

VANDERMONDE FACTORIZATIONS OF A REGULAR HANKEL MATRIX AND THEIR APPLICATION ON THE COMPUTATION OF BÉZIER CURVES

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Abstract. In this paper, a new method to compute a Bézier curve of degree $n = 2m - 1$ is introduced, here formulated as a Bernstein-Hankel form in \mathbb{C}^m , that is, each coordinate of the curve is of the form $e_m^T B_m^e(s) H B_m^e(s)^T e_m$, where $B_m^e(s)$ is a $m \times m$ lower triangular Bernstein matrix and H is a Hankel matrix. The method depends on Vandermonde factorizations of a regular Hankel matrix, and so we begin with a proof, which utilizes Pascal matrices techniques, that given a regular Hankel matrix H , there is a finite set of complex numbers γ such that $x^m - p_{m-1}x^{m-1} - \dots - p_0$ has multiple roots, where $(p_0 \dots p_{m-1}) = (h_{m+1} \dots h_n \gamma) H^{-1}$. Therefore, a Vandermonde factorization of H can be accomplished by taking a complex number at random, and the Bernstein-Hankel form can be easily calculated, thus yielding points on the Bézier curve. We also see that even when H is nearly singular, the method still works by shifting the skew-diagonal of H . By comparing this new method with a Pascal matrix method and Casteljaou's, we see that the results suggest that this new method is very effective with regard to accuracy and time of computation for various values of n .

Key words. Pascal matrix, Bernstein matrix, Bézier curve, Hankel form, Vandermonde factorization

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1. Introduction. Let H be a Hankel matrix of order n , i.e., $(\forall i, j \in \{1, \dots, n\}) H_{ij} = h_{i+j-1}$. A very known theorem says that, if H is nonsingular, then a Vandermonde matrix V and a diagonal matrix D exist such that $H = VDV^T$. There is a proof of this fact in [9], which utilizes a class of matrices arisen in the theory of root separation of algebraic polynomials, namely the class of Bezoutians. Here, in section § 2, from a procedure that is currently utilized in linear prediction to estimate parameters in exponential modeling, it is showed that the spectrum of the companion matrix $C = C(x_\gamma)$, where x_γ is the solution of the linear prediction system $Hx = y_\gamma$, with $y_\gamma = (h_{n+1} \dots h_{2n-1} \gamma)^T$, is simple for all but a finite set of γ . For the values belonging to this finite set, there is a more general factorization: $H = V_c D V_c^T$, where V_c is a confluent Vandermonde matrix and D is a block diagonal matrix, as it can be seen in [4]. Our approach to the proof of the Vandermonde factorization of a nonsingular Hankel matrix is very similar to the one found in [7], but the proofs are distinct. For instance, we make here use of generalized Pascal matrices to quickly obtain some general properties of Hankel matrices.

In section §3, we see that a Bézier curve of degree $n - 1$, where $n = 2m - 1$, can be described as a Bernstein-Hankel form on \mathbb{C}^m . Also, in this section a new algorithm to compute Bézier curves is proposed, from a Vandermonde factorization of the associated Hankel matrix. In section § 4, results of numerical experiments are presented, which strongly suggest that we can compute those curves in a very fast and precise way. That is corroborated from the comparisons done with the Casteljaou's method ([5]) with various values of n . On the other hand, however, several experiments indicate that the computation of Vandermonde factorization of a Hankel matrix is sensitive to its condition with respect to inversion. However, once its skew-diagonal entries are shifted toward skew-diagonal dominance the precision of the computation

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improves, which is a simple and efficient way to deal with the instability of Vandermonde factorization of ill-conditioned Hankel matrices, at least for the computation of Bézier curves from this approach.

2. Vandermonde factorizations of a nonsingular Hankel matrix.

Let $H = \begin{pmatrix} h_1 & h_2 & \dots & h_n \\ \vdots & \vdots & \vdots & \vdots \\ h_{n-1} & h_n & \dots & h_{2n-2} \\ h_n & h_{n+1} & \dots & h_{2n-1} \end{pmatrix}$. Suppose H is nonsingular. Let x_γ be the

solution of the linear prediction system $Hx = y_\gamma$, where $y_\gamma = (h_{n+1} \dots h_{2n-1} \gamma)^T$. We want to show that the set of $\gamma \in \mathbb{C}$ for which the companion matrix $C_\gamma = \text{compan}(x_\gamma)$ is not diagonalizable is finite. Since C_γ is a nonderogatory matrix, it suffices to show that S , the set of scalars γ such that the spectrum of C_γ is not simple, is finite. This means that, out of this set, the characteristic polynomial of C_γ , $p_\gamma(x)$, doesn't have multiple roots. If $a = (a_0 \dots a_{n-1})^T$ and $b = (b_0 \dots b_{n-1})^T$ are the respective solutions of $Hx = e_n = (0 \dots 0 1)^T$ and $Hx = (h_{n+1} \dots h_{2n-1} 0)^T$, then $p_\gamma(x) = r(x) - \gamma s(x)$, where $r(x) = x^n - b_{n-1}x^{n-1} - \dots - b_1x - b_0$ and $s(x) = a_{n-1}x^{n-1} + \dots + a_1x + a_0$. It is not difficult to see that S is finite iff $r(x)$ and $s(x)$ don't have any common root.

LEMMA 2.1. *Let H be a $n \times n$ nonsingular Hankel matrix. If $a = (a_0 \dots a_{n-1})^T$ and $b = (b_0 \dots b_{n-1})^T$ are the respective solutions of $Hx = e_n$ and $Hx = (h_{n+1} \dots h_{2n-1} 0)^T$, then $a_0 \neq 0$ or $b_0 \neq 0$.*

Proof.

