CONJUGACY CLASSES IN WEYL GROUPS AND Q-W ALGEBRAS

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Abstract. We define noncommutative deformations $W_q^s(G)$ of algebras of regular functions on certain transversal slices to the set of conjugacy classes in an algebraic group G which play the role of Slodowy slices in algebraic group theory. The algebras $W_q^s(G)$ called q-W algebras are labeled by (conjugacy classes) of elements s of the Weyl group of G. The algebra $W_q^s(G)$ is a quantization of a Poisson structure defined on the corresponding transversal slice in G with the help of Poisson reduction of a Poisson bracket associated to a Poisson–Lie group G^* dual to a quasitriangular Poisson–Lie group. We also define a quantum group counterpart of the category of generalized Gelfand–Graev representations and establish an equivalence between this category and the category of representations of the corresponding q-W algebra. The algebras $W_q^s(G)$ can be regarded as quantum group counterparts of W–algebras. However, in general they are not deformations of the usual W–algebras.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G' the adjoint group of $\mathfrak{g}$ and $e \in \mathfrak{g}$ a nonzero nilpotent element in $\mathfrak{g}$. By the Jacobson–Morozov theorem there is an $\mathfrak{sl}_2$–triple (e, h, f) associated to e, i.e. elements $f, h \in \mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Fix such an $\mathfrak{sl}_2$–triple.

Let χ be the element of $\mathfrak{g}^*$ which corresponds to e under the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by the Killing form. Under the action of ad h we have a decomposition

$$\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}(i), \text{ where } \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}. \tag{1.1}$$

The skew–symmetric bilinear form ω on $\mathfrak{g}(-1)$ defined by $\omega(x, y) = \chi([x, y])$ is nondegenerate. Fix an isotropic Lagrangian subspace l of $\mathfrak{g}(-1)$ with respect to ω.

Let

$$\mathfrak{m} = l \oplus \bigoplus_{i\leq -2}\mathfrak{g}(i). \tag{1.2}$$

Note that $\mathfrak{m}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ and $\chi \in \mathfrak{g}^*$ restricts to a character $\chi : \mathfrak{m} \to \mathbb{C}$. Denote by $\mathbb{C}_\chi$ the corresponding 1–dimensional $U(\mathfrak{m})$–module.

The associative algebra $W^e(\mathfrak{g}) = \mathrm{End}_{U(\mathfrak{g})}\,(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{opp}$ is called the W–algebra associated to the nilpotent element e. The algebra $W^e(\mathfrak{g})$ was introduced in [17] in case when e is principal nilpotent and in [19] when the grading (1.1) is even. In paper [6] the algebras $W^e(\mathfrak{sl}_n)$ are defined using cohomological BRST reduction, and the simple equivalent algebraic definition for arbitrary nilpotent element e given above first appeared in [20]. The equivalence of these two definitions follows, for instance, from a general property of homological Hecke–type algebras (see [28, 31]). An explicit computation establishing this equivalence can also be found in the Appendix in [8].

If we denote by $z(f)$ the centralizer of f in $\mathfrak{g}$ then the algebra $W^e(\mathfrak{g})$ can be regarded as a noncommutative deformation of the algebra of regular functions on the Slodowy slice $s(e) = e + z(f)$ which is transversal to the set of adjoint orbits in $\mathfrak{g}$ (see [6, 11, 20]). Note also that the center $Z(U(\mathfrak{g}))$ is naturally a subalgebra of the center of $W^e(\mathfrak{g})$, and for any character $\eta : Z(U(\mathfrak{g})) \to \mathbb{C}$ the algebra $W^e(\mathfrak{g})/W^e(\mathfrak{g})\mathrm{ker}\ \eta$ can be regarded as a noncommutative deformation of the algebra of regular

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functions defined on a fiber of the adjoint quotient map $\delta_{\mathfrak{g}} : s(e) \rightarrow \mathfrak{h}/W$, where $\delta_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{h}/W$ is induced by the inclusion $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^{G'} \hookrightarrow \mathbb{C}[\mathfrak{g}]$, $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$ and W is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. In particular, for singular fibers one obtains noncommutative deformations of the coordinate rings of the corresponding singularities (see [20]).

W–algebras are primarily important in the theory of Whittaker or, more generally, generalized Gelfand–Graev representations (see [14, 17]). Namely, to each nilpotent element e in $\mathfrak{g}$ one can associate the corresponding category of Gelfand–Graev representations which are $\mathfrak{g}$–modules on which $x - \chi(x)$ acts locally nilpotently for each $x \in \mathfrak{m}$. The category of generalized Gelfand–Graev representations associated to a nilpotent element $e \in \mathfrak{g}$ is equivalent to the category left modules over the W–algebra associated to e. This remarkable result was proved by Kostant in case of regular nilpotent $e \in \mathfrak{g}$ (see [17]) and by Skryabin in the general case (see Appendix to [20]). A more direct proof of Skryabin's theorem was obtained in [11].

In this paper we construct quantum group analogues of the algebras $W^e(\mathfrak{g})$. Namely, we define certain noncommutative deformations of algebras of regular functions defined on transversal slices to the set of conjugacy classes in an algebraic group G associated to the Lie algebra $\mathfrak{g}$. Such slices associated to (conjugacy classes of) Weyl group elements $s \in W$ were defined in [33] where we also introduced some natural Poisson structures on them. The algebras $W_q^s(G)$, $s \in W$ introduced in this paper are quantizations of those Poisson structures.

Technically, in order to define the algebras $W_q^s(G)$ one should first construct quantum group analogues of nilpotent Lie subalgebras $\mathfrak{m} \subset \mathfrak{g}$ and of their nontrivial characters χ. Nilpotent subalgebras in $\mathfrak{g}$ can be naturally described in terms of root vectors. It it well known that one can define analogues of root vectors in the standard Drinfeld-Jimbo quantum group $U_h(\mathfrak{g})$ in such a way that their ordered products form a Poincaré–Birkhoff-Witt basis of $U_h(\mathfrak{g})$ (see [5], Ch. 8). The definition of the quantum group analogues of root vectors is given in terms of a certain braid group action on $U_h(\mathfrak{g})$ and depends on the choice of a normal ordering of the system of positive roots. Our first task is to associate to each Weyl group element $s \in W$ a system of simple roots, a normal ordering of the corresponding system of positive roots Δ_+ and an ordered segment $\Delta_{\mathfrak{m}_+} \subset \Delta_+$. The definition of the normal ordering relies on some distinguished normal orderings compatible with involutions in Weyl groups. Such normal orderings are defined in the Appendix. Next, for each element $s \in W$ we also define a new realization $U_h^s(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ in such a way that the subalgebra $U_h^s(\mathfrak{m}_+)$ generated by the quantum root vectors associated to roots from $\Delta_{\mathfrak{m}_+}$ has a nontrivial character χ_h^s. Note that in general position the subalgebras $U_h^s(\mathfrak{m}_+)$ are not deformations of the algebras $U(\mathfrak{m})$. In case when s is a Coxeter element the subalgebras $U_h^s(\mathfrak{m}_+)$ were defined in [26].

Then we recall that the standard quantum group $U_h(\mathfrak{g})$ contains a certain Hopf subalgebra defined over the ring $\mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$, $q = e^h$, $d \in \mathbb{N}$ such that its specialization at $q = 1$ is the Poisson–Hopf algebra of regular functions on an algebraic Poisson–Lie group dual in the sense of Poisson–Lie groups to the quasirtiangular Poisson Lie group associated to the standard (Drinfeld-Jimbo) bialgebra structure on $\mathfrak{g}$ (see Section 8). Similarly, we define a certain Hopf subalgebra $\mathbb{C}_{\mathcal{A}'}[G^*]$ defined over the ring $\mathcal{A}' = \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}, \frac{1-q^{\frac{1}{2d}}}{1-q_i^{-2}}]_{i=1,\ldots,\mathrm{rank}(\mathfrak{g})}$, where $q_i = q^{d_i}$, and d_i are some positive integers, such that its specialization at $q = 1$ is the Poisson–Hopf algebra of regular functions on an algebraic Poisson–Lie group G^* dual in the sense of Poisson Lie groups to the quasirtiangular Poisson–Lie group associated to the nonstandard bialgebra structure on $\mathfrak{g}$ with the r–matrix

$$r_+^s = \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_\beta \otimes X_{-\beta} + \frac{1}{2}\left((\frac{1+s}{1-s} P_{\mathfrak{h}'} + 1) \otimes id \right) t_0,$$

where $X_{\pm\beta}$ are root vectors of $\mathfrak{g}$ corresponding to roots $\pm\beta$, $\beta \in \Delta_+$, $(\cdot, \cdot)$ is the normalized Killing form of $\mathfrak{g}$, $t_0 \in \mathfrak{h} \otimes \mathfrak{h}$ is the Cartan part of the Casimir element of $\mathfrak{g}$ and $P_{\mathfrak{h}'}$ is the orthogonal, with respect to the Killing form, projection operator onto the orthogonal complement $\mathfrak{h}'$ to the set of fixpoints of s in $\mathfrak{h}$.

The algebra $\mathbb{C}_{\mathcal{A}'}[G^*]$ defined in Section 10 contains a subalgebra $\mathbb{C}_{\mathcal{A}'}[M_-] = \mathbb{C}_{\mathcal{A}}[G^*] \cap U_h^s(\mathfrak{m}_+)$ which can be equipped with a nontrivial character χ_q^s. If we denote by $\mathbb{C}_{\chi_q^s}$ the corresponding rank one representation of $\mathbb{C}_{\mathcal{A}}[M_-]$ then the associative algebra

$$W_q^s(G) = \mathrm{End}_{\mathbb{C}_{\mathcal{A}'}[G^*]}(\mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s})^{opp}$$

is the deformation of the algebra of regular functions defined on a transversal slice to the set of conjugacy classes in an algebraic group G associated to the Lie algebra $\mathfrak{g}$. This slice can be equipped with a Poisson structure using Poisson reduction in the Poisson–Lie group G^* and a dense embedding $G^* \subset G$ (see [33]). The algebra $W_q^s(G)$ is a quantization of the Poisson structure defined on the slice. The proof of these facts occupies Sections 11 and 12. Note that in general the algebras $W_q^s(G)$ are not deformations of the usual W–algebras $W^e(\mathfrak{g})$. In case when s is a Coxeter element of W the algebras $W_q^s(G)$ were introduced in [27, 30].

For $\varepsilon \in \mathbb{C}$ one can also consider specializations $W_\varepsilon^s(G) = \mathrm{End}_{\mathbb{C}_\varepsilon[G^*]}(\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s})^{opp}$ of algebras $W_q^s(G)$ defined with the help of the specializations $\mathbb{C}_\varepsilon[G^*] = \mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})\mathbb{C}_{\mathcal{A}'}[G^*]$, $\mathbb{C}_\varepsilon[M_-] = \mathbb{C}_{\mathcal{A}'}[M_-]/(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})\mathbb{C}_{\mathcal{A}'}[M_-]$ of the algebras $\mathbb{C}_{\mathcal{A}'}[G^*]$, $\mathbb{C}_{\mathcal{A}'}[M_-]$ and the natural specialization χ_ε^s of the character χ_q^s.

In Section 12 we remark that for generic ε the center $Z(\mathbb{C}_\varepsilon[G^*])$ of the algebra $\mathbb{C}_\varepsilon[G^*]$ is naturally a subalgebra in $W_\varepsilon^s(G)$, and if $\eta : Z(\mathbb{C}_\varepsilon[G^*]) \to \mathbb{C}$ is a character then the algebra $W_\varepsilon^s(G)/W_\varepsilon^s(G)\mathrm{ker}\,\eta$ can be regarded as a noncommutative deformation of the algebra of regular functions defined on a fiber of the conjugation quotient map $\delta_G : G \to H/W$ restricted to a transversal slice in G. (Recall that $\delta_G : G \to H/W$ is generated by the inclusion $\mathbb{C}[H]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where H is the maximal torus of G corresponding to the Cartan subalgebra $\mathfrak{h}$ and W is the Weyl group of the pair (G, H)). In particular, for singular fibers we obtain noncommutative deformations of the coordinate rings of the corresponding singularities.

Next we discuss a homological realization of algebras $W_\varepsilon^s(G)$ similar to that suggested in [6] for the usual W–algebras. The latter one is based on the BRST cohomological reduction procedure for Lie algebras. In case of arbitrary associative algebras an analogue of the BRST reduction technique was suggested in [28, 31]. We apply this technique in the situation considered in this paper. This yields a graded associative algebra $\mathrm{Hk}^\bullet((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s))$ with trivial negatively graded components and such that $\mathrm{Hk}^0((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s)) = W_\varepsilon^s(G)^{opp}$. For every left $\mathbb{C}_\varepsilon[G^*]$–module V and right $\mathbb{C}_\varepsilon[G^*]$–module W the algebra $\mathrm{Hk}^\bullet((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s))$ naturally acts in the spaces $\mathrm{Ext}^\bullet_{\mathbb{C}_\varepsilon[M_-]}(\mathbb{C}_{\chi_\varepsilon^s}, V)$ and $\mathrm{Tor}^\bullet_{\mathbb{C}_\varepsilon[M_-]}(W, \mathbb{C}_{\chi_\varepsilon^s})$, from the right and from the left, respectively. Note that, as the example given in Section 13 shows, in contrast to the Lie algebra case the positively graded components of the algebra $\mathrm{Hk}^\bullet((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s))$ do not vanish even for generic ε.

Observe that it is impossible to define an analogue of the category of Gelfand–Graev representations for quantum groups in the same way as in case of Lie algebras. As an example consider the module $\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}$ in case when $\mathfrak{g} = \mathfrak{sl}_2$ and s is the only nontrivial element of the Weyl group. Then the subalgebra $\mathbb{C}_\varepsilon[M_-]$ is generated by a single element e which is a counterpart of the only nontrivial positive root vector $X^+ \in \mathfrak{sl}_2$. Fix a character χ_ε^s of $\mathbb{C}_\varepsilon[M_-]$ defined by $\chi_\varepsilon^s(e) = 1$. The Lie algebra counterpart of this module belongs to the category of Gelfand–Graev representations in the sense that the element $X^+ - \chi(X^+)$, where $\chi(X^+) \neq 0$, acts locally nilpotently on it. But the example given in the end of Section 13 shows that the action of the operator $e - \chi_\varepsilon^s(e)$ on $\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}$ is semisimple. However, the module $\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}$ is the most natural candidate to become an element of the quantum group counterpart of the category of Gelfand–Graev representations.

Indeed, in [30] in a slightly different setting of quantum deformations of the usual W–algebras it was shown that there is one–to–one correspondence between $U_h(\mathfrak{sl}_2)$–modules finitely generated by Whittaker vectors v (i.e. the vectors for which $ev = v$) and finite–dimensional representations of the corresponding W–algebra which is isomorphic to the center of $U_h(\mathfrak{sl}_2)$ in that case. This result also holds in case of an arbitrary complex simple Lie algebra and the deformed W–algebras associated to Coxeter elements of the Weyl group. This suggests that the quantum group analogue of the category of Gelfand–Graev representations should be defined as the category $\mathcal{C}_q^s$ of $\mathbb{C}_{\mathcal{A}'}[G^*]$–modules generated by Whittaker vectors, i.e. the vectors v for which $xv = \chi_\varepsilon^s v$ for any $x \in \mathbb{C}_{\mathcal{A}'}[M_-]$. However, it is not even obvious that such modules form an abelian category since a priori kernels and cokernels in modules from this category do not need to be Gelfand–Graev representations. In Section 14 we show that this is indeed the case and establish an equivalence between $\mathcal{C}_q^s$ and the category of $W_q^s(G)$–modules.

Another interesting problem related to algebras $W_\varepsilon^s(G)$ is concerned with representation theory of quantum groups at roots of unity. Recall that this representation theory is similar to the representation theory of semisimple Lie algebras over a field of prime characteristic p. Let $\mathfrak{g}_p$ be such an algebra. In [20] it was shown that to each element $\xi \in \mathfrak{g}_p^*$ one can associate a W-algebra in such a way that the corresponding reduced universal enveloping algebra $U(\mathfrak{g}_p)_\xi$ is isomorphic to the algebra of matrices of size $d(\xi)$ with entries being elements of the W–algebra, where $d(\xi) = p^{\frac{1}{2}\dim\,\mathcal{O}_\xi}$, and $\mathcal{O}_\xi$ is the coadjoint orbit of ξ. Since each finite–dimensional representation V of $\mathfrak{g}_p$ is a representation of an algebra $U(\mathfrak{g}_p)_\xi$ for some ξ the last statement implies the Kac–Weisfeiler conjecture which states the dimension of V is divisible by $d(\xi)$.

In [7] De Concini, Kac and Procesi formulated a similar conjecture for quantum groups at roots of unity. However, in case of quantum groups at roots of unity irreducible finite–dimensional representations are parameterized by conjugacy classes in algebraic groups. We expect that there is a quantum group counterpart of the correspondence between reduced enveloping algebras and W–algebras mentioned above, and the algebras $W_\varepsilon^s(G)$, where ε is a root of unity, will play in this correspondence the same role as the usual W–algebras in the Lie algebra case. This is quite natural from the geometric point of view since the algebras $W_\varepsilon^s(G)$ are noncommutative deformations of algebras of functions on transversal slices to the set of conjugacy classes in G. The conjectural correspondence mentioned above would also imply the De Concini–Kac–Procesi conjecture. This will be explained in a subsequent paper.

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2. Notation

Fix the notation used throughout of the text. Let G be a connected finite–dimensional complex simple Lie group, $\mathfrak{g}$ its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Let $\alpha_i,\ i = 1, \ldots l,\ l = rank(\mathfrak{g})$ be a system of simple roots, $\Delta_+ = \{\beta_1, \ldots, \beta_N\}$ the set of positive roots. Let $H_1, \ldots, H_l$ be the set of simple root generators of $\mathfrak{h}$.

Let a_{ij} be the corresponding Cartan matrix, and let $d_1, \ldots, d_l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non–degenerate invariant symmetric bilinear form $(,)$ on $\mathfrak{g}$ such that $(H_i, H_j) = d_j^{-1} a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by $\alpha^\vee$ the element of $\mathfrak{h}$ that corresponds to $\alpha \in \mathfrak{h}^*$ under this isomorphism. The induced bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = b_{ij}$.

Let W be the Weyl group of the root system Δ. W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $s_1, \ldots, s_l$,

$$s_i(h) = h - \alpha_i(h) H_i,\ h \in \mathfrak{h}.$$

The action of W preserves the bilinear form $(,)$ on $\mathfrak{h}$. We denote a representative of $w \in W$ in G by the same letter. For $w \in W, g \in G$ we write $w(g) = wgw^{-1}$. For any root $\alpha \in \Delta$ we also denote by s_α the corresponding reflection.

Let $\mathfrak{b}_+$ be the positive Borel subalgebra and $\mathfrak{b}_-$ the opposite Borel subalgebra; let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be their nilradicals. Let $H = \exp \mathfrak{h}, N_+ = \exp \mathfrak{n}_+, N_- = \exp \mathfrak{n}_-, B_+ = HN_+, B_- = HN_-$ be the Cartan subgroup, the maximal unipotent subgroups and the Borel subgroups of G which correspond to the Lie subalgebras $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{b}_+$ and $\mathfrak{b}_-$, respectively.

We identify $\mathfrak{g}$ and its dual by means of the canonical invariant bilinear form. Then the coadjoint action of G on $\mathfrak{g}^*$ is naturally identified with the adjoint one. We also identify ${\mathfrak{n}_+}^* \cong \mathfrak{n}_-,\ {\mathfrak{b}_+}^* \cong \mathfrak{b}_-$.

Let $\mathfrak{g}_\beta$ be the root subspace corresponding to a root $\beta \in \Delta$, $\mathfrak{g}_\beta = \{x \in \mathfrak{g} | [h, x] = \beta(h)x \text{ for every } h \in \mathfrak{h}\}$. $\mathfrak{g}_\beta \subset \mathfrak{g}$ is a one–dimensional subspace. It is well–known that for $\alpha \neq -\beta$ the root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the canonical invariant bilinear form. Moreover $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non–degenerately paired by this form.

Root vectors $X_\alpha \in \mathfrak{g}_\alpha$ satisfy the following relations:

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})\alpha^\vee.$$

Note also that in this paper we denote by $\mathbb{N}$ the set of nonnegative integer numbers, $\mathbb{N} = \{0, 1, \ldots\}$.

3. Quantum groups

In this section we recall some basic facts about quantum groups. We follow the notation of [5].

Let h be an indeterminate, $\mathbb{C}[[h]]$ the ring of formal power series in h. We shall consider $\mathbb{C}[[h]]$–modules equipped with the so–called h–adic topology. For every such module V this topology is characterized by requiring that $\{h^n V \mid n \geq 0\}$ is a base of the neighborhoods of 0 in V, and that translations in V are continuous. It is easy to see that, for modules equipped with this topology, every $\mathbb{C}[[h]]$–module map is automatically continuous.

A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$–module A equipped with a structure of $\mathbb{C}[[h]]$–Hopf algebra (see [5], Definition 4.3.1), the algebraic tensor products entering the axioms of the Hopf algebra are replaced by their completions in the h–adic topology. We denote by $\mu, \imath, \Delta, \varepsilon, S$ the multiplication, the unit, the comultiplication, the counit and the antipode of A, respectively.

The standard quantum group $U_h(\mathfrak{g})$ associated to a complex finite–dimensional simple Lie algebra $\mathfrak{g}$ is the algebra over $\mathbb{C}[[h]]$ topologically generated by elements $H_i,\ X_i^+,\ X_i^-,\ i = 1, \ldots, l$, and with the following defining relations:

$$\begin{gathered} [H_i, H_j] = 0,\ [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \\ X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \text{where } K_i = e^{d_i h H_i},\ e^h = q,\ q_i = q^{d_i} = e^{d_i h}, \end{gathered} \tag{3.1}$$

and the quantum Serre relations:

$$\begin{gathered} \sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} (X_i^\pm)^{1-a_{ij}-r} X_j^\pm (X_i^\pm)^r = 0,\ i \neq j, \\ \text{where} \\ \left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![n-m]_q!},\ [n]_q! = [n]_q \ldots [1]_q,\ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \end{gathered} \tag{3.2}$$

$U_h(\mathfrak{g})$ is a topological Hopf algebra over $\mathbb{C}[[h]]$ with comultiplication defined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_h(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,$$

$$\Delta_h(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,$$

antipode defined by

$$S_h(H_i) = -H_i,\ S_h(X_i^+) = -X_i^+ K_i^{-1},\ S_h(X_i^-) = -K_i X_i^-,$$

and counit defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^{\pm}) = 0.$$

We shall also use the weight–type generators

$$Y_i = \sum_{j=1}^{l} d_i (a^{-1})_{ij} H_j,$$

and the elements $L_i = e^{hY_i}$. They commute with the root vectors $X_i^{\pm}$ as follows:

$$L_i X_j^{\pm} L_i^{-1} = q_i^{\pm\delta_{ij}} X_j^{\pm}. \tag{3.3}$$

We also obviously have

$$L_i L_j = L_j L_i. \tag{3.4}$$

The Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the standard bialgebra structure on $\mathfrak{g}$, i.e. $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g}),\ \Delta_h = \Delta \pmod h$, where Δ is the standard comultiplication on $U(\mathfrak{g})$, and

$$\frac{\Delta_h - \Delta_h^{opp}}{h} \pmod h = \delta,$$

where $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the standard cocycle on $\mathfrak{g}$. Recall that

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2r_+,\ r_+ \in \mathfrak{g} \otimes \mathfrak{g},$$

$$r_+ = \frac{1}{2} \sum_{i=1}^{l} Y_i \otimes X_i + \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_\beta \otimes X_{-\beta}. \tag{3.5}$$

Here $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$ are root vectors of $\mathfrak{g}$. The element $r_+ \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r–matrix.

$U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra, i.e. there exists an invertible element $\mathcal{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, called a universal R–matrix, such that

$$\Delta_h^{opp}(a) = \mathcal{R} \Delta_h(a) \mathcal{R}^{-1} \text{ for all } a \in U_h(\mathfrak{g}), \tag{3.6}$$

where $\Delta^{opp} = \sigma\Delta$, σ is the permutation in $U_h(\mathfrak{g})^{\otimes 2}$, $\sigma(x \otimes y) = y \otimes x$, and

$$\begin{aligned} (\Delta_h \otimes id)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \\ (id \otimes \Delta_h)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}, \end{aligned} \tag{3.7}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1,\ \mathcal{R}_{23} = 1 \otimes \mathcal{R},\ \mathcal{R}_{13} = (\sigma \otimes id)\mathcal{R}_{23}$.

From (3.6) and (3.7) it follows that $\mathcal{R}$ satisfies the quantum Yang–Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.8}$$

For every quasitriangular Hopf algebra we also have (see Proposition 4.2.7 in [5]):

$$(S \otimes id)\mathcal{R} = (id \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1},$$

and

$$(S\otimes S)\mathcal{R}=\mathcal{R}. \tag{3.9}$$

We shall explicitly describe the element $\mathcal{R}$. First following [5] we recall the construction of root vectors of $U_h(\mathfrak{g})$ in terms of a braid group action on $U_h(\mathfrak{g})$. Let $m_{ij},\ i\neq j$ be equal to $2,3,4,6$ if $a_{ij}a_{ji}$ is equal to $0,1,2,3$. The braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ has generators $T_i,\ i=1,\ldots,l$, and defining relations

$$T_iT_jT_iT_j\ldots=T_jT_iT_jT_i\ldots$$

for all $i\neq j$, where there are m_{ij} T's on each side of the equation.

There is an action of the braid group $\mathcal{B}_\mathfrak{g}$ by algebra automorphisms of $U_h(\mathfrak{g})$ defined on the standard generators as follows:

$$T_i(X_i^+)=-X_i^-e^{hd_iH_i},\ T_i(X_i^-)=-e^{-hd_iH_i}X_i^+,\ T_i(H_j)=H_j-a_{ji}H_i,$$

$$T_i(X_j^+)=\sum_{r=0}^{-a_{ij}}(-1)^{r-a_{ij}}q_i^{-r}(X_i^+)^{(-a_{ij}-r)}X_j^+(X_i^+)^{(r)},\ i\neq j,$$

$$T_i(X_j^-)=\sum_{r=0}^{-a_{ij}}(-1)^{r-a_{ij}}q_i^{r}(X_i^-)^{(r)}X_j^-(X_i^-)^{(-a_{ij}-r)},\ i\neq j,$$

where

$$(X_i^+)^{(r)}=\frac{(X_i^+)^r}{[r]_{q_i}!},\ (X_i^-)^{(r)}=\frac{(X_i^-)^r}{[r]_{q_i}!},\ r\geq 0,\ i=1,\ldots,l.$$

Recall that an ordering of a set of positive roots Δ_+ is called normal if all simple roots are written in an arbitrary order, and for any three roots $\alpha,\ \beta,\ \gamma$ such that $\gamma=\alpha+\beta$ we have either $\alpha<\gamma<\beta$ or $\beta<\gamma<\alpha$.

