sign "+" in (11) for u^1, we would obtain $\alpha = \frac{X-Z}{m} \to 0$ for any particle irrespective of the relation between the parameters in accordance with what was said above while discussing case 1.

B. Axially-symmetric rotating black hole

Now, let us consider the generic metric describing an axially-symmetric black hole

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2. \tag{14}$$

that includes the Kerr and Kerr-Newman black holes. However, the configuration is more general due to the possible presence of matter (dirty black holes). It follows from equations of motion that

$$\dot{t} = u^0 = \frac{X}{N^2},\ X = E - \omega L \tag{15}$$

(for simplicity, here we assume that the rest mass $m = 1$).

$$\dot{\phi} = \frac{L}{g_{\phi\phi}} + \frac{\omega X}{N^2}, \tag{16}$$

$$\dot{l} = -\frac{Z}{N},\ Z^2 = X^2 - N^2(1 + \frac{L^2}{g_{\phi\phi}}) \tag{17}$$

where $u_0 = -E$ is the energy, $u_\phi = L$ is the angular momentum. For motion "forward in time", we must have $\dot{t} > 0$, so $E - \omega L > 0$. We imply that $\dot{l} < 0$.

Now, the relevant lightlike vectors are

$$l_\mu = (-N^2, N, 0, 0) \tag{18}$$

$$N_\mu = \frac{1}{2N^2}(-N^2, -N, 0, 0) \tag{19}$$

$$(Nl) = -1. \tag{20}$$

The vector (6) reads

$$\xi^\mu = \xi_1^\mu + \omega \xi_2^\mu \tag{21}$$

where $\xi_1^\mu = (1, 0, 0, 0)$ is the Killing vector that generates translations in time, $\xi_2^\mu = (0, 0, 1, 0)$ generates rotations. On the horizon $N = 0$ the vector ξ^μ becomes lightlike.

One can check that eq. (1) is indeed satisfied, where nonzero components of vectors a_μ and b_μ equal $b_z = \sqrt{g_{zz}}.$, $a_\phi = \sqrt{g_{\phi\phi}}$, $a_0 = -\omega a_\phi$, The scalar product $(ua) = \frac{L}{\sqrt{g_{\phi\phi}}}$ is finite. Then, $-(ul) = Z + X$, $(uN) = \frac{X-Z}{2}$, $-(u\xi) = X$. The critical value is singled out by the condition $X = 0$ on the horizon ($E = \omega_H L$) that indeed coincides with (5). Then, we again obtain that $E^2_{c.m.} \to \infty$ in accordance with the previous discussion and [12]. For a particle moving away from the horizon ($\dot{l} > 0$), we would have $\alpha = X - Z \to 0$ near the horizon independently of the relationship between the energy and the angular momentum. In the intermediate case $\dot{l} = 0$ one should have $Z = 0$.Such a type of orbits can be realized in the vicinity of the extremal horizon [17]. Then, $N \to 0$, $X \to 0$ and we again return to the condition (5).

V. CONCLUSIONS

Thus, we elucidated the generic nature of the effect and showed that diversity of different metrics and even classes of metric has the same underlying reason in this context. In doing so, we did not use explicitly equations of motion of particles at all, did not rely on explicit form of the metric, field equations from which it is obtained, etc. Actually, the nature of the effect turned out to be surprisingly simple and stemming from the mutual properties of lightlike and timelike vectors in the vicinity of the future horizon. It may happen that the condition (1) is not realized in some particular cases (say, for some classes of trajectories [8]). Nonetheless, if (i) the horizon exists and (ii) the condition (5) is indeed satisfied, the effect of unbound $E_{c.m.}$ can manifest itself in general. Moreover, it follows from our derivation that these reasonings apply not only to the horizons of static or stationary black holes. As a matter of fact, the effect is valid even if the aforementioned condition is obeyed for some portion of the surface only. Moreover, these portions can shrink to the point. In particular, the results of the present work seem to apply to dynamic or isolated horizons [16]. The fact that the essence of the effect of infinite $E_{c.m.}$ reveals itself in so general setting, lends support to the idea that it can survive notwithstanding model-dependent factors (electromagnetic radiation, gravitational radiation, etc.).

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