

Theory of interacting topological superfluids and superconductors

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We investigate the three-dimensional, time-reversal invariant topological superconductors with generic interaction by their response to external fields. The first description is a gravitational topological field theory, which gives a Z_2 classification of topological superconductors, and predicts a half-quantized thermal Hall effect on the surface. The second description introduces a s -wave proximity pairing field on the surface, and the associated topological defects give an integer Z classification of the topological superconductor phases. Generally, charge conserving, time reversal breaking fields probe the Z_2 sector, and charge conservation breaking and time reversal breaking fields probe the Z sector of the interacting topological superconductor.

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Recently, topological insulators(TI) and topological superconductors(TSC) have attracted great attention in condensed matter physics[1–4]. Topological insulators are states which are incompressible in the bulk but have topological surface states protected by the bulk topological properties which remain robustly metallic upon arbitrary perturbation on the surface, within the given symmetry class. Time-reversal invariant topological insulators in two and three dimensions (2D and 3D) have been theoretically predicted and experimentally realized in recent years[5–11]. Topological superconductors are analog of topological insulators in superconductors, which has a full superconducting gap, and gapless Majorana edge states propagating on the boundary. Time-reversal breaking topological superconductors in 2D[12] and time-reversal invariant topological superconductors in 2D and 3D[13–15] have been theoretically predicted but have not been verified experimentally. The most promising candidate for 3D topological superconductor (more precisely, superfluid) is the He³B phase.

There are two approaches for the classification and description of TIs and TSCs: topological field theory(TFT) and topological band theory(TBT). In the TFT approach, topological insulators are defined by the quantized responses derived from low-energy effective actions[16]. TFT also reveals the relations among TIs in various dimensions. For instance, the 4D Chern-Simons insulator[17] is the root state of 3D and 2D time reversal invariant(TRI) TIs. The 3D topological insulator is characterized by the topological effective theory $S = \frac{\theta}{32\pi^2} F_{\mu\nu} F_{\sigma\tau} \epsilon^{\mu\nu\sigma\tau}$ with $\theta = \pi \bmod 2\pi$, which describes the electromagnetic response of the topological insulator when time-reversal symmetry is broken on the surface.[16, 18] The TBT was first developed in the work of Kane and Mele[19, 20], who proposed a Z_2 topological invariant within the non-interacting band theory. This TBT was later generalized to 3D TRI TIs[21–23] and more recently to all the ten symmetry classes[14, 24]. It was also shown that TFT reduces to TBT in the non-interacting limit for time-reversal invariant TIs[25].

Interestingly, in 1D it has been shown that the non-interacting classifications are sometimes unstable to interaction.[26]. Therefore, in order to define TSCs in dimensions higher than 1, it is helpful to apply the topological field theory (TFT) approach of TIs[16, 17], since the TFT describes physical response properties of the system, which is applicable to interacting systems. All topological insulators can be described in this way by TFT[16], including the 2D and 3D TIs. An explicit expression of the topological invariant of these classes in interacting systems are obtained in the form of Green's function[27]. However, there are some symmetry classes in which this approach does not apply. For example, the 3D time-reversal invariant TSC. Due to the absence of charge conservation symmetry, no TFT can be obtained by the coupling to the electromagnetic field.

In this paper, we introduce two new approaches to the classification of 3D interacting time-reversal invariant TSCs. The general idea behind the TFT description is to find the correct physical external field which can generate a physical observable topological response of the system. Such topological response can then be used to classify the topological states. TIs and TSCs in different topological classes and different dimensions require different external field to probe their topological response properties. We present two different approaches with different probe field used, which applies to different physical settings. The first approach is the gravitational TFT, which describes the topological response of the TSC to a gravitational field. Since the coupling to gravitational field does not require charge conservation, the gravitational TFT can be applied to TSC similar to the electromagnetic TFT approach of 3D TI[16]. The topological term is the gravitational analogy of the θ -term for the gauge field, which is the Pontryagin invariant of the spacetime manifold[28]. However, similar to the electromagnetic θ term for 3D TI, the gravitational TFT only provides a Z_2 classification of the 3D TSC. In the non-interacting limit, the nontrivial class with $\theta = \pi$ corresponds to the TSC states with odd topological quantum number, while

the trivial class with $\theta = 0$ corresponds to those with even topological quantum number.