Any two normal orderings in Δ_+ can be reduced to each other by the so–called elementary transpositions (see [35], Theorem 1). The elementary transpositions for rank 2 root systems are inversions of the following normal orderings (or the inverse normal orderings):

$$\begin{array}{ll} \alpha,\ \beta & A_1+A_1 \\ \alpha,\ \alpha+\beta,\ \beta & A_2 \\ \alpha,\ \alpha+\beta,\ \alpha+2\beta,\ \beta & B_2 \\ \alpha,\ \alpha+\beta,\ 2\alpha+3\beta,\ \alpha+2\beta,\ \alpha+3\beta,\ \beta & G_2 \end{array} \tag{3.10}$$

where it is assumed that $(\alpha,\alpha)\geq(\beta,\beta)$. Moreover, any normal ordering in a rank 2 root system is one of orderings (3.10) or one of the inverse orderings.

In general an elementary inversion of a normal ordering in a set of positive roots Δ_+ is the inversion of an ordered segment of form (3.10) (or of a segment with the inverse ordering) in the ordered set Δ_+, where $\alpha-\beta\notin\Delta$.

For any reduced decomposition $w_0=s_{i_1}\ldots s_{i_D}$ of the longest element w_0 of the Weyl group W of $\mathfrak{g}$ the set

$$\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_D=s_{i_1}\ldots s_{i_{D-1}}\alpha_{i_D}$$

is a normal ordering in Δ_+, and there is one to one correspondence between normal orderings of Δ_+ and reduced decompositions of w_0 (see [36]).

Now fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_D}$ of the longest element w_0 of the Weyl group W of $\mathfrak{g}$ and define the corresponding root vectors in $U_h(\mathfrak{g})$ by

$$X^{\pm}_{\beta_k} = T_{i_1} \dots T_{i_{k-1}} X^{\pm}_{i_k}. \tag{3.11}$$

Proposition 3.1. *For $\beta = \sum_{i=1}^{l} m_i \alpha_i,\ m_i \in \mathbb{N}\ X^{\pm}_{\beta}$ is a polynomial in the noncommutative variables $X^{\pm}_i$ homogeneous in each $X^{\pm}_i$ of degree m_i.*

The root vectors X^{+}_{β} satisfy the following relations:

$$X^{+}_{\alpha} X^{+}_{\beta} - q^{(\alpha,\beta)} X^{+}_{\beta} X^{+}_{\alpha} = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C(k_1, \dots, k_n) {X^{+}_{\delta_1}}^{k_1} {X^{+}_{\delta_2}}^{k_2} \dots {X^{+}_{\delta_n}}^{k_n}, \tag{3.12}$$

where $C(k_1, \dots, k_n) \in \mathbb{C}[q, q^{-1}]$. They also commute with elements of the subalgebra $U_h(\mathfrak{h})$ as follows:

$$[H_i, X^{\pm}_{\beta}] = \pm \beta(H_i) X^{\pm}_{\beta},\ i = 1, \dots, l. \tag{3.13}$$

Note that by construction

$$X^{+}_{\beta} \pmod h = X_{\beta} \in \mathfrak{g}_{\beta},$$

$$X^{-}_{\beta} \pmod h = X_{-\beta} \in \mathfrak{g}_{-\beta}$$

are root vectors of $\mathfrak{g}$.

Let $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$ be the $\mathbb{C}[[h]]$–subalgebras of $U_h(\mathfrak{g})$ topologically generated by the X^{+}_i, by the X^{-}_i and by the H_i, respectively.

Now using the root vectors $X^{\pm}_{\beta}$ we can construct a topological basis of $U_h(\mathfrak{g})$. Define for $\mathbf{r} = (r_1, \dots, r_D) \in \mathbb{N}^D$,

$$(X^+)^{\mathbf{r}} = (X^{+}_{\beta_1})^{r_1} \dots (X^{+}_{\beta_D})^{r_D},$$
$$(X^-)^{\mathbf{r}} = (X^{-}_{\beta_D})^{r_D} \dots (X^{-}_{\beta_1})^{r_1},$$

and for $\mathbf{s} = (s_1, \dots s_l) \in \mathbb{N}^l$,

$$H^{\mathbf{s}} = H_1^{s_1} \dots H_l^{s_l}.$$

Proposition 3.2. **([15], Proposition 3.3)** *The elements $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$ and $H^{\mathbf{s}}$, for $\mathbf{r},\ \mathbf{t} \in \mathbb{N}^N$, $\mathbf{s} \in \mathbb{N}^l$, form topological bases of $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$, respectively, and the products $(X^+)^{\mathbf{r}} H^{\mathbf{s}} (X^-)^{\mathbf{t}}$ form a topological basis of $U_h(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}[[h]]$ modules:*

$$U_h(\mathfrak{n}_-) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_+) \to U_h(\mathfrak{g}).$$

An explicit expression for $\mathcal{R}$ may be written by making use of the q-exponential

$$exp_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}$$

in terms of which the element $\mathcal{R}$ takes the form:

$$\mathcal{R} = exp\left[h \sum_{i=1}^{l} (Y_i \otimes H_i) \right] \prod_{\beta} exp_{q_\beta} [(1 - q_\beta^{-2}) X^{+}_{\beta} \otimes X^{-}_{\beta}], \tag{3.14}$$

where $q_\beta = q^{d_i}$ if the positive root β is Weyl group conjugate to the simple root α_i; the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the α–term appears to the left of the β–term if $\alpha < \beta$ with respect to the normal ordering of Δ_+.

Remark 3.1. *The r–matrix $r_+ = \frac{1}{2} h^{-1} (\mathcal{R} - 1 \otimes 1) \ (mod\, h)$, which is the classical limit of $\mathcal{R}$, coincides with the classical r–matrix (3.5).*

4. Realizations of quantum groups associated to Weyl group elements

The subalgebras of $U_h(\mathfrak{g})$ which possess nontrivial characters are defined in terms of the new realizations $U_h^s(\mathfrak{g})$ of $U_h(\mathfrak{g})$ associated to Weyl group elements, and we start by defining these new realizations.

Let s be an element of the Weyl group W of the pair $(\mathfrak{g},\mathfrak{h})$, and $\mathfrak{h}'$ the orthogonal complement, with respect to the Killing form, to the subspace of $\mathfrak{h}$ fixed by the natural action of s on $\mathfrak{h}$. The restriction of the natural action of s on $\mathfrak{h}^*$ to the subspace $\mathfrak{h}'^*$ has no fixed points. Therefore one can define the Cayley transform $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ of the restriction of s to $\mathfrak{h}'^*$, where $P_{\mathfrak{h}'^*}$ is the orthogonal projection operator onto $\mathfrak{h}'^*$ in $\mathfrak{h}^*$, with respect to the Killing form.

Now we suggest a new realization of the quantum group $U_h(\mathfrak{g})$ associated to $s\in W$. Let $U_h^s(\mathfrak{g})$ be the associative algebra over $\mathbb{C}[[h]]$ topologically generated by elements $e_i, f_i, H_i,\ i=1,\ldots l$ subject to the relations:

$$\begin{aligned}
&[H_i,H_j]=0,\ [H_i,e_j]=a_{ij}e_j,\ [H_i,f_j]=-a_{ij}f_j,\\
&e_if_j-q^{c_{ij}}f_je_i=\delta_{i,j}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},\, c_{ij}=\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)\\
&K_i=e^{d_ihH_i},\\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^rq^{rc_{ij}}\left[\begin{array}{c}1-a_{ij}\\ r\end{array}\right]_{q_i}(e_i)^{1-a_{ij}-r}e_j(e_i)^r=0,\ i\neq j,\\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^rq^{rc_{ij}}\left[\begin{array}{c}1-a_{ij}\\ r\end{array}\right]_{q_i}(f_i)^{1-a_{ij}-r}f_j(f_i)^r=0,\ i\neq j.
\end{aligned}\tag{4.1}$$

Theorem 4.1. *For every solution $n_{ij}\in\mathbb{C},\ i,j=1,\ldots,l$ of equations*

$$d_jn_{ij}-d_in_{ji}=c_{ij}\tag{4.2}$$

there exists an algebra isomorphism $\psi_{\{n\}}:U_h^s(\mathfrak{g})\to U_h(\mathfrak{g})$ defined by the formulas:

$$\begin{aligned}
&\psi_{\{n\}}(e_i)=X_i^+\textstyle\prod_{p=1}^lL_p^{n_{ip}},\\
&\psi_{\{n\}}(f_i)=\textstyle\prod_{p=1}^lL_p^{-n_{ip}}X_i^-,\\
&\psi_{\{n\}}(H_i)=H_i.
\end{aligned}$$

Proof. The proof of this theorem is direct verification of defining relations (4.1). The most nontrivial part is to verify deformed quantum Serre relations, i.e. the last two relations in (4.1). The defining relations of $U_h(\mathfrak{g})$ imply the following relations for $\psi_{\{n\}}(e_i)$,

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\left[\begin{array}{c}1-a_{ij}\\ k\end{array}\right]_{q_i}q^{k(d_jn_{ij}-d_in_{ji})}\psi_{\{n\}}(e_i)^{1-a_{ij}-k}\psi_{\{n\}}(e_j)\psi_{\{n\}}(e_i)^k=0,$$

for any $i\neq j$. Now using equation (4.2) we arrive to the quantum Serre relations for e_i in (4.1). □

Remark 4.2. *The general solution of equation (4.2) is given by*

$$n_{ji}=\frac{1}{2}(c_{ij}+\frac{s_{ij}}{d_i}),\tag{4.3}$$

where $s_{ij}=s_{ji}$.

We call the algebra $U_h^s(\mathfrak{g})$ the realization of the quantum group $U_h(\mathfrak{g})$ corresponding to the element $s \in W$.

Remark 4.3. *Let n_{ij} be a solution of the homogeneous system that corresponds to (4.2),*

$$d_i n_{ji} - d_j n_{ij} = 0.$$

Then the map defined by

$$X_i^+ \mapsto X_i^+ \prod_{p=1}^l L_p^{n_{ip}},$$

$$X_i^- \mapsto \prod_{p=1}^l L_p^{-n_{ip}} X_i^-, \tag{4.4}$$

$$H_i \mapsto H_i$$

is an automorphism of $U_h(\mathfrak{g})$. Therefore for given element $s \in W$ the isomorphism $\psi_{\{n\}}$ is defined uniquely up to automorphisms (4.4) of $U_h(\mathfrak{g})$.

Now we shall study the algebraic structure of $U_h^s(\mathfrak{g})$. Denote by $U_h^s(\mathfrak{n}_\pm)$ the subalgebra in $U_h^s(\mathfrak{g})$ generated by e_i $(f_i), i = 1, \ldots l$. Let $U_h^s(\mathfrak{h})$ be the subalgebra in $U_h^s(\mathfrak{g})$ generated by $H_i,\ i = 1, \ldots, l$.

We shall construct a Poincaré–Birkhoff-Witt basis for $U_h^s(\mathfrak{g})$. It is convenient to introduce an operator $K \in \mathrm{End}\ \mathfrak{h}$ such that

$$KH_i = \sum_{j=1}^l \frac{n_{ij}}{d_i} Y_j. \tag{4.5}$$

In particular, we have

$$\frac{n_{ji}}{d_j} = (KH_j, H_i).$$

Equation (4.2) is equivalent to the following equation for the operator K:

$$K - K^* = \frac{1+s}{1-s} P_{\mathfrak{h}'^*}.$$

Proposition 4.2. *(i) For any solution of equation (4.2) and any normal ordering of the root system Δ_+ the elements $e_\beta = \psi_{\{n\}}^{-1}(X_\beta^+ e^{hK\beta^\vee})$ and $f_\beta = \psi_{\{n\}}^{-1}(e^{-hK\beta^\vee} X_\beta^-)$, $\beta \in \Delta_+$ lie in the subalgebras $U_h^s(\mathfrak{n}_+)$ and $U_h^s(\mathfrak{n}_-)$, respectively. The elements e_β, $\beta \in \Delta_+$ satisfy the following commutation relations*

$$e_\alpha e_\beta - q^{(\alpha,\beta) + (\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} e_\beta e_\alpha = \sum_{\alpha < \delta_1 < \ldots < \delta_n < \beta} C'(k_1, \ldots, k_n) e_{\delta_1}^{k_1} e_{\delta_2}^{k_2} \ldots e_{\delta_n}^{k_n}. \tag{4.6}$$

(ii) Moreover, the elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \ldots e_{\beta_D}^{r_D}$, $f^{\mathbf{t}} = f_{\beta_D}^{t_D} \ldots f_{\beta_1}^{t_1}$ and $H^{\mathbf{s}} = H_1^{s_1} \ldots H_l^{s_l}$ for $\mathbf{r}$, $\mathbf{t}$, $\mathbf{s} \in \mathbb{N}^l$ form topological bases of $U_h^s(\mathfrak{n}_+)$, $U_h^s(\mathfrak{n}_-)$ and $U_h^s(\mathfrak{h})$, and the products $f^{\mathbf{t}} H^{\mathbf{s}} e^{\mathbf{r}}$ form a topological basis of $U_h^s(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}[[h]]$ modules

$$U_h^s(\mathfrak{n}_-) \otimes U_h^s(\mathfrak{h}) \otimes U_h^s(\mathfrak{n}_+) \to U_h^s(\mathfrak{g}).$$

Proof. Let $\beta = \sum_{i=1}^l m_i \alpha_i \in \Delta_+$ be a positive root, $X_\beta^+ \in U_h(\mathfrak{g})$ the corresponding root vector. Then $\beta^\vee = \sum_{i=1}^l m_i d_i H_i$, and so $K\beta^\vee = \sum_{i,j=1}^l m_i n_{ij} Y_j$. Now the proof of the first statement follows immediately from Proposition 3.1, commutation relations (3.3), (3.12) and the definition of the isomorphism $\psi_{\{n\}}$. The second assertion is a consequence of Proposition 3.2. □

The realizations $U_h^s(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ are connected with quantizations of some nonstandard bialgebra structures on $\mathfrak{g}$. At the quantum level changing bialgebra structure corresponds to the so–called Drinfeld twist. We shall consider a particular class of such twists described in the following proposition.

Proposition 4.3. **([5], Proposition 4.2.13)** *Let $(A,\mu,\imath,\Delta,\varepsilon,S)$ be a Hopf algebra over a commutative ring. Let $\mathcal{F}$ be an invertible element of $A\otimes A$ such that*

$$\mathcal{F}_{12}(\Delta\otimes id)(\mathcal{F})=\mathcal{F}_{23}(id\otimes\Delta)(\mathcal{F}),$$
$$(\varepsilon\otimes id)(\mathcal{F})=(id\otimes\varepsilon)(\mathcal{F})=1. \tag{4.7}$$

Then, $v=\mu(id\otimes S)(\mathcal{F})$ is an invertible element of A with

$$v^{-1}=\mu(S\otimes id)(\mathcal{F}^{-1}).$$

Moreover , if we define $\Delta^{\mathcal{F}}:A\rightarrow A\otimes A$ and $S^{\mathcal{F}}:A\rightarrow A$ by

$$\Delta^{\mathcal{F}}(a)=\mathcal{F}\Delta(a)\mathcal{F}^{-1},\ S^{\mathcal{F}}(a)=vS(a)v^{-1},$$

then $(A,\mu,\imath,\Delta^{\mathcal{F}},\varepsilon,S^{\mathcal{F}})$ is a Hopf algebra denoted by $A^{\mathcal{F}}$ and called the twist of A by $\mathcal{F}$.

Corollary 4.4. **([5], Corollary 4.2.15)** *Suppose that A and $\mathcal{F}$ as in Proposition 4.3, but assume in addition that A is quasitriangular with universal R–matrix $\mathcal{R}$. Then $A^{\mathcal{F}}$ is quasitriangular with universal R–matrix*

$$\mathcal{R}^{\mathcal{F}}=\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \tag{4.8}$$

where $\mathcal{F}_{21}=\sigma\mathcal{F}$.

Fix an element $s\in W$. Consider the twist of the Hopf algebra $U_h(\mathfrak{g})$ by the element

$$\mathcal{F}=exp(-h\sum_{i,j=1}^{l}\frac{n_{ji}}{d_j}Y_i\otimes Y_j)\in U_h(\mathfrak{h})\otimes U_h(\mathfrak{h}), \tag{4.9}$$

where n_{ij} is a solution of the corresponding equation (4.2).

This element satisfies conditions (4.7), and so $U_h(\mathfrak{g})^{\mathcal{F}}$ is a quasitriangular Hopf algebra with the universal R–matrix $\mathcal{R}^{\mathcal{F}}=\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$, where $\mathcal{R}$ is given by (3.14). We shall explicitly calculate the element $\mathcal{R}^{\mathcal{F}}$. Substituting (3.14) and (4.9) into (4.8) and using (3.13) we obtain

$$\begin{array}{l}\mathcal{R}^{\mathcal{F}}=exp\left[h(\sum_{i=1}^{l}(Y_i\otimes H_i)+\sum_{i,j=1}^{l}(-\frac{n_{ij}}{d_i}+\frac{n_{ji}}{d_j})Y_i\otimes Y_j)\right]\times\\ \prod_\beta exp_{q_\beta^{-1}}[(1-q_\beta^{-2})X_\beta^+e^{hK\beta^\vee}\otimes e^{-hK^*\beta^\vee}X_\beta^-],\end{array}$$

where K is defined by (4.5).

Equip $U_h^s(\mathfrak{g})$ with the comultiplication given by : $\Delta_s(x)=(\psi_{\{n\}}^{-1}\otimes\psi_{\{n\}}^{-1})\Delta_h^{\mathcal{F}}(\psi_{\{n\}}(x))$. Then $U_h^s(\mathfrak{g})$ becomes a quasitriangular Hopf algebra with the universal R–matrix $\mathcal{R}^s=\psi_{\{n\}}^{-1}\otimes\psi_{\{n\}}^{-1}\mathcal{R}^{\mathcal{F}}$. Using equation (4.2) this R–matrix may be written as follows

$$\begin{array}{l}\mathcal{R}^s=exp\left[h(\sum_{i=1}^{l}(Y_i\otimes H_i)+\sum_{i=1}^{l}\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i\otimes Y_i)\right]\times\\ \prod_\beta exp_{q_\beta^{-1}}[(1-q_\beta^{-2})e_\beta\otimes e^{h\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee}f_\beta],\end{array} \tag{4.10}$$

where $P_{\mathfrak{h}'}$ is the orthogonal projection operator onto $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the Killing form.

The element $\mathcal{R}^s$ may be also represented in the form

$$\begin{array}{l}\mathcal{R}^s=\prod_\beta exp_{q_\beta^{-1}}[(1-q_\beta^{-2})e_\beta e^{-h(\frac{1+s}{1-s}P_{\mathfrak{h}'}+1)\beta^\vee}\otimes e^{h\beta^\vee}f_\beta]\times\\ exp\left[h(\sum_{i=1}^{l}(Y_i\otimes H_i)+\sum_{i=1}^{l}\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i\otimes Y_i)\right].\end{array} \tag{4.11}$$

The comultiplication Δ_s is given on generators by

$$\Delta_s(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_s(e_i) = e_i \otimes e^{hd_i(\frac{2}{1-s}P_{\mathfrak{h}'}+P_{\mathfrak{h}'^\perp})H_i} + 1 \otimes e_i,$$

$$\Delta_s(f_i) = f_i \otimes e^{-hd_i\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i} + e^{-hd_iH_i} \otimes f_i,$$

where $P_{\mathfrak{h}'^\perp}$ is the orthogonal projection operator onto the orthogonal complement $\mathfrak{h}'^\perp$ to $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the Killing form.

Finally, the new antipode $S_s(x) = \psi_{\{n\}}^{-1}S_h^{\mathcal{F}}(\psi_{\{n\}}(x))$ is given by

$$S_s(e_i) = -e_ie^{-hd_i(\frac{2}{1-s}P_{\mathfrak{h}'}+P_{\mathfrak{h}'^\perp})H_i},\ S_s(f_i) = -e^{hd_iH_i}f_ie^{hd_i\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i},\ S_s(H_i) = -H_i.$$

Note that the Hopf algebra $U_h^s(\mathfrak{g})$ is a quantization of the bialgebra structure on $\mathfrak{g}$ defined by the cocycle

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)2r_+^s,\ r_+^s \in \mathfrak{g}\otimes\mathfrak{g}, \tag{4.12}$$

where $r_+^s = r_+ + \frac{1}{2}\sum_{i=1}^{l}\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i$, and r_+ is given by (3.5).

5. Normal orderings of root systems related to Weyl group elements

In this section we define certain normal orderings of root systems associated to Weyl group elements. The definition of subalgebras of $U_h(\mathfrak{g})$ having nontrivial characters will be given in terms of root vectors associated to such normal orderings.

Let s, as in the previous section, be an element of the Weyl group W of the pair $(\mathfrak{g},\mathfrak{h})$ and $\mathfrak{h}_\mathbb{R}$ the real form of $\mathfrak{h}$, the real linear span of simple coroots in $\mathfrak{h}$. The set of roots Δ is a subset of the dual space $\mathfrak{h}_\mathbb{R}^*$.

The Weyl group element s naturally acts on $\mathfrak{h}_\mathbb{R}$ as an orthogonal transformation with respect to the scalar product induced by the Killing form of $\mathfrak{g}$. Using the spectral theory of orthogonal transformations we can decompose $\mathfrak{h}_\mathbb{R}$ into a direct orthogonal sum of s–invariant subspaces,

$$\mathfrak{h}_\mathbb{R} = \bigoplus_{i=0}^{K}\mathfrak{h}_i, \tag{5.1}$$

where we assume that $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_\mathbb{R}$ fixed by the action of s, and each of the other subspaces $\mathfrak{h}_i \subset \mathfrak{h}_\mathbb{R}$, $i = 1,\ldots,K$, is either two–dimensional or one–dimensional and the Weyl group element s acts on it as rotation with angle θ_i, $0 < \theta_i \leq \pi$ or as reflection with respect to the origin (which also can be regarded as rotation with angle π). Note that since s has finite order $\theta_i = \frac{2\pi}{m_i}$, $m_i \in \mathbb{N}$.

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0,\ldots,K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}_i$ with respect to the natural pairing between $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$.

Now we consider certain s–invariant subsets of roots $\overline{\Delta}_i$, $i = 0,\ldots,K$, defined as follows

$$\overline{\Delta}_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i,\ h_i(\alpha) \neq 0\}, \tag{5.2}$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets $\overline{\Delta}_i$ are empty, and that the definition of these subsets depends on the order of terms in direct sum (5.1).

Now consider the nonempty s–invariant subsets of roots $\overline{\Delta}_{i_k}$, $k = 0,\ldots,M$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$. According to this definition $\overline{\Delta}_{i_0} = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s. Observe also that the root system Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$,

$$\Delta = \bigcup_{k=0}^{M} \overline{\Delta}_{i_k}.$$

Now assume that

$$|h_{i_k}(\alpha)| > |\sum_{l\leq j<k} h_{i_j}(\alpha)|, \text{ for any } \alpha \in \overline{\Delta}_{i_k},\ k = 0, \ldots, M,\ l < k. \tag{5.3}$$

Condition (5.3) can be always fulfilled by suitable rescalings of the elements h_{i_k}.

Consider the element

$$\bar{h} = \sum_{k=0}^{M} h_{i_k} \in \mathfrak{h}_{\mathbb{R}}. \tag{5.4}$$

From definition (5.2) of the sets $\overline{\Delta}_i$ we obtain that for $\alpha \in \overline{\Delta}_{i_k}$

$$\bar{h}(\alpha) = \sum_{j\leq k} h_{i_j}(\alpha) = h_{i_k}(\alpha) + \sum_{j<k} h_{i_j}(\alpha) \tag{5.5}$$

Now condition (5.3), the previous identity and the inequality $|x + y| \geq ||x| - |y||$ imply that for $\alpha \in \overline{\Delta}_{i_k}$ we have

$$|\bar{h}(\alpha)| \geq ||h_{i_k}(\alpha)| - |\sum_{j<k} h_{i_j}(\alpha)|| > 0.$$

Since Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$, $\Delta = \bigcup_{k=0}^{M} \overline{\Delta}_{i_k}$, the last inequality ensures that $\bar{h}$ belongs to a Weyl chamber of the root system Δ, and one can define the subset of positive roots Δ_+ and the set of simple positive roots Γ with respect to that chamber. From condition (5.3) and formula (5.5) we also obtain that a root $\alpha \in \overline{\Delta}_{i_k}$ is positive if and only if

$$h_{i_k}(\alpha) > 0. \tag{5.6}$$

We denote by $(\overline{\Delta}_{i_k})_+$ the set of positive roots contained in $\overline{\Delta}_{i_k}$, $(\overline{\Delta}_{i_k})_+ = \Delta_+ \bigcap \overline{\Delta}_{i_k}$.

We shall also need a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ associated to the semisimple element $\bar{h}_0 = \sum_{k=1}^{M} h_{i_k} \in \mathfrak{h}_{\mathbb{R}}$ associated to $s \in W$. This subalgebra is defined with the help of the linear eigenspace decomposition of $\mathfrak{g}$ with respect to the adjoint action of $\bar{h}_0$ on $\mathfrak{g}$, $\mathfrak{g} = \bigoplus_m (\mathfrak{g})_m$, $(\mathfrak{g})_m = \{x \in \mathfrak{g} \mid [\bar{h}_0, x] = mx\}$, $m \in \mathbb{R}$. By definition $\mathfrak{p} = \bigoplus_{m\geq 0}(\mathfrak{g})_m$ is a parabolic subalgebra in $\mathfrak{g}$, $\mathfrak{n} = \bigoplus_{m>0}(\mathfrak{g})_m$ and $\mathfrak{l} = \{x \in \mathfrak{g} \mid [\bar{h}_0, x] = 0\}$ are the nilradical and the Levi factor of $\mathfrak{p}$, respectively. Note that we have natural inclusions of Lie algebras $\mathfrak{p} \supset \mathfrak{b}_+ \supset \mathfrak{n}$, where $\mathfrak{b}_+$ is the Borel subalgebra of $\mathfrak{g}$ corresponding to the system Γ of simple roots, and Δ_{i_0} is the root system of the reductive Lie algebra $\mathfrak{l}$.

For every element $w \in W$ one can introduce the set $\Delta_w = \{\alpha \in \Delta_+ : w(\alpha) \in -\Delta_+\}$, and the number of the elements in the set Δ_w is equal to the length $l(w)$ of the element w with respect to the system Γ of simple roots in Δ_+.

Now recall that in the classification theory of conjugacy classes in the Weyl group W of the complex simple Lie algebra $\mathfrak{g}$ the so-called primitive (or semi–Coxeter in another terminology) elements play a primary role. The primitive elements $w \in W$ are characterized by the property $\det(1 - w) = \det a$, where a is the Cartan matrix of $\mathfrak{g}$. According to the results of [4] the element s of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ is primitive in the Weyl group W' of a regular semisimple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of the form

$$\mathfrak{g}' = \mathfrak{h}' + \sum_{\alpha\in\Delta'} \mathfrak{g}_\alpha,$$

where Δ' is a root subsystem of the root system Δ of $\mathfrak{g}$, $\mathfrak{g}_\alpha$ is the root subspace of $\mathfrak{g}$ corresponding to root α, and $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$ (it coincides with $\mathfrak{h}'$ introduced in Section 4).

Moreover, by Theorem C in [4] s can be represented as a product of two involutions,

$$s = s^1 s^2, \tag{5.7}$$

where $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \ldots \gamma_n$ and $\gamma_{n+1} \ldots \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'$, in particular l' is the rank of $\mathfrak{g}'$.