Suppose $|H(1 : n-1, 2 : n)| \neq 0$. Therefore, from Cramer's rule, $a_0 \neq 0$. Let x_1, \dots, x_{n-1} be the unique scalars such that

$$x_1 \begin{pmatrix} h_2 \\ \vdots \\ h_n \end{pmatrix} + \dots + x_{n-1} \begin{pmatrix} h_n \\ \vdots \\ h_{2n-2} \end{pmatrix} = \begin{pmatrix} h_{n+1} \\ \vdots \\ h_{2n-1} \end{pmatrix}.$$

Hence, $x = (x_0 \ x_1 \ \dots \ x_{n-1})^T = \gamma a + b$ is the solution of $Hx = (h_{n+1} \dots h_{2n-1} \gamma)^T$, with $x_0 = 0$, iff $\gamma = x_1 h_{n+1} + \dots + x_{n-1} h_{2n-1}$. For other complex numbers γ , $x_0 = \gamma a_0 + b_0 \neq 0$, that is, $a_0 \neq 0$ or $b_0 \neq 0$. Notice that $a_0 \neq 0$, and $b_0 = 0$ iff $x_1 h_{n+1} + \dots + x_{n-1} h_{2n-1} = 0$.

Now, suppose $H(1 : n-1, 2 : n) = H(2 : n, 1 : n-1)$ is singular. First, since H is nonsingular, the dimension of $\text{span}\{H(2 : n, 1), \dots, H(2 : n, n-1), H(2 : n, n)\}$ is $(n-1)$, as well as the dimension of $\text{span}\{H(1 : n-1, 1), \dots, H(1 : n-1, n-1), H(1 : n-1, n)\}$. Hence, $H(2 : n, n) \notin \text{span}\{H(2 : n, 1), \dots, H(2 : n, n-1)\}$, whose dimension is $n-2$. On the other side, $H(2 : n, n) \in \text{span}\{H(1 : n-1, 1), \dots, H(1 : n-1, n)\} = \text{span}\{H(1 : n-1, 1), H(2 : n, 1), \dots, H(2 : n, n-1)\}$, and so, there exist x_0, \dots, x_{n-1} , where x_0 is different from zero and unique, such that

$$\begin{pmatrix} h_{n+1} \\ \vdots \\ h_{2n-1} \end{pmatrix} = x_0 \begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \end{pmatrix} + x_1 \begin{pmatrix} h_2 \\ \vdots \\ h_n \end{pmatrix} + \dots + x_{n-1} \begin{pmatrix} h_n \\ \vdots \\ h_{2n-2} \end{pmatrix}.$$

Observe that, in this case, for all $\gamma \in \mathbb{C}$, $x_0 = b_0 \neq 0$, and $a_0 = 0$. \square

From the above proof, there can be at most one complex number γ such that $p_\gamma(0) = -b_0 - \gamma a_0 = 0$. We can also conclude from the lemma 2.1 that zero is not a

common root of $r(x) = x^n - b_{n-1}x^{n-1} - \dots - b_1x - b_0$ and $s(x) = a_{n-1}x^{n-1} + \dots + a_1x + a_0$.

Now, define $H_\gamma^\kappa = \begin{pmatrix} h_1 & \dots & h_n & h_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ h_n & \dots & h_{2n-1} & \gamma \\ h_{n+1} & \dots & \gamma & \kappa \end{pmatrix}$. Since H is nonsingular, H_γ^κ

is also nonsingular iff $\kappa \neq \kappa_0 = (h_{n+1} \dots h_{2n-1} \gamma) H^{-1}(h_{n+1} \dots h_{2n-1} \gamma)^T$, which is equal to $(h_{n+1} \dots h_{2n-1} \gamma)(b_0 + \gamma a_0 \dots b_{n-2} + \gamma a_{n-2} \ b_{n-1} + \gamma a_{n-1})^T$.

Note that $H_\gamma^\kappa \begin{pmatrix} -b_0 - \gamma a_0 \\ \vdots \\ -b_{n-1} - \gamma a_{n-1} \\ 1 \end{pmatrix} = (\kappa - \kappa_0) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. Therefore, lemma 2.1 can

be rewritten as the following lemma:

LEMMA 2.2. Let $H_\gamma^\kappa = \begin{pmatrix} h_1 & \dots & h_n & h_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ h_n & \dots & h_{2n-1} & \gamma \\ h_{n+1} & \dots & \gamma & \kappa \end{pmatrix}$ be a Hankel matrix,

where $H = H_\gamma^\kappa(1 : n, 1 : n)$ is nonsingular. Suppose that H_γ^κ is also nonsingular, that is, $\kappa \neq (h_{n+1} \dots h_{2n-1} \gamma) H^{-1}(h_{n+1} \dots h_{2n-1} \gamma)^T$. Let p be the solution of $H_\gamma^\kappa x = e_{n+1}$. Then, except for one possible complex number γ , $p_0 \neq 0$.

Now, let α be any complex number and $q_\gamma(x) = p_\gamma(x + \alpha) = r(x + \alpha) - \gamma s(x + \alpha)$. In an analogous way to the proof for $\alpha = 0$, it will be shown that $r(\alpha)$ and $s(\alpha)$ cannot be both null because there can be only one complex number γ such that $q_\gamma(0) = 0$. To prove this, we introduce some notations and definitions in the following.

DEFINITION 2.3. Let $\alpha \in \mathbb{C}$. $P_n[\alpha]$ be the $n \times n$ is the lower triangular matrix defined for each $i, j \in \{1, 2, \dots, n\}$ by

$$(P_n[\alpha])_{ij} = \begin{cases} \alpha^{i-j} \binom{i-1}{j-1} & , \text{ for } i \geq j; \\ 0 & , \text{ otherwise.} \end{cases}$$

$P_n[\alpha]$ is said a generalized lower triangular Pascal matrix. If $\alpha = 1$, $P_n[1] = P_n$ is called the $n \times n$ lower triangular Pascal matrix.

