INTERPOLATION FUNCTION OF THE GENOCCHI TYPE POLYNOMIALS

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Abstract The main purpose of this paper is to construct not only generating functions of the new approach Genocchi type numbers and polynomials but also interpolation function of these numbers and polynomials which are related to a, b, c arbitrary positive real parameters. We prove multiplication theorem of these polynomials. Furthermore, we give some identities and applications associated with these numbers, polynomials and their interpolation functions.

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1 Introduction, Definitions and Notations

The history of the Euler numbers and the Genocchi numbers go beck to Euler on16th century and Genocchi on 19th century, respectively. From Euler and Genocchi to this time, these numbers can be defined in many other ways. These numbers and polynomials play an important role in many branch of Mathematics, for instance, Number Theory, Finite differences. Therefore, applications of these numbers and their generating functions have been investigated by many authors in the literature. Many kind of functions are used to obtain generating functions of the Euler numbers and the Genocchi numbers cf. ([1]-[28]).

The classical Euler numbers E_n are defined by means of the following generating function.

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\ |t|<\pi,$$

cf. ([1]-[28]).

The classics Genocchi numbers G_n are defined by means of the following generating function

$$f_G(t)=\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\ |t|<\pi. \tag{1}$$

The Genocchi numbers, named after Angelo Genocchi, are a sequence of integers. This numbers are satisfies the following relations. By the *umbral* *calculus* convention in (1), we have the recurrence relations of the Genocchi numbers as follows:

$$G_0 = 0,\ (G+1)^n + G_n = \begin{cases} 2, & n = 1 \\ 0, & n \neq 1 \end{cases} \tag{2}$$

where G^n is replaced by G_n.

Relations between Genocchi numbers, Euler numbers and Bernoulli numbers are given by:

$$E_n = \frac{G_{n+1}}{n+1}, \tag{3}$$

$$\begin{aligned} G_{2n} &= 2(1-2^{2n})B_{2n} \\ &= 2nE_{2n-1}(0), \end{aligned}$$

where B_n and E_n are Bernoulli numbers and Euler Numbers respectively, cf. ([4], [11], [14], [7], [16], [13], [24], [28], [26]).

The ordinary Genocchi Polynomials are defined by means of the following generating function:

$$\mathrm{f}_G(t,x) = f_G(t)e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{4}$$

From (1) and (4), we easily see that

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}.$$

Observe that $G_0 = 0$, $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = G_{2n+1} = 0$, $n \in \mathbb{Z}^+$ cf. ([4], [5], [6], [11], [13], [18]).

In [19] and [20], Luo et al defined new type generalized Bernoulli polynomials and Euler polynomials depending on three positive arbitrary real parameters. They proved many identities and relations related to these polynomials. Main motivation of the work is to define generating functions of Genocchi type numbers and polynomials depending on three positive arbitrary real parameters.

Luo et al ([19], [20]) did not define interpolation functions of their numbers and polynomials. On the other hand, in this present paper, we can construct interpolation functions of our new numbers and polynomials which depending on three positive arbitrary real parameters. We also prove some new relations and properties associated with these numbers, polynomials and intepolation functions.

We now summarize our paper as follows:

In Section 2, we construct generating functions of the Genocchi type numbers and polynomials. We give reoccurrence relations of these numbers. We prove multiplication theorem of these polynomials. We also give some properties of these numbers and polynomials.

In Section 3, we find formula of the alternating sums of powers of consecutive integers.

In section 4, by using derivative operator $\frac{d^k f(t)}{dt^k}\mid_{t=0}$ to the generating function of the Genocchi type numbers and polynomials, we construct interpolation functions of these numbers and polynomials.

In Section 5, we give further remarks and observations on interpolation functions.

2 Genocchi Type Numbers and Polynomials

In this section, by using same method that of , we define generalized Genocchi number and polynomial depending on three positive arbitrary real parameters. We investigate fundamental properties of these numbers and polynomials.

Throughout of this paper a, b and c are positive real parameters with $a \neq b$ and $x \in \mathbb{R}$.