Proposition 5.1. *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots defined with the help of element (5.4), $\Delta_+ = \{\alpha \in \Delta | \bar{h}(\alpha) > 0\}$.*

Then there is a normal ordering of the root system Δ_+ of the following form

$$\begin{gathered}\beta^1_1, \ldots, \beta^1_t, \beta^1_{t+1}, \ldots, \beta^1_{t+\frac{p-n}{2}}, \gamma_1, \beta^1_{t+\frac{p-n}{2}+2}, \ldots, \beta^1_{t+\frac{p-n}{2}+n_1}, \gamma_2, \\ \beta^1_{t+\frac{p-n}{2}+n_1+2} \ldots, \beta^1_{t+\frac{p-n}{2}+n_2}, \gamma_3, \ldots, \gamma_n, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)}, \ldots, \\ \beta^2_1, \ldots, \beta^2_q, \gamma_{n+1}, \beta^2_{q+2}, \ldots, \beta^2_{q+m_1}, \gamma_{n+2}, \beta^2_{q+m_1+2}, \ldots, \beta^2_{q+m_2}, \gamma_{n+3}, \ldots, \\ \gamma_{l'}, \beta^2_{q+m_{l(s^2)}+1}, \ldots, \beta^2_{2q+2m_{l(s^2)}-(l'-n)}, \beta^2_{2q+2m_{l(s^2)}-(l'-n)+1}, \ldots, \beta^2_{l(s^2)}, \\ \beta^0_1, \ldots, \beta^0_{D_0},\end{gathered} \tag{5.8}$$

where

$$\{\beta^1_1, \ldots, \beta^1_t, \beta^1_{t+1}, \ldots, \beta^1_{t+\frac{p-n}{2}}, \gamma_1, \beta^1_{t+\frac{p-n}{2}+2}, \ldots, \beta^1_{t+\frac{p-n}{2}+n_1}, \gamma_2,$$
$$\beta^1_{t+\frac{p-n}{2}+n_1+2} \ldots, \beta^1_{t+\frac{p-n}{2}+n_2}, \gamma_3, \ldots, \gamma_n, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)}\} = \Delta_{s^1},$$

$$\{\beta^1_{t+1}, \ldots, \beta^1_{t+\frac{p-n}{2}}, \gamma_1, \beta^1_{t+\frac{p-n}{2}+2}, \ldots, \beta^1_{t+\frac{p-n}{2}+n_1}, \gamma_2,$$
$$\beta^1_{t+\frac{p-n}{2}+n_1+2} \ldots, \beta^1_{t+\frac{p-n}{2}+n_2}, \gamma_3, \ldots, \gamma_n\} = \{\alpha \in \Delta_+ | s^1(\alpha) = -\alpha\},$$

$$\{\beta^2_1, \ldots, \beta^2_q, \gamma_{n+1}, \beta^2_{q+2}, \ldots, \beta^2_{q+m_1}, \gamma_{n+2}, \beta^2_{q+m_1+2}, \ldots, \beta^2_{q+m_2}, \gamma_{n+3}, \ldots,$$
$$\gamma_{l'}, \beta^2_{q+m_{l(s^2)}+1}, \ldots, \beta^2_{2q+2m_{l(s^2)}-(l'-n)}, \beta^2_{2q+2m_{l(s^2)}-(l'-n)+1}, \ldots, \beta^2_{l(s^2)}\} = \Delta_{s^2},$$

$$\{\gamma_{n+1}, \beta^2_{q+2}, \ldots, \beta^2_{q+m_1}, \gamma_{n+2}, \beta^2_{q+m_1+2}, \ldots, \beta^2_{q+m_2}, \gamma_{n+3}, \ldots,$$
$$\gamma_{l'}, \beta^2_{q+m_{l(s^2)}+1}, \ldots, \beta^2_{2q+2m_{l(s^2)}-(l'-n)}\} = \{\alpha \in \Delta_+ | s^2(\alpha) = -\alpha\},$$

$$\{\beta^0_1, \ldots, \beta^0_{D_0}\} = \Delta_0 = \{\alpha \in \Delta_+ | s(\alpha) = \alpha\},$$

and s^1, s^2 are the involutions entering decomposition (5.7), $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \ldots, \gamma_n$ and $\gamma_{n+1}, \ldots, \gamma_{l'}$ are positive and mutually orthogonal.

The length of the ordered segment $\Delta_{\mathfrak{m}_+} \subset \Delta$ in normal ordering (5.8),

$$\begin{gathered}\Delta_{\mathfrak{m}_+} = \gamma_1, \beta^1_{t+\frac{p-n}{2}+2}, \ldots, \beta^1_{t+\frac{p-n}{2}+n_1}, \gamma_2, \beta^1_{t+\frac{p-n}{2}+n_1+2} \ldots, \beta^1_{t+\frac{p-n}{2}+n_2}, \\ \gamma_3, \ldots, \gamma_n, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)}, \ldots, \beta^2_1, \ldots, \beta^2_q, \\ \gamma_{n+1}, \beta^2_{q+2}, \ldots, \beta^2_{q+m_1}, \gamma_{n+2}, \beta^2_{q+m_1+2}, \ldots, \beta^2_{q+m_2}, \gamma_{n+3}, \ldots, \gamma_{l'},\end{gathered} \tag{5.9}$$

is equal to

$$D - (\frac{l(s) - l'}{2} + D_0), \tag{5.10}$$

where D is the number of roots in Δ_+, $l(s)$ is the length of s and D_0 is the number of positive roots fixed by the action of s.

Moreover, for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^{q} c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$.

Proof. First observe that by Proposition 10.4.3. in [3] each of the planes $\mathfrak{h}_i$ is invariant not only with respect to the action of the Weyl group element s but also with respect to the action of both involutions s^1 and s^2 which act as reflections in the plane $\mathfrak{h}_i$.

Now consider the planes $\mathfrak{h}_{i_k}$ related to the definition of the set of positive roots. The plane $\mathfrak{h}_{i_k}$ is shown at Figure 1.

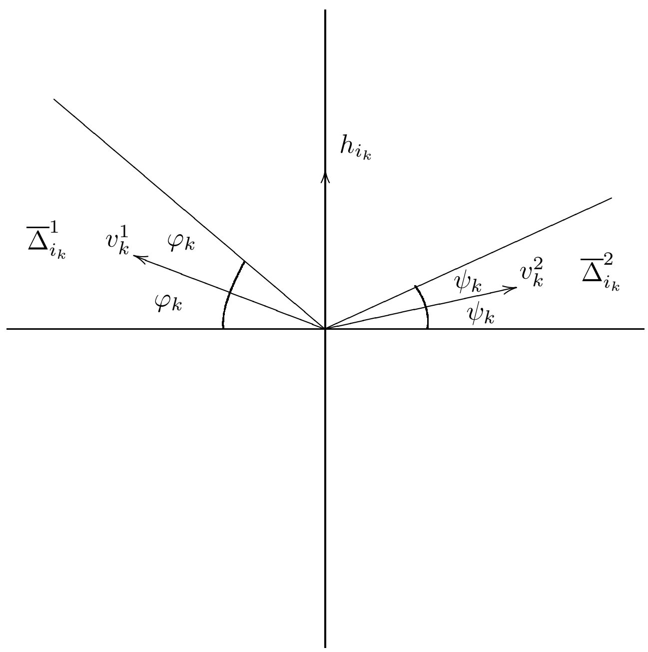


Fig.1

The vector h_{i_k} is directed upwards at the picture, and the orthogonal projections of elements from $(\overline{\Delta}_{i_k})_+$ onto $\mathfrak{h}_{i_k}$ are contained in the upper half plane. The involutions s^1 and s^2 act in $\mathfrak{h}_{i_k}$ as reflections with respect to the lines orthogonal to the vectors labeled by v^1_k and v^2_k, respectively, at Figure 1, the angle between v^1_k and v^2_k being equal to $\pi - \theta_{i_k}/2$. The nonzero projections of the roots from the set $\{\gamma_1, \ldots \gamma_n\} \bigcap \overline{\Delta}_{i_k}$ onto the plane $\mathfrak{h}_{i_k}$ have the same (or the opposite) direction as the vector v^1_k, and the nonzero projections of the roots from the set $\{\gamma_{n+1}, \ldots \gamma_{l'}\} \bigcap \overline{\Delta}_{i_k}$ onto the plane $\mathfrak{h}_{i_k}$ have the same (or the opposite) direction as the vector v^2_k.

For each of the involutions s^1 and s^2 we obviously have decompositions $\Delta_{s^{1,2}} = \bigcup_{k=1}^{M} \overline{\Delta}^{1,2}_{i_k}$, where $\overline{\Delta}^{1,2}_{i_k} = \overline{\Delta}_{i_k} \bigcap \Delta_{s^{1,2}}$. In the plane $\mathfrak{h}_{i_k}$, the elements from the sets $\overline{\Delta}^{1,2}_{i_k}$ are projected onto the interiors of the sectors labeled by $\overline{\Delta}^{1,2}_{i_k}$. Therefore the sets $\overline{\Delta}^1_{i_k}$ and $\overline{\Delta}^2_{i_k}$ have empty intersection, and hence the sets Δ_{s^1} and Δ_{s^2} have empty intersection as well. In particular, by the results of §3 in [36] the decomposition $s = s^1 s^2$ is reduced in the sense that $l(s) = l(s^2) + l(s^1)$, and $\Delta_s = \Delta_{s^2} \bigcup s^2(\Delta_{s^1})$ (disjoint union).

Recall that by the results of §3 in [36] for any element $w \in W$ one can always find a reduced decomposition $w_0 = ww'$, where w_0 is the longest element of the Weyl group and $w' \in W$ is some element of the Weyl group. This implies, in particular, that any normal ordering of the root system Δ_+ can be reduced by applying a number of elementary transpositions to one of the forms

$$\beta_1, \ldots, \beta_m, \ldots, \beta_D, \tag{5.11}$$

$$\beta_D, \ldots, \beta_m, \ldots, \beta_1, \tag{5.12}$$

where $\Delta_w = \{\beta_1, \ldots, \beta_m\}$.

Applying the last observation to the Weyl group element s^1 we obtain a normal ordering of the root system Δ_+ of the form

$$\beta_1^1, \ldots, \beta_{l(s^1)}^1, \ldots, \beta_D, \tag{5.13}$$

where $\Delta_{s^1} = \{\beta_1^1, \ldots, \beta_{l(s^1)}^1\}$. Recalling that $\Delta_{s^1} \bigcap \Delta_{s^2} = \{\emptyset\}$ and using (5.12) one can reduce normal ordering (5.13) by applying a number of elementary transpositions to the form

$$\beta_1^1, \ldots, \beta_{l(s^1)}^1, \ldots, \beta_1^2, \ldots, \beta_{l(s^2)}^2, \tag{5.14}$$

where $\Delta_{s^2} = \{\beta_1^2, \ldots, \beta_{l(s^2)}^2\}$.

Indeed, observe that the set Δ_{s^2} must contain at least one simple root since for any reduced decomposition $s^2 = s_{i_1} \ldots s_{i_k}$ we have $\Delta_{s^2} = \{\alpha_{i_k}, s_{i_k}\alpha_{i_{k-1}}, \ldots, s_{i_k} \ldots s_{i_2}\alpha_{i_1}\}$, and $\alpha_{i_k} \in \Delta_{s^2}$. Since $\Delta_{s^1} \cap \Delta_{s^2} = \{\emptyset\}$ we can move all the simple roots in Δ_{s^2} and their linear combinations to the right in normal ordering (5.13) using elementary transpositions. Denote the ordered segment which consists of such linear combinations by $\Delta_{s^2}^c = \{\beta_1^{2'}, \ldots, \beta_k^{2'}\}$. Thus we reduced (5.13) to the following form

$$\beta_1^1, \ldots, \beta_{l(s^1)}^1, \ldots, \beta_1^{2'}, \ldots, \beta_k^{2'}.$$

Now assume that $\alpha \in \Delta_{s^2},\ \alpha \notin \Delta_{s^2}^c$ is a root such that there are no other roots to the right from it in the normal ordering above which belong to Δ_{s^2} except for those from $\Delta_{s^2}^c$. Then α can be moved to the right by a number of elementary transpositions. Indeed, since if α belongs to one of segments (3.10) then by the choice of α either the entire segment belongs to Δ_{s^2} or the other end of the segment does not belong to Δ_{s^2}. In the latter case we can apply the elementary transposition to the segment to move α to the right. Applying this procedure several times we obtain a new normal ordering of the form

$$\beta_1^1, \ldots, \beta_{l(s^1)}^1, \ldots, \alpha, \beta_1^{2'}, \ldots, \beta_k^{2'}.$$

One can now proceed by induction to obtain normal ordering (5.14).

Now observe that since Δ_{i_0} is the root system of the Levi factor $\mathfrak{l}$ of the parabolic subalgebra $\mathfrak{p}$ the elements of Δ_{i_0} are linear combinations of roots from a subset $\Gamma_0 \subset \Gamma$. Therefore noting that $\Delta_{i_0} \bigcap (\Delta_{s^1} \bigcup \Delta_{s^2}) = \{\emptyset\}$ and applying elementary transpositions to normal ordering (5.13) one can reduce it to a form similar to (5.13) in which the roots from the closed subset $(\Delta_{i_0})_+ = \Delta_{i_0} \bigcap \Delta_+ = \{\beta_1^0, \ldots, \beta_{N_0}^0\}$ form a segment. Moreover, we claim that one can reduce normal ordering (5.14) to the following form

$$\beta_1^1, \ldots, \beta_{l(s^1)}^1, \ldots, \beta_1^0, \ldots, \beta_{D_0}^0, \beta_1^2, \ldots, \beta_{l(s^2)}^2. \tag{5.15}$$

In order to prove the last statement it suffices to verify that for any $\alpha \in (\Delta_{i_0})_+$ and $\beta \in \Delta_{s^2}$ such that $\alpha + \beta = \gamma \in \Delta_+$ we have $\gamma \in \Delta_{s^2}$. Indeed, if $s^2\gamma \in \Delta_+$ then $s^2(\alpha + \beta) = \alpha + s^2\beta = s^2\gamma \in \Delta_+$, and hence $\alpha = s^2\gamma + (-s^2\beta)$ with $s^2\gamma, -s^2\beta \in \Delta_+$ and $s^2\gamma, -s^2\beta \notin (\Delta_{i_0})_+$. This is impossible since Δ_{i_0} is the root system of the Levi factor $\mathfrak{l}$ of the parabolic subalgebra $\mathfrak{p}$.

Using the previous assertion and applying elementary transpositions one can also reduce normal ordering (5.15) to the form

$$\beta_1^1, \ldots, \beta_{l(s^1)}^1, \ldots, \beta_1^2, \ldots, \beta_{l(s^2)}^2, \beta_1^0, \ldots, \beta_{D_0}^0. \tag{5.16}$$

Now we look at the segments $\beta_1^1,\ldots,\beta_{l(s^1)}^1$ and $\beta_1^2,\ldots,\beta_{l(s^2)}^2$ of normal ordering (5.16). We consider the case of the segment $\beta_1^1,\ldots,\beta_{l(s^1)}^1$ in detail. The other segment is treated in a similar way.

Recall that by Theorem A in [22] every involution w in the Weyl group W is the longest element of the Weyl group of a Levi subalgebra in $\mathfrak{g}$, and w acts by multiplication by -1 at the Cartan subalgebra $\mathfrak{h}_w\subset\mathfrak{h}$ of the semisimple part $\mathfrak{m}_w$ of that Levi subalgebra. By Lemma 5 in [4] the involution w can also be expressed as a product of $\dim\mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w,\mathfrak{h}_w)$, with respect to mutually orthogonal roots. In case of the involution s^1, $s^1=s_{\gamma_1}\ldots s_{\gamma_n}$ is such an expression, and the roots $\gamma_1,\ldots,\gamma_n$ span the Cartan subalgebra $\mathfrak{h}_{s^1}$.

Since $\mathfrak{m}_{s^1}$ is the semisimple part of a Levi subalgebra, using elementary transpositions one can reduce the normal ordering of the segment $\beta_1^1,\ldots,\beta_{l(s^1)}^1$ to the form

$$\beta_1^1,\ldots,\beta_t^1,\beta_{t+1}^1,\ldots,\beta_{t+p}^1,\beta_{t+p+1},\ldots,\beta_{l(s^1)}^1, \tag{5.17}$$

where $\beta_{t+1}^1,\ldots,\beta_{t+p}^1$ is a normal ordering of the system $\Delta_+(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$ of positive roots of the pair $(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$. Now applying elementary transpositions we can reduce the ordering $\beta_{t+1}^1,\ldots,\beta_{t+p}^1$ to the form compatible with the decomposition $s^1=s_{\gamma_1}\ldots s_{\gamma_n}$ (see Appendix).

Applying similar arguments to the involution s^2 and using the normal ordering of the positive roots of the pair $(\mathfrak{m}_{s^2},\mathfrak{h}_{s^2})$ inverse to that compatible with the decomposition $s^2=s_{\gamma_{n+1}}\ldots s_{\gamma_{l'}}$ we finally obtain the following normal ordering of the set Δ_+

$$\begin{gathered}\beta_1^1,\ldots,\beta_t^1,\beta_{t+1}^1,\ldots,\beta_{t+\frac{p-n}{2}}^1,\gamma_1,\beta_{t+\frac{p-n}{2}+2}^1,\ldots,\beta_{t+\frac{p-n}{2}+n_1}^1,\gamma_2,\\ \beta_{t+\frac{p-n}{2}+n_1+2}^1\ldots,\beta_{t+\frac{p-n}{2}+n_2}^1,\gamma_3,\ldots,\gamma_n,\beta_{t+p+1},\ldots,\beta_{l(s^1)}^1,\ldots,\\ \beta_1^2,\ldots,\beta_q^2,\gamma_{n+1},\beta_{q+2}^2,\ldots,\beta_{q+m_1}^2,\gamma_{n+2},\beta_{q+m_1+2}^2,\ldots,\beta_{q+m_2}^2,\gamma_{n+3},\ldots,\\ \gamma_{l'},\beta_{q+m_{l(s^2)}+1}^2,\ldots,\beta_{2q+2m_{l(s^2)}-(l'-n)}^2,\beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2,\ldots,\beta_{l(s^2)}^2,\\ \beta_1^0,\ldots,\beta_{N_0}^0,\end{gathered} \tag{5.18}$$

where

$$\begin{gathered}\gamma_{n+1},\beta_{q+2}^2,\ldots,\beta_{q+m_1}^2,\gamma_{n+2},\beta_{q+m_1+2}^2,\ldots,\beta_{q+m_2}^2,\gamma_{n+3},\ldots,\\ \gamma_{l'},\beta_{q+m_{l(s^2)}+1}^2,\ldots,\beta_{2q+2m_{l(s^2)}-(l'-n)}^2\end{gathered}$$

is the normal ordering of the system of positive roots of the pair $(\mathfrak{m}_{s^2},\mathfrak{h}_{s^2})$ inverse to that compatible with the decomposition $s^2=s_{\gamma_{n+1}}\ldots s_{\gamma_{l'}}$. By construction ordering (5.18) is the required ordering (5.8).

We claim that $t=l(s^1)-(t+p)$, i.e. there are equal numbers of roots on the left and on the right from the segment $\beta_{t+1}^1,\ldots,\beta_{t+p}^1$ in the segment

$$\beta_1^1,\ldots,\beta_t^1,\beta_{t+1}^1,\ldots,\beta_{t+p}^1,\beta_{t+p+1},\ldots,\beta_{l(s^1)}^1,$$

and

$$t=\frac{l(s^1)-p}{2}. \tag{5.19}$$

Recall that by formula (3.5) in [36], given a reduced decomposition $w=s_{i_1}\ldots s_{i_m}$ of a Weyl group element w, one can represent w as a product of reflections with respect to the roots from the set

$$\Delta_{w^{-1}}=\{\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_m=s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_m}\},$$

$$w=s_{i_1}\ldots s_{i_m}=s_{\beta_m}\ldots s_{\beta_1} \tag{5.20}$$

Note that

$$\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_m=s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_m}$$

is the initial segment of a normal ordering of Δ_+.

Applying this observation to the segment of the normal ordering (5.17) consisting of elements from the set Δ_{s^1} one can represent $(s^1)^{-1}=s^1$ as follows

$$s^1=s_{\beta^1_{l(s^1)}}\ldots s_{\beta^1_{t+p+1}}s_{\beta^1_{t+p}}\ldots s_{\beta^1_{t+1}}s_{\beta^1_t}\ldots s_{\beta^1_1}, \tag{5.21}$$

where if $s^1=s_{i_1}\ldots s_{i_{l(s^1)}}$ is the corresponding reduced decomposition of s^1 then

$$\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_m=s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_{l(s^1)}}. \tag{5.22}$$

Since $\beta^1_{t+1},\ldots,\beta^1_{t+p}$ is a normal ordering of the system of positive roots of the pair $(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$ and s^1 is the longest element in the Weyl group of the pair $(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$ we also have

$$s^1=s_{\beta^1_{t+p}}\ldots s_{\beta^1_{t+1}}, \tag{5.23}$$

and hence by (5.21)

$$s^1=s_{\beta^1_{l(s^1)}}\ldots s_{\beta^1_{t+p+1}}s^1s_{\beta^1_t}\ldots s_{\beta^1_1}. \tag{5.24}$$

From the last formula we deduce that

$$s_{\beta^1_1}\ldots s_{\beta^1_t}=(s_{\beta^1_t}\ldots s_{\beta^1_1})^{-1}(s^1)^{-1}s_{\beta^1_{l(s^1)}}\ldots s_{\beta^1_{t+p+1}}s^1s_{\beta^1_t}\ldots s_{\beta^1_1}. \tag{5.25}$$

Now formula (5.20) implies that $s_{\beta^1_1}\ldots s_{\beta^1_t}=s_{i_t}\ldots s_{i_1}$, and relations (5.22) combined with (5.20) yield

$$u(s^1)^{-1}s_{\beta^1_{l(s^1)}}\ldots s_{\beta^1_{t+p+1}}s^1u^{-1}=s_{u(s^1)^{-1}(\beta^1_{l(s^1)})}\ldots s_{u(s^1)^{-1}(\beta^1_{t+p+1})}=$$
$$=s_{s_{i_{t+p+1}}\ldots s_{i_{l(s^1)-1}}\alpha_{i_{l(s^1)}}}\ldots s_{i_{t+p+1}}=s_{i_{t+p+1}}\ldots s_{i_{l(s^1)}},$$

where $u=s_{\beta^1_1}\ldots s_{\beta^1_t}=s_{i_t}\ldots s_{i_1}$. Therefore from formula (5.25) we deduce that

$$s_{i_t}\ldots s_{i_1}=s_{i_{t+p+1}}\ldots s_{i_{l(s^1)}}. \tag{5.26}$$

Since the decompositions in both sides of (5.26) are parts of reduced decompositions they are reduced as well, and we have $t=l(s^1)-(t+p)$. This is equivalent to formula (5.19).

Using a similar formula for the involution s^2 and recalling the definition of the orderings of positive roots of the pairs $(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$, $(\mathfrak{m}_{s^2},\mathfrak{h}_{s^2})$ compatible with decompositions $s^1=s_{\gamma_1}\ldots s_{\gamma_n}$ and $s^2=s_{\gamma_{n+1}}\ldots s_{\gamma_{l'}}$ (see Appendix) we deduce that the number of roots in the segment $\Delta_{\mathfrak{m}_+}$ of normal ordering (5.18),

$$\Delta_{\mathfrak{m}_+}=\gamma_1,\beta^1_{t+\frac{p-n}{2}+2},\ldots,\beta^1_{t+\frac{p-n}{2}+n_1},\gamma_2,\beta^1_{t+\frac{p-n}{2}+n_1+2}\ldots,\beta^1_{t+\frac{p-n}{2}+n_2},$$
$$\gamma_3,\ldots,\gamma_n,\beta_{t+p+1},\ldots,\beta^1_{l(s^1)},\ldots,\beta^2_1,\ldots,\beta^2_q,$$
$$\gamma_{n+1},\beta^2_{q+2},\ldots,\beta^2_{q+m_1},\gamma_{n+2},\beta^2_{q+m_1+2},\ldots,\beta^2_{q+m_2},\gamma_{n+3},\ldots,\gamma_{l'}$$

is equal to $D-(\frac{l(s)-l'}{2}+D_0)$, where $l(s)=l(s^1)+l(s^2)$ is the length of s and D_0 is the number of positive roots fixed by the action of s. This proves the second statement of the proposition.

Now let $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$, be any two roots such that $\alpha<\beta$. We shall show that the sum $\alpha+\beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k\gamma_{i_k}$, where $c_k\in\mathbb{N}$ and $\alpha<\gamma_{i_1}<\ldots<\gamma_{i_k}<\beta$.

Suppose that such a decomposition exists, $\alpha+\beta=\sum_{k=1}^q c_k\gamma_{i_k}$. Obviously at least one of the roots α,β must belong to the set $\Delta_+(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})\bigcap\Delta_{\mathfrak{m}_+}$ or to the set $\Delta_+(\mathfrak{m}_{s^2},\mathfrak{h}_{s^2})\bigcap\Delta_{\mathfrak{m}_+}$ for otherwise the set of roots γ_{i_k} such that $\alpha<\gamma_{i_k}<\beta$ is empty.

Suppose that $\alpha\in\Delta_+(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})\bigcap\Delta_{\mathfrak{m}_+}$. The other cases are considered in a similar way.

If $\beta \not\in \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \bigcap \Delta_{\mathfrak{m}_+}$ then $\alpha + \beta = \sum_{k=1}^{q} c_k \gamma_{i_k}$, and $\gamma_{i_k} \leq \gamma_n$. In particular, since $\alpha, \in \mathfrak{h}_{s^1}$ and $\gamma_{i_k} \in \mathfrak{h}_{s^1}$ if $\gamma_{i_k} \leq \gamma_n$, we have $\beta = \sum_{k=1}^{q} c_k \gamma_{i_k} - \alpha \in \mathfrak{h}_{s^1}$. This is impossible by the definition of the ordering of the set $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ compatible with the decomposition $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$.

If $\beta \in \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \bigcap \Delta_{\mathfrak{m}_+}$ then $\alpha + \beta = \sum_{k=1}^{q} c_k \gamma_{i_k} = \sum_{i_k \leq n} c_k \gamma_{i_k} + \sum_{i_k > n} c_k \gamma_{i_k}$. This implies

$$\alpha - \sum_{i_k \leq n} c_k \gamma_{i_k} = \sum_{i_k > n} c_k \gamma_{i_k} - \beta.$$

The l.h.s. of the last formula is an element of $\mathfrak{h}_{s^1}$ and the r.h.s. is an element $\mathfrak{h}_{s^2}$. Since $\mathfrak{h}' = \mathfrak{h}_{s^1} + \mathfrak{h}_{s^2}$ is a direct vector space decomposition we infer that

$$\alpha = \sum_{i_k \leq n, \alpha < \gamma_{i_k}} c_k \gamma_{i_k}$$

and

$$\beta = \sum_{i_k > n, \gamma_{i_k} < \beta} c_k \gamma_{i_k}.$$

This is impossible by the definition of the orderings of the sets $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ and $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ compatible with the decompositions $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$ and $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, respectively.