Some results about these matrices (see [6], [1]) are listed in the following lemma:

LEMMA 2.4. Let $P_n[\alpha]$ a generalized lower triangular Pascal matrix. Then,

- (a) $P_n[0] = I_n$;
- (b) $P_n[\alpha]P_n[\beta] = P_n[\alpha + \beta]$;
- (c) $(P_n[\alpha])^{-1} = P_n[-\alpha]$;
- (d) Let $\alpha \neq 0$ and let $G_n(\alpha)$ be the $n \times n$ diagonal matrix such that, for all $k \in \{1, \dots, n\}$, $(G_n(\alpha))_{kk} = \alpha^{k-1}$. Then $P_n[\alpha] = G_n(\alpha)P_nG_n(\alpha)^{-1} = G_n(\alpha)P_nG_n(\alpha^{-1})$. In particular, $P_n^{-1} = G_n(-1)P_nG_n(-1)$.

DEFINITION 2.5. For $s \in [0, 1]$, the $n \times n$ Bernstein matrix $B_n^e(s)$ is the matrix

defined for each $i, j \in \{1, 2, \dots, n\}$ as follows:

$$[B_n^e(s)]_{ij} = \begin{cases} \binom{i-1}{j-1} s^{j-1} (1-s)^{i-j} & , \text{ for } i \geq j; \\ 0 & , \text{ otherwise.} \end{cases}$$

A very important fact about Bernstein matrices, which will be used here later, is the following proposition, whose proof can be found in [1]:

PROPOSITION 2.6. *Let $s \in [0, 1]$ and let $B_e(s)$ be a $n \times n$ Bernstein matrix. Then, $B_n^e(s) = P_n G_n(s) P_n^{-1}$, where P_n is the $n \times n$ lower triangular Pascal matrix and $G_n(s) = \text{diag}([1, s, \dots, s^{n-1}])$.*

In the following, we present some relations between Pascal and Hankel matrices.

LEMMA 2.7. *Let H be a $n \times n$ Hankel matrix and let P_n be the $n \times n$ lower triangular Pascal matrix. Then $P_n H P_n^T$ is still a Hankel matrix.*

Proof. The lemma obviously holds when $n = 1$. Suppose it holds for all Hankel matrices H of order $n \geq 1$. Now, let H be a $(n+1) \times (n+1)$ Hankel matrix and consider $P_{n+1} H P_{n+1}^T$. Since $P_{n+1} H P_{n+1}^T$ is symmetric and $P_{n+1} H P_{n+1}^T = [P_n H P_n^T v; v^T \kappa]$, for some $v \in \mathbb{C}^n$, by induction it suffices to show that, for all $k \in \{1, \dots, n-1\}$, $(P_{n+1} H P_{n+1}^T)_{n+1, k} = (P_{n+1} H P_{n+1}^T)_{n, k+1}$. Now,

$$\begin{aligned} (P_{n+1} H P_{n+1}^T)_{n+1, k} &= e_{n+1}^T P_{n+1} \sum_{j=0}^{k-1} \binom{k-1}{j} H e_{j+1} = \\ &= \sum_{i=0}^n \sum_{j=0}^{k-1} \binom{n}{i} \binom{k-1}{j} e_{i+1}^T H e_{j+1} = \sum_{s=2}^{n+k-1} h_{s-1} \sum_{i=0}^s \binom{n}{i} \binom{k-1}{s-i}, \end{aligned}$$

which is equal, from Vandermonde convolution ([8]), to

$$\begin{aligned} \sum_{s=2}^{n+k-1} h_{s-1} \sum_{i=0}^s \binom{n-1}{i} \binom{k}{s-i} &= \sum_{i=0}^{n-1} \sum_{j=0}^k \binom{n-1}{i} \binom{k}{j} e_{i+1}^T H e_{j+1} = \\ &= e_n^T P_{n+1} \sum_{j=0}^k \binom{k}{j} H e_{j+1} = (P_{n+1} H P_{n+1}^T)_{n, k+1}. \end{aligned}$$

□

COROLLARY 2.8. *Let H be a $n \times n$ Hankel matrix and α be a complex number. Then, $P_n[\alpha] H P_n[\alpha]^T$ is still a Hankel matrix.*

Proof. For $\alpha = 0$, the result follows from lemma 2.7. Let $\alpha \neq 0$. Since from lemma 2.4 $P_n[\alpha] = G_n(\alpha) P_n G_n(\alpha^{-1})$, where $G_n(\alpha) = \text{diag}(1, \alpha, \dots, \alpha^{n-1})$, it suffices to show that $G(\alpha) H G(\alpha)$ is a Hankel matrix. But this is obviously true, for $(G(\alpha) H G(\alpha))_{ij} = h_{i+j-1} \alpha^{i+j-2}$. □

Next we give a proof that $r(x)$ and $s(x)$ don't have any common root by using a generalized Pascal matrix technique.

PROPOSITION 2.9. *Let H be a $n \times n$ nonsingular Hankel matrix. Let $a = (a_0 a_1 \dots a_{n-1})^T$ and $b = (b_0 b_1 \dots b_{n-1})^T$ be the solutions of $Ha = e_n$ and $Hb =$*

$(h_{n+1} \dots h_{2n-1} 0)^T$, respectively. Then $r(x) = x^n - b_{n-1}x^{n-1} - \dots - b_1x - b_0$ and $s(x) = a_{n-1}x^{n-1} + \dots + a_1x + a_0$ don't have any common root.