A natural question is whether this indicates that with interaction only a Z_2 classification is stable, or there are other ways we can define the Z classification in interacting TSC. To answer this question we propose the second approach. We consider the proximity effect of the surface of 3D TSC with an s-wave superconductor. The proximity effect introduce an s-wave pairing field to the surface states. When the pairing field has a π phase domain wall on the surface, there are N number of chiral Majorana fermions propagating on the domain wall. Since the chiral Majorana fermions are stable without any symmetry requirement, and can be characterized by the thermal current it carries, one can use the number of Majorana zero mode as a generic definition of TSC in 3D. From this approach we conclude that the integer classification of 3D TSC remains robust when interaction is considered. This approach also directly leads to an experimental proposal for the measurement of the topological quantum number. Moreover, the idea of defining interacting TIs and TSCs by studying the defects obtained from some symmetry-breaking field configuration can be generalized to generic symmetry classes.

Gravitational topological field theory of 3D TSC First we consider gravitational TFT of time-reversal invariant (TRI) TSC in 3D. For simplicity, we first investigate continuous models using the example of the $^3\text{He-B}$ phase as a 3D TRI TSC.

In the flat space-time, the Hamiltonian of $^3\text{He-B}$ phase is given by[13, 29]

$$H(\mathbf{p}) = m(\mathbf{p})\tau^3 + \Delta\sigma \cdot \mathbf{p}\tau^1 \quad (1)$$

in the Nambu space $\psi = (c_\uparrow, c_\downarrow, c_\downarrow^\dagger, -c_\uparrow^\dagger)$, where \uparrow, \downarrow denote spin orientation. $\sigma^{1,2,3}$ and $\tau^{1,2,3}$ labels the Pauli matrices in spin and particle-hole indices, respectively. $m(\mathbf{p}) = p^2/2M - \mu$ is the kinetic energy term. The weak pairing phase $M > 0, \mu > 0$ is topologically nontrivial and corresponds to the physical He^3B phase[13–15], while the strong pairing phase $\mu < 0$ is a trivial superfluid phase. After rescaling the momenta $\Delta p_i \rightarrow p_i$, the corresponding Lagrangian can be written as

$$L = \bar{\psi}[p_\mu\Gamma^\mu + m(p_i)]\psi \quad (2)$$

where $\mu = 0, 1, 2, 3$ (We use $(0, 1, 2, 3)$ interchangeably with (t, x, y, z)), $\Gamma^0 = \tau^3$ and $\Gamma^i = i\sigma^i\tau^2; i = 1, 2, 3$. This is exactly the Majorana-Dirac Lagrangian if $m(p_i) = m$ is independent of p_i .

Now we turn to Lagrangian of $^3\text{He-B}$ in curved space-time. It is convenient to take the space-time as a closed 4-manifolds M . To consistently define the Lagrangian globally, M is required to be a spin manifold[30]. Locally the low energy action is

$$S_\psi = \int_M d^4x \sqrt{-g} \bar{\psi} [i\Gamma^\alpha e_\alpha^\mu (\partial_\mu + \frac{1}{2}i\omega_\mu^{\beta\gamma}\Sigma_{\beta\gamma}) + m]\psi \quad (3)$$

where $\Sigma_{\alpha\beta} \equiv \frac{1}{4}i[\Gamma^\alpha, \Gamma^\beta]$ is the generator of Lorentz transformation in the spinor representation, and e_α^μ is the vielbeins[28]. Physically we can interpret the gravitational field as an external source coupling to the energy-momentum tensor of the fluid. In a deeply insightful paper, Volovik points out that the order parameter of the $^3\text{He-B}$ phase also couples to fermions like a gravitational vielbein[31]. Therefore, we can also interpret e_α^μ as the internal order parameter of the $^3\text{He-B}$ phase.

Because the Majorana fermions are gapped, we can integrate out them to obtain a gravitational effective action. The term we are interested is the gravitational theta term, which can be easily obtained by calculating chiral anomaly due to gravity. A chiral transformation $m \rightarrow m \exp(i\Gamma^5\theta)$ generates the topological term in the effective action[32, 33] $S_\theta = \frac{\theta}{2} \int_M \hat{A}(M)$ where $\hat{A}(M)$ is the Dirac genus of M [28]. Since the Hamiltonian with positive mass $m = -\mu > 0$ is adiabatically connected to a trivial superconductor by taking $\mu \rightarrow -\infty$, by a chiral rotation of $\theta = \pi$ we can obtain the action of the He^3B phase with $\mu > 0$ as $S_\theta = \frac{\pi}{2} \int_M \hat{A}(M)$. It is worth noting that Majorana fermion has a $1/2$ coefficient compared to Dirac fermion. Written explicitly, we obtain the following Pontryagin invariant