Therefore the sum $\alpha + \beta$, $\alpha < \beta$, $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ can not be represented as a linear combination $\sum_{k=1}^{q} c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$. This completes the proof of the proposition. □

We call the system of positive roots Δ_+ ordered as in (5.8) the normally ordered system of positive roots associated to the (conjugacy class) of the Weyl group element $s \in W$. We shall also need the circular ordering in the root system Δ corresponding to normal ordering (5.8) of the positive root system Δ_+.

Let $\beta_1, \beta_2, \ldots, \beta_N$ be a normal ordering of a positive root system Δ_+. Then following [16] one can introduce the corresponding circular normal ordering of the root system Δ where the roots in Δ are located on a circle in the following way

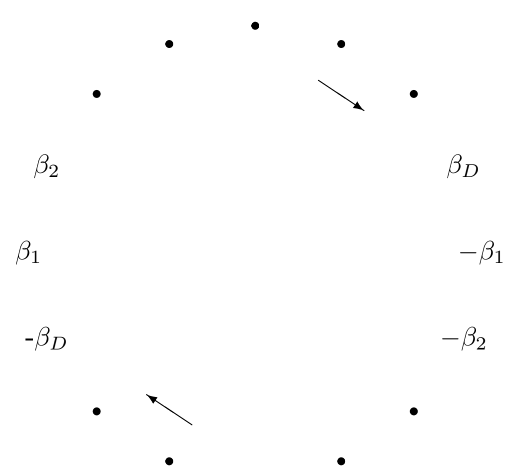


Fig.2

Let $\alpha, \beta \in \Delta$. One says that the segment $[\alpha, \beta]$ of the circle is minimal if it does not contain the opposite roots $-\alpha$ and $-\beta$ and the root β follows after α on the circle above, the circle being oriented clockwise. In that case one also says that $\alpha < \beta$ in the sense of the circular normal ordering,

$$\alpha < \beta \Leftrightarrow \text{the segment } [\alpha, \beta] \text{ of the circle is minimal.} \tag{5.27}$$

Later we shall need the following property of minimal segments which is a direct consequence of Proposition 3.3 in [15].

Lemma 5.2. *Let $[\alpha, \beta]$ be a minimal segment in a circular normal ordering of a root system Δ. Then if $\alpha + \beta$ is a root we have*

$$\alpha < \alpha + \beta < \beta.$$

6. Nilpotent subalgebras and quantum groups

Now we can define the subalgebras of $U_h(\mathfrak{g})$ which resemble nilpotent subalgebras in $\mathfrak{g}$ and possess nontrivial characters.

Theorem 6.1. *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Fix a decomposition (5.7) of s and let Δ_+ be the system of positive roots associated to s. Let $U_h^s(\mathfrak{g})$ be the realization of the quantum group $U_h(\mathfrak{g})$ associated to s. Let $e_\beta \in U_h^s(\mathfrak{n}_+)$, $\beta \in \Delta_+$ be the root vectors associated to the corresponding normal ordering (5.8) of Δ_+.*

Then elements $e_\beta \in U_h^s(\mathfrak{n}_+)$, $\beta \in \Delta_{\mathfrak{m}_+}$, where $\Delta_{\mathfrak{m}_+} \subset \Delta$ is ordered segment (5.9), generate a subalgebra $U_h^s(\mathfrak{m}_+) \subset U_h^s(\mathfrak{g})$ such that $U_h^s(\mathfrak{m}_+)/hU_h^s(\mathfrak{m}_+) \simeq U(\mathfrak{m}_+)$, where $\mathfrak{m}_+$ is the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors X_α, $\alpha \in \Delta_{\mathfrak{m}_+}$. The elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \ldots e_{\beta_D}^{r_D}$, $r_i \in \mathbb{N}$, $i = 1, \ldots D$ and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, form a topological basis of $U_h^s(\mathfrak{m}_+)$.

Moreover the map $\chi_h^s : U_h^s(\mathfrak{m}_+) \to \mathbb{C}[[h]]$ defined on generators by

$$\chi_h^s(e_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \ldots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, c_i \in \mathbb{C}[[h]] \end{cases} \tag{6.1}$$

is a character of $U_h^s(\mathfrak{m}_+)$.

Proof. The first statement of the theorem follows straightforwardly from commutation relations (4.6) and Proposition 4.2.

In order to prove that the map $\chi_h^s : U_h^s(\mathfrak{m}_+) \to \mathbb{C}[[h]]$ defined by (6.1) is a character of $U_h^s(\mathfrak{m}_+)$ we show that all relations (4.6) for e_α, e_β with $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$, which are obviously defining relations in the subalgebra $U_h^s(\mathfrak{m}_+)$, belong to the kernel of χ_h^s. By definition the only generators of $U_h^s(\mathfrak{m}_+)$ on which χ_h^s does not vanish are e_{γ_i}, $i = 1, \ldots, l'$. By the last statement in Proposition 5.1 for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$. Hence for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the value of the map χ_h^s on the l.h.s. of the corresponding commutation relation (4.6) is equal to zero.

Therefore it suffices to prove that

$$\chi_h^s(e_{\gamma_i} e_{\gamma_j} - q^{(\gamma_i, \gamma_j) + (\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j)} e_{\gamma_j} e_{\gamma_j}) = c_i c_j (1 - q^{(\gamma_i, \gamma_j) + (\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j)}) = 0, \; i < j.$$

The last identity holds provided $(\gamma_i, \gamma_j) + (\frac{1+s}{1-s} P_{\mathfrak{h}'}^* \gamma_i, \gamma_j) = 0$ for $i < j$. As we shall see in the next Lemma this is indeed the case.

Recall that $\gamma_1, \ldots, \gamma_{l'}$ form a basis of a subspace $\mathfrak{h}'^* \subset \mathfrak{h}^*$ on which s acts without fixed points. We shall study the matrix elements of the Cayley transform of the restriction of s to $\mathfrak{h}'^*$ with respect to this basis.

Lemma 6.2. *Let $P_{\mathfrak{h}'^*}$ be the orthogonal projection operator onto ${\mathfrak{h}'}^*$ in $\mathfrak{h}^*$, with respect to the Killing form. Then the matrix elements of the operator $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ in the basis $\gamma_1, \ldots, \gamma_{l'}$ are of the form:*

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i, \gamma_j\right) = \varepsilon_{ij}(\gamma_i, \gamma_j), \tag{6.2}$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}$$

Proof. (Compare with [1], Ch. V, §6, Ex. 3). First we calculate the matrix of the Coxeter element s with respect to the basis $\gamma_1, \ldots, \gamma_{l'}$. We obtain this matrix in the form of the Gauss decomposition of the operator $s : {\mathfrak{h}'}^* \to {\mathfrak{h}'}^*$.

Let $z_i = s\gamma_i$. Recall that $s_{\gamma_i}(\gamma_j) = \gamma_j - A_{ij}\gamma_i$, $A_{ij} = (\gamma_i^\vee, \gamma_j)$. Using this definition the elements z_i may be represented as:

$$z_i = y_i - \sum_{k \geq i} A_{ki} y_k,$$

where

$$y_i = s_{\gamma_1} \ldots s_{\gamma_{i-1}} \gamma_i. \tag{6.3}$$

Using the matrix notation we can rewrite the last formula as follows:

$$z_i = (I + V)_{ki} y_k, \quad \text{where } V_{ki} = \begin{cases} A_{ki} & k \geq i \\ 0 & k < i \end{cases} \tag{6.4}$$

To calculate the matrix of the operator $s : {\mathfrak{h}'}^* \to {\mathfrak{h}'}^*$ with respect to the basis $\gamma_1, \ldots, \gamma_{l'}$ we have to express the elements y_i via $\gamma_1, \ldots, \gamma_{l'}$. Applying the definition of reflections to (6.3) we can pull out the element γ_i to the right:

$$y_i = \gamma_i - \sum_{k < i} A_{ki} y_k.$$

Therefore

$$\gamma_i = (I + U)_{ki} y_k \ , \ \text{where } U_{ki} = \begin{cases} A_{ki} & k < i \\ 0 & k \geq i \end{cases}$$

Thus

$$y_k = (I + U)^{-1}_{jk} \gamma_j. \tag{6.5}$$

Summarizing (6.5) and (6.4) we obtain:

$$s\gamma_i = \left((I + U)^{-1}(I - V)\right)_{ki} \gamma_k. \tag{6.6}$$

This implies:

$$\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i = \left(\frac{2I + U - V}{U + V}\right)_{ki} \gamma_k. \tag{6.7}$$

Observe that $(U + V)_{ki} = A_{ki}$ and $(2I + U - V)_{ij} = -A_{ij}\varepsilon_{ij}$. Substituting these expressions into (6.7) we get :

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i, \gamma_j\right) = -(A^{-1})_{kp}\varepsilon_{pi}A_{pi}(\gamma_j, \gamma_k) = \varepsilon_{ij}(\gamma_i, \gamma_j). \tag{6.8}$$

This completes the proof of the lemma, and thus Theorem 6.1 is proved. □

□

The matrix A_{ij} is called the Carter matrix of s. We shall also use the Lie subalgebra $\mathfrak{m}_-$ of $\mathfrak{g}$ generated by the root vectors $X_{-\alpha}$, $\alpha \in \Delta_{\mathfrak{m}_+}$.

7. Some specializations of the algebra $U_h^s(\mathfrak{g})$

In this section we introduce some forms of the quantum group $U_h^s(\mathfrak{g})$ which are similar to the rational form, the restricted integral form and to its specialization for the standard quantum group $U_h(\mathfrak{g})$. The motivations of the definitions given below will be clear in Section 10. The results in this section are slight modifications of similar statements for $U_h(\mathfrak{g})$, and we refer to [5], Ch. 9 for the proofs.

We start with the observation that the numbers

$$p_{ij} = \left(\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i, Y_j\right) + (Y_i, Y_j) \tag{7.9}$$

are rational, $p_{ij} \in \mathbb{Q}$.

Indeed, let γ_i^*, $i = 1, \ldots, l'$ be the basis of $\mathfrak{h}'^*$ dual to γ_i, $i = 1, \ldots, l'$ with respect to the restriction of the bilinear form $(\cdot,\cdot)$ to $\mathfrak{h}'^*$. Since the numbers (γ_i, γ_j) are integer each element γ_i^* has the form $\gamma_i^* = \sum_{j=1}^{l'} m_{ij}\gamma_j$, where $m_{ij} \in \mathbb{Q}$. Now we have

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i, Y_j\right) + (Y_i, Y_j) =$$
$$= \sum_{k,l,p,q=1}^{l'} \gamma_k(Y_i)\gamma_l(Y_j)\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_p, \gamma_q\right) m_{kp}m_{lq} + (Y_i, Y_j).$$

All the terms in the r.h.s. of the last identity are rational since $\gamma_i(Y_j) \in \mathbb{Z}$ for any $i = 1, \ldots l'$ and $j = 1, \ldots, l$ because Y_i are the fundamental weights, the numbers $\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_p, \gamma_q\right)$ are integer by Lemma 6.2, the coefficients m_{ij} are rational as we observed above, and the scalar products (Y_i, Y_j) of the fundamental weights are rational. Therefore the numbers p_{ij} are rational.

Denote by d the smallest integer number divisible by all the denominators of the rational numbers $p_{ij}/2$, $i, j = 1, \ldots, l$.

Let $U_q^s(\mathfrak{g})$ be the $\mathbb{C}(q^{\frac{1}{2d}})$-subalgebra of $U_h^s(\mathfrak{g})$ generated by the elements $e_i, f_i, t_i^{\pm 1} = \exp(\pm\frac{h}{2d}H_i)$, $i = 1, \ldots l$.

The defining relations for the algebra $U_q^s(\mathfrak{g})$ are

$$\begin{gathered}
t_it_j = t_jt_i,\ t_it_i^{-1} = t_i^{-1}t_i = 1,\ t_ie_jt_i^{-1} = q^{\frac{a_{ij}}{2d}}e_j,\ t_if_jt_i^{-1} = q^{-\frac{a_{ij}}{2d}}f_j, \\
e_if_j - q^{c_{ij}}f_je_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},\ c_{ij} = \left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right) \\
K_i = t_i^{2dd_i}, \\
\sum_{r=0}^{1-a_{ij}}(-1)^r q^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (e_i)^{1-a_{ij}-r}e_j(e_i)^r = 0,\ i \neq j, \\
\sum_{r=0}^{1-a_{ij}}(-1)^r q^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (f_i)^{1-a_{ij}-r}f_j(f_i)^r = 0,\ i \neq j.
\end{gathered} \tag{7.10}$$

Note that by the choice of d we have $q^{c_{ij}} \in \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$.

The second form of $U_h^s(\mathfrak{g})$ is the $\mathcal{A}=\mathbb{C}[q^{\frac{1}{2d}},q^{-\frac{1}{2d}}]$–subalgebra $U_{\mathcal{A}}^s(\mathfrak{g})$ in $U_q^s(\mathfrak{g})$ generated by the elements $t_i^{\pm 1},\ (e_i)^{(r)}=\frac{(e_i)^r}{[r]_{q_i}!},\ (f_i)^{(r)}=\frac{(f_i)^r}{[r]_{q_i}!},\ i=1,\ldots,l,\ r\geq 1$. $U_{\mathcal{A}}^s(\mathfrak{g})$ is called the restricted form of $U_q^s(\mathfrak{g})$.

The most important for us is the specialization $U_\varepsilon^s(\mathfrak{g})$ of $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_\varepsilon^s(\mathfrak{g})=U_{\mathcal{A}}^s(\mathfrak{g})/(q^{\frac{1}{2d}}-\varepsilon^{\frac{1}{2d}})U_{\mathcal{A}}^s(\mathfrak{g})$, $\varepsilon\in\mathbb{C}^*$.

$U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_\varepsilon^s(\mathfrak{g})$ are Hopf algebras with the comultiplication induced from $U_h^s(\mathfrak{g})$.

If $\varepsilon^{2d_i}\neq 1$ for $i=1,\ldots,l$ then $U_\varepsilon^s(\mathfrak{g})$ is generated over $\mathbb{C}$ by $t_i^{\pm 1},\ e_i,\ f_i,\ i=1,\ldots,l$ subject to relations (7.10) where $q=\varepsilon$.

The algebra $U_{\mathcal{A}}^s(\mathfrak{g})$ is more difficult to describe. But we do not need its explicit description in this paper.

The elements t_i are central in the algebra $U_1^s(\mathfrak{g})$, and the quotient of $U_1^s(\mathfrak{g})$ by the two–sided ideal generated by t_i-1 is isomorphic to $U(\mathfrak{g})$.

If V is a $U_q^s(\mathfrak{g})$–module then its weight spaces are all the non–zero $\mathbb{C}(q^{\frac{1}{2d}})$–linear subspaces of the form

$$V_{\mathbf{c}}=\{v\in V, t_iv=c_iv,\ c_i\in\mathbb{C}(q^{\frac{1}{2d}})^*,\ i=1,\ldots,l.$$

The l-tuple $\mathbf{c}=(c_1,\ldots,c_l)\in(\mathbb{C}(q^{\frac{1}{2d}})^*)^l$ is called a weight.

If $\mathbf{c}'=(c_1',\ldots,c_l')$ is another weight one says that $\mathbf{c}'\leq\mathbf{c}$ if $c_i'c_i=q^{\frac{1}{2d}\beta(H_i)}$ for some $\beta\in Q^+=\bigoplus_{i=1}^l\mathbb{N}\alpha_i$ and all $i=1,\ldots,l$.

A highest weight $U_q^s(\mathfrak{g})$–module is a $U_q^s(\mathfrak{g})$–module V which contains a weight vector $v\in V_{\mathbf{c}}$ annihilated by the action of all elements e_i and such that $V=U_q^s(\mathfrak{g})v$. In that case we also have a weight space decomposition

$$V=\bigoplus_{\mathbf{c}'\leq\mathbf{c}}V_{\mathbf{c}'},$$

and $\dim_{\mathbb{C}(q^{\frac{1}{2d}})}V_{\mathbf{c}}=1$. In particular, $\mathbf{c}$ is uniquely defined by V. It is called the highest weight of V, and v is called the highest weight vector.

Verma and finite–dimensional irreducible $U_q^s(\mathfrak{g})$–modules are defined in the usual way. For instance, the Verma module $M_q(\lambda)$ corresponding to a highest weight $\lambda\in\mathfrak{h}^*$ is the quotient of $U_q^s(\mathfrak{g})$ by the right ideal generated by e_i and $t_i-q^{\frac{1}{2d}\lambda(H_i)}$, where $i=1,\ldots,l$.

The image of 1 in $M_q(\lambda)$ is the highest weight vector v_λ in $M_q(\lambda)$. For $\lambda\in P_+=\{\mu\in\mathfrak{h}^*,\mu(H_i)\in\mathbb{N}$ for all $i\}$ the unique irreducible quotient $V_q(\lambda)$ of $M_q(\lambda)$ is a finite–dimensional irreducible representation of $U_q^s(\mathfrak{g})$.

If V is a highest weight $U_q(\mathfrak{g})$–module with highest weight vector v then $V_{\mathcal{A}}=U_{\mathcal{A}}^s(\mathfrak{g})v$ is a $U_{\mathcal{A}}^s$–submodule of V which has weight decomposition induced by that of V.

Moreover, $V_{\mathcal{A}}\otimes_{\mathcal{A}}\mathbb{C}(q^{\frac{1}{2d}})\simeq V$, $V_{\mathcal{A}}$ is the direct sum of its intersections with the weight spaces of V, each such intersection is a free $\mathcal{A}$–module of finite rank, and $\overline{V}=V_{\mathcal{A}}/(q^{\frac{1}{2d}}-1)V_{\mathcal{A}}$ is naturally a $U(\mathfrak{g})$-module. In particular for $\lambda\in P_+$ $M(\lambda)=\overline{M}_q(\lambda)$ and $V(\lambda)=\overline{V}_q(\lambda)$ are the Verma and the finite–dimensional irreducible $U(\mathfrak{g})$–modules with highest weight λ.

For Verma and finite–dimensional representations every nonzero weight subspace have weight of the form $(q^{\frac{1}{2d}\lambda(H_1)},\ldots,q^{\frac{1}{2d}\lambda(H_l)})$, where $\lambda\in P=\{\mu\in\mathfrak{h}^*,\mu(H_i)\in\mathbb{Z}$ for all $i\}$. One simply calls such a subspace a subspace of weight λ.

Similarly one can define highest weight, Verma and finite–dimensional $U_\varepsilon^s(\mathfrak{g})$–modules in case when ε is transcendental; one should just replace q with ε in the definitions above for the algebra $U_q^s(\mathfrak{g})$.

For the solution $n_{ji}=\frac{1}{2}c_{ij}$ to equations (4.2) the root vectors e_β,f_β belong to all the above introduced subalgebras of $U_h(\mathfrak{g})$, and one can define Poincaré–Birkhoff–Witt bases for them in a similar way. From now on we shall assume that the solution to equations (4.2) is fixed as above, $n_{ji} = \frac{1}{2}c_{ij}$.

If we define for $\mathbf{s} = (s_1, \ldots s_l) \in \mathbb{Z}^l$

$$t^{\mathbf{s}} = t_1^{s_1} \ldots t_l^{s_l}$$

and denote by $U_q^s(\mathfrak{n}_+), U_q^s(\mathfrak{n}_-)$ and $U_q^s(\mathfrak{h})$ the subalgebras of $U_q^s(\mathfrak{g})$ generated by the e_i, f_i and by the t_i, respectively, then the elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \ldots e_{\beta_D}^{r_D}$, $f^{\mathbf{t}} = f_{\beta_D}^{t_D} \ldots f_{\beta_1}^{t_1}$ and $t^{\mathbf{s}}$, for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$, $\mathbf{s} \in \mathbb{Z}^l$, form bases of $U_q^s(\mathfrak{n}_+), U_q^s(\mathfrak{n}_-)$ and $U_q^s(\mathfrak{h})$, respectively, and the products $e^{\mathbf{r}}t^{\mathbf{s}}f^{\mathbf{t}}$ form a basis of $U_q^s(\mathfrak{g})$. In particular, multiplication defines an isomorphism:

$$U_q^s(\mathfrak{n}_-) \otimes U_q^s(\mathfrak{h}) \otimes U_q^s(\mathfrak{n}_+) \to U_q^s(\mathfrak{g}).$$

By specializing the above constructed basis for $q = \varepsilon$ we obtain a similar basis for $U_\varepsilon^s(\mathfrak{g})$.

One should also note that the algebra $U_{\mathcal{A}}^s(\mathfrak{g})$ is invariant under the action of automorphisms T_i, and one can define the powers of the root vectors in $U_{\mathcal{A}}^s(\mathfrak{g})$ similarly to (3.11),

$$(e_{\beta_k})^{(r)} = T_{i_1} \ldots T_{i_{k-1}}(e_{i_k})^{(r)} \in U_{\mathcal{A}}^s(\mathfrak{g}), (f_{\beta_k})^{(r)} = T_{i_1} \ldots T_{i_{k-1}}(f_{i_k})^{(r)} \in U_{\mathcal{A}}^s(\mathfrak{g}).$$

Let $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$ be the subalgebras of $U_{\mathcal{A}}^s(\mathfrak{g})$ generated by the $(e_i)^{(r)}$ and by the $(f_i)^{(r)}$, $i = 1, \ldots, l$, $r \in \mathbb{N}$, respectively. Using the root vectors $(e_\beta)^{(r)}$ and $(f_\beta)^{(r)}$ we can construct a basis of $U_{\mathcal{A}}^s(\mathfrak{g})$. Define for $\mathbf{r} = (r_1, \ldots, r_D) \in \mathbb{N}^D$,

$$(e)^{(\mathbf{r})} = (e_{\beta_1})^{(r_1)} \ldots (e_{\beta_D})^{(r_D)},$$

$$(f)^{(\mathbf{r})} = (f_{\beta_D})^{(r_D)} \ldots (f_{\beta_1})^{(r_1)},$$

The elements $(e)^{(\mathbf{r})}$, $(f)^{(\mathbf{t})}$ for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^N$ form bases of $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$, respectively.

The elements

$$\left[\begin{array}{l} K_i; c \\ r \end{array} \right]_{q_i} = \prod_{s=1}^{r} \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}} \ , \ i = 1, \ldots, l, \ c \in \mathbb{Z}, \ r \in \mathbb{N}$$

belong to $U_{\mathcal{A}}^s(\mathfrak{g})$. Denote by $U_{\mathcal{A}}^s(\mathfrak{h})$ the subalgebra of $U_{\mathcal{A}}^s(\mathfrak{g})$ generated by those elements and by $t_i^{\pm 1}$, $i = 1, \ldots, l$.

Then multiplication defines an isomorphism of $\mathcal{A}$ modules:

$$U_{\mathcal{A}}^s(\mathfrak{n}_-) \otimes U_{\mathcal{A}}^s(\mathfrak{h}) \otimes U_{\mathcal{A}}^s(\mathfrak{n}_+) \to U_{\mathcal{A}}^s(\mathfrak{g}).$$

A basis for $U_{\mathcal{A}}^s(\mathfrak{h})$ is a little bit more difficult to describe. We do not need its explicit description (see [5], Proposition 9.3.3 for details).

None of the subalgebras of $U_h^s(\mathfrak{g})$ introduced above is quasitriangular. However, one can define an action of R–matrix (4.10) in the finite–dimensional representations of $U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_\varepsilon^s(\mathfrak{g})$. Indeed, observe that one can write R–matrix (4.10) in the factorized form

$$\mathcal{R}^s = \mathcal{E}\widetilde{\mathcal{R}}, \tag{7.11}$$

where

$$\mathcal{E} = exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) + \sum_{i=1}^{l}\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i) \right]$$

and

$$\widetilde{\mathcal{R}} = \sum_{u_1, \ldots, u_D = 0}^{\infty} \prod_{r=1}^{D} q_{\beta_r}^{\frac{1}{2}u_r(u_r+1)}(1 - q_{\beta_r}^{-2})^{u_r} e_{\beta_r}^{u_r} \otimes e^{u_r h \frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee}(f_{\beta_r})^{(u_r)},$$

where the order of the factors in the product is such that the β_r–term appears to the right of the β_s–term if $r > s$.

Using the fact that the numbers p_{ij} defined by (7.9) are of the form $p_{ij} = \frac{2v_{ij}}{d}$, $v_{ij} \in \mathbb{Z}$ one can check that actually $e^{u_r h \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee} \in U_{\mathcal{A}}^s(\mathfrak{g})$. Therefore $e_\beta^{u_r} \otimes e^{u_r h \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee}(f_\beta)^{(u_r)} \in U_{\mathcal{A}}^s(\mathfrak{g}) \otimes U_{\mathcal{A}}^s(\mathfrak{g})$.

For every two finite–dimensional $U_{\mathcal{A}}^s(\mathfrak{g})$-modules V and W only finitely many terms in the expression for $\widetilde{\mathcal{R}}$ act nontrivially on $V \otimes W$ since the action of root vectors on V and W is nilpotent. Therefore the action of the element $\widetilde{\mathcal{R}}$ in the space $V \otimes W$ is well defined.

Moreover, if V_μ and W_λ are two weight subspaces of V and W of weights $\mu, \lambda \in P$ then one can define an action of $\mathcal{E}$ in $V_\mu \otimes W_\lambda$ as multiplication by the scalar $q^{(\lambda,\mu)+(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \lambda, \mu)}$. Since the numbers p_{ij} defined by (7.9) are of the form $p_{ij} = \frac{2v_{ij}}{d}$ this scalar is an element of $\mathcal{A} = \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$.

If we define an action of the element $\mathcal{R}^s$ in $V \otimes W$ as the composition of the above defined action of the operators $\mathcal{E}$ and $\widetilde{\mathcal{R}}$ in $V \otimes W$ and denote the obtained operator by $R^{V,W}$ then one can check that

$$R^{V,W}(\pi_V \otimes \pi_W)\Delta_s(x) {R^{V,W}}^{-1} = (\pi_W \otimes \pi_V)\Delta_s^{opp}(x),$$

where π_V, π_W are the representations V and W and Δ_s is the comultiplication on $U_{\mathcal{A}}^s(\mathfrak{g})$. Moreover, $R^{V,W}$ satisfies the quantum Yang-Baxter equation.

By specializing q to a particular value $q = \varepsilon$ one can obtain an operator with similar properties acting in the tensor product of any two finite–dimensional $U_\varepsilon^s(\mathfrak{g})$-modules. Obviously, the above construction can be applied in case of the algebra $U_q^s(\mathfrak{g})$ as well.

Finally we discuss an obvious analogue of the subalgebra $U_h^s(\mathfrak{m}_+) \subset U_h^s(\mathfrak{g})$ for $U_{\mathcal{A}}^s(\mathfrak{g})$.

Let $U_{\mathcal{A}}^s(\mathfrak{m}_+) \subset U_{\mathcal{A}}^s(\mathfrak{g})$ be the subalgebra generated by elements $e_\beta \in U_{\mathcal{A}}^s(\mathfrak{n}_+)$, $\beta \in \Delta_{\mathfrak{m}_+}$, where $\Delta_{\mathfrak{m}_+} \subset \Delta$ is the ordered segment (5.9). Obviously, the defining relations in the subalgebra $U_{\mathcal{A}}^s(\mathfrak{m}_+)$ are given by formula (4.6),

$$e_\alpha e_\beta - q^{(\alpha,\beta)+(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha, \beta)} e_\beta e_\alpha = \sum_{\alpha < \gamma_1 < \ldots < \gamma_n < \beta} C'(k_1, \ldots, k_n) e_{\gamma_1}^{k_1} e_{\gamma_2}^{k_2} \ldots e_{\gamma_n}^{k_n}, \tag{7.12}$$

where $C'(k_1, \ldots, k_n) \in \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$.

The elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \ldots e_{\beta_D}^{r_D}$, $r_i \in \mathbb{N}$, $i = 1, \ldots D$, and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, form a basis of $U_{\mathcal{A}}^s(\mathfrak{m}_+)$.

Obviously $U_{\mathcal{A}}^s(\mathfrak{m}_+)/(q^{\frac{1}{2d}} - 1)U_{\mathcal{A}}^s(\mathfrak{m}_+) \simeq U(\mathfrak{m}_+)$, where $\mathfrak{m}_+$ is the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors X_α, $\alpha \in \Delta_{\mathfrak{m}_+}$.

Moreover, the map $\chi_{\mathcal{A}}^s : U_{\mathcal{A}}^s(\mathfrak{m}_+) \to \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$ defined on generators by

$$\chi_{\mathcal{A}}^s(e_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \ldots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, c_i \in \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}] \end{cases}$$

is a character of $U_{\mathcal{A}}^s(\mathfrak{m}_+)$.

By specializing q to a particular value $q = \varepsilon$ one can obtain a subalgebra $U_\varepsilon^s(\mathfrak{m}_+) \subset U_\varepsilon^s(\mathfrak{g})$ with similar properties.

8. Poisson–Lie groups

In this section we recall some notions concerned with Poisson–Lie groups (see [5], [9], [21], [25]). These facts will be used in Section 10 to define q-W–algebras.

Let G be a finite–dimensional Lie group equipped with a Poisson bracket, $\mathfrak{g}$ its Lie algebra. G is called a Poisson–Lie group if the multiplication $G \times G \to G$ is a Poisson map. A Poisson bracket satisfying this axiom is degenerate and, in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element defines the structure of a Lie algebra in the space $T_e^* G \simeq \mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called the tangent bialgebra of G.

Lie brackets in $\mathfrak{g}$ and $\mathfrak{g}^*$ satisfy the following compatibility condition:

Let $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be the dual of the commutator map $[,]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$. Then δ is a 1-cocycle on $\mathfrak{g}$ (with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$).

Let c_{ij}^k, f_c^{ab} be the structure constants of $\mathfrak{g}, \mathfrak{g}^*$ with respect to the dual bases $\{e_i\}, \{e^i\}$ in $\mathfrak{g}, \mathfrak{g}^*$. The compatibility condition means that

$$c_{ab}^s f_s^{ik} \; - \; c_{as}^i f_b^{sk} \; + \; c_{as}^k f_b^{si} \; - \; c_{bs}^k f_a^{si} \; + \; c_{bs}^i f_a^{sk} \; = \; 0.$$

This condition is symmetric with respect to exchange of c and f. Thus if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra.

The following proposition shows that the category of finite–dimensional Lie bialgebras is isomorphic to the category of finite–dimensional connected simply connected Poisson–Lie groups.

Proposition 8.1. **([5], Theorem 1.3.2)** *If G is a connected simply connected finite–dimensional Lie group, every bialgebra structure on $\mathfrak{g}$ is the tangent bialgebra of a unique Poisson structure on G which makes G into a Poisson–Lie group.*

Let G be a finite–dimensional Poisson–Lie group, $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent bialgebra of G. The connected simply connected finite–dimensional Poisson–Lie group corresponding to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called the dual Poisson–Lie group and denoted by G^*.

$(\mathfrak{g}, \mathfrak{g}^*)$ is called a factorizable Lie bialgebra if the following conditions are satisfied (see [9], [21]):

(1) $\mathfrak{g}$ *is equipped with a non–degenerate invariant scalar product* $(\cdot, \cdot)$.

We shall always identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by means of this scalar product.

(2) *The dual Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by*

$$[X, Y]_* = \frac{1}{2}\left([rX, Y] + [X, rY]\right), X, Y \in \mathfrak{g}, \tag{8.1}$$

where $r \in \mathrm{End}\ \mathfrak{g}$ is a skew symmetric linear operator (classical r-matrix).

(3) *r satisfies the modified classical Yang-Baxter identity:*

$$[rX, rY] - r\left([rX, Y] + [X, rY]\right) = -\left[X, Y\right],\ X, Y \in \mathfrak{g}. \tag{8.2}$$

Define operators $r_\pm \in \mathrm{End}\ \mathfrak{g}$ by

$$r_\pm = \frac{1}{2}\left(r \pm id\right).$$

We shall need some properties of the operators $r_\pm$. Denote by $\mathfrak{b}_\pm$ and $\mathfrak{n}_\mp$ the image and the kernel of the operator $r_\pm$:

$$\mathfrak{b}_\pm = Im\ r_\pm,\ \mathfrak{n}_\mp = Ker\ r_\pm. \tag{8.3}$$

Proposition 8.2. **([2], [23])** *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then*

(i) $\mathfrak{b}_\pm \subset \mathfrak{g}$ is a Lie subalgebra, the subspace $\mathfrak{n}_\pm$ is a Lie ideal in $\mathfrak{b}_\pm$, $\mathfrak{b}_\pm^\perp = \mathfrak{n}_\pm$.

(ii) $\mathfrak{n}_\pm$ is an ideal in $\mathfrak{g}^*$.

(iii) $\mathfrak{b}_\pm$ is a Lie subalgebra in $\mathfrak{g}^*$. Moreover $\mathfrak{b}_\pm = \mathfrak{g}^*/\mathfrak{n}_\pm$.

(iv) $(\mathfrak{b}_\pm, \mathfrak{b}_\pm^*)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{b}_\pm, \mathfrak{b}_\pm^*) \simeq (\mathfrak{b}_\pm, \mathfrak{b}_\mp)$. The canonical paring between $\mathfrak{b}_\mp$ and $\mathfrak{b}_\pm$is given by

$$(X_\mp, Y_\pm)_\pm = (X_\mp, r_\pm^{-1} Y_\pm),\ X_\mp \in \mathfrak{b}_\mp;\ Y_\pm \in \mathfrak{b}_\pm. \tag{8.4}$$

The classical Yang–Baxter equation implies that $r_\pm$, regarded as a mapping from $\mathfrak{g}^*$ into $\mathfrak{g}$, is a Lie algebra homomorphism. Moreover, $r_+^* = -r_-$, and $r_+ - r_- = id$.

Put $\mathfrak{d} = \mathfrak{g} \oplus g$ (direct sum of two copies). The mapping

$$\mathfrak{g}^* \to \mathfrak{d}\ : X \mapsto (X_+,\ X_-),\ X_\pm \ = \ r_\pm X \tag{8.5}$$

is a Lie algebra embedding. Thus we may identify $\mathfrak{g}^*$ with a Lie subalgebra in $\mathfrak{d}$.

Naturally, embedding (8.5) extends to a homomorphism

$$G^* \to G \times G, \ L \mapsto (L_+, L_-).$$

We shall identify G^* with the corresponding subgroup in $G \times G$.

9. Poisson reduction

In this section we recall basic facts on Poisson reduction (see [34], [24]).

Let $M, \ B, \ B'$ be Poisson manifolds. Two Poisson surjections

$$\begin{array}{ccccc} & & M & & \\ & \overset{\pi'}{\swarrow} & & \overset{\pi}{\searrow} & \\ B' & & & & B \end{array}$$

form a dual pair if the pullback ${\pi'}^* C^\infty(B')$ is the centralizer of $\pi^* C^\infty(B)$ in the Poisson algebra $C^\infty(M)$. In that case the sets $B'_b = \pi'\left(\pi^{-1}(b)\right), \ b \in B$ are Poisson submanifolds in B' (see [34]) called reduced Poisson manifolds.

Fix an element $b \in B$. Then the algebra of functions $C^\infty(B'_b)$ may be described as follows. Let I_b be the ideal in $C^\infty(M)$ generated by elements $\pi^*(f), \ f \in C^\infty(B), \ f(b) = 0$. Denote $M_b = \pi^{-1}(b)$. Then the algebra $C^\infty(M_b)$ is simply the quotient of $C^\infty(M)$ by I_b. Denote by $P_b : C^\infty(M) \to C^\infty(M)/I_b = C^\infty(M_b)$ the canonical projection onto the quotient.

Lemma 9.1. *Suppose that the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$. Then one can define an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$ by*

$$f \cdot \varphi = P_b(\{\pi^*(f), \tilde{\varphi}\}), \tag{9.1}$$

where $f \in C^\infty(B)$, $\varphi \in C^\infty(M_b)$ and $\tilde{\varphi} \in C^\infty(M)$ is a representative of φ in $C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Moreover, $C^\infty(B'_b)$ is the subspace of invariants in $C^\infty(M_b)$ with respect to this action.

Proof. Let $\varphi \in C^\infty(M_b)$. Choose a representative $\tilde{\varphi} \in C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Since the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, Hamiltonian vector fields of functions $\pi^*(f), \ f \in C^\infty(B)$ are tangent to the surface M_b. Therefore the r.h.s. of (9.1) only depends on φ but not on the representative $\tilde{\varphi}$, and hence formula (9.1) defines an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$.

Using the definition of the dual pair we obtain that $\varphi = {\pi'}^*(\psi)$ for some $\psi \in C^\infty(B'_b)$ if and only if $P_b(\{\pi^*(f), \tilde{\varphi}\}) = 0$ for every $f \in C^\infty(B)$.

Finally we obtain that $C^\infty(B'_b)$ is exactly the subspace of invariants in $C^\infty(M_b)$ with respect to action (9.1). □

Definition 9.1. *The algebra $C^\infty(B'_b)$ is called a reduced Poisson algebra. We also denote it by $C^\infty(M_b)^{C^\infty(B)}$.*

Remark 9.4. *Note that the description of the algebra $C^\infty(M_b)^{C^\infty(B)}$ obtained in Lemma 9.1 is independent of both the manifold B' and the projection π'. Observe also that the reduced space B'_b may be identified with a cross–section of the action of the Poisson algebra $C^\infty(B)$ on M_b by Hamiltonian vector fields in case when this action is free. In particular, in that case B'_b may be regarded as a submanifold in M_b.*

An important example of dual pairs is provided by Poisson group actions. Recall that a (local) Poisson group action of a Poisson–Lie group A on a Poisson manifold M is a (local) group action $A \times M \to M$ which is also a Poisson map (as usual, we suppose that $A \times M$ is equipped with the product Poisson structure).

In [24] it is proved that if the space M/A is a smooth manifold, there exists a unique Poisson structure on M/A such that the canonical projection $M \to M/A$ is a Poisson map.

Let $\mathfrak{a}$ be the Lie algebra of A. Denote by $\langle \cdot, \cdot \rangle$ the canonical paring between $\mathfrak{a}^*$ and $\mathfrak{a}$. A map $\mu : M \to A^*$ is called a moment map for a (local) right Poisson group action $A \times M \to M$ if (see [18])

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X \rangle (\xi_\varphi), \tag{9.2}$$

where θ_{A^*} is the universal right–invariant Maurer–Cartan form on A^*, $X \in \mathfrak{a}$, $\widehat{X}$ is the corresponding vector field on M and ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$.

By Theorem 4.9 in [18] one can always equip A^* with a Poisson structure in such a way that μ becomes a Poisson mapping. From the definition of the moment map it follows that if M/A is a smooth manifold then the canonical projection $M \to M/A$ and the moment map $\mu : M \to A^*$ form a dual pair (see [18] for details).

The main example of Poisson group actions is the so–called dressing action. The dressing action may be described as follows (see [18], [24]).

Proposition 9.2. *Let G be a connected simply connected Poisson–Lie group with factorizable tangent Lie bialgebra, G^* the dual group. Then there exists a unique right local Poisson group action*

$$G^* \times G \to G^*, \ ((L_+, L_-), g) \mapsto g \circ (L_+, L_-),$$

such that the identity mapping $\mu : G^* \to G^*$ is the moment map for this action.

Moreover, let $q : G^* \to G$ be the map defined by

$$q(L_+, L_-) = L_- L_+^{-1}.$$

Then

$$q(g \circ (L_+, L_-)) = g^{-1} L_- L_+^{-1} g.$$

The notion of Poisson group actions may be generalized as follows. Let $A \times M \to M$ be a Poisson group action of a Poisson–Lie group A on a Poisson manifold M. A subgroup $K \subset A$ is called admissible if the set $C^\infty(M)^K$ of K-invariants is a Poisson subalgebra in $C^\infty(M)$. If space M/K is a smooth manifold, we may identify the algebras $C^\infty(M/K)$ and $C^\infty(M)^K$. Hence there exists a Poisson structure on M/K such that the canonical projection $M \to M/K$ is a Poisson map.

Proposition 9.3. **([24], Theorem 6; [18], §2)** *Let $(\mathfrak{a}, \mathfrak{a}^*)$ be the tangent Lie bialgebra of a Poisson–Lie group A. A connected Lie subgroup $K \subset A$ with Lie algebra $\mathfrak{k} \subset \mathfrak{a}$ is admissible if the annihilator $\mathfrak{k}^\perp$ of $\mathfrak{k}$ in $\mathfrak{a}^*$ is a Lie subalgebra $\mathfrak{k}^\perp \subset \mathfrak{a}^*$.*

We shall need the following particular example of dual pairs arising from Poisson group actions.

Let $A \times M \to M$ be a right (local) Poisson group action of a Poisson–Lie group A on a manifold M. Suppose that this action possesses a moment map $\mu : M \to A^*$. Let K be an admissible subgroup in A. Denote by $\mathfrak{k}$ the Lie algebra of K. Assume that $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra in $\mathfrak{a}^*$. Suppose also that there is a splitting $\mathfrak{a}^* = \mathfrak{t} \oplus \mathfrak{k}^\perp$, and that $\mathfrak{t}$ is a Lie subalgebra in $\mathfrak{a}^*$. Then the linear space $\mathfrak{k}^*$ is naturally identified with $\mathfrak{t}$. Assume that $A^* = K^\perp T$ as a manifold, where $K^\perp, T$ are the Lie subgroups of A^* corresponding to the Lie subalgebras $\mathfrak{k}^\perp, \mathfrak{t} \subset \mathfrak{a}^*$, respectively. Denote by $\pi_{K^\perp}, \pi_T$ the projections onto $K^\perp$ and T in this decomposition. Suppose that $K^\perp$ is a connected subgroup in A^* and that for any $k^\perp \in K^\perp$ the transformation

$$\begin{gathered} \mathfrak{t} \to \mathfrak{t}, \\ t \mapsto (\mathrm{Ad}(k^\perp) t)_{\mathfrak{t}}, \end{gathered} \tag{9.3}$$

where the subscript $\mathfrak{t}$ stands for the $\mathfrak{t}$–component with respect to the decomposition $\mathfrak{a}^* = \mathfrak{t} \oplus \mathfrak{k}^\perp$, is invertible. The following proposition is a slight generalization of Theorem 14 in [29]. The proof given in [29] still applies under the conditions imposed on $K, K^\perp$ and T above.

Proposition 9.4. *Define a map* $\overline{\mu} : M \to T$ *by*

$$\overline{\mu} = \pi_T \mu.$$

Then

(i) $\overline{\mu}^* (C^\infty (T))$ *is a Poisson subalgebra in* $C^\infty (M)$*, and hence one can equip* T *with a Poisson structure such that* $\overline{\mu} : M \to T$ *is a Poisson map.*

(ii)Moreover, the algebra $C^\infty (M)^K$ *is the centralizer of* $\overline{\mu}^* (C^\infty (T))$ *in the Poisson algebra* $C^\infty (M)$*. In particular, if* M/K *is a smooth manifold the maps*

$$\begin{array}{ccccc} & & M & & \\ & \overset{\pi}{\swarrow} & & \overset{\overline{\mu}}{\searrow} & , \\ M/K & & & & T \end{array} \qquad (9.4)$$

form a dual pair.

Remark 9.5. *Let* $t \in T$ *be as in Lemma 9.1. Assume that* $\pi(\overline{\mu}^{-1}(t))$ *is a smooth manifold (*M/K *does not need to be smooth). Then the algebra* $C^\infty(\pi(\overline{\mu}^{-1}(t)))$ *is isomorphic to the reduced Poisson algebra* $C^\infty(\overline{\mu}^{-1}(t))^{C^\infty(T)}$*.*

Remark 9.6. *In the proof of Theorem 14 in* [29] *we obtained a formula which relates the action of the Poisson algebra* $\overline{\mu}^* (C^\infty (T))$ *and the action of* K *on* $C^\infty (M)$*. Let* $X \in \mathfrak{k}$ *and* $\widehat{X}$ *be the corresponding vector field on* M*,* ξ_φ *the Hamiltonian vector field of* $\varphi \in C^\infty(M)$*. Then*

$$\begin{aligned} L_{\widehat{X}}\varphi &= \langle \mathrm{Ad}(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T), X\rangle(\xi_\varphi) = \\ & \langle \mathrm{Ad}(\pi_{K^\perp}\mu)(\theta_T), X\rangle(\overline{\mu}_*(\xi_\varphi)), \end{aligned} \qquad (9.5)$$

where θ_T *is the universal right invariant Cartan form on* T*.*

10. Quantization of Poisson–Lie groups and q-W–algebras

Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots associated to s. Observe that cocycle (4.12) equips $\mathfrak{g}$ with the structure of a factorizable Lie bialgebra. Using the identification End $\mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}$ the corresponding r–matrix may be represented as

$$r^s = P_+ - P_- + \frac{1+s}{1-s} P_{\mathfrak{h}'},$$

where P_+, P_- and $P_{\mathfrak{h}'}$ are the projection operators onto $\mathfrak{n}_+, \mathfrak{n}_-$ and $\mathfrak{h}'$ in the direct sum

$$\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h}' + {\mathfrak{h}'}^\perp + \mathfrak{n}_-,$$

where ${\mathfrak{h}'}^\perp$ is the orthogonal complement to $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the Killing form.

Let G be the connected simply connected simple Poisson–Lie group with the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, G^* the dual group. Observe that G is an algebraic group (see §104, Theorem 12 in [35]).

Note also that

$$r^s_+ = P_+ + \frac{1}{1-s} P_{\mathfrak{h}'} + \frac{1}{2} P_{{\mathfrak{h}'}^\perp},\ r^s_- = -P_- + \frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{{\mathfrak{h}'}^\perp},$$

and hence the subspaces $\mathfrak{b}_\pm$ and $\mathfrak{n}_\pm$ defined by (8.3) coincide with the Borel subalgebras in $\mathfrak{g}$ and their nil–radicals, respectively. Therefore every element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(L_+, L_-) = (h_+, h_-)(n_+, n_-), \qquad (10.1)$$

where $n_\pm \in N_\pm$, $h_+ = exp((\frac{1}{1-s} P_{\mathfrak{h}'} + \frac{1}{2} P_{{\mathfrak{h}'}^\perp})x)$, $h_- = exp((\frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{{\mathfrak{h}'}^\perp})x)$, $x \in \mathfrak{h}$. In particular, G^* is a solvable algebraic subgroup in $G \times G$.

For every algebraic variety V we denote by $\mathbb{C}[V]$ the algebra of regular functions on V. Our main object will be the algebra of regular functions on G^*, $\mathbb{C}[G^*]$. This algebra may be explicitly described as follows. Let π_V be a finite–dimensional representation of G. Then matrix elements of $\pi_V(L_\pm)$ are well–defined functions on G^*, and $\mathbb{C}[G^*]$ is the subspace in $C^\infty(G^*)$ generated by matrix elements of $\pi_V(L_\pm)$, where V runs through all finite–dimensional representations of G.

The elements $L^{\pm,V} = \pi_V(L_\pm)$ may be viewed as elements of the space $\mathbb{C}[G^*] \otimes \mathrm{End}V$. For every two finite–dimensional $\mathfrak{g}$ modules V and W we denote ${r^s_+}^{VW} = (\pi_V \otimes \pi_W) r^s_+$, where r^s_+ is regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$.

Proposition 10.1. **([25], Section 2)** *$\mathbb{C}[G^*]$ is a Poisson subalgebra in the Poisson algebra $C^\infty(G^*)$, the Poisson brackets of the elements $L^{\pm,V}$ are given by*

$$(10.2)\qquad \begin{aligned} \{L_1^{\pm,W}, L_2^{\pm,V}\} &= 2[{r^s_+}^{VW}, L_1^{\pm,W} L_2^{\pm,V}], \\ \{L_1^{-,W}, L_2^{+,V}\} &= 2[{r^s_+}^{VW}, L_1^{-,W} L_2^{+,V}], \end{aligned}$$

where

$$L_1^{\pm,W} = L^{\pm,W} \otimes I_V,\ L_2^{\pm,V} = I_W \otimes L^{\pm,V},$$

and I_X is the unit matrix in X.

Moreover, the map $\Delta : \mathbb{C}[G^*] \to \mathbb{C}[G^*] \otimes \mathbb{C}[G^*]$ dual to the multiplication in G^*,

$$(10.3)\qquad \Delta(L_{ij}^{\pm,V}) = \sum_k L_{ik}^{\pm,V} \otimes L_{kj}^{\pm,V},$$

is a homomorphism of Poisson algebras, and the map $S : \mathbb{C}[G^*] \to \mathbb{C}[G^*]$,

$$S(L_{ij}^{\pm,V}) = (L^{\pm,V})^{-1}_{ij}$$

is an antihomomorphism of Poisson algebras.

Remark 10.7. *Recall that a Poisson–Hopf algebra is a Poisson algebra which is also a Hopf algebra such that the comultiplication is a homomorphism of Poisson algebras and the antipode is an antihomomorphism of Poisson algebras. According to Proposition 10.1 $\mathbb{C}[G^*]$ is a Poisson–Hopf algebra.*

Now we construct a quantization of the Poisson–Hopf algebra $\mathbb{C}[G^*]$. For technical reasons we shall need an extension of the algebra $U^s_{\mathcal{A}}(\mathfrak{g})$ to an algebra $U^s_{\mathcal{A}'}(\mathfrak{g}) = U^s_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathcal{A}'$, where $\mathcal{A}' = \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}, \frac{1-q^{\frac{1}{2d}}}{1-q_i^{-2}}]_{i=1,\ldots,l}$. Note that the ratios $\frac{1-q^{\frac{1}{2d}}}{1-q_i^{-2}}$ have no singularities when $q=1$, and we can define a localization, $\mathcal{A}'/(1-q^{\frac{1}{2d}})\mathcal{A}' = \mathbb{C}$ as well as similar localizations for other generic values of ε, $\mathcal{A}'/(\varepsilon^{\frac{1}{2d}} - q^{\frac{1}{2d}})\mathcal{A}' = \mathbb{C}$ and similar localizations of algebras over $\mathcal{A}'$. $U^s_{\mathcal{A}'}(\mathfrak{g})$ is naturally a Hopf algebra with the comultiplication and the antipode induced from $U^s_{\mathcal{A}}(\mathfrak{g})$.

First, using arguments similar to those applied in the end of Section 7 where we defined the action of the element $\mathcal{R}^s$ in tensor products of finite–dimensional representations, one can show that for any finite–dimensional $U^s_{\mathcal{A}}(\mathfrak{g})$ module V the invertible elements ${}^qL^{\pm,V}$ given by

$${}^qL^{+,V} = (id \otimes \pi_V){\mathcal{R}^s_{21}}^{-1} = (id \otimes \pi_V S^s)\mathcal{R}^s_{21},\ {}^qL^{-,V} = (id \otimes \pi_V)\mathcal{R}^s.$$

are well defined elements of $U^s_{\mathcal{A}}(\mathfrak{g}) \otimes \mathrm{End}V$ (compare with [10]). If we fix a basis in V, ${}^qL^{\pm,V}$ may be regarded as matrices with matrix elements $({}^qL^{\pm,V})_{ij}$ being elements of $U^s_{\mathcal{A}}(\mathfrak{g})$. We also recall that one can define an operator $R^{VW} = (\pi_V \otimes \pi_W)\mathcal{R}^s$ (see Section 7).

From the Yang–Baxter equation for $\mathcal{R}$ we get relations between ${}^qL^{\pm,V}$:

$$(10.4)\qquad R^{VW}\,{}^qL_1^{\pm,W}\,{}^qL_2^{\pm,V} = {}^qL_2^{\pm,V}\,{}^qL_1^{\pm,W} R^{VW},$$

$$(10.5)\qquad R^{VW}\,{}^qL_1^{-,W}\,{}^qL_2^{+,V} = {}^qL_2^{+,V}\,{}^qL_1^{-,W} R^{VW}.$$

By ${}^qL_1^{\pm,W}$, ${}^qL_2^{\pm,V}$ we understand the following matrices in $V\otimes W$ with entries being elements of $U_{\mathcal{A}}^s(\mathfrak{g})$:

$$ {}^qL_1^{\pm,W}={}^qL^{\pm,W}\otimes I_V,\ {}^qL_2^{\pm,V}=I_W\otimes {}^qL^{\pm,V}, $$

where I_X is the unit matrix in X.

From (3.7) we can obtain the action of the comultiplication on the matrices ${}^qL^{\pm,V}$:

$$ \Delta_s({}^qL_{ij}^{\pm,V})=\sum_k {}^qL_{ik}^{\pm,V}\otimes {}^qL_{kj}^{\pm,V} \tag{10.6} $$

and the antipode,

$$ S_s({}^qL_{ij}^{\pm,V})=({}^qL^{\pm,V})_{ij}^{-1}. \tag{10.7} $$

We denote by $\mathbb{C}_{\mathcal{A}'}[G^*]$ the Hopf subalgebra in $U_{\mathcal{A}'}^s(\mathfrak{g})$ generated by matrix elements of $({}^qL^{\pm,V})^{\pm1}$, where V runs through all finite–dimensional representations of $U_{\mathcal{A}}^s(\mathfrak{g})$.

Since $\mathcal{R}^s=1\otimes 1 \pmod h$ relations (10.4) and (10.5) imply that the quotient algebra $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}}-1)\mathbb{C}_{\mathcal{A}'}[G^*]$ is commutative, and one can equip it with a Poisson structure given by

$$ \{x_1,x_2\}=\frac{1}{2d}\frac{[a_1,a_2]}{q^{\frac{1}{2d}}-1} \pmod{(q^{\frac{1}{2d}}-1)}, \tag{10.8} $$

where $a_1,a_2\in\mathbb{C}_{\mathcal{A}'}[G^*]$ reduce to $x_1,x_2\in\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}}-1)\mathbb{C}_{\mathcal{A}'}[G^*] \pmod{(q^{\frac{1}{2d}}-1)}$. Obviously, the maps (10.6) and (10.7) induce a comultiplication and an antipode on $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}}-1)\mathbb{C}_{\mathcal{A}'}[G^*]$ compatible with the introduced Poisson structure, and the quotient $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}}-1)\mathbb{C}_{\mathcal{A}'}[G^*]$ becomes a Poisson–Hopf algebra.