Proof. Let $\gamma \in \mathbb{C}$ and let $p_\gamma = (-b_0 - \gamma a_0 \dots - b_{n-1} - \gamma a_{n-1} 1)^T$. Let $q_\gamma = (q_0 \dots q_{n-1} 1)^T$ be the vector of coefficients of the polynomial $r(x + \alpha) - \gamma s(x + \alpha)$. We note that $q_\gamma = P_{n+1}[\alpha]^T p_\gamma = P_{n+1}[\alpha]^T (H_\gamma^\kappa)^{-1} e_{n+1}$, for $\kappa = 1 + \kappa_0$. Thus, $H_\gamma^\kappa P_{n+1}[\alpha]^{-T} q_\gamma = e_{n+1}$, and so,

$$\widehat{H}_\gamma^\kappa q_\gamma = P_{n+1}[\alpha]^{-1} H_\gamma^\kappa P_{n+1}[\alpha]^{-T} q_\gamma = P_{n+1}[-\alpha] H_\gamma^\kappa P_{n+1}[-\alpha]^T q_\gamma = e_{n+1}.$$

$\widehat{H}_\gamma^\kappa$ is also nonsingular and, from corollary 2.8, is a Hankel matrix. Since $H_\gamma^\kappa = H_0^0 + \gamma(e_{n+1}e_n^T + e_n e_{n+1}^T) + \kappa e_{n+1}e_{n+1}^T$, we see that $\widehat{H}_\gamma^\kappa = \widehat{H}_0^0 + \gamma(e_{n+1}e_n^T + e_n e_{n+1}^T) + (\kappa - 2n\alpha)e_{n+1}e_{n+1}^T$. That is,

$$\widehat{H}_\gamma^\kappa = \begin{pmatrix} \hat{h}_1 & \dots & \hat{h}_n & \hat{h}_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ \hat{h}_n & \dots & \hat{h}_{2n-1} & \hat{\gamma} \\ \hat{h}_{n+1} & \dots & \hat{\gamma} & \hat{\kappa} \end{pmatrix},$$

where $\widehat{H}_\gamma^\kappa(1 : n, 1 : n) = \widehat{H} = P_n[-\alpha] H P_n[-\alpha]^T$ is nonsingular and $\hat{\gamma} = \gamma + (\widehat{H}_0^0)_{n+1, n}$. Thus, from lemma 2.2, except for one possible complex number γ , $(q_\gamma)_0 \neq 0$. \square

Note that $(q_\gamma)_0 = 0$ only when $s(\alpha) \neq 0$, that is, when $|\widehat{H}_\gamma^\kappa(1 : n - 1, 2 : n)| = |\widehat{H}(1 : n - 1, 2 : n)| \neq 0$. In this case, $\gamma = r(\alpha)/s(\alpha)$.

PROPOSITION 2.10. *Let $\gamma \in \mathbb{C}$. Let $p_\gamma(x) = x^n - b_{n-1}x^{n-1} - \dots - b_0 - \gamma(a_{n-1}x^{n-1} + \dots + a_0) = r(x) - \gamma s(x)$ the characteristic polynomial of $C_\gamma = H_1(\gamma)H^{-1}$, where $H_1(\gamma)$ is the Hankel matrix defined by $H_1(\gamma)e_k = H e_{k+1}$ for $k = 1, \dots, n - 1$ and $H_1(\gamma)e_n = (h_{n+1} \dots h_{2n-1} \gamma)^T$, that is, $C_\gamma = [e_2^T; \dots; e_n^T; (h_{n+1} \dots h_{2n-1} \gamma)H^{-1}]$. Then the set of scalars γ such that C_γ is not diagonalizable is finite.*

Proof. C_γ is a companion matrix, and hence, a nonderogatory matrix. Thus, it suffices to show that the set of scalars γ such that the spectrum of C_γ is not simple is finite.

Let $\alpha \in \mathbb{C}$ be an eigenvalue of C_γ , that is, a root of $p_\gamma(x)$. Therefore, $r(\alpha) = \gamma s(\alpha)$. Then, from proposition 2.9, $s(\alpha) \neq 0$. So, there are two cases:

(i) $r(\alpha) = 0$, and this occurs iff $\gamma = 0$. In this case, C_0 is not diagonalizable iff $r'(\alpha) = 0$.

(ii) $r(\alpha) \neq 0$, which means that $\gamma = r(\alpha)/s(\alpha)$. Therefore, $p'_\gamma(\alpha) = 0$ iff $r'(\alpha) = s'(\alpha) = 0$, or $s'(\alpha) \neq 0$ and $r'(\alpha) = \gamma s'(\alpha)$.

Therefore, since $s \neq 0$ and r/s is not a constant, α is contained in the set of the roots of $r's - rs'$, which has at most $2(n - 1)$ elements. Hence, we can conclude that $\{\gamma \in \mathbb{C} \mid C_\gamma \text{ is not diagonalizable}\}$ is finite and has at most $2(n - 1)$ elements. \square

We can now state the following theorem:

THEOREM 2.11. *Let H be a $n \times n$ nonsingular Hankel matrix. Let $r(x) = x^n - b_{n-1}x^{n-1} - \dots - b_0$ and $s(x) = a_{n-1}x^{n-1} + \dots + a_0$, where $a = (a_0 a_1 \dots a_{n-1})^T$ and $b = (b_0 b_1 \dots b_{n-1})^T$ are such that $Ha = e_n$ and $Hb = (h_{n+1} \dots h_{2n-1} 0)^T$. Let $S = \{\alpha \in \mathbb{C} \mid (rs' - r's)(\alpha) = 0 \text{ and } s(\alpha) \neq 0\}$ and $T = \{r(\alpha)/s(\alpha) \mid \alpha \in S\}$.*

Then, for all $\gamma \in \mathbb{C} - T$, $H = V_\gamma D_\gamma V_\gamma^T$, where $V_\gamma = \text{vander}(\alpha_1, \dots, \alpha_n)$, $D_\gamma = \text{diag}(V_\gamma^{-1} H e_1)$, $\{\alpha_1, \dots, \alpha_n\} = \lambda(C_\gamma)$, and C_γ is the companion matrix whose last row is $(b_0 + \gamma a_0 \dots b_{n-1} + \gamma a_{n-1})$.