$$S_\theta = -\frac{\theta}{1536\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} R_{\beta\mu\nu}^\alpha R_{\alpha\rho\sigma}^\beta \quad (4)$$

with $\theta = \pi$. For a generic 4-manifold, $S_\theta = n\pi/48$ with n integer. However, for spin manifolds the value of Dirac genus is restricted to $\int_M \hat{A}(M) = 2n$ with integer n [34], so that $S_\theta = n\pi$, as is expected from TRI. On a domain wall between $\theta = 0$ and $\theta = \pi$, we get the Chern-Simons term from the theta term as

$$S_{2D} = \frac{1}{384\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr}(\omega_\mu \partial_\nu \omega_\rho + \frac{2}{3}\omega_\mu \omega_\nu \omega_\rho) \quad (5)$$

which describes the surface of $\text{He}^3\text{-B}$ phase in the presence of surface magnetization.

For a generic TSC with a non-relativistic Hamiltonian, the direct coupling to gravity may be complicated. However, physically one can always consider the non-relativistic Hamiltonian as a low energy limit of a underlying relativistic theory, so that the coupling to gravity is in principle always well-defined. The topological term (4) is insensitive to an arbitrary deformation of the fermion Hamiltonian as long as the fermions remains gapped, so that the TFT description for a generic TSC can always be obtained by deforming the Hamiltonian to several decoupled copies of the He^3B model(1). For the TSC with topological quantum number $N \in \mathbb{Z}$, the resulting action is S_θ in Eq. (4) with $\theta = N\pi$. However, for even integer N , $\exp(iS_\theta) = 1$ so that S_θ has no physical effect. We conclude from this fact that the gravitational TFT gives a Z_2 classification of 3D TRI TSC, which is weaker than the integer classification of the non-interacting system.

Surface state description. To understand such a difference between Z_2 and Z classification, it is helpful to study the topological surface states. For $^3\text{He-B}$ phase, the surface state is a single flavor of 2D massless Majorana fermion[13, 29]. For our discussion here, it is most convenient to use the Majorana basis in which the surface state effective Hamiltonian is given by

$$H = \sum_{\mathbf{k}} \eta_{-\mathbf{k}}^T v (\sigma_z k_x + \sigma_x k_y) \eta_{\mathbf{k}} \quad (6)$$

with $\eta_{\mathbf{k}} = \eta_{-\mathbf{k}}^\dagger$ the two-component Majorana fermion field. The two components carry opposite spin, and the time-reversal operation is defined $T^{-1}\eta_{\mathbf{k}}T = i\sigma_y\eta_{-\mathbf{k}}$. It should be noticed that only Pauli matrices σ_z and σ_x are allowed in this Hamiltonian since the single particle Hamiltonian $h(\mathbf{k}) = v(\sigma_z k_x + \sigma_x k_y)$ satisfies the anti-symmetry condition $h(-\mathbf{k}) = -h^T(\mathbf{k})$. No mass term is allowed by time-reversal symmetry since the only possible mass term $i\sigma_y$ breaks time-reversal symmetry. More generically for a topological superconductor with topological quantum number N , the surface state consists of multiple copies of the Majorana fermions with different chirality:

$$H = \sum_{s=1}^{N_+} \sum_{\mathbf{k}} \eta_{s,-\mathbf{k}}^T v (\sigma_z k_x + \sigma_x k_y) \eta_{s,\mathbf{k}} + \sum_{s=1}^{N_-} \sum_{\mathbf{k}} \psi_{s,-\mathbf{k}}^T v (\sigma_z k_x - \sigma_x k_y) \psi_{s,\mathbf{k}} \quad (7)$$

where $\eta_{s,\mathbf{k}}$ and $\psi_{s,\mathbf{k}}$ are the ‘‘left-handed’’ and ‘‘right-handed’’ Majorana fermions, and the integer $N_+ - N_- = N$ is determined by the bulk topological invariant. Such a definition of ‘‘chirality’’ for Majorana fermion is possible because σ_y is not allowed to appear in the linear- k terms, so that the two kinds of Majorana fermions $\eta_{\mathbf{k}}$ and $\psi_{\mathbf{k}}$ cannot be deformed to each other. More rigorously, the topological invariant $N = N_+ - N_-$ can be defined by the following formula:

$$N = -\frac{1}{2} \text{Ind} \left(\mathcal{T} \left[\frac{\partial h(\mathbf{k})}{\partial k_x}, \frac{\partial h(\mathbf{k})}{\partial k_y} \right] \Big|_{\mathbf{k}=0} \right) \quad (8)$$

with $\mathcal{T} = i\sigma_y I$ the time-reversal transformation matrix for all Majorana fermions. $\text{Ind}(M)$ for a Hermitian matrix M is defined as the number of positive eigenvalues minus the number of negative eigenvalues.

From the discussion above we have seen from the surface state point of view why the topological classification is Z for non-interacting theory. The relation of this surface state picture to the gravitational TFT (4) is similar to that between the Dirac surface state and the topological electromagnetic action of 3D TI[16]. Consider a generic T-breaking mass term of the surface states $H_m = \sum_s m_s \eta_{s,-\mathbf{k}}^T \sigma_y \eta_{s,\mathbf{k}} + \sum_s m'_s \psi_{s,-\mathbf{k}}^T \sigma_y \psi_{s,\mathbf{k}}$. With the

mass terms the surface states are gapped but obtained a quantized thermal Hall coefficient

$$\kappa_{xy} = \frac{\pi k_B^2 T}{24\hbar} \left(\sum_{s=1}^{N_+} \text{sgn}(m_s) - \sum_{s=1}^{N_-} \text{sgn}(m'_s) \right) \quad (9)$$

Each Majorana cone contributes half the thermal Hall coefficient of that of a 2D ($p+ip$) superconductor[12, 35–37]. Such a thermal Hall effect on the surface is consistent with the description of the bulk TFT (4) since the Pontryagin invariant can be reduced to a gravitational Chern-Simons term on the boundary as eq. 5. However, due to the dependence on the sign of mass terms, the thermal Hall coefficient is only determined by the surface state theory (7) mod $2 \times \frac{\pi k_B^2 T}{24\hbar}$. This is consistent with the observation that the coefficient of the bulk TFT (4) takes Z_2 value 0 or π mod 2π . In other words, only the topological superconductors with odd N necessarily has thermal Hall effect on the surface, while those with even N can have zero thermal Hall coefficient for some choice of mass terms. It’s interesting to note that the same gravitational TFT approach can be applied to time-reversal invariant TI, for which the surface thermal Hall conductivity will be doubled. Thus we see that the gravitational TFT description is trivial for 3D TI.

Integer Z classification of 3D TRI superconductor by topological defects The analysis above seems indicating that only the topological superconductors with odd topological quantum number are stable. However, in the following we will show that the Z classification is actually robust with interaction, although it cannot be captured by the gravitational TFT. As has been discussed in the introduction part, a general philosophy of classifying interacting topological phases is by defining a physical topological response of the system to a proper external field. The gravitational TFT can only probe the Z_2 part of the topological invariant, but the full Z can be probed by another physical field—an s-wave pairing field. We consider the generic 3D time-reversal invariant Majorana fermion system with a Hamiltonian $H = \int d^3\mathbf{r} d^3\mathbf{r}' \eta^T(\mathbf{r}) h(\mathbf{r}, \mathbf{r}') \eta(\mathbf{r}') + H_{\text{int}}$. Here h is an anti-symmetric Hermitian single-particle operator, and H_{int} describes non-quadratic terms in Majorana fermion η . The Hamiltonian is time-reversal invariant satisfying $T^{-1}HT = H$, with the anti-unitary time-reversal operator defined by $T^{-1}\eta(\mathbf{r})T = \mathcal{T}\eta(\mathbf{r})$ and $\mathcal{T}^2 = -1$. Now we introduce the following perturbation term to the Hamiltonian

$$H_\Delta = -i \int d^3\mathbf{r} \Delta(\mathbf{r}) \eta^T(\mathbf{r}) [T^{-1}\eta(\mathbf{r})T] = \int d^3\mathbf{r} i \Delta(\mathbf{r}) \eta^T(\mathbf{r}) \mathcal{T} \eta(\mathbf{r}) \quad (10)$$

with $\Delta(\mathbf{r})$ being a real field. To see the physical meaning of this term, we can take the He³B Hamiltonian (1)

as an example. In the complex fermion basis $H_\Delta = -\int d^3\mathbf{r}i\Delta(\mathbf{r})\left(c_{\uparrow\mathbf{k}}^\dagger(\mathbf{r})c_{\downarrow\mathbf{k}}^\dagger(\mathbf{r}) - h.c.\right)$ which is an s -wave pairing term with imaginary pairing order parameter. It is important to notice that such a T-breaking pairing term (10) is completely determined by the representation of time-reversal transformation and is independent from the basis choice and any other detail of the system.