Proposition 10.2. *The Poisson–Hopf algebra $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}}-1)\mathbb{C}_{\mathcal{A}'}[G^*]$ is isomorphic to $\mathbb{C}[G^*]$ as a Poisson–Hopf algebra.*

Proof. Denote by $p:\mathbb{C}_{\mathcal{A}'}[G^*]\to\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}}-1)\mathbb{C}_{\mathcal{A}'}[G^*]=\mathbb{C}[G^*]'$ the canonical projection, and let $\tilde{L}^{\pm,V}=(p\otimes p_V)({}^qL^{\pm,V})\in\mathbb{C}[G^*]'\otimes \mathrm{End}V$, where $p_V:V\to\overline{V}=V/(q^{\frac{1}{2d}}-1)V$ be the projection of finite–dimensional $U_{\mathcal{A}}^s(\mathfrak{g})$–module V onto the corresponding $\mathfrak{g}$–module $\overline{V}$.

First observe that the map

$$ \imath:\mathbb{C}[G^*]'\to\mathbb{C}[G^*],\ (\imath\otimes id)\tilde{L}^{\pm,V}=L^{\pm,\overline{V}} $$

is a well–defined linear isomorphism. Indeed, consider, for instance, element $\tilde{L}^{-,V}$. From (4.10) it follows that

$$ \begin{array}{l} \tilde{L}_{ij}^{-,V}=\{(p\otimes id)exp\left[\sum_{i=1}^l hH_i\otimes\pi_{\overline{V}}((-\frac{2s}{1-s}P_{\mathfrak{h}'}+P_{\mathfrak{h}'^\perp})Y_i)\right]\times \\ \prod_\beta exp[p((1-q_\beta^{-2})e_\beta)\otimes\pi_{\overline{V}}(X_{-\beta})]\}_{ij}. \end{array} \tag{10.9} $$

On the other hand (10.1) implies that every element L_- may be represented in the form

$$ \begin{array}{l} L_-=exp\left[\sum_{i=1}^l b_i(\frac{s}{1-s}P_{\mathfrak{h}'}-\frac{1}{2}P_{\mathfrak{h}'^\perp})Y_i\right]\times \\ \prod_\beta exp[b_\beta X_{-\beta}],\ b_i,b_\beta\in\mathbb{C}, \end{array} \tag{10.10} $$

and hence

$$ \begin{array}{l} L_{ij}^{-,V}=\{exp\left[\sum_{i=1}^l b_i\otimes\pi_V((\frac{s}{1-s}P_{\mathfrak{h}'}-\frac{1}{2}P_{\mathfrak{h}'^\perp})Y_i)\right]\times \\ \prod_\beta exp[b_\beta\otimes\pi_V(X_{-\beta})]\}_{ij}. \end{array} \tag{10.11} $$

Therefore $\imath$ is a linear isomorphism. We have to prove that $\imath$ is an isomorphism of Poisson–Hopf algebras.

Recall that $\mathcal{R}^s=1\otimes 1+2hr_+^s \pmod{h^2}$. Therefore from commutation relations (10.4), (10.5) it follows that $\mathbb{C}[G^*]'$ is a commutative algebra, and the Poisson brackets of matrix elements $\tilde{L}_{ij}^{\pm,V}$

(see (10.8)) are given by (10.2), where $L^{\pm,V}$ are replaced by $\tilde{L}^{\pm,V}$. The factor $\frac{1}{2d}$ in formula (10.8) normalizes the Poisson bracket in such a way that bracket (10.8) is in agreement with (10.2).

From (10.6) we also obtain that the action of the comultiplication on the matrices $\tilde{L}^{\pm,V}$ is given by (10.3), where $L^{\pm,V}$ are replaced by $\tilde{L}^{\pm,V}$. This completes the proof. □

We shall call the map $p:\mathbb{C}_{\mathcal{A}'}[G^*]\rightarrow \mathbb{C}[G^*]$ the quasiclassical limit.

From the definition of the elements ${}^qL^{\pm,V}$ it follows that $\mathbb{C}_{\mathcal{A}'}[G^*]$ is the subalgebra in $U_{\mathcal{A}'}^s(\mathfrak{g})$ generated by the elements $\prod_{j=1}^l t_j^{\pm 2dp_{ij}},\ \prod_{j=1}^l t_j^{\pm 2dp_{ji}},\ i=1,\ldots l,\ \tilde{e}_\beta=(1-q_\beta^{-2})e_\beta,\ \tilde{f}_\beta=(1-q_\beta^{-2})e^{h\beta^\vee}f_\beta,\ \beta\in\Delta_+$.

Now using the Hopf algebra $\mathbb{C}_{\mathcal{A}'}[G^*]$ we shall define quantum versions of W–algebras. From the definition of the elements ${}^qL^{\pm,V}$ it follows that $\mathbb{C}_{\mathcal{A}'}[G^*]$ contains the subalgebra $\mathbb{C}_{\mathcal{A}'}[N_-]$ generated by elements $\tilde{e}_\beta=(1-q_\beta^{-2})e_\beta,\ \beta\in\Delta_+$.

Suppose that the ordering of the root system Δ_+ is fixed as in formula (5.8). Denote by $\mathbb{C}_{\mathcal{A}'}[M_-]$ the subalgebra in $\mathbb{C}_{\mathcal{A}'}[N_-]$ generated by elements $\tilde{e}_\beta,\ \beta\in\Delta_{\mathfrak{m}_+}$.

By construction $\mathbb{C}_{\mathcal{A}'}[N_-]$ is a quantization of the algebra of regular functions on the algebraic subgroup $N_-\subset G^*$ corresponding to the Lie subalgebra $\mathfrak{n}_-\subset\mathfrak{g}^*$, and $\mathbb{C}_{\mathcal{A}'}[M_-]$ is a quantization of the algebra of regular functions on the algebraic subgroup $M_-\subset G^*$ corresponding to the Lie subalgebra $\mathfrak{m}_-\subset\mathfrak{g}^*$ in the sense that $p(\mathbb{C}_{\mathcal{A}'}[N_-])=\mathbb{C}[N_-]$ and $p(\mathbb{C}_{\mathcal{A}'}[M_-])=\mathbb{C}[M_-]$. We also denote by M_+ the algebraic subgroup $M_+\subset G^*$ corresponding to the Lie subalgebra $\mathfrak{m}_+\subset\mathfrak{g}^*$.

We claim that the defining relations in the subalgebra $\mathbb{C}_{\mathcal{A}'}[M_-]$ are given by a formula similar to (7.12). Indeed, consider the defining relations in the subalgebra $U_{\mathcal{A}}^s(\mathfrak{m}_+)$,

$$e_\alpha e_\beta - q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}e_\beta e_\alpha=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C'(k_1,\ldots,k_n)e_{\delta_1}^{k_1}e_{\delta_2}^{k_2}\ldots e_{\delta_n}^{k_n},$$

where $C'(k_1,\ldots,k_n)\in\mathbb{C}[q^{\frac{1}{2d}},q^{-\frac{1}{2d}}]$. Commutation relations between quantum analogues of root vectors obtained in Proposition 4.2 in [15] imply that each function $C'(k_1,\ldots,k_n)$ has a zero of order $k_1+\ldots+k_n-1$ at point $q=1$. Therefore one can write the following defining relations for the generators $\tilde{e}_\beta=(1-q_\beta^{-2})e_\beta,\ \beta\in\Delta_{\mathfrak{m}_+}$ in the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$,

$$\tilde{e}_\alpha\tilde{e}_\beta - q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}\tilde{e}_\beta\tilde{e}_\alpha=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C''(k_1,\ldots,k_n)\tilde{e}_{\delta_1}^{k_1}\tilde{e}_{\delta_2}^{k_2}\ldots\tilde{e}_{\delta_n}^{k_n},$$

where $C''(k_1,\ldots,k_n)\in\mathcal{A}'$, and each function $C''(k_1,\ldots,k_n)$ has a zero of order 1 at point $q=1$.

Now arguments similar to those used in the proof of Theorem 6.1 show that the map $\chi_q^s:\mathbb{C}_{\mathcal{A}'}[M_-]\rightarrow\mathcal{A}'$,

$$\chi_q^s(\tilde{e}_\beta)=\left\{\begin{array}{ll}0 & \beta\notin\{\gamma_1,\ldots,\gamma_{l'}\}\\ k_i & \beta=\gamma_i,k_i\in\mathcal{A}'\end{array}\right.,\tag{10.12}$$

is a character of $\mathbb{C}_{\mathcal{A}'}[M_-]$. Denote by $\mathbb{C}_{\chi_q^s}$ the rank one representation of the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$ defined by the character χ_q^s.

We call the algebra

$$W_q^s(G)=\mathrm{End}_{\mathbb{C}_{\mathcal{A}'}[G^*]}(\mathbb{C}_{\mathcal{A}'}[G^*]\otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]}\mathbb{C}_{\chi_q^s})^{opp}\tag{10.13}$$

the q-W–algebra associated to the (conjugacy class) of the Weyl group element $s\in W$.

Observe that by Frobenius reciprocity we also have

$$W_q^s(G)=\mathrm{Hom}_{\mathbb{C}_{\mathcal{A}'}[M_-]}(\mathbb{C}_{\chi_q^s},\mathbb{C}_{\mathcal{A}'}[G^*]\otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]}\mathbb{C}_{\chi_q^s}).$$

Therefore if we denote by I_q the left ideal in $\mathbb{C}_{\mathcal{A}'}[G^*]$ generated by the kernel of χ_q^s then $W_q^s(G)$ can be defined as the subspace of all $x+I_q\in Q_{\chi_q^s},\ Q_{\chi_q^s}=\mathbb{C}_{\mathcal{A}'}[G^*]\otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]}\mathbb{C}_{\chi_q^s}=\mathbb{C}_{\mathcal{A}'}[G^*]/I_q$, such that $[m,x]=mx-xm\in I_q$ for any $m\in\mathbb{C}_{\mathcal{A}'}[M_-]$.

Now consider the Lie algebra $\mathfrak{L}_{\mathcal{A}'}$ associated to the associative algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$, i.e. $\mathfrak{L}_{\mathcal{A}'}$ is the Lie algebra which is isomorphic to $\mathbb{C}_{\mathcal{A}'}[M_-]$ as a linear space, and the Lie bracket in $\mathfrak{L}_{\mathcal{A}'}$ is given by the usual commutator of elements in $\mathbb{C}_{\mathcal{A}'}[M_-]$.

If we denote by $\rho_{\chi_q^s}$ the canonical projection $\mathbb{C}_{\mathcal{A}'}[G^*] \to \mathbb{C}_{\mathcal{A}'}[G^*]/I_q$ then the algebra $W_q^s(G)$ can be regarded as the algebra of invariants with respect to the following action of the Lie algebra $\mathfrak{L}_{\mathcal{A}'}$ on the space $\mathbb{C}_{\mathcal{A}'}[G^*]/I_q$:

$$m \cdot (x + I_q) = \rho_{\chi_q^s}([m, x]). \tag{10.14}$$

where $x \in \mathbb{C}_{\mathcal{A}'}[G^*]$ is any representative of $x + I_q \in \mathbb{C}_{\mathcal{A}'}[G^*]/I_q$ and $m \in \mathbb{C}_{\mathcal{A}'}[M_-]$.

In terms of this description the multiplication in $W_s^q(G)$ takes the form $(x+I_q)(y+I_q) = xy+I_q$, $x+I_q, y+I_q \in W_q^s(G)$. Note also that since χ_q^s is a character of $\mathbb{C}_{\mathcal{A}'}[M_-]$ the ideal I_q is stable under that action of $\mathbb{C}_{\mathcal{A}'}[M_-]$ on $\mathbb{C}_{\mathcal{A}'}[G^*]$ by commutators.

In conclusion we remark that by specializing q to a particular value $\varepsilon \in \mathbb{C}$ one can define a complex associative algebra $\mathbb{C}_\varepsilon[G^*] = \mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})\mathbb{C}_{\mathcal{A}'}[G^*]$, its subalgebra $\mathbb{C}_\varepsilon[M_-]$ with a nontrivial character χ_ε^s and the corresponding W–algebra

$$W_\varepsilon^s(G) = \mathrm{End}_{\mathbb{C}_\varepsilon[G^*]}(\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s})^{opp} \tag{10.15}$$

Obviously, for generic ε we have $W_\varepsilon^s(G) = W_q^s(G)/(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})W_q^s(G)$.

11. Poisson reduction and q-W algebras

In this section we shall analyze the quasiclassical limit of the algebra $W_q^s(G)$. Using results of Section 9 we realize this limit algebra as the algebra of functions on a reduced Poisson manifold.

Denote by χ^s the character of the Poisson subalgebra $\mathbb{C}[M_-]$ such that $\chi^s(p(x)) = \chi_q^s(x) \pmod{(q^{\frac{1}{2d}} - 1)}$ for every $x \in \mathbb{C}_{\mathcal{A}'}[M_-]$.

Let $I = p(I_q)$ be the ideal in $\mathbb{C}[G^*]$ generated by the kernel of χ^s. Then the Poisson algebra $W^s(G) = W_q^s(G)/(q^{\frac{1}{2d}} - 1)W_q^s(G)$ is the subspace of all $x + I \in Q_{\chi^s}$, $Q_{\chi^s} = \mathbb{C}[G^*]/I$, such that $\{m, x\} \in I$ for any $m \in \mathbb{C}[M_-]$, and the Poisson bracket in $W^s(G)$ takes the form $\{(x+I), (y+I)\} = \{x, y\} + I$, $x + I, y + I \in W^s(G)$. We shall also write $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} = (Q_{\chi^s})^{\mathbb{C}[M_-]}$.

Denote by ρ_{χ^s} the canonical projection $\mathbb{C}[G^*] \to \mathbb{C}[G^*]/I$. The discussion above implies that $W^s(G)$ is the algebra of invariants with respect to the following action of the Poisson algebra $\mathbb{C}[M_-]$ on the space $\mathbb{C}[G^*]/I$:

$$x \cdot (v + I) = \rho_{\chi^s}(\{x, v\}), \tag{11.1}$$

where $v \in \mathbb{C}[G^*]$ is any representative of $v + I \in \mathbb{C}[G^*]/I$ and $x \in \mathbb{C}[M_-]$.

We shall describe the space of invariants $(\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]}$ with respect to this action by analyzing "dual geometric objects". First observe that algebra $(\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]}$ is a particular example of the reduced Poisson algebra introduced in Lemma 9.1.

Indeed, recall that according to (10.1) any element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(L_+, L_-) = (h_+, h_-)(n_+, n_-), \tag{11.2}$$

where $n_\pm \in N_\pm$, $h_+ = exp((\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $h_- = exp((\frac{s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $x \in \mathfrak{h}$.

Formula (10.1) and decomposition of N_- into products of one–dimensional subgroups corresponding to roots also imply that every element L_- may be represented in the form

$$\begin{aligned} L_- = exp&\left[\textstyle\sum_{i=1}^{l} b_i(\frac{s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}'^\perp})H_i\right] \times \\ &\textstyle\prod_\beta exp[b_\beta X_{-\beta}],\ b_i, b_\beta \in \mathbb{C}, \end{aligned} \tag{11.3}$$

where the product over roots is taken in the same order as in normal ordering (5.8).

Now define a map $\mu_{M_+}: G^* \to M_-$ by

$$\mu_{M_+}(L_+, L_-) = m_-, \tag{11.4}$$

where for L_- given by (11.3) m_- is defined as follows

$$m_- = \prod_{\beta \in \Delta_{\mathfrak{m}_+}} exp[b_\beta X_{-\beta}],$$

and the product over roots is taken in the same order as in the normally ordered segment $\Delta_{\mathfrak{m}_+}$.

By definition μ_{M_+} is a morphism of algebraic varieties. We also note that by definition $\mathbb{C}[M_-] = \{\varphi \in \mathbb{C}[G^*] : \varphi = \varphi(m_-)\}$. Therefore $\mathbb{C}[M_-]$ is generated by the pullbacks of regular functions on M_- with respect to the map μ_{M_+}. Since $\mathbb{C}[M_-]$ is a Poisson subalgebra in $\mathbb{C}[G^*]$, and regular functions on M_- are dense in $C^\infty(M_-)$ on every compact subset, we can equip the manifold M_- with the Poisson structure in such a way that μ_{M_+} becomes a Poisson mapping.

Let u be the element defined by

$$u = \prod_{i=1}^{l'} exp[t_i X_{-\gamma_i}] \in M_-, t_i = k_i \ (\mathrm{mod}\ (q^{\frac{1}{2d}} - 1)), \tag{11.5}$$

where the product over roots is taken in the same order as in the normally ordered segment $\Delta_{\mathfrak{m}_+}$. From (10.9) and the definition of χ^s it follows that $\chi^s(\varphi) = \varphi(u)$ for every $\varphi \in \mathbb{C}[M_-]$. χ^s naturally extends to a character of the Poisson algebra $C^\infty(M_-)$.

Now applying Lemma 9.1 for $M = G^*$, $B = M_-$, $\pi = \mu_{M_+}$, $b = u$ we can define the reduced Poisson algebra $C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}$ (see also Remark 9.4). Denote by I_u the ideal in $C^\infty(G^*)$ generated by elements $\mu_{M_+}^*\psi$, $\psi \in C^\infty(M_-)$, $\psi(u) = 0$. Let $P_u : C^\infty(G^*) \to C^\infty(G^*)/I_u = C^\infty(\mu_{M_+}^{-1}(u))$ be the canonical projection. Then the action (9.1) of $C^\infty(M_-)$ on $C^\infty(\mu_{N_+}^{-1}(u))$ takes the form:

$$\psi \cdot \varphi = P_u(\{\mu_{M_+}^*\psi, \tilde{\varphi}\}), \tag{11.6}$$

where $\psi \in C^\infty(M_-)$, $\varphi \in C^\infty(\mu_{M_+}^{-1}(u))$ and $\tilde{\varphi} \in C^\infty(G^*)$ is a representative of φ such that $P_u\tilde{\varphi} = \varphi$.

Lemma 11.1. *$\mu_{M_+}^{-1}(u)$ is a subvariety in G^*. Moreover, the algebra $\mathbb{C}[G^*]/I$ is isomorphic to the algebra of regular functions on $\mu_{M_+}^{-1}(u)$, $\mathbb{C}[G^*]/I = \mathbb{C}[\mu_{M_+}^{-1}(u)]$, and the algebra $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]}$ is isomorphic to the algebra of regular functions on $\mu_{M_+}^{-1}(u)$ which are invariant with respect to the action (11.6) of $C^\infty(M_-)$ on $C^\infty(\mu_{M_+}^{-1}(u))$, i.e.*

$$W^s(G) = \mathbb{C}[\mu_{M_+}^{-1}(u)] \cap C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}.$$

Proof. By definition $\mu_{M_+}^{-1}(u)$ is a subvariety in G^*. Next observe that $I = \mathbb{C}[G^*] \cap I_u$. Therefore the algebra $\mathbb{C}[G^*]/I$ is identified with the algebra of regular functions on $\mu_{M_+}^{-1}(u)$.

Since $\mathbb{C}[M_-]$ is dense in $C^\infty(M_-)$ on every compact subset in M_- we have:

$$C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)} \cong C^\infty(\mu_{M_+}^{-1}(u))^{\mathbb{C}[M_-]}.$$

Finally observe that action (11.6) coincides with action (11.1) when restricted to regular functions. □

In case when the roots $\gamma_1, \ldots, \gamma_n$ are simple, and hence the segment Δ_{s^1} is of the form $\Delta_{s^1} = \{\gamma_1, \ldots, \gamma_n\}$, we shall realize the algebra $C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}$ as the algebra of functions on a reduced Poisson manifold. In the spirit of Lemma 9.1 we shall construct a map that forms a dual pair together with the mapping μ_{M_+}. In this construction we use the dressing action of the Poisson–Lie group G on G^* (see Proposition 9.2).

Consider the restriction of the dressing action $G^* \times G \to G^*$ to the subgroup $M_+ \subset G$. According to part (iv) of Proposition 8.2 $(\mathfrak{b}_+, \mathfrak{b}_-)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$. Therefore B_+ is a Poisson–Lie subgroup in G. We claim that $M_+ \subset B_+$ is an admissible subgroup.

Indeed, observe that if the roots $\gamma_1, \ldots, \gamma_n$ are simple then $\Delta_{s^1} = \{\gamma_1, \ldots, \gamma_n\}$, and the complementary subset to $\Delta_{\mathfrak{m}_+}$ in Δ_+ is a minimal segment $\Delta^0_{\mathfrak{m}_+}$ with respect to normal ordering (5.8). Now using Proposition 8.2 (iv) the subspace $\mathfrak{m}_+^\perp$ in $\mathfrak{b}_-$ can be identified with the linear subspace in $\mathfrak{b}_-$ spanned by the Cartan subalgebra $\mathfrak{h}$ and the root subspaces corresponding to the roots from the minimal segment $-\Delta^0_{\mathfrak{m}_+}$. Using the fact that the adjoint action of $\mathfrak{h}$ normalizes root subspaces and Lemma 5.2 we deduce that $\mathfrak{m}_+^\perp \subset \mathfrak{b}_-$ is a Lie subalgebra, and hence $M_+ \subset B_+$ is an admissible subgroup. Therefore $C^\infty(G^*)^{M_+}$ is a Poisson subalgebra in the Poisson algebra $C^\infty(G^*)$.

Proposition 11.2. *Assume that the roots $\gamma_1, \ldots, \gamma_n$ are simple. Then the algebra $C^\infty(G^*)^{M_+}$ is the centralizer of $\mu^*_{M_+}(C^\infty(M_-))$ in the Poisson algebra $C^\infty(G^*)$.*

Proof. First recall that, as we observed above, B_+ is a Poisson–Lie subgroup in G. By Proposition 9.2 for $X \in \mathfrak{b}_+$ we have:

$$L_{\widehat{X}}\varphi(L_+, L_-) = (\theta_{G^*}(L_+, L_-), X)(\xi_\varphi) = (r_-^{-1}\mu^*_{B_+}(\theta_{B_-}), X)(\xi_\varphi), \tag{11.7}$$

where $\widehat{X}$ is the corresponding vector field on G^*, ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(G^*)$, and the map $\mu_{B_+} : G^* \to B_-$ is defined by $\mu_{B_+}(L_+, L_-) = L_-$. Now from Proposition 8.2 (iv) and the definition of the moment map it follows that μ_{B_+} is a moment map for the dressing action of the subgroup B_+ on G^*.

We also proved above that M_+ is an admissible subgroup in the Lie–Poisson group B_+. Moreover the dual group B_- can be uniquely factorized as $B_- = M_+^\perp M_-$, where $M_+^\perp \subset B_-$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{m}_+^\perp \subset \mathfrak{b}_-$.

We conclude that all the conditions of Proposition 9.4 are satisfied with $A = B_+, K = M_+, A^* = B_-, T = M_-, K^\perp = M_+^\perp, \mu = \mu_{B_+}$. It follows that the algebra $C^\infty(G^*)^{M_+}$ is the centralizer of $\mu^*_{M_+}(C^\infty(M_-))$ in the Poisson algebra $C^\infty(G^*)$. This completes the proof. □

Let G^*/M_+ be the quotient of G^* with respect to the dressing action of M_+, $\pi : G^* \to G^*/M_+$ the canonical projection. Note that the space G^*/M_+ is not a smooth manifold. However, in the next section we will see that the subspace $\pi(\mu^{-1}_{M_+}(u)) \subset G^*/M_+$ is a smooth manifold. Therefore by Remark 9.5 the algebra $C^\infty(\pi(\mu^{-1}_{M_+}(u)))$ is isomorphic to $C^\infty(\mu^{-1}_{M_+}(u))^{C^\infty(M_-)}$. Moreover we will see that $\pi(\mu^{-1}_{M_+}(u))$ has a structure of algebraic variety. Using Lemma 11.1 we will obtain that the algebra $W^s(G)$ is the algebra of regular functions on this variety.

12. Cross–section theorem

In this section we describe the reduced space $\pi(\mu^{-1}_{M_+}(u)) \subset G^*/M_+$ and the algebra $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]}$ assuming that the roots $\gamma_1, \ldots, \gamma_n$ are simple.

First observe that in the considered case the map $q : G^* \to G$ (see Proposition 9.2) is an embedding of G^* into G as a manifold. Using this embedding one can reduce the study of the dressing action to the study of the action of G on itself by conjugations. This simplifies many geometric problems. Consider the restriction of this action to the subgroup M_+. Denote by $\pi_q : G \to G/M_+$ the canonical projection onto the quotient with respect to this action. Then we can identify the reduced space $\pi(\mu^{-1}_{M_+}(u))$ with the subspace $\pi_q(q(\mu^{-1}_{M_+}(u)))$ in G/M_+. Using this identification we shall explicitly describe the reduced space $\pi(\mu^{-1}_{M_+}(u))$. We start with description of the image of the "level surface" $\mu^{-1}_{M_+}(u)$ under the embedding q.

Let $X_\alpha(t) = \exp(tX_\alpha) \in G$, $t \in \mathbb{C}$ be the one–parametric subgroup in the algebraic group G corresponding to root $\alpha \in \Delta$. Recall that for any $\alpha \in \Delta_+$ and any $t \neq 0$ the element $s_\alpha(t) = X_\alpha(-t)X_{-\alpha}(t)X_\alpha(-t) \in G$ is a representative for the reflection s_α corresponding to the root α. Denote by $s \in G$ the following representative of the Weyl group element $s \in W$,

$$s = s_{\gamma_1}(t_1)\dots s_{\gamma_{l'}}(t_{l'}), \tag{12.1}$$

where the numbers t_i are defined in (11.5), and we assume that $t_i \neq 0$ for any i.

We shall also use the following representatives for s^1 and s^2

$$s^1 = s_{\gamma_1}(t_1)\dots s_{\gamma_n}(t_n),\ s^2 = s_{\gamma_{n+1}}(t_{n+1})\dots s_{\gamma_{l'}}(t_{l'}).$$

Let Z be the subgroup of G corresponding to the Lie subalgebra $\mathfrak{z}$ generated by the semisimple part $\mathfrak{m}$ of the Levi subalgebra $\mathfrak{l}$ and by the centralizer of s in $\mathfrak{h}$. Denote by N the subgroup of G corresponding to the Lie subalgebra $\mathfrak{n}$ and by $\overline{N}$ the opposite unipotent subgroup in G with the Lie algebra $\overline{\mathfrak{n}} = \bigoplus_{m<0}(\mathfrak{g})_m$. By definition we have that $N_+ \subset ZN$.

Proposition 12.1. *Let $q : G^* \to G$ be the map introduced in Proposition 9.2,*

$$q(L_+, L_-) = L_-L_+^{-1}.$$

Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple and that the numbers t_i defined in (11.5) are not equal to zero for all i. Then $q(\mu_{M_+}^{-1}(u))$ is a subvariety in $NsZN$ and the closure $\overline{q(\mu_{M_+}^{-1}(u))}$ of $q(\mu_{M_+}^{-1}(u))$ is also contained in $NsZN$.