Proof. From proposition 2.10, for all $\gamma \in \mathbb{C} - T$, $\lambda(C_\gamma)$ is simple. Suppose $\{\alpha_1, \dots, \alpha_n\} = \lambda(C_\gamma)$. Let $v = (h_{n+1} \dots h_{2n-1} \gamma)^T$ and $H_1 = [H(2 : n, :); v]$. Then,

$$C_\gamma = H_1 H^{-1} = V_\gamma \text{diag}([\alpha_1, \dots, \alpha_n]) V_\gamma^{-1},$$

where $V_\gamma e_i = (1 \alpha_i \dots \alpha_i^{n-1})^T$, for all $i \in \{1, \dots, n\}$. So,

$$V_\gamma^{-1} H_1 = \text{diag}([\alpha_1, \dots, \alpha_n]) V_\gamma^{-1} H.$$

Let $d = (d_1 \dots d_n)^T = V_\gamma^{-1} H e_1$. Hence, for all $i \in \{1, \dots, n\}$,

$$V_\gamma^{-1} H e_i = \text{diag}([\alpha_1, \dots, \alpha_n])^{i-1} d = (d_1 \alpha_1^{i-1} \dots d_n \alpha_n^{i-1})^T = D_\gamma V_\gamma^T e_i.$$

□

3. Bézier curve as a Hankel form. Efficient methods to compute Bézier curves of degree $n - 1$ ([3]) are fundamental tools in Computed-Aided Geometric Design area. The Casteljau's algorithm is a widespread method for this computation. However, for each $s \in (0, 1)$ it demands $\mathcal{O}(n^2)$ multiplications. For n not very large, there are more efficient methods, like the ones introduced in [10], where the computation of points on these curves is carried out by generalized Ball curves, or the ones presented in [2], which use fast Pascal matrix-multiplication. Here we show that we can describe a Bézier curve as a Hankel form and, hence, we see that we can easily compute points of the curve from a Vandermonde factorization of the associated Hankel matrix.

Let $Q_0 = (x_0, y_0)$, $Q_1 = (x_1, y_1)$, ..., $Q_{n-1} = (x_{n-1}, y_{n-1})$ be n points in \mathbb{R}^2 . Bézier has his name on the curve B defined from these n points as follows:

$$B(s) = \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix} = \sum_{i=0}^{n-1} \binom{n-1}{i} s^i (1-s)^{n-1-i} Q_i, \quad s \in [0, 1].$$

Let $x = (x_0 \dots x_{n-1})^T$ and $y = (y_0 \dots y_{n-1})^T$. Then, for each $s \in [0, 1]$,

$$b_1(s) = e_n^T B_n^e(s) x \text{ and } b_2(s) = e_n^T B_n^e(s) y,$$

where $B_n^e(s)$ is a $n \times n$ Bernstein matrix. Thus, from lemma 2.4, for each $s \in [0, 1]$,

$$b_1(s) = e_n^T P_n G_n(s) P_n^{-1} x \text{ and } b_2(s) = e_n^T P_n G_n(s) P_n^{-1} y. \quad (3.1)$$

In the following, we discuss different approaches that make use of (3.1) to compute a Bézier curve.

We can notice that, if $B(s) = B_{Q_0 Q_1 \dots Q_{n-1}}(s)$ denotes the Bézier curve determined by the points Q_0, Q_1, \dots, Q_{n-1} , then

$$B(s) = (1-s) B_{Q_0 Q_1 \dots Q_{n-2}}(s) + s B_{Q_1 Q_2 \dots Q_{n-1}}(s).$$

Without loss of generality, from now on we will suppose that n , the number of control points of a Bézier curve, is odd: $n = 2m - 1$, $m > 1$. In this case, it is easy to conclude by induction that, for all $k = 0, \dots, m - 1$,

$$B(s) = \sum_{j=0}^k \binom{k}{j} (1-s)^{k-j} s^j B_{Q_j Q_{j+1} \dots Q_{j+n-k-1}}(s).$$

Particularly, for $k = m - 1$ we have

$$B(s) = \sum_{j=0}^{m-1} \binom{m-1}{j} (1-s)^{m-1-j} s^j B_{Q_j Q_{j+1} \dots Q_{j+m-1}}(s),$$

and so,

$$b_1(s) = \sum_{j=0}^{m-1} \binom{m-1}{j} (1-s)^{m-1-j} s^j e_m^T P_m G(t) P_m^{-1} x_{j \dots j+m-1},$$

$$b_2(s) = \sum_{j=0}^{m-1} \binom{m-1}{j} (1-s)^{m-1-j} s^j e_m^T P_m G(t) P_m^{-1} y_{j \dots j+m-1},$$

where $x_{j \dots j+m-1}$ and $y_{j \dots j+m-1}$ denote the column vectors $(x_j \dots x_{j+m-1})^T$ and $(y_j \dots y_{j+m-1})^T$, respectively, for $j = 0, \dots, m - 1$. However,

$$\begin{aligned} & \sum_{j=0}^{m-1} \binom{m-1}{j} (1-s)^{m-1-j} s^j e_m^T P_m G(t) P_m^{-1} x_{j \dots j+m-1} = \\ & = e_m^T P_m G(t) P_m^{-1} \left(\sum_{j=0}^{m-1} \binom{m-1}{j} (1-s)^{m-1-j} s^j x_{j \dots j+m-1} \right), \end{aligned}$$

and $\sum_{j=0}^{m-1} \binom{m-1}{j} (1-s)^{m-1-j} s^j x_{j \dots j+m-1}$ is a column vector whose i th coordinate is $e_m^T P_m G(t) P_m^{-1} x_{i-1 \dots m+i-2}$. Thus, we can state the following lemma, from which we can conclude that each coordinate of a Bézier curve is a Hankel form:

LEMMA 3.1. *Let $n = 2m - 1$, where m is an integer greater than 1 and let $B(s) = (b_1(s) \ b_2(s))^T$ be a Bézier curve of degree $n - 1$ defined from n points $Q_0 = (x_0, y_0)$, $Q_1 = (x_1, y_1)$, ..., $Q_{n-1} = (x_{n-1}, y_{n-1})$ in \mathbb{R}^2 . Then*

$$b_1(s) = e_m^T B_m^e(s) H_x (B_m^e(s))^T e_m \text{ and } b_2(s) = e_m^T B_m^e(s) H_y (B_m^e(s))^T e_m,$$

where $H_x = \text{hankel}(C_x, R_x)$ and $H_y = \text{hankel}(C_y, R_y)$ are $m \times m$ Hankel matrices whose first columns are $C_x = (x_0 \dots x_{m-1})^T$ and $C_y = (y_0 \dots y_{m-1})^T$ respectively, and whose last rows are $R_x = (x_{m-1}, \dots, x_{n-1})$ and $R_y = (y_{m-1}, \dots, y_{n-1})$ respectively.

COROLLARY 3.2. *Let $n = 2m - 1$, where m is an integer greater than 1. Let B be a Bézier curve of degree $n - 1$ defined from n control points, and let $x = (x_0 \dots x_{n-1})^T$ and $y = (y_0 \dots y_{n-1})^T$ be their respective vector of coordinates. Let $H_x = \text{hankel}(C_x, R_x)$*

and $H_y = \text{hankel}(C_y, R_y)$, where $C_x = (x_0 \dots x_{m-1})^T$, $R_x = (x_{m-1}, \dots, x_{n-1})$, $C_y = (y_0 \dots y_{m-1})^T$ and $R_y = (y_{m-1}, \dots, y_{n-1})$. If H_x and H_y are nonsingular, then there exist complex numbers d_1, \dots, d_n , t_1, \dots, t_n , $\hat{d}_1, \dots, \hat{d}_n$ and $\hat{t}_1, \dots, \hat{t}_n$ such that

$$b_1(s) = \sum_{i=1}^m d_i (1 - s + s.t_i)^{n-1} \text{ and } b_2(s) = \sum_{i=1}^m \hat{d}_i (1 - s + s.\hat{t}_i)^{n-1}. \quad (3.2)$$

Proof. If H_x is nonsingular, from theorem 2.11, there is a Vandermonde matrix $V = \text{vander}([t_1, \dots, t_n])$ and a diagonal matrix $D = \text{diag}([d_1, \dots, d_n])$ such that $H_x = VDV^T$. So,

$$\begin{aligned} b_1(s) &= e_m^T B_m^e(s) H_x (B_m^e(s))^T e_m = e_m^T B_m^e(s) V D V^T (B_m^e(s))^T e_m = \\ &= \sum_{i=1}^m d_i (1 - s + s.t_i)^{2m-2} = \sum_{i=1}^m d_i (1 - s + s.t_i)^{n-1}, \end{aligned}$$

for $e_m^T B_m^e(s) V e_i = \sum_{j=0}^{m-1} (1-s)^{m-1-j} .s^j .t_i^j = (1-s+s.t_i)^{m-1}$ for all $i \in \{1, \dots, m\}$. In an analogous way, we conclude that

$$b_2(s) = \sum_{i=1}^m \hat{d}_i (1 - s + s.\hat{t}_i)^{n-1},$$

for some $\hat{d}_1, \dots, \hat{d}_n$ and $\hat{t}_1, \dots, \hat{t}_n$. \square

The following proposition is about another representation of a Bézier curve of degree $n - 1$.

PROPOSITION 3.3. *Let $n = 2m - 1$, where m is an integer greater than 1 and let $B(s) = (b_1(s) b_2(s))^T$ be a Bézier curve of degree $n - 1$ defined from n points $Q_0 = (x_0, y_0)$, $Q_1 = (x_1, y_1)$, \dots , $Q_{n-1} = (x_{n-1}, y_{n-1})$ of \mathbb{R}^2 . Then*

$$b_1(s) = \sum_{k=0}^{n-1} a_k \binom{n-1}{k} s^k \text{ and } b_2(s) = \sum_{k=0}^{n-1} b_k \binom{n-1}{k} s^k, \text{ where}$$

$$\begin{pmatrix} a_0 & \dots & a_{n-1} \\ \vdots & \ddots & \vdots \\ a_{m-1} & \dots & a_{n-1} \end{pmatrix} = P_m^{-1} H_x P_m^{-T} \text{ and } \begin{pmatrix} b_0 & \dots & b_{n-1} \\ \vdots & \ddots & \vdots \\ b_{m-1} & \dots & b_{n-1} \end{pmatrix} = P_m^{-1} H_y P_m^{-T}.$$

Proof. Let $A = P_m^{-1} H_x P_m^{-T}$ and $B = P_m^{-1} H_y P_m^{-T}$. From lemma 3.1, it follows that

$$b_1(s) = e_m^T P_m G(s) A G(s) P_m^T e_m \text{ and } b_2(s) = e_m^T P_m G(s) B G(s) P_m^T e_m.$$