Now we are ready to define generating function of Genocchi type number depending on three positive arbitrary real parameters as follows:

$$F(t;a,b) = \frac{2t}{b^t + a^t} = \sum_{n=0}^{\infty} \mathcal{G}_n(a,b)\frac{t^n}{n!},\ \ |t| < \frac{\pi}{|\ln a - \ln b|}. \tag{5}$$

By using (5) and the *umbral calculus* convention, we obtain

$$\frac{2te^{-t\ln a}}{e^{t(\ln b - \ln a)} + 1} = e^{\mathcal{G}_n(a,b)t}$$

After some calculations, we get the following reoccurrence relations for the number $\mathcal{G}_n(a,b)$ as follows:

Let $\mathcal{G}_0(a,b) = 0$ and $\mathcal{G}_1(a,b) = 1$. For $n \geq 2$,

$$\mathcal{G}_n(a,b) + \sum_{k=0}^{n} \binom{n}{k} (\ln b - \ln a)^{k-n} \mathcal{G}_k(a,b) = 2n \ln^{n-1}\left(\frac{1}{a}\right). \tag{6}$$

By using (6), we give few Genocchi-type numbers as follows:

$$\begin{aligned} \mathcal{G}_2(a,b) &= -\ln a - \ln b, \\ \mathcal{G}_3(a,b) &= -6\ln^2 a + 3\ln a \ln b. \end{aligned}$$

Remark 1 *By substituting $a = 1$, $b = e$ into (5) and (2), then we arrive at (1) and (6), respectively. That is, the number $\mathcal{G}_n(1,e)$ reduces to the number G_n.*

Lemma 2 *Let a, b be arbitrary positive real parameters. Then we have*

$$\mathcal{G}_n(a,b) = (\ln b - \ln a)^{n-1} G_n\left(\frac{\ln a}{\ln a - \ln b}\right), \tag{7}$$

and

$$\mathcal{G}_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\ln a)^{n-k} (\ln b - \ln a)^{k-1} G_k. \tag{8}$$

Proof. We firstly give proof of (7). From (5), we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n(a,b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (\ln b - \ln a)^{n-1} G_n \left(\frac{\ln a}{\ln a - \ln b} \right) \frac{t^n}{n!}.$$

By comparing the coefficient $\frac{z^n}{n!}$ in the both sides of the above equation, we easily arrive at (7). Secondly, we give proof of (8). By using (5), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{G}_n(a,b) \frac{t^n}{n!} \\ = & \frac{1}{(\ln b - \ln a)} \sum_{n=0}^{\infty} G_n (\ln b - \ln a)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} (-\ln a)^n \frac{t^n}{n!}. \end{aligned}$$

By using Cauchy product in the above, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{G}_n(a,b) \frac{t^n}{n!} \\ = & \frac{1}{\ln b - \ln a} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} G_k (-1)^{n-k} (\ln a)^{n-k} (\ln b - \ln a)^k \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficient $\frac{t^n}{n!}$ in the both sides of the above equation, we easily arrive at (8). ■

Genocchi type polynomials, depending on three positive arbitrary real parameters, are defined by means of the following generating function:

Let

$$\mathcal{F}(t,x;a,b,c) = F(t;a,b) c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n(x;a,b,c) \frac{t^n}{n!}, \tag{9}$$

where $|t| < \frac{\pi}{|\ln a - \ln b|}$.

Remark 3 *If $x = 0$, then (9) reduces to (5). By substituting $a = 1$, $b = c = e$ into (9), then we have*

$$\begin{aligned} \mathcal{F}(t,x;1,e,e) &= \mathrm{f}_G(t,x). \\ F(t;1,e) &= f_G(t). \end{aligned}$$

From the above, we have

$$\mathcal{G}_n(x;1,e,e) = G_n(x),$$

$$\mathcal{G}_n(x;a,b,1) = \mathcal{G}(a,b),$$

$$\mathcal{G}_n(0;a,b,c) = \mathcal{G}_n(a,b),$$

and

$$\mathcal{G}_n(0;1,e,e) = G_n.$$

Remark 4 *Recently, many kind generating functions related to Bernoulli, Euler and Genocchi type polynomials have been found. Srivastava et al. [29, pp. 254, Eq. (20)], introduced and investigated the new type generalization of the Bernoulli polynomials order α, $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$, which are defined by means of the following generating functions:*

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \ \left(\left|t\ln(\frac{a}{b}) + \ln\lambda\right| < 2\pi; 1^{\alpha} := 1; x \in \mathbb{R}\right). \tag{10}$$

If we set $\alpha = 1$ and $\lambda = -1$ in (10), then we have

$$\mathfrak{B}_n^{(1)}(x;-1;a,b,c) = -\frac{1}{2}\mathcal{G}_n(x;a,b,c).$$

The numbers $\mathfrak{B}_n^{(1)}(x;-1;a,b,c)$ are related to Apostol-Bernoulli numbers. Ozden et al. [21] have unifed and extend the generating functions of the generalized Bernoulli polynomials, the generalized Euler polynomials and the generalized Genocchi polynomials associated with the positive real parameters a and b and the complex parameter β. By applying the Mellin transformation to the generating function of the unification of Bernoulli, Euler and Genocchi polynomials, they defined a unification of the zeta functions.