Proof. First, using definition (11.4) of the map μ_{M_+} and the relation $h_- = s(h_+)$ following from (10.1) we can describe the space $\mu_{M_+}^{-1}(u)$ as follows:

$$\mu_{M_+}^{-1}(u) = \{(h_+n_+, s(h_+)ux) | n_+ \in N_+, h_+ \in H, x \in M_-^0\}, \tag{12.2}$$

where M_-^0 is the subgroup of G generated by the one–parametric subgroups corresponding to the roots from the segment $-\Delta^0_{\mathfrak{m}_+}$. Therefore

$$q(\mu_{M_+}^{-1}(u)) = \{s(h_+)uxn_+^{-1}h_+^{-1} | n_+ \in N_+, h_+ \in H, x \in M_-^0\}. \tag{12.3}$$

Now we show that uxn_+^{-1} belongs to $NsZN$. Indeed, we have

$$uxn_+^{-1} = uxX_{\gamma_1}(-t_1)\dots X_{\gamma_{l'}}(-t_{l'})X_{\gamma_{l'}}(t_{l'})\dots X_{\gamma_1}(t_1)n_+^{-1}. \tag{12.4}$$

Now observe that the segment $\Delta' = -\gamma_{n+1}, \dots, \gamma_n$ is minimal with respect to the circular normal ordering, $-\Delta^0_{\mathfrak{m}_+} \subset \Delta'$ and $\Delta' \bigcap -\Delta_+ \subset -\Delta_{s^2} \bigcup -\Delta_0$. Combining these facts with Lemma 5.2, recalling the definition of the element u and using the commutation relations between one–parametric subgroups corresponding to roots and the commutation relations $X_{\gamma_i}(t)X_{-\gamma_j}(t') = X_{-\gamma_j}(t')X_{\gamma_i}(t)$, $i \neq j$, $i, j = 1, \dots, n$ for one–parametric subgroups corresponding to the simple roots γ_i, $i = 1, \dots, n$ one can rewrite formula (12.4) in the following form

$$uxn_+^{-1} = X_{-\gamma_1}(t_1)X_{\gamma_1}(-t_1)\dots X_{-\gamma_{l'}}(t_{l'})X_{\gamma_{l'}}(-t_{l'})x'k, \tag{12.5}$$

where $x' \in \overline{N}$ is such that $s^2x's^{2^{-1}} \in N$, and $k \in ZN$. Finally observe that the decomposition $s = s^1s^2$ is reduced, and hence $sx's^{-1} \in N$. Therefore using the relations $X_{-\gamma_i}(t_i)X_{\gamma_i}(-t_i) = X_{\gamma_i}(t_i)s_{\gamma_i}(t_i)$ we obtain

$$uxn_+^{-1} = x''sk, \tag{12.6}$$

where

$$x'' = X_{\gamma_1}(t_1)s_{\gamma_1}X_{\gamma_2}(t_2)s_{\gamma_1}^{-1}\ldots (s_{\gamma_1}\ldots s_{\gamma_{l'-1}})X_{\gamma_{l'}}(t_{l'})(s_{\gamma_1}\ldots s_{\gamma_{l'-1}})^{-1}sx's^{-1}\in N, \tag{12.7}$$

$s_{\gamma_i} = s_{\gamma_i}(t_i) \in G$ and $k \in ZN$. Hence $uxn_+^{-1} \in NsZN$.

Next, the space $NsZN$ is invariant with respect to the following action of H:

$$h \circ L = s(h)Lh^{-1}. \tag{12.8}$$

Indeed, let $L = vszw,\ v, w \in N, z \in Z$ be an element of $NsZN$. Then

$$h \circ L = s(h)vs(h)^{-1}s(h)sh^{-1}hzwh^{-1} = s(h)vs(h)^{-1}shzwh^{-1}. \tag{12.9}$$

The r.h.s. of the last equality belongs to $NsZN$ because H normalizes N and Z.

Comparing action (12.8) with (12.3) and recalling that $uxn_+^{-1} \in NsZN$ we deduce $q(\mu_{M_+}^{-1}(u)) \subset NsZN$. Since q is an embedding, $q(\mu_{M_+}^{-1}(u))$ is a subvariety in $NsZN$.

The variety $q(\mu_{M_+}^{-1}(u))$ is not closed in $NsZN$. Indeed, from formulas (12.3) and (12.9) it follows that $q(\mu_{M_+}^{-1}(u))$ contains elements of the form $x''sk$ with arbitrary $k \in ZN$ and x'' given by (12.7), where t_i are arbitrary nonzero complex numbers. Obviously its closure $\overline{q(\mu_{M_+}^{-1}(u))}$ is obtained by adding elements of the same form with some t_i equal to 0. Clearly, such elements also belong to $NsZN$ This completes the proof. □

We identify $\mu_{M_+}^{-1}(u)$ with the subvariety in $NsZN$ described in the previous proposition. As we observed in the beginning of this section the reduced space $\pi(\mu_{M_+}^{-1}(u))$ is isomorphic to $\pi_q(q(\mu_{M_+}^{-1}(u)))$. Note that by Proposition 12.1 $q(\mu_{M_+}^{-1}(u)) \subset NsZN$.

Proposition 12.2. **([33], Propositions 2.1 and 2.2)** *Let $N_s = \{v \in N | svs^{-1} \in \overline{N}\}$. Then the conjugation map*

$$N \times sZN_s \to NsZN \tag{12.10}$$

is an isomorphism of varieties. Moreover, the variety sZN_s is a transversal slice to the set of conjugacy classes in G.

Theorem 12.3. *Assume that the roots $\gamma_1, \ldots, \gamma_n$ are simple and that the numbers t_i defined in (11.5) are not equal to zero for all i. Then the (locally defined) conjugation action of M_+ on $q(\mu_{M_+}^{-1}(u))$ is (locally) free, the quotient $\pi_q(q(\mu_{M_+}^{-1}(u)))$ is a smooth variety and the algebra of regular functions on $\pi_q(q(\mu_{M_+}^{-1}(u)))$ is isomorphic to the algebra of regular functions on the slice sZN_s.*

The Poisson algebra $W^s(G)$ is isomorphic to the Poisson algebra of regular functions on $\pi(\mu_{M_+}^{-1}(u))$, $W^s(G) = \mathbb{C}[\pi(\mu_{M_+}^{-1}(u))] = \mathbb{C}[sZN_s]$, and the algebra $\mathbb{C}[\mu_{M_+}^{-1}(u)]$ is isomorphic to $\mathbb{C}[M_+] \otimes W^s(G) \cong \mathbb{C}[M_+] \otimes \mathbb{C}[sZN_s]$. Thus the algebra W_q^s is a noncommutative deformation of the algebra of regular functions on the transversal slice sZN_s.

Proof. First observe that by construction $\mu_{M_+}^{-1}(u) \cong q(\mu_{M_+}^{-1}(u)) \subset NsZN$ is (locally) stable under the action of $M_+ \subset N$ on $NsZN$ by conjugations. Since the conjugation action of N on $NsZN$ is free the (locally defined) conjugation action of M_+ on $q(\mu_{M_+}^{-1}(u))$ is (locally) free as well. Therefore the quotient $\pi_q(q(\mu_{M_+}^{-1}(u)))$ is a smooth variety.

Since by Proposition 12.1 $\overline{q(\mu_{M_+}^{-1}(u))} \subset NsZN$ the induced (local) action of M_+ on $\overline{q(\mu_{M_+}^{-1}(u))}$ is (locally) free as well. Now observe that from the description of the set $\overline{q(\mu_{M_+}^{-1}(u))}$ given in the end Proposition 12.1 it follows that $sZN_s \subset \overline{q(\mu_{M_+}^{-1}(u))}$. Since by the previous proposition any two points of sZN_s are not N–conjugate they are not M_+–conjugate as well, and hence we have an embedding $sZN_s \subset \pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$. From formula (5.10) for the cardinality $\sharp\Delta_{\mathfrak{m}_+}$ of the set $\Delta_{\mathfrak{m}_+}$ and from the definition of $q(\mu_{M_+}^{-1}(u))$ we deduce that the dimension of the quotient $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is equal to the dimension of the variety sZN_s,

$$\dim\, \pi_q(\overline{q(\mu_{M_+}^{-1}(u))}) = \dim\, G - 2\dim\, M_+ = 2D + l - 2\sharp\Delta_{\mathfrak{m}_+} == 2D + l -$$
$$-2(D - \frac{l(s) - l'}{2} - D_0) = l(s) + 2D_0 + l - l' = \dim\, N_s + \dim\, Z = \dim\, sZN_s.$$

Note also that both sZN_s and $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ are connected. Therefore any regular function on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is completely defined by its restriction to sZN_s. Such restrictions also span the space of regular functions on sZN_s since by the previous proposition the restriction map induces an isomorphism of the space of N–invariant regular functions on $NsZN$ and the space of regular functions on sZN_s, and N–invariant regular functions on $NsZN$ can be restricted to M_+–invariant functions on $\overline{q(\mu_{M_+}^{-1}(u))}$. Obviously, any regular function on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is completely defined by its restriction to $\pi_q(q(\mu_{M_+}^{-1}(u)))$. Therefore the algebra of regular functions on $\pi_q(q(\mu_{M_+}^{-1}(u)))$ is isomorphic to the algebra of regular functions on the slice sZN_s. This proves the first statement of the proposition.

Now observe that by Remark 9.5 the map

$$C^\infty(\pi(\mu_{M_+}^{-1}(u))) \to C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)},\ \psi \mapsto \pi^*\psi$$

is an isomorphism. By construction the map $\pi : \mu_{N_+}^{-1}(u) \to \pi(\mu_{N_+}^{-1}(u))$ is a morphism of varieties. Therefore the map

$$\mathbb{C}[\pi(\mu_{M_+}^{-1}(u))] \to \mathbb{C}[\mu_{M_+}^{-1}(u)] \cap C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)},\ \psi \mapsto \pi^*\psi$$

is an isomorphism.

Finally observe that by Lemma 11.1 the algebra $\mathbb{C}[\mu_{M_+}^{-1}(u)] \cap C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}$ is isomorphic to $W^s(G)$, and hence $W^s(G) \cong \mathbb{C}[sZN_s]$.

The first three statements of the theorem also imply isomorphisms, $\mathbb{C}[\mu_{M_+}^{-1}(u)] \cong \mathbb{C}[M_+] \otimes W^s(G) \cong \mathbb{C}[M_+] \otimes \mathbb{C}[sZN_s]$. This completes the proof. ☐

Remark 12.8. *A similar theorem can be proved in case when the roots $\gamma_{n+1}, \ldots, \gamma_{l'}$ are simple. In that case instead of the map $q : G^* \to G$ one should use another map $q' : G^* \to G$, $q'(L_+, L_-) = L_-^{-1}L_+$ which has the same properties as q, see* [25], *Section 2.*

Theorem 12.3 implies that the algebra $W^s(G)$ coincides with the deformed Poisson W–algebra introduced in [33].

In conclusion we discuss a simple property of the algebra $W^s_\varepsilon(G)$ which allows to construct noncommutative deformations of coordinate rings of singularities arising in the fibers of the conjugation quotient map $\delta_G : G \to H/W$ generated by the inclusion $\mathbb{C}[H]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where H is the maximal torus of G corresponding to the Cartan subalgebra $\mathfrak{h}$ and W is the Weyl group of the pair (G, H).

Observe that each central element $z \in Z(\mathbb{C}_\varepsilon[G^*])$ obviously gives rise to an element $\rho_{\chi^s_\varepsilon}(z) \in \mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi^s_\varepsilon}$, and since z is central

$$\rho_{\chi^s_\varepsilon}(z) \in \mathrm{Hom}_{\mathbb{C}_\varepsilon[M_-]}(\mathbb{C}_{\chi^s_\varepsilon}, \mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi^s_\varepsilon})^{opp} =$$
$$= \mathrm{End}_{\mathbb{C}_\varepsilon[G^*]}(\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi^s_\varepsilon})^{opp} = W^s_\varepsilon(G).$$

The proof of the following proposition is similar to that of Theorem $\mathrm{A_h}$ in [30].

Proposition 12.4. *Let $\varepsilon \in \mathbb{C}$ be generic. Then the restriction of the linear map $\rho_{\chi^s_\varepsilon} : \mathbb{C}_\varepsilon[G^*] \to \mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi^s_\varepsilon}$ to the center $Z(\mathbb{C}_\varepsilon[G^*])$ of $\mathbb{C}_\varepsilon[G^*]$ gives rise to an injective homomorphism of algebras,*

$$\rho_{\chi^s_\varepsilon} : Z(\mathbb{C}_\varepsilon[G^*]) \to W^s_\varepsilon(G).$$

Now if $\eta : Z(\mathbb{C}_\varepsilon[G^*]) \to \mathbb{C}$ is a character then from Theorem 12.3 and the results of Section 6 in [33] it follows that the algebra $W^s_\varepsilon(G)/W^s_\varepsilon(G)\mathrm{ker}\ \eta$ can be regarded as a noncommutative deformation of the algebra of regular functions defined on a fiber of the conjugation quotient map $\delta_G : sZN_s \to H/W$. In particular, for singular fibers we obtain noncommutative deformations of the coordinate rings of the corresponding singularities.

13. Homological realization of q-W algebras

In this section we realize the algebra $W^s_q(G)$ in the spirit of homological BRST reduction.

Let A be an associative algebra over a unital ring $\mathbf{k}$, $A_0 \subset A$ a subalgebra with augmentation $\epsilon : A_0 \to \mathbf{k}$. We denote this rank one A_0–module by $\mathbf{k}_\epsilon$.

Let $A-\mathrm{mod}$ ($A_0-\mathrm{mod}$) be the category of left A (A_0) modules. Denote by $\mathrm{Ind}^A_{A_0}$ the functor of induction,

$$\mathrm{Ind}^A_{A_0} : A_0 - \mathrm{mod} \to A - \mathrm{mod}$$

defined on objects by

$$\mathrm{Ind}^A_{A_0}(V) = A \otimes_{A_0} V,\ V \in A_0 - \mathrm{mod}.$$

Let $D^-(A)$ ($D^-(A_0)$) be the derived category of the abelian category $A-\mathrm{mod}$ ($A_0-\mathrm{mod}$) whose objects are bounded from above complexes of left A (A_0) modules. Let $(\mathrm{Ind}^A_{A_0})^L : D^-(A_0) \to D^-(A)$ be the left derived functor of the functor of induction. Recall that if $V^\bullet \in D^-(A_0)$ then $(\mathrm{Ind}^A_{A_0})^L(V^\bullet) = A \otimes_{A_0} X^\bullet$, where $X^\bullet$ is a projective resolution of the complex $V^\bullet$.

The $\mathbb{Z}$–graded algebra

$$\mathrm{Hk}^\bullet(A, A_0, \epsilon) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^-(A)}((\mathrm{Ind}^A_{A_0})^L(\mathbf{k}_\epsilon), T^n((\mathrm{Ind}^A_{A_0})^L(\mathbf{k}_\epsilon))), \tag{13.11}$$

where T is the grading shift functor, is called in [28, 31] the Hecke algebra of the triple (A, A_0, ϵ).

For every left A–module V and right A–module W the algebra $\mathrm{Hk}^\bullet(A, A_0, \epsilon)$ naturally acts in the spaces $\mathrm{Ext}^\bullet_{A_0}(\mathbf{k}_\epsilon, V)$ and $\mathrm{Tor}^\bullet_{A_0}(W, \mathbf{k}_\epsilon)$, from the right and from the left, respectively (see [28, 31] for details).

Note that if $H^\bullet((\mathrm{Ind}^A_{A_0})^L(\mathbf{k}_\epsilon)) = \mathrm{Tor}^\bullet_{A_0}(A, \mathbf{k}_\epsilon) = A \otimes_{A_0} \mathbf{k}_\epsilon$ the object $(\mathrm{Ind}^A_{A_0})^L(\mathbf{k}_\epsilon) \in D^-(A)$ is isomorphic in $D^-(A)$ to the complex $\ldots \to 0 \to A \otimes_{A_0} \mathbf{k}_\epsilon \to 0 \to \ldots$ (with $A \otimes_{A_0} \mathbf{k}_\epsilon$ at the 0-th place) and hence

$$\mathrm{Hk}^\bullet(A, A_0, \epsilon) = \mathrm{Ext}^\bullet_A(A \otimes_{A_0} \mathbf{k}_\epsilon, A \otimes_{A_0} \mathbf{k}_\epsilon). \tag{13.12}$$

In particular,

$$\mathrm{Hk}^n(A, A_0, \epsilon) = 0,\ n < 0, \tag{13.13}$$

and the zeroth graded component of the algebra $\mathrm{Hk}^\bullet(A, A_0, \epsilon)$ takes the form

$$\mathrm{Hk}^0(A, A_0, \epsilon) = \mathrm{Hom}_A(A \otimes_{A_0} \mathbf{k}_\epsilon, A \otimes_{A_0} \mathbf{k}_\epsilon). \tag{13.14}$$

In view of definition (10.13) and the last formula it is natural to consider the Hecke algebra associated to the triple $(\mathbb{C}_\mathcal{A}[G^*], \mathbb{C}_\mathcal{A}[M_-], \chi^s_q)$ and its specialization $(\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi^s_\varepsilon)$.

Theorem 13.5. *Assume that the roots $\gamma_1, \ldots, \gamma_n$ (or $\gamma_{n+1}, \ldots, \gamma_{l'}$) are simple. Then*

$$\mathrm{Hk}^n((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s)) = 0, \ n < 0,$$

and

$$\mathrm{Hk}^0((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s)) = \mathrm{End}_{\mathbb{C}_\varepsilon[G^*]}(\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}) = W_\varepsilon^s(G)^{opp}.$$

Proof. First observe that the definition of the algebra $\mathbb{C}_\varepsilon[G^*]$ implies that it has a Poincaré–Birkhoff–Witt basis similar to that for $U_\varepsilon(\mathfrak{g})$. Since the roots $\gamma_1, \ldots, \gamma_n$ are simple the segment $\Delta_{\mathfrak{m}_+}$ has a complementary segment $\Delta^0_{\mathfrak{m}_+}$ in Δ_+. This fact and the existence of the Poincaré–Birkhoff–Witt basis for $\mathbb{C}_\varepsilon[G^*]$ imply that $\mathbb{C}_\varepsilon[G^*]$ is a free $\mathbb{C}_\varepsilon[M_-]$–module with respect to multiplication by elements from $\mathbb{C}_\varepsilon[M_-]$ on $\mathbb{C}_\varepsilon[G^*]$ from the right. Therefore

$$\mathrm{Tor}^\bullet_{\mathbb{C}_\varepsilon[M_-]}(\mathbb{C}_\varepsilon[G^*], \mathbb{C}_{\chi_\varepsilon^s}) = \mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s},$$

and the discussion of the general properties of Hecke algebras in the beginning of this section gives

$$\mathrm{Hk}^n(\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s) = 0, \ n < 0,$$

and

$$\mathrm{Hk}^0(\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s) = \mathrm{End}_{\mathbb{C}_\varepsilon[G^*]}(\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}) = W_\varepsilon^s(G)^{opp}.$$

(see formulas (13.13) and (13.14)). This completes the proof.

□

Note that the higher components

$$\mathrm{Hk}^n((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s)), n > 0$$

of the algebra $\mathrm{Hk}^\bullet((\mathbb{C}_\varepsilon[G^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s))$ are not equal to zero.

Consider, for instance, the case when $\mathfrak{g} = \mathfrak{sl}_2$. In that case the only nontrivial element of the Weyl group is $s = -1$, and hence $\frac{1+s}{1-s} = 0$. Assume that $\varepsilon^2 \neq 1$. Then the algebra $\mathbb{C}_\varepsilon[SL_2^*]$ is the complex associative algebra with generators $e, f, t^{\pm 1}$ subject to the relations

$$tt^{-1} = t^{-1}t = 1, \ tet^{-1} = \varepsilon e_j, \ tft^{-1} = \varepsilon^{-1} f, \ ef - fe = \frac{t^2 - t^{-2}}{\varepsilon - \varepsilon^{-1}}. \tag{13.15}$$

The subalgebra $\mathbb{C}_\varepsilon[M_-]$ is generated by the element e. Fix a character χ_ε^s of $\mathbb{C}_\varepsilon[M_-]$ defined by $\chi_\varepsilon^s(e)$=1.

The center of the algebra $\mathbb{C}_\varepsilon[SL_2^*]$ contains the element

$$\Omega = \frac{\varepsilon t^2 + \varepsilon^{-1} t^{-2}}{(\varepsilon - \varepsilon^{-1})^2} + fe.$$

Denote by v the image of $1 \in \mathbb{C}_\varepsilon[SL_2^*]$ in the left $\mathbb{C}_\varepsilon[SL_2^*]$–module $Q = \mathbb{C}_\varepsilon[SL_2^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}$. One checks straightforwardly that the elements $v_{mk} = t^m \Omega^k v$, $m \in \mathbb{Z}, k \in \mathbb{N}$ form a linear basis of Q. It follows that as a $\mathbb{C}_\varepsilon[M_-]$–module Q is the direct sum of one–dimensional modules $\mathbb{C}_{mk} = \mathbb{C} v_{mk}$, and the element e acts on $\mathbb{C}_{mk}$ by multiplication by ε^{-m}. Now by Frobenius reciprocity

$$\mathrm{Hk}^n(\mathbb{C}_\varepsilon[SL_2^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s) = \mathrm{Ext}^n_{\mathbb{C}_\varepsilon[M_-]}(\mathbb{C}_{\chi_\varepsilon^s}, \mathbb{C}_\varepsilon[SL_2^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}),$$

and from the description of Q as a left $\mathbb{C}_\varepsilon[M_-]$–module given above we obtain linear space isomorphisms

$$\mathrm{Hk}^0(\mathbb{C}_\varepsilon[SL_2^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s) \simeq \bigoplus_{m \in \mathbb{Z}, \varepsilon^m = 1, k \in \mathbb{N}} \mathbb{C}_{mk},$$

$$\mathrm{Hk}^1(\mathbb{C}_\varepsilon[SL_2^*], \mathbb{C}_\varepsilon[M_-], \chi_\varepsilon^s) \simeq \bigoplus_{m \in \mathbb{Z}, k \in \mathbb{N}} \mathbb{C}_{mk}/(\varepsilon^m - 1)\mathbb{C}_{mk} = \bigoplus_{m \in \mathbb{Z}, \varepsilon^m = 1, k \in \mathbb{N}} \mathbb{C}_{mk}.$$

The other components of the algebra $\mathrm{Hk}^\bullet(\mathbb{C}_\varepsilon[SL_2^*],\mathbb{C}_\varepsilon[M_-],\chi_\varepsilon^s)$ vanish. But as we see

$$\mathrm{Hk}^1(\mathbb{C}_\varepsilon[SL_2^*],\mathbb{C}_\varepsilon[M_-],\chi_\varepsilon^s)\neq 0$$

for any ε. This is related to the fact that the action of $\mathbb{C}_\varepsilon[M_-]$ on Q is semisimple while in the Lie algebra case a similar action is nilpotent.

14. Skryabin equivalence for quantum groups

In this section we establish a remarkable equivalence between the category of $W_\varepsilon^s(G)$–modules and a certain category of $\mathbb{C}_\varepsilon[G^*]$–modules. This equivalence is a quantum group counterpart of Skryabin equivalence established in the Appendix to [20].

Let V be a $\mathbb{C}_{\mathcal{A}'}[G^*]$–module. An element $v\in V$ is called a Whittaker vector (with respect to the subalgebra $\mathbb{C}_{\mathcal{A}'}[M_-]$) if $mv=\chi_q^s v$ for any $m\in\mathbb{C}_{\mathcal{A}'}[M_-]$. For any $\mathbb{C}_{\mathcal{A}'}[G^*]$–module V the space $\mathrm{Wh}(V)=\{v\in V:\ mv=\chi_q^s v \text{ for any } m\in\mathbb{C}_{\mathcal{A}'}[M_-]\}$ is called the space of Whittaker vectors of V. Let $\mathcal{C}_q^s$ be the category of $\mathbb{C}_{\mathcal{A}'}[G^*]$–modules which are free as $\mathcal{A}'$–modules and generated by Whittaker vectors, with morphisms being homomorphisms of $\mathbb{C}_{\mathcal{A}'}[G^*]$–modules. We also denote by $\mathcal{C}_\varepsilon^s$ the category of $\mathbb{C}_\varepsilon[G^*]$–modules which are specializations of modules from $\mathcal{C}_q^s$ at $q=\varepsilon\in\mathbb{C}$. The spaces of Whittaker vectors for modules from $\mathcal{C}_\varepsilon^s$ are defined similarly to the case of modules from $\mathcal{C}_q^s$.

Note that it is not obvious that the category $\mathcal{C}_q^s$ is abelian. As we shall see from the next theorem this is indeed the case.

Theorem 14.6. *Assume that the roots $\{\gamma_1,\ldots,\gamma_n\}$ (or $\gamma_{n+1},\ldots,\gamma_{l'}$) are simple and that the numbers t_i defined in (11.5) are not equal to zero for all i. Then the functor $E\mapsto Q_{\mathcal{A}'}\otimes_{W_q^s(G)}E$, where $Q_{\mathcal{A}'}=\mathbb{C}_{\mathcal{A}'}[G^*]\otimes_{\mathbb{C}_\varepsilon[M_-]}\mathbb{C}_{\chi_q^s}$, is an equivalence of the category of left $W_q^s(G)$–modules which are free as $\mathcal{A}'$–modules and the category $\mathcal{C}_q^s$. The inverse equivalence is given by the functor $V\mapsto\mathrm{Wh}(V)$. In particular, the latter functor is exact and the category $\mathcal{C}_q^s$ is abelian.*

For generic $\varepsilon\in\mathbb{C}$ the functor $E\mapsto Q_\varepsilon\otimes_{W_\varepsilon^s(G)}E$, where $Q_\varepsilon=\mathbb{C}_\varepsilon[G^*]\otimes_{\mathbb{C}_\varepsilon[M_-]}\mathbb{C}_{\chi_\varepsilon^s}$, is an equivalence of the category of left $W_\varepsilon^s(G)$–modules and the category $\mathcal{C}_\varepsilon^s$. The inverse equivalence is given by the functor $V\mapsto\mathrm{Wh}(V)$. In particular, the latter functor is exact and the category $\mathcal{C}_\varepsilon^s$ is abelian.

Proof. Let E be a $W_q^s(G)$–module. First we observe that by the definition of the algebra W_q^s we have $\mathrm{Wh}(Q_{\mathcal{A}'}\otimes_{W_q^s(G)}E)=E$. Therefore to prove the theorem it suffices to check that for any $V\in\mathcal{C}_q^s$ the canonical map $f:Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)\to V$ is an isomorphism. Since by construction all the Whittaker vectors from $\mathrm{Wh}(V)$ do not belong to the kernel of f and V is generated by the Whittaker vectors the map f is surjective. We have to prove that f is injective.

Let $V'\subset Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)$ be the kernel of f. Consider the $\mathcal{A}'$–subalgebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ in $U_q^s(\mathfrak{g})$ generated by elements $e_\beta'=e_\beta-\frac{\chi_q^s(\tilde{e}_\beta)}{1-q_\beta^{-2}}$, $\beta\in\Delta_{\mathfrak{m}_+}$.

We claim that the defining relations in the subalgebra $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ are given by formula similar to (4.6),

$$e_\alpha' e_\beta' - q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}e_\beta' e_\alpha' = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}\tilde{C}(k_1,\ldots,k_n)(e_{\delta_1}')^{k_1}(e_{\delta_2}')^{k_2}\ldots(e_{\delta_n}')^{k_n}, \qquad (14.16)$$

where $\tilde{C}(k_1,\ldots,k_n)\in\mathcal{A}'$ and that $U_{\mathcal{A}'}^s(\mathfrak{m}_+)/(1-q^{\frac{1}{2d}})U_{\mathcal{A}'}^s(\mathfrak{m}_+)=U(\mathfrak{m}_+)$.