Now, $e_m^T P_m G(s) = \left(\binom{m-1}{0} \binom{m-1}{1} s \dots \binom{m-1}{m-1} s^{m-1} \right)$. Therefore,

$$b_1(s) = \sum_{k=0}^{n-1} a_k \left(\sum_{j=0}^k \binom{m-1}{j} \binom{m-1}{k-j} \right) s^k,$$

$$b_2(s) = \sum_{k=0}^{n-1} b_k \left(\sum_{j=0}^k \binom{m-1}{j} \binom{m-1}{k-j} \right) s^k,$$

and the conclusion now follows from Vandermonde convolution

$$\sum_{j=0}^k \binom{m-1}{j} \binom{m-1}{k-j} = \binom{2(m-1)}{k} = \binom{n-1}{k}.$$

□

We have just proved a property of the Pascal matrix-vector multiplication, which is remarked in the following corollary:

COROLLARY 3.4. *Let $n = 2m - 1$, where $m \geq 1$, let $x = (x_0 \dots x_{n-1})^T$ be a vector of \mathbb{C}^n and let H_x the Hankel matrix defined by $(H_x)_{ij} = x_{i+j-2}$. Then, $a = P_n x$, where $a = (a_0 \dots a_{n-1})^T$ is such that $(P_m H_x P_m^T)_{ij} = a_{i+j-2}$.*

Proof. Let $y = G_n(-1)x$. Let $a = P_n x = G_n(-1)P_n^{-1}G_n(-1)x$. Then,

$$e_n^T P_n G_n(s) P_n^{-1} y = e_n^T P_n G_n(s) G_n(-1) a = \sum_{k=0}^{n-1} (-1)^k a_k \binom{n-1}{k} s^k.$$

On the other hand, from proposition 3.3, $(-1)^{i+j-2} a_{i+j-2} = (P_m^{-1} H_y P_m^{-T})_{ij}$. Therefore, $a_{i+j-2} = (P_m H_x P_m^T)_{ij}$. □

4. Numerical experiments. We are going to compare Bézier curves of degree $(n-1)$ computed from the classical Casteljaou's algorithm as well as from two other descriptions of the curve: as a Hankel form and by using the spectral decomposition $B_n^e(s) = P_n G_n(s) P_n^{-1}$. We first observe that an uniform scaling of the control points of a Bézier curve yields an uniform scaling of the curve and if those points are translated by a vector $v = (p, q)$, then the Bézier curve is also translated by v . Hence, without loss of generality, we are going to assume that the coordinates of the control points are all real positive and also less than or equal to 1. So, we are going to use the MATLAB function *rand* to generate n test control points: $A = \text{rand}(n, 2)$.

The Casteljaou's algorithm is a very accurate algorithm to evaluate Bézier curves, for it is based on a numerically stable Bernstein matrix-vector multiplication:

Algorithm 1 Casteljaou's algorithm

```

n = length(x);
x = x(:);
ss = 1 - s;
for k = 2 : n do
  for t = n : -1 : k do
    x(t) = ss*x(t-1) + s*x(t);
  end for
end for
b(s) = x(n)

```

This multiplication can be seen as a sequence of bi-diagonal matrix-vector multiplications, which becomes well explicit from the following lemma [2]:

LEMMA 4.1. *Let $B_n^e(t)$ be a $n \times n$ Bernstein matrix. Then*

$$B_n^e(s) = E_{n-1}^e(s) \dots E_1^e(s) \text{ where, for } 1 \leq k \leq n-1,$$

$$E_k^e(s) = e_1 e_1^T + \dots + e_k e_k^T + e_{k+1} [(1-s)e_k + s e_{k+1}]^T + \dots + e_n [(1-s)e_{n-1} + s e_n]^T.$$

Another way of calculating a Bézier curve is from its description as a Hankel form, which allows us to utilize a Vandermonde factorization of the associated Hankel matrix, and its algorithm is as follows: We are supposing here that H^x and H^y

Algorithm 2 Bézier curve as a Hankel form

- given $n = 2m - 1$ distinct points of \mathbb{R}^2 , let H^x and H^y be two $m \times m$ Hankel matrices formed from their coordinates;
 - choose a number γ at random and define the vectors $x_\gamma = (h_{m+1}^x \dots h_n^x \gamma)^T$ and $y_\gamma = (h_{m+1}^y \dots h_n^y \gamma)^T$;
 - solve the systems $H^x z_\gamma = x_\gamma$ and $H^y w_\gamma = y_\gamma$ and consider the companion matrices C_{z_γ} and C_{w_γ} ;
 - find the spectra of C_{z_γ} and C_{w_γ} ;
 - define $d_x = (V_\gamma^x)^{-1} H^x e_1$ and $d_y = (V_\gamma^y)^{-1} H^y e_1$, where V_γ^x and V_γ^y are Vandermonde matrices formed from the spectrum of C_{z_γ} and from the spectrum C_{w_γ} , respectively;
 - for each $s \in [0, 1]$, the Bézier curve $B(s)$ is then defined from the equation (3.2).
-

are both nonsingular and that γ is not one of those numbers which yield in non-diagonalizable companion matrices.