Theorem 5 *Let a, b, c be arbitrary positive real parameters. Then we have*

$$\mathcal{G}_n(x;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}(x\ln c)^{n-k}\mathcal{G}_k(a,b), \tag{11}$$

or

$$\mathcal{G}_n(x;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}(x\ln c)^{n-k}(\ln b - \ln a)^{n-1}G_k\left(\frac{\ln a}{\ln a - \ln b}\right). \tag{12}$$

Proof of (11). By (7), we have

$$\sum_{n=0}^{\infty}\mathcal{G}_n(x;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\mathcal{G}_n(a,b)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x\ln c)^n\frac{t^n}{n!}.$$

By Cauchy product in the above, we easily see that

$$\sum_{n=0}^{\infty}\mathcal{G}_n(x;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(a,b)x^{n-k}(\ln c)^{n-k}\right)\frac{t^n}{n!}.$$

By comparing the coefficient $\frac{z^n}{n!}$ in the both sides of the above equation, we easily arrive at the desire result. By substituting (7) into (11) and (8) into (11), after some elementary calculations, we arrive at the proofs of (12). ∎

By (12), we easily obtain the following corollary.

Corollary 6 *Let a, b, c be arbitrary positive real parameters. Then we have*

$$\begin{aligned}\mathcal{G}_n(x;a,b,c) &= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j,n-k,k-j}(-1)^{k-j}x^{n-k} \\ &\quad\times(\ln c)^{n-k}(\ln a)^{k-j}(\ln b-\ln a)^{n+j-k-1}G_j,\end{aligned}$$

where G_j denotes classical Genocchi numbers and

$$\binom{n}{j,n-k,k-j}=\frac{n!}{j!(n-k)!(k-j)!}.$$

We now give application of Theorem 5 as follows:

$$\begin{aligned}\mathcal{G}_0(x;a,b,c) &= 0,\\ \mathcal{G}_1(x;a,b,c) &= 1,\\ \mathcal{G}_2(x;a,b,c) &= 2x\ln c-\ln a-\ln b,\end{aligned}$$

If we take $a=1$, and $b=c=e$ in the above, then we obtain ordinary Genocchi polynomials as follows:

$$\begin{aligned}\mathcal{G}_0(x;1,e,e) &= 0,\\ \mathcal{G}_1(x;1,e,e) &= 1,\\ \mathcal{G}_2(x;1,e,e) &= 2x-1,\\ \mathcal{G}_3(x;1,e,e) &= 3(x^2-x).\end{aligned}$$

By using (9), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_n(x+1;a,b,c)\frac{t^n}{n!}=\frac{2tc^{(x+1)t}}{b^t+a^t}\\ =\;& 2tc^{xt}+\frac{2tc^{xt}(c^t-a^t-b^t)}{b^t+a^t}=2\sum_{n=0}^{\infty}\frac{(\ln c)^n x^n t^{n+1}}{n!}\\ &+2\left(\sum_{n=0}^{\infty}\frac{\mathcal{G}_n(x;a,b,c)t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\left((\ln c)^n-(\ln a)^n-(\ln b)^n\right)t^n}{n!}\right).\end{aligned}$$

After some elementary calculations in the above, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_n(x+1;a,b,c)\frac{t^n}{n!}\\ =\;& -2\mathcal{G}_0+t\left(2+2(\ln c-\ln a-\ln b)\mathcal{G}_0-2\mathcal{G}_1(x;a,b,c)\right)\\ &+\sum_{n=2}^{\infty}\left(2n(\ln c)^{n-1}x^{n-1}-\mathcal{G}_n(x;a,b,c)\right)\frac{t^n}{n!}\\ &+\sum_{n=2}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l}(\ln c)^{n-l}-(\ln a)^{n-l}-(\ln b)^{n-l}\right)\mathcal{G}_n(x;a,b,c)\frac{t^n}{n!}. \qquad (13)\end{aligned}$$

Thus, by using the above equation, we arrive at the following corollary:

Corollary 7 *Let a, b, c be arbitrary positive real parameters. Then we have*

$$\mathcal{G}_n(x+1;a,b,c)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(x;a,b,c)(\ln c)^{n-k}, \tag{14}$$

or

$$\mathcal{G}_n(x+1;a,b,c)=\mathcal{G}_n(x;\frac{a}{c},\frac{b}{c},c). \tag{15}$$

By substituting $a=1$, $b=c$ into (13) with $n\geq 1$, we have

$$G_n(x+1;1,b,b)=2n(\ln b)^{n-1}x^{n-1}-G_n(x;1,b,b).$$

By substituting $b=e$ into (13) with $n\geq 1$, we obtain

$$G_n(x+1)=2nx^{n-1}-G_n(x).$$

By using (9), we easily arrive at the following result:

Corollary 8 *The Generalized Genocchi polynomial is satisfying following relations*

$$\mathcal{G}_n(x+y;a,b,c)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(x;a,b,c)(\ln c)^{n-k}y^{n-k},$$

or

$$\mathcal{G}_n(x+y;a,b,c)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(y;a,b,c)(\ln c)^{n-k}x^{n-k}.$$

Theorem 9 *(Multiplication Theorem) Let a, b, c be arbitrary positive real parameters. Then we have*

$$\sum_{n=0}^{\infty}\mathcal{G}_n(x;a,b,c)\frac{t^n}{n!}=y^{n-1}\sum_{j=0}^{y-1}(-1)^j\mathcal{G}_n\left(\frac{j}{y};a,b,\frac{c^{\left(\frac{x}{y}\right)}b^{\left(\frac{j}{y}\right)}}{a^{\left(\frac{j+1}{y}\right)}}\right).$$

Proof. By using (9), we have

$$\sum_{n=0}^{\infty}\mathcal{G}_n(x;a,b,c)\frac{t^n}{n!}=y^{n-1}\sum_{n=0}^{\infty}\sum_{j=0}^{y-1}(-1)^j\mathcal{G}_n\left(\frac{j}{y};a,b,\frac{c^{\left(\frac{x}{y}\right)}b^{\left(\frac{j}{y}\right)}}{a^{\left(\frac{j+1}{y}\right)}}\right)\frac{t^n}{n!}.$$

After some calculations in the above, we arrive at the desired result. ∎

We now define Genocchi type polynomial of higher order as follows:

$$\mathcal{F}^{(k)}(t,x;a,b,c)=\left(\frac{2t}{b^t+a^t}\right)^k c^{xt}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}(x;a,b,c)\frac{t^n}{n!}, \tag{16}$$

where $\mathcal{G}_n^{(k)}(x;a,b,c)$ denotes the Genocchi type polynomial of higher order and k is positive integer.

Observe that $\mathcal{F}^{(1)}(t,x;a,b,c)=\mathcal{F}(t,x;a,b,c)$ and $\mathcal{G}_n^{(1)}(x;a,b,c)=\mathcal{G}_n(x;a,b,c)$.

By using (16), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_n^{(l+k)}(x+y;a,b,c)\frac{t^n}{n!}\\ =\;&(\frac{2t}{a^t+b^t})^{l+k}c^{(x+y)t}\\ =\;&\sum_{n=0}^{\infty}\mathcal{G}_n^{(l)}(x;a,b,c)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}(y;a,b,c)\frac{t^n}{n!}.\end{aligned}$$

From the above, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{G}_n^{(l+k)}(x+y;a,b,c)\frac{t^n}{n!}\\ =\;&\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathcal{G}_j^{(l)}(x;a,b,c)\mathcal{G}_{n-j}^{(k)}(y;a,b,c)\right)\frac{t^n}{n!}.\end{aligned}$$

After some elementary calculations, we arrive at the following theorem:

Theorem 10 *Let l and k be positive integers. Let a, b, c be arbitrary positive real parameters. Then we have*

$$\mathcal{G}_n^{(l+k)}(x+y;a,b,c)=\sum_{j=0}^{n}\binom{n}{j}\mathcal{G}_j^{(l)}(x;a,b,c)\mathcal{G}_{n-j}^{(k)}(y;a,b,c). \qquad (17)$$

3 The alternating sums of powers of consecutive integers

In 1713, J. Bernoulli discovered a formula for the sum $\sum_{b=0}^{n} b^j$ for $j\in\mathbb{Z}^+$. In this section we prove the alternating sums of powers of consecutive integers for $\mathcal{G}_n(x;a,b,c)$.