Indeed, consider the defining relations in the subalgebra $U_{\mathcal{A}}^s(\mathfrak{m}_+)$,

$$e_\alpha e_\beta - q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}e_\beta e_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C'(k_1,\ldots,k_n)e_{\delta_1}^{k_1}e_{\delta_2}^{k_2}\ldots e_{\delta_n}^{k_n}.$$

where $C'(k_1,\ldots,k_n)\in\mathbb{C}[q^{\frac{1}{2d}},q^{-\frac{1}{2d}}]$. Commutation relations between quantum analogues of root vectors obtained in Proposition 4.2 in [15] imply that each function $C'(k_1,\ldots,k_n)$ has a zero of order $k_1+\ldots+k_n-1$ at point $q=1$. Therefore, as we observed in Section 10, one can write the following defining relations for the generators $\tilde{e}_\beta=(1-q_\beta^{-2})e_\beta$, $\beta\in\Delta_{\mathfrak{m}_+}$ in the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$,

$$\tilde{e}_\alpha\tilde{e}_\beta-q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}\tilde{e}_\beta\tilde{e}_\alpha=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C''(k_1,\ldots,k_n)\tilde{e}_{\delta_1}^{k_1}\tilde{e}_{\delta_2}^{k_2}\ldots\tilde{e}_{\delta_n}^{k_n},\tag{14.17}$$

where $C''(k_1,\ldots,k_n)\in\mathcal{A}'$, and each function $C''(k_1,\ldots,k_n)$ has a zero of order 1 at point $q=1$. Now we rewrite the last relations in terms of the new generators $\tilde{e}'_\beta=\tilde{e}_\beta-\chi_q^s(\tilde{e}_\beta)$ of $\mathbb{C}_{\mathcal{A}'}[M_-]$. We claim that the defining relations for the new generators $\tilde{e}'_\beta$ in the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$ have the same form,

$$\tilde{e}'_\alpha\tilde{e}'_\beta-q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}\tilde{e}'_\beta\tilde{e}'_\alpha=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C'''(k_1,\ldots,k_n)(\tilde{e}'_{\delta_1})^{k_1}(\tilde{e}'_{\delta_2})^{k_2}\ldots(\tilde{e}'_{\delta_n})^{k_n},\tag{14.18}$$

and that the terms linear in $\tilde{e}'_{\delta_i}$ in the r.h.s. of these relations are the same as in (14.17). Obviously, relations (14.17) rewritten in terms of the new generators $\tilde{e}'_\beta$ have the same l.h.s. as relations (14.17) and can not contain constant terms in the r.h.s. since the character χ_q^s vanishes on all generators $\tilde{e}'_\beta$. It is also clear that the only terms in the r.h.s. of relations (14.18) for $\tilde{e}'_\alpha$ and $\tilde{e}'_\beta$ which may contain root vectors corresponding to roots γ not satisfying the restriction $\alpha<\gamma<\beta$ are linear terms, and the coefficients in front of them have zeroes of order 1 at point $q=1$. Such terms would give a nontrivial contribution to the linearization of the natural Poisson bracket of the elements $p(\tilde{e}'_\beta)$ in $\mathbb{C}[M_-]$, the linearization being taken with respect to the grading by the powers of the elements $p(\tilde{e}'_\beta)$. By construction this linearization is the linearization of the Poisson bracket on the Poisson manifold M_- at point u. But since the map $\mathbb{C}[M_-]\to\mathbb{C}$, $f\mapsto f(u)$ is a character of the Poisson algebra $\mathbb{C}[M_-]$ we deduce from (10.2) for $L^-=u\exp(l), l\in\mathfrak{m}_-$ that up to terms of higher order in l

$$\{l_1^{-,W},l_2^{-,V}\}=2[{r_+^s}^{VW},l_1^{-,W}+l_2^{-,V}],$$

where the notation as in (10.2), and hence the linearization of the Poisson bracket on the Poisson manifold M_- at point u coincides with the linearization at point 1. Thus the linear terms in the r.h.s. of the relations for $\tilde{e}'_\beta$ are the same as in (14.17), and we arrive at (14.18). Dividing (14.18) by $(1-q_\beta^{-2})(1-q_\alpha^{-2})$ we obtain (14.16) and the discussion of the linear terms in the r.h.s. of (14.18) also implies that $U^s_{\mathcal{A}'}(\mathfrak{m}_+)/(1-q^{\frac{1}{2d}})U^s_{\mathcal{A}'}(\mathfrak{m}_+)=U(\mathfrak{m}_+)$.

Now using the fact that $e'_\beta=\frac{1}{(1-q_\beta^{-2})}\tilde{e}'_\beta$ and that for any $x\in\mathbb{C}_{\mathcal{A}'}[G^*]$ we have $\frac{1}{(1-q_\beta^{-2})}[\tilde{e}'_\beta,x]\in\mathbb{C}_{\mathcal{A}'}[G^*]$ one verifies that the formula $e'_\beta\cdot(x+I'_q)=\frac{1}{(1-q_\beta^{-2})}\rho_{\chi_q^s}(\tilde{e}'_\beta x)=\frac{1}{(1-q_\beta^{-2})}\rho_{\chi_q^s}([\tilde{e}'_\beta,x])$, $\beta\in\Delta_{\mathfrak{m}_+}$, $x+I_q\in Q_{\mathcal{A}'}$ defines a representation of the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ in $Q_{\mathcal{A}'}$.

Observe that the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ is naturally augmented in such a way all the generators $\tilde{e}'_\beta$, $\beta\in\Delta_{\mathfrak{m}_+}$ act on the corresponding rank one module in the trivial way.

Since the annihilator of $\mathrm{Wh}(V)\subset Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)$ in $\mathbb{C}_{\mathcal{A}'}[G^*]$ contains the kernel of the character χ_q^s we can define an action of the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)$ by $e'_\beta\cdot xv=\frac{1}{(1-q_\beta^{-2})}\tilde{e}'_\beta xv=\frac{1}{(1-q_\beta^{-2})}[\tilde{e}'_\beta,x]v$, $\beta\in\Delta_{\mathfrak{m}_+}$, $x\in\mathbb{C}_{\mathcal{A}'}[G^*]$, $v\in\mathrm{Wh}(V)$, where we also used the fact that $e'_\beta=\frac{1}{(1-q_\beta^{-2})}\tilde{e}'_\beta$ and that for any $x\in\mathbb{C}_{\mathcal{A}'}[G^*]$ we have $\frac{1}{(1-q_\beta^{-2})}[\tilde{e}'_\beta,x]\in\mathbb{C}_{\mathcal{A}'}[G^*]$.

Now consider the specialization $Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)_1=\mathbb{C}[G^*]/I\otimes_{W^s(G)}\mathrm{Wh}(\overline{V})$, $\overline{V}=V/(q^{\frac{1}{2d}}-1)V$, $\mathrm{Wh}(\overline{V})=\mathrm{Wh}(V)(q^{\frac{1}{2d}}-1)$ of $Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)$ at $q=1$. The action of the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ in the space $Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)$ generates a representation of $U(\mathfrak{m}_+)$ in the space $Q_{\mathcal{A}'}\otimes_{W_q^s(G)}\mathrm{Wh}(V)_1$. We denote this representation by ϱ.

The action of the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'}$ also generates an action of $U(\mathfrak{m}_+)$ in $\mathbb{C}[G^*]/I$ which by definition coincides with action (11.1) on generators. Namely, for the action of $U(\mathfrak{m}_+)$ in $\mathbb{C}[G^*]/I$ we have

$$X_\beta \cdot (x+I) = \frac{1}{2d_\beta}\rho_{\chi^s}(\{p(\tilde{e}_\beta), x\}), \tag{14.19}$$

where $\beta \in \Delta_{\mathfrak{m}_+}$, $X_\beta \in \mathfrak{m}_+$ is the corresponding root vector, $X_\beta = e'_\beta \pmod{(q^{\frac{1}{2d}}-1)}$, $x+I \in \mathbb{C}[G^*]/I$, $x \in \mathbb{C}[G^*]$ and $d_\beta = d_i$ if β is Weyl group conjugate to the simple root α_i. Observe also that the action of $U(\mathfrak{m}_+)$ in the space $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$ can be obtained from the action of $U(\mathfrak{m}_+)$ in $\mathbb{C}[G^*]/I$ if we identify $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1 = \mathbb{C}[G^*]/I \otimes_{W^s(G)} \mathrm{Wh}(\overline{V})$ with a subquotient of $\mathbb{C}[G^*]/I \otimes \mathrm{Wh}(\overline{V}) = \mathbb{C}[M_+] \otimes W^s(G) \otimes \mathrm{Wh}(\overline{V})$ equipped with the induced $U(\mathfrak{m}_+)$–action.

Note that action (11.1) is related to the action of M_+ on $\mathbb{C}[q(\mu_{M_+}^{-1}(u))]$ induced by the action of M_+ on $q(\mu_{M_+}^{-1}(u))$ by conjugations; the exact relation is given by formula (9.5) where $A, A^*, K, T, K^\perp$ and μ are specified as in the proof of Proposition 11.2. Recall also that by Proposition 12.2 and Theorem 12.3 the action of M_+ on $\mathbb{C}[q(\mu_{M_+}^{-1}(u))]$ induced by the action of M_+ on $q(\mu_{M_+}^{-1}(u))$ by conjugations is generated by the action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+]$ by left invariant differential operators and by the isomorphism $\mathbb{C}[M_+] \otimes W^s(G) = \mathbb{C}[q(\mu_{M_+}^{-1}(u))]$.

Now let $V_1' \subset Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$ be the specialization of V' at $q=1$. Recall that V is $\mathcal{A}'$–free and $\mathcal{A}'$ has no zero divisors. Therefore without loss of generality we can assume, using the definition of the the above constructed action of $U_{\mathcal{A}'}(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)$, that V' is invariant under this action, and hence V_1' is invariant under the representation ϱ of $U(\mathfrak{m}_+)$. By construction V_1' is invariant under the action by multiplication by elements from $\mathbb{C}[G^*]$ on $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$. Formula (9.5), with $A, A^*, K, T, K^\perp$ and μ specified as in the proof of Proposition 11.2, only contains operators of multiplication by functions from $\mathbb{C}[G^*]$ and action operators for action (14.19) of $U(\mathfrak{m}_+)$ on $\mathbb{C}[G^*]/I$. Since $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$ is a subquotient of $\mathbb{C}[G^*]/I \otimes \mathrm{Wh}(\overline{V}) = \mathbb{C}[M_+] \otimes W^s(G) \otimes \mathrm{Wh}(\overline{V})$ the last two facts imply that V_1' is invariant under the action ϱ_1 of $U(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$ induced by the action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+]$ by left invariant differential operators.

Since the last action of the nilpotent Lie algebra $\mathfrak{m}_+$ on $\mathbb{C}[M_+]$ is locally nilpotent the induced action ϱ_1 of $\mathfrak{m}_+$ on $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$ is locally nilpotent as well. Therefore by Engel theorem the invariant subspace $V_1' \subset Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)_1$ (if it is nontrivial) contains a nonzero vector annihilated by any element from $\mathfrak{m}_+$ with respect to the action ϱ_1. From formula (9.5) one can immediately deduce that the actions ϱ and ϱ_1 have the same invariant elements (compare with Proposition 11.2). Therefore V_1' (if it is nontrivial) contains a nonzero vector annihilated by any element from $\mathfrak{m}_+$ with respect to the action ϱ as well.

Now observe that by construction V_1' is contained in the kernel of the specialization f_1 of the homomorphism f, $f_1 : \mathbb{C}[G^*]/I \otimes_{W^s(G)} \mathrm{Wh}(\overline{V}) \to \overline{V}$, where $\overline{V} = V/(q^{\frac{1}{2d}}-1)V$ and $\mathrm{Wh}(\overline{V}) = \mathrm{Wh}(V)(q^{\frac{1}{2d}}-1)$. As we proved above the specialization V_1' of V' at $q=1$ must contain (if V_1' is nontrivial) a nonzero vector which is invariant under the action ϱ of $U(\mathfrak{m}_+)$. Such vectors must belong to the subspace $\mathrm{Wh}(\overline{V}) \subset \mathbb{C}[G^*]/I \otimes_{W^s(G)} \mathrm{Wh}(\overline{V})$ since by the definition of ϱ the space $\mathrm{Wh}(\overline{V}) \cap Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V) \subset Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)$ consists of all vectors invariant under the action ϱ. But by the definition of the map f_1 the subspace $\mathrm{Wh}(\overline{V})$ does not contain nontrivial elements from the kernel of f_1. Therefore $V_1' = 0$, and hence $V' = (q^{\frac{1}{2d}}-1)W'$, $W' \subset Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \mathrm{Wh}(V)$. Since V is $\mathcal{A}'$–free and $\mathcal{A}'$ has no zero divisors we also have $W' \subset V'$. Iterating this process we deduce that any element $w \in V'$ can be represented in the form $w = (q^{\frac{1}{2d}}-1)^B w', w' \in V'$ with arbitrary large $B \in \mathbb{N}$ which is possible only in case when $V' = 0$. Therefore f is injective.

By specializing q to a generic value $\varepsilon \in \mathbb{C}$ and observing that by definition the specialization of the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ at a generic ε is isomorphic to $\mathbb{C}_\varepsilon[M_-]$, and the specialization of the $U_{\mathcal{A}'}(\mathfrak{m}_+)$–module $Q_{\mathcal{A}'}$ at ε is isomorphic to $\mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi^s_\varepsilon}$ we obtain a similar result for the category $\mathcal{C}^s_\varepsilon$. This completes the proof of the theorem.

□

Appendix. Normal orderings of root systems compatible with involutions in Weyl groups

By Theorem A in [22] every involution w in the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ is the longest element of the Weyl group of a Levi subalgebra in $\mathfrak{g}$ with respect to some system of positive roots, and w acts by multiplication by -1 in the Cartan subalgebra $\mathfrak{h}_w \subset \mathfrak{h}$ of the semisimple part $\mathfrak{m}_w$ of that Levi subalgebra. By Lemma 5 in [4] the involution w can also be expressed as a product of $\dim \mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal roots, $w = s_{\gamma_1} \ldots s_{\gamma_n}$, and the roots $\gamma_1, \ldots, \gamma_n$ span the subalgebra $\mathfrak{h}_w$.

If w is the longest element in the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$ with respect to some system of positive roots, where $\mathfrak{m}_w$ is a simple Lie algebra and $\mathfrak{h}_w$ is a Cartan subalgebra of $\mathfrak{m}_w$, then w is an involution acting by multiplication by -1 in $\mathfrak{h}_w$ if and only if $\mathfrak{m}_w$ is of one of the following types: $A_1, B_l, C_l, D_{2n}, E_7, E_8, F_4, G_2$.

Fix a system of positive roots $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$. Let $w = s_{\gamma_1} \ldots s_{\gamma_n}$ be a representation of w as a product of $\dim \mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal positive roots. A normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ is called compatible with the decomposition $w = s_{\gamma_1} \ldots s_{\gamma_n}$ if it is of the following form

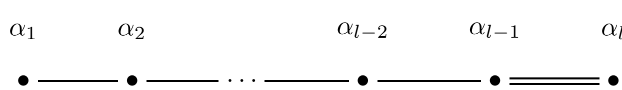
$$\beta_1, \ldots, \beta_{\frac{p-n}{2}}, \gamma_1, \beta^1_{\frac{p-n}{2}+2}, \ldots, \beta^1_{\frac{p-n}{2}+n_1}, \gamma_2, \beta^1_{\frac{p-n}{2}+n_1+2} \ldots, \beta^1_{\frac{p-n}{2}+n_2}, \gamma_3, \ldots, \gamma_n, \quad (14.20)$$

where p is the number of positive roots, and for any two positive roots $\alpha, \beta \in \Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ such that $\gamma_1 \leq \alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$.

Existence of such compatible normal orderings is checked straightforwardly for all simple Lie algebras of types $A_1, B_l, C_l, D_{2n}, E_7, E_8, F_4$ and G_2. In case A_1 this is obvious since there is only one positive root. In the other cases normal orderings defined by the properties described below for each of the types $B_l, C_l, D_{2n}, E_7, E_8, F_4, G_2$ exist and are compatible with decompositions of nontrivial involutions in Weyl group. We use Bourbaki notation for the systems of positive and simple roots (see [1]).

- B_l

Dynkin diagram:



Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$.

Positive roots: ε_i $(1 \leq i \leq l)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$.

The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\varepsilon_1} \ldots s_{\varepsilon_l}$.

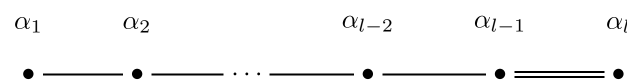
Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\varepsilon_1} \ldots s_{\varepsilon_l}$:

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_l,$$

where the roots $\varepsilon_i - \varepsilon_j$ $(1 \le i < j \le l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(B_l)$ are situated to the left from ε_1, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \le i < j \le l)$ are situated to the right from ε_1.

- C_l

Dynkin diagram:



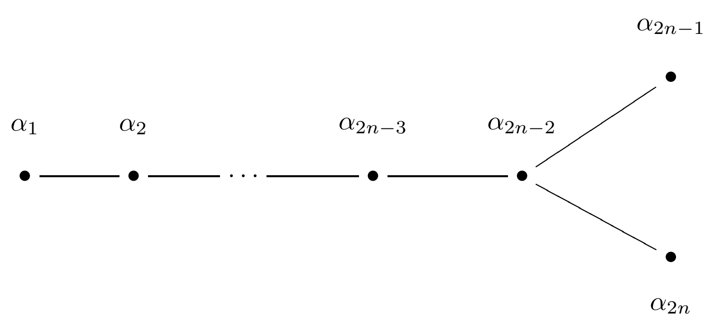
Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$.
Positive roots: $2\varepsilon_i$ $(1 \le i \le l)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \le i < j \le l)$.
The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{2\varepsilon_1} \ldots s_{2\varepsilon_l}$.
Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{2\varepsilon_1} \ldots s_{2\varepsilon_l}$:

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_1, \ldots, 2\varepsilon_2, \ldots, 2\varepsilon_l,$$

where the roots $\varepsilon_i - \varepsilon_j$ $(1 \le i < j \le l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(C_l)$ are situated to the left from $2\varepsilon_1$, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \le i < j \le l)$ are situated to the right from $2\varepsilon_1$.

- D_{2n}

Dynkin diagram:



Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{2n-1} = \varepsilon_{2n-1} - \varepsilon_{2n}, \alpha_{2n} = \varepsilon_{2n-1} + \varepsilon_{2n}$.
Positive roots: $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \le i < j \le 2n)$.
The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \ldots s_{\varepsilon_{2n-1} - \varepsilon_{2n}} s_{\varepsilon_{2n-1} + \varepsilon_{2n}}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

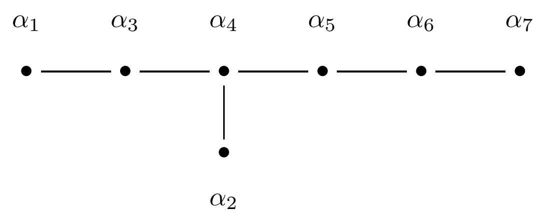
$$w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \ldots s_{\varepsilon_{2n-1} - \varepsilon_{2n}} s_{\varepsilon_{2n-1} + \varepsilon_{2n}} :$$

$$\varepsilon_2 - \varepsilon_3, \varepsilon_4 - \varepsilon_5, \ldots, \varepsilon_{2n-2} - \varepsilon_{2n-1}, \ldots, \varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_{2n-1} - \varepsilon_{2n-2},$$
$$\varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_3 + \varepsilon_4, \ldots, \varepsilon_{2n-1} + \varepsilon_{2n},$$

where the roots $\varepsilon_i-\varepsilon_j$ $(1\leq i<j\leq l)$ forming the subsystem $\Delta_+(A_{l-1})\subset\Delta_+(C_l)$ are situated to the left from $\varepsilon_1+\varepsilon_2$, and the roots $\varepsilon_i+\varepsilon_j$ $(1\leq i<j\leq l)$ are situated to the right from $\varepsilon_1+\varepsilon_2$.

- E_7

  Dynkin diagram:



Simple roots: $\alpha_1=\frac{1}{2}(\varepsilon_1+\varepsilon_8)-\frac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7)$, $\alpha_2=\varepsilon_1+\varepsilon_2,\alpha_3=\varepsilon_2-\varepsilon_1$, $\alpha_4=\varepsilon_3-\varepsilon_2,\alpha_5=\varepsilon_4-\varepsilon_3,\alpha_6=\varepsilon_5-\varepsilon_4,\alpha_7=\varepsilon_6-\varepsilon_5$.

Positive roots: $\pm\varepsilon_i+\varepsilon_j$ $(1\leq i<j\leq 6)$, $\varepsilon_8-\varepsilon_7$, $\frac{1}{2}(\varepsilon_8-\varepsilon_7+\sum_{i=1}^{6}(-1)^{\nu(i)}\varepsilon_i)$ with $\sum_{i=1}^{6}\nu(i)$ odd.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w=s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w,\mathfrak{h}_w)$ compatible with expression

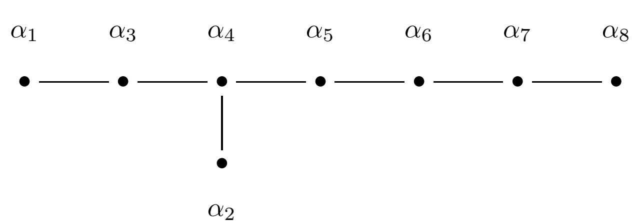
$$w=s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}:$$

$$\alpha_1,\varepsilon_3-\varepsilon_2,\varepsilon_5-\varepsilon_4,\ldots,\varepsilon_8-\varepsilon_7,\ldots,\varepsilon_2-\varepsilon_1,\varepsilon_4-\varepsilon_3,\varepsilon_6-\varepsilon_5,\ldots,$$
$$\varepsilon_6+\varepsilon_5,\ldots,\varepsilon_4+\varepsilon_3,\ldots,\varepsilon_2+\varepsilon_1,$$

where the roots $\pm\varepsilon_i+\varepsilon_j$ $(1\leq i<j\leq 6)$ forming the subsystem $\Delta_+(D_6)\subset\Delta_+(E_7)$ are placed as in case of the compatible normal ordering of the system $\Delta_+(D_6)$, the only roots from the subsystem $\Delta_+(A_5)\subset\Delta_+(D_6)$ situated to the right from the maximal root $\varepsilon_8-\varepsilon_7$ are $\varepsilon_2-\varepsilon_1,\varepsilon_4-\varepsilon_3,\varepsilon_6-\varepsilon_5$, the roots $\varepsilon_i+\varepsilon_j$ $(1\leq i<j\leq 6)$ are situated to the right from $\varepsilon_6+\varepsilon_5$, and a half of the positive roots which do not belong to the subsystem $\Delta_+(D_6)\subset\Delta_+(E_7)$ are situated to the left from $\varepsilon_8-\varepsilon_7$ and the other half of those roots are situated to the right from $\varepsilon_8-\varepsilon_7$.

- E_8

  Dynkin diagram:



Simple roots: $\alpha_1 = \frac{1}{2}(\varepsilon_1+\varepsilon_8) - \frac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7)$, $\alpha_2 = \varepsilon_1+\varepsilon_2, \alpha_3 = \varepsilon_2-\varepsilon_1$, $\alpha_4 = \varepsilon_3-\varepsilon_2, \alpha_5 = \varepsilon_4-\varepsilon_3, \alpha_6 = \varepsilon_5-\varepsilon_4, \alpha_7 = \varepsilon_6-\varepsilon_5, \alpha_8 = \varepsilon_7-\varepsilon_6$.

Positive roots: $\pm\varepsilon_i+\varepsilon_j$ $(1 \leq i < j \leq 8)$, $\frac{1}{2}(\varepsilon_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}\varepsilon_i)$ with $\sum_{i=1}^{7}\nu(i)$ even.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}s_{\varepsilon_8+\varepsilon_7}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

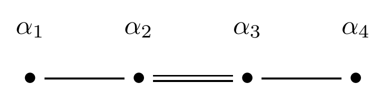
$$w = s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}s_{\varepsilon_8-\varepsilon_7}:$$

$$\alpha_1, \varepsilon_3-\varepsilon_2, \varepsilon_5-\varepsilon_4, \varepsilon_7-\varepsilon_6, \ldots, \varepsilon_2-\varepsilon_1, \varepsilon_4-\varepsilon_3, \varepsilon_6-\varepsilon_5,$$
$$\varepsilon_8-\varepsilon_7, \ldots, \varepsilon_8+\varepsilon_7, \ldots, \varepsilon_6+\varepsilon_5, \ldots, \varepsilon_4+\varepsilon_3, \ldots, \varepsilon_2+\varepsilon_1,$$

where the roots $\pm\varepsilon_i+\varepsilon_j$ $(1 \leq i < j \leq 8)$ forming the subsystem $\Delta_+(D_8) \subset \Delta_+(E_8)$ are placed as in case of the compatible normal ordering of the system $\Delta_+(D_8)$, the roots $\varepsilon_i+\varepsilon_j$ $(1 \leq i < j \leq 8)$ are situated to the right from $\varepsilon_8+\varepsilon_7$; the positive roots which do not belong to the subsystem $\Delta_+(D_8) \subset \Delta_+(E_8)$ can be split into two groups: the roots from the first group contain $\frac{1}{2}(\varepsilon_8+\varepsilon_7)$ in their decompositions with respect to the basis $\varepsilon_i, i = 1, \ldots, 8$, and the roots from the second group contain $\frac{1}{2}(\varepsilon_8-\varepsilon_7)$ in their decompositions with respect to the basis $\varepsilon_i, i = 1, \ldots, 8$; a half of the roots from the first group are situated to the left from $\varepsilon_2-\varepsilon_1$ and the other half of those roots are situated to the right from $\varepsilon_8+\varepsilon_7$; a half of the roots from the second group are situated to the left from $\varepsilon_2-\varepsilon_1$ and the other half of those roots are situated to the right from $\varepsilon_8-\varepsilon_7$.

- F_4

Dynkin diagram:



Simple roots: $\alpha_1 = \varepsilon_2-\varepsilon_3, \alpha_2 = \varepsilon_3-\varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$.

Positive roots: ε_i $(1 \leq i \leq 4)$, $\varepsilon_i-\varepsilon_j, \varepsilon_i+\varepsilon_j$ $(1 \leq i < j \leq 4)$, $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\varepsilon_1}s_{\varepsilon_2}s_{\varepsilon_3}s_{\varepsilon_4}$.

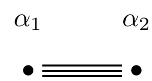
Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\varepsilon_1}s_{\varepsilon_2}s_{\varepsilon_3}s_{\varepsilon_4}$:

$$\alpha_4, \varepsilon_1-\varepsilon_2, \ldots, \varepsilon_3-\varepsilon_4, \ldots, \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_4,$$

where the roots $\varepsilon_i \pm \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(B_4) \subset \Delta_+(F_4)$ are situated as in case of B_4, and a half of the positive roots which do not belong to the subsystem $\Delta_+(B_4) \subset \Delta_+(F_4)$ are situated to the left from ε_1 and the other half of those roots are situated to the right from ε_1.

- G_2

Dynkin diagram:



Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Positive roots: $\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2$.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\alpha_1} s_{3\alpha_1+2\alpha_2}$.

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\alpha_1} s_{3\alpha_1+2\alpha_2}$:

$$\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, \alpha_1.$$

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