Now we consider the time-reversal invariant TSC with the perturbation (10) only near the surface. Physically, such a field can be induced by the proximity effect between the TSC and an s -wave superconductor on top of it. Since this term breaks time-reversal symmetry, the surface states of the TSC can become gapped. The effect of this s -wave pairing field to the surface state can be seen straightforwardly from the surface effective theory (7). Since $T^{-1}\eta_s T = i\sigma_y\eta_s$, $T^{-1}\psi_t T = i\sigma_y\psi_t$ for the surface Majorana fermions, the s -wave pairing term (10) is reduced to $H_{\Delta_{\text{surf}}} = \int d^2\mathbf{r}\Delta(\mathbf{r})\left(\sum_s \eta_s^T \sigma_y \eta_s + \sum_t \psi_t^T \sigma_y \psi_t\right)$ on the surface. In other words, a uniform pairing field $\Delta(\mathbf{r}) = \Delta > 0$ corresponds to a uniform mass term $m_s = m_t = \Delta$ for all the surface Majorana fermions. Consequently the thermal Hall conductivity given by Eq. (9) is $\kappa_{xy} = N\frac{\pi k_B^2 T}{24\hbar}$ with N the bulk topological invariant, so that by introducing the pairing field, the topological quantum number N can be obtained without ambiguity.

The statement above can be made more explicit by consider a domain wall on the surface where $\Delta(\mathbf{r})$ changes sign, as shown in Fig. 1. Physically, such a domain wall is a π -Josephson junction of the s -wave superconductor on top of the time-reversal invariant TSC. For the surface Majorana fermions, the mass term changes sign across the domain wall, so that there are chiral Majorana fermion propagating along the domain wall[12]. Due to the uniform mass for all Majorana fermions, the number of chiral Majorana fermion is given by N_+ while the number of anti-chiral Majorana fermion is given by N_- . Consequently there are $N = N_+ - N_-$ copies of chiral Majorana fermions which are robust gapless states on the domain wall. It is important to notice that a chiral Majorana fermion in 1D is well-defined and robust even in an interacting system, since it has a chiral central charge $c = 1/2$ and thus carries a chiral thermal current $I^E = \frac{\pi k_B^2 T^2}{24\hbar}$. Thus one can use the chiral thermal current as a physical observable way to define chiral Majorana fermions, so that the chiral Majorana fermions on the domain wall can be used as a generic definition of the bulk 3D TSC:

- For a generic 3D time-reversal invariant superconductor, apply a small pairing field in the form of Eq. (10) with a domain wall of $\Delta(\mathbf{r})$, as shown in Fig. 1. The superconductor is a TSC with topological quantum number $N \in \mathbb{Z}$ if the thermal current propagating along the domain wall at temperature

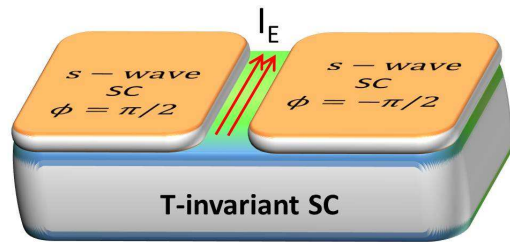


FIG. 1: Illustration of the experimental setting for the measurement of the integer-valued topological quantum number of 3D TSC. s -wave pairing is introduced on the surface by proximity effect, with a π -Josephson junction. The topological quantum number N of the bulk is characterized by the number of chiral Majorana fermions propagating along the junction illustrated by the red arrows, which carry a quantized thermal current $I^E = N\frac{\pi k_B^2 T^2}{24\hbar}$ for low temperature $T \rightarrow 0$.

$$T \text{ is } I^E = N\frac{\pi k_B^2 T^2}{24\hbar} \text{ in the } T \rightarrow 0 \text{ limit.}$$

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