By using (5) and (9), we obtain

$$\begin{aligned}&F(t;1,b)-(-1)^m\mathcal{F}(t,m;1,b,b)\\ =\;&\sum_{n=0}^{\infty}\left(\mathcal{G}_n(1,b)-(-1)^m\mathcal{G}_n(m;1,b,b)\right)\frac{t^n}{n!}.\end{aligned}$$

From the above, we have

$$\sum_{k=0}^{m-1}(-1)^k b^{kt}=\sum_{n=0}^{\infty}\left(\frac{\mathcal{G}_n(1,b)-(-1)^m\mathcal{G}_n(m;1,b,b)}{2}\right)\frac{t^{n-1}}{n!}.$$

After some calculations, we arrive at the following theorem:

Theorem 11 *Let m and n be positive integers. Let b be arbitrary positive real parameter. Then we have*

$$\sum_{k=0}^{m-1}(-1)^k k^n = \frac{\mathcal{G}_n(1,b)-(-1)^m\mathcal{G}_n(m;1,b,b)}{2n\ln^n(b)}.$$

Remark 12 *By substituting $b=e$ into Theorem 11, then we have*

$$\begin{aligned}\sum_{k=0}^{m-1}(-1)^k k^n &= \frac{\mathcal{G}_n(1,e)-(-1)^m\mathcal{G}_n(m;1,e,e)}{2n} \\ &= \frac{G_n-(-1)^m G_n(m)}{2n}.\end{aligned}$$

Thus, by (3), Theorem 11 reduces to

$$\sum_{k=0}^{m-1}(-1)^k k^n = \frac{E_n-(-1)^m E_n(m)}{2},$$

cf. ([4], [5], [14], [22], [20], [19], [28]).

4 Interpolation Functions

In this section, we construct interpolation function of the generalized Genocchi type numbers and polynomials on $\mathbb{C}$.

By using (9), we have

$$\begin{aligned}\mathcal{F}(t,x;a,b,c) &= 2t\sum_{n=0}^{\infty}(-1)^n e^{(x\ln c-\ln a+n\ln\frac{b}{a})t} \\ &= \sum_{n=0}^{\infty}\mathcal{G}_n(x;a,b,c)\frac{t^n}{n!}.\end{aligned}$$

By applying derivative operator $\frac{d^k\mathcal{F}(t,x;a,b,c)}{dt^k}|_{t=0}$ to the above, we obtain

$$\mathcal{G}_k(x;a,b,c) = 2k\sum_{n=0}^{\infty}(-1)^n(n\ln c-\ln a+n\ln\frac{b}{a})^{k-1}.$$

Therefore, by using the above relation, we obtain the following theorem.

Theorem 13 *Let $k\in\mathbb{Z}^+$. Let a, b, c be arbitrary positive real parameters. Then we have*

$$\frac{\mathcal{G}_k(x;a,b,c)}{k} = 2\sum_{n=0}^{\infty}(-1)^n(x\ln c-\ln a+n\ln\frac{b}{a})^{k-1}.$$

By Theorem 13, we can derive Hurwitz type generalized Genocchi zeta function, which interpolates Genocchi polynomials at negative integers, as follows.

Definition 14 *Let $s \in \mathbb{C}$. Let a, b, c be arbitrary positive real parameters. We define*

$$\mathfrak{z}_{\mathcal{G}}(s,x;a,b,c) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(x\ln c - \ln a + n\ln\frac{b}{a})^s}.$$

From Definition 14, we see that

$$\mathcal{Z}_{\mathcal{G}}(s,1;a,b,1) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(-\ln a + n\ln\frac{b}{a})^s}. \tag{18}$$

By substituting $s = -n$, with $n \in \mathbb{Z}^+$, into Definition 14 and using Theorem 13, we arrive at the following Theorem.

Theorem 15 *Let $n \in \mathbb{Z}^+$. Let a, b, c be arbitrary positive real parameters. Then we have*

$$\mathfrak{z}_{\mathcal{G}}(-n,x;a,b,c) = \frac{\mathcal{G}_n(x;a,b,c)}{n}.$$

Observe that setting $x = 1$ in Theorem 15, we get interpolation function of the numbers $\mathcal{G}_n(a,b)$ as follows:

$$\mathcal{Z}_{\mathcal{G}}(-n,1;a,b,1) = \frac{\mathcal{G}_n(a,b)}{n}.$$

By setting $n = j + my$ with $j = 1,2,\cdots,y$, y is an odd integer, and $m = 0,1,\cdots,\infty$ in (18), we obtain

$$\mathcal{Z}_{\mathcal{G}}(s,1;a,b,1) = \frac{1}{y^s}\sum_{j=1}^{y}(-1)^j\sum_{m=0}^{\infty}\frac{(-1)^m}{\left(-\frac{\ln a}{y} + \frac{j\ln\frac{b}{a}}{y} + m\ln\frac{b}{a}\right)^s}.$$