The third way of computing a Bézier curve will be carried out by a Pascal matrix method, which computes a Bézier curve $B(s)$ of degree $n - 1$ via the decomposition $B_n^e(s) = P_n G_n(-s) P_n G_n(-1)$:

Algorithm 3 Pascal matrix algorithm

- given n , take $t \geq 1$ such that $P_n(t)$ is similar to a lower triangular Toeplitz matrix $T = T(t)$ with maximum $\min T_{ij} / \max T_{ij}$;
 - multiply $z = P_n G_n(-1)x = P_n x_-$ and $w = P_n G_n(-1)y = P_n y_-$ via fast Toeplitz matrix-vector multiplication;
 - from a Horner-like scheme, evaluate the polynomials $e_n^T P_n G_n(-s)z$ and $e_n^T P_n G_n(-s)w$.
-

We have used a fast Pascal matrix-vector multiplication done from the similar Toeplitz matrix $T(t)$ (see [11]), where t has been found by a procedure described in [2], plus the $B(s)$ evaluation given by a Horner-like scheme that evaluates the polynomial concomitantly with the binomial coefficients. Since the $B(s)$ -evaluation becomes unstable when s approaches to 1, we have introduced a simple procedure to improve the evaluation, that is to divide the process of evaluation in two independent steps:

- (a) evaluate $e_n^T P_n G_n(-s)z$ and $e_n^T P_n G_n(-s)w$ for $0 \leq s \leq 1/2$;

- (b) evaluate $e_n^T P_n G_n(-s) z_r$ and $e_n^T P_n G_n(-s) w_r$ for $1/2 > s \geq 0$, which is equivalent to evaluate $e_n^T P_n G_n(-s) z$ and $e_n^T P_n G_n(-s) w$ for $1/2 < s \leq 1$.

TABLE 4.1

Mean run time of computation of 129 points of a Bézier curve of degree $N-1$ by three different methods: Casteljau's (C), Hankel form (H) and direct Pascal matrix method (P). The results of the second and third methods are compared to the ones obtained by Casteljau's via norm of the difference of the computed points by the respective method and by Casteljau's.

N	Time (Casteljau)	Time (Hankel)	Time (Pascal)	$\ B_C - B_H\ $	$\ B_C - B_P\ $
15	0.005s	0.007s	0.004s	1.3399e-13	2.6782e-12
23	0.009s	0.009s	0.004s	1.0540e-11	2.2427e-09
31	0.015s	0.011s	0.005s	2.3082e-09	1.4962e-06
39	0.022s	0.016s	0.006s	9.7593e-11	5.6283e-04
47	0.030s	0.019s	0.006s	6.6642e-05	0.1035
55	0.040s	0.023s	0.007s	4.9873e-08	457.2366
63	0.053s	0.027s	0.007s	1.8852e-05	2.2703e+04
71	0.066s	0.029s	0.008s	6.0574e-07	1.7485e+07
79	0.082s	0.036s	0.009s	1.0117e-06	5.6499e+09

4.1. Conditioning a Hankel matrix. It is not rare $n = 2m - 1$ numbers taken in the interval $[0,1]$ at random result in an ill-conditioned $m \times m$ Hankel matrix H . A simple way of handling this is to shift its skew diagonal in order to turn it into a skew-diagonal dominant matrix, $\tilde{H} = H + \sigma C$, where C is the reciprocal matrix. Let B_H and $B_{\tilde{H}}$ be the Bézier curves corresponding to H and \tilde{H} , respectively. Then, for each $s \in [0,1]$, we compute $B_H(s)$ by subtracting σ times $e_m^T B_m^e(s) C_m (B_m^e(s))^T e_m$ from $B_{\tilde{H}}$. Moreover, this quadratic form has a simple formulation as can be seen in the next lemma.

LEMMA 4.2. Let $C_m = \text{hankel}(e_m, e_1^T)$, which is called the reciprocal matrix. Then, if $w = e^{2\pi i/m}$,

$$e_m^T B_m^e(s) C_m (B_m^e(s))^T e_m = \frac{1}{m} \sum_{j=1}^m w^{j-1} (1 - s + s.w^{j-1})^{n-1}.$$

Proof. It is easy to see that $C_m = V D V^T$, where $V = \text{vander}(1, w, \dots, w^{m-1})$ and $D = \text{diag}(1/m, w/m, \dots, w^{m-1}/m)$. From the proof of Corollary 3.2,

$$e_m^T B_m^e(s) V D V^T (B_m^e(s))^T e_m = \frac{1}{m} \sum_{j=1}^m w^{j-1} (1 - s + s.w^{j-1})^{n-1}.$$

□

In table 4.2, we can see that this simple technique of preconditioning have improved the computation of Bézier curves when their control points yield ill-conditioned Hankel matrices (cond(H) is the maximum condition number of the two Hankel matrices formed by the coordinates of the control points). For each Hankel matrix H , σ was taken as the sum of the absolute values of its entries. Since our Vandermonde factorization of a Hankel matrix depends on a value chosen at random, the error between the curve computed by Casteljau's and the one computed from that factorization varied enormously when the Hankel matrices associated with the coordinates were ill-conditioned. In table 4.2, for each n , we can see the maximum error among

several experiments done. However, sometimes it happened to have a big error followed by a tiny one. Notice that all our experiments have been run in a 32-bits AMD Athlon XP 1700+ (1467 MHz).

TABLE 4.2

Mean run time of computation of 129 points of a Bézier curve of degree $N-1$ by three different methods: Casteljau's (C), Hankel form (H) and preconditioning Hankel form (PH). The results of the second and third methods are compared to the ones obtained by Casteljau's via norm of the difference of the computed points by the respective method and by Casteljau's.

N	cond(H)	Time (C)	Time (H)	Time (PH)	$\ B_C - B_H\ _2$	$\ B_C - B_{PH}\ _2$
31	1.5379e+03	0.015s	0.012s	0.019s	3.3983e+11	2.9510e-11
39	760.4605	0.022s	0.016s	0.024s	2.6541e+09	1.1134e-10
47	2.9956e+03	0.030s	0.019s	0.031s	1.3731e+13	1.0189e-10
55	577.1450	0.041s	0.023s	0.036s	4.3750e+03	1.7107e-08
63	4.2415e+03	0.053s	0.027s	0.042s	9.8000e+70	2.5894e-08
71	907.6247	0.066s	0.029s	0.049s	3.7813e+06	3.2318e-07
79	1.1167e+03	0.081s	0.033s	0.057s	4.1314e+04	2.1604e-05

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