After some calculations in the above, we arrive at the following corollary:

Corollary 16 *Let y be an odd integer. Let a, b, c be arbitrary positive real parameters. Then we have*

$$\mathcal{Z}_{\mathcal{G}}(s,1;a,b,1) = \frac{1}{y^s}\sum_{j=1}^{y}(-1)^j\mathfrak{z}_{\mathcal{G}}\left(s,1;a,b,\frac{b^{\left(\frac{j}{y}\right)}}{a^{\left(\frac{y+j-1}{y}\right)}}\right).$$

5 Further Remarks and Observations

The function $\mathfrak{z}_{\mathcal{G}}(s,x;a,b,c)$ and $\mathcal{Z}_{\mathcal{G}}(s,1;a,b,1)$ are related to the *Lerch trancendent* $\Phi(z,s,a)$ which is the analytic continuation of the series

$$\begin{aligned}\Phi(z,s,a) &= \frac{1}{a^s} + \frac{z}{(a+1)^s} + \frac{z}{(a+2)^s} + \cdots q \\ &= \sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},\end{aligned}$$

cf. see [31, p. 121 et seq.], [2]. The series $\sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$ converge for $a \in \mathbb{C} \diagdown \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$ where

$$\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}\,,\ \mathbb{Z}^- = \{-1, -2, -3, ...\}\,.$$

Φ denotes the familiar Hurwitz-Lerch Zeta function (cf. [31, p. 121 et seq.], [2], [27], [11],[18]).

The Lerch zeta function $\Phi(z, s, u)$ is the analytic continuation of the following series

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+u)^s} \tag{19}$$

which converge for any real number $u > 0$ if z and s are any complex numbers with either $|z| < 1$, or $|z| = 1$ and $\Re(s) > 1$.

The functio $\Phi(z, s, u)$ is related to many special functions, some of them are given as follows, cf. ([31, p. 124], [2], [27]):

Special cases include the analytic continuations of the Riemann zeta function

$$\Phi(1, s, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ \Re(s) > 1,$$

the Hurwitz zeta function

$$\Phi(1, s, a) = \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},\ \Re(s) > 1,$$

the alternating zeta function (also called Dirichlet's eta function $\eta(s)$)

$$\Phi(-1, s, 1) = \zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

the Dirichlet beta function

$$\frac{\Phi(-1, s, \frac{1}{2})}{2^s} = \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s},$$

the Legendre chi function

$$\frac{z\Phi(z^2, s, \frac{1}{2})}{2^s} = \chi_s(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s},\ |z| \le 1; \Re(s) > 1,$$

the polylogarithm

$$z\Phi(z, n, 1) = Li_m(z) = \sum_{n=0}^{\infty} \frac{z^k}{n^m}$$

and the Lerch zeta function (sometimes called the Hurwitz-Lerch zeta function)

$$L(\lambda, \alpha, s) = \Phi(e^{2\pi i \lambda}, s, \alpha),$$

which is a special function and generalizes the Hurwitz zeta function and polylogarithm, cf. ([2], [31], [4], [6], [7], [9], [8], [12], [17], [27]) and see also the references cited in each of these earlier works.

Setting $a=1, b=c=e$ in Definition 14, then we have

$$\mathfrak{z}_{\mathcal{G}}(s,x;1,e,e)=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(x+n)^s},$$

and

$$\mathcal{Z}_{\mathcal{G}}(s,1;1,e,e)=2\sum_{n=0}^{\infty}\frac{(-1)^n}{n^s}.$$

By using (19), the function $\mathfrak{z}_{\mathcal{G}}(s,x;1,e,e)$ and $\mathcal{Z}_{\mathcal{G}}(s,1;1,e,e)$ satisfies the following identities:

$$\mathfrak{z}_{\mathcal{G}}(s,x;1,e,e)=-2\Phi(-1,s,x).$$

and

$$\begin{aligned}\mathcal{Z}_{\mathcal{G}}(s,1;1,e,e) &= 2\Phi(-1,s,1)\\ &= -2\zeta^*(s).\end{aligned}$